

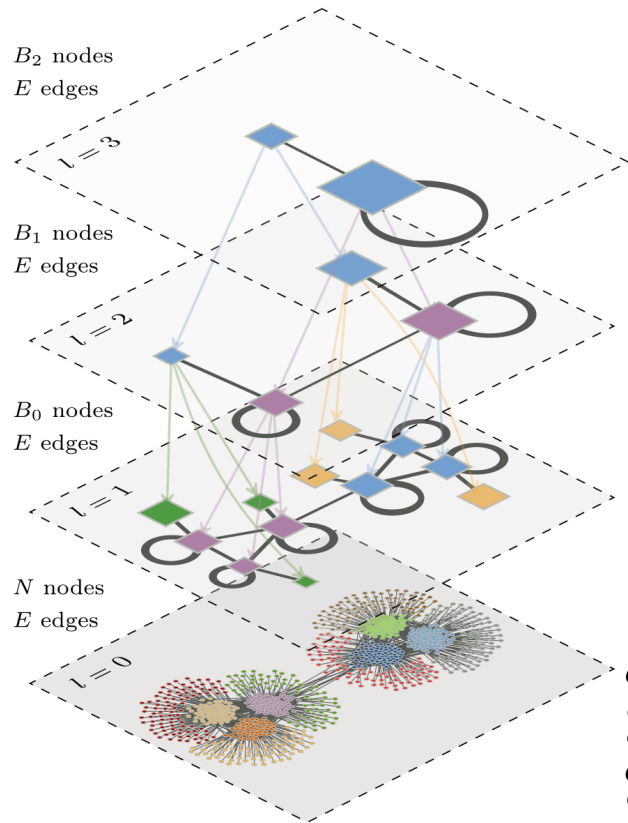
Efficient detection of hierarchical block structures in networks

Leto Peel

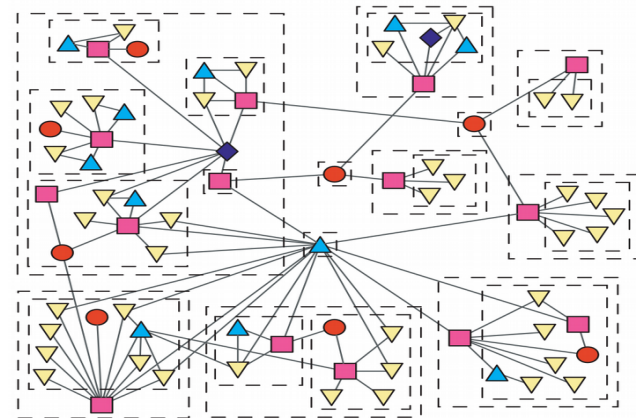
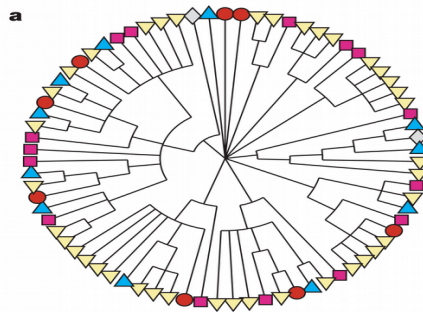
Université catholique de Louvain

@PiratePeel

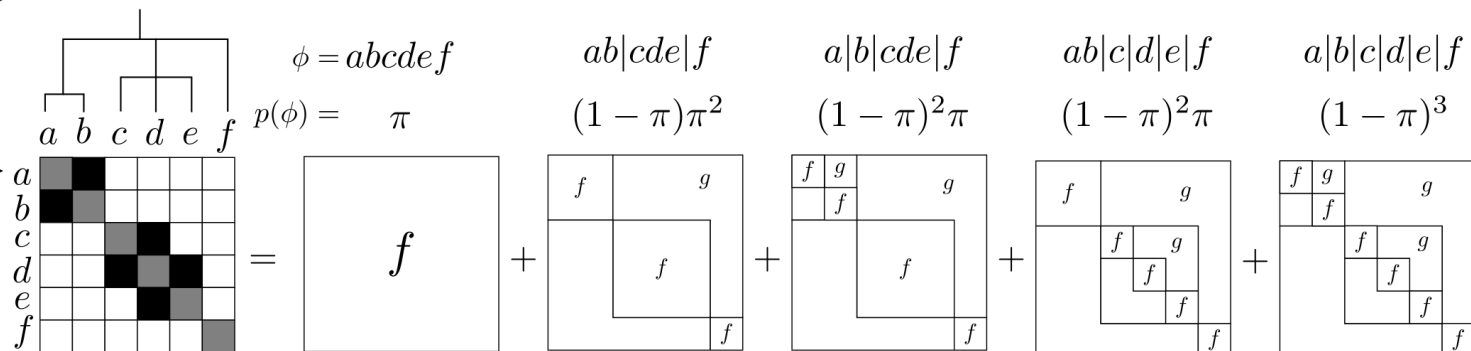
Hierarchical block structures



[Peixoto 2014]



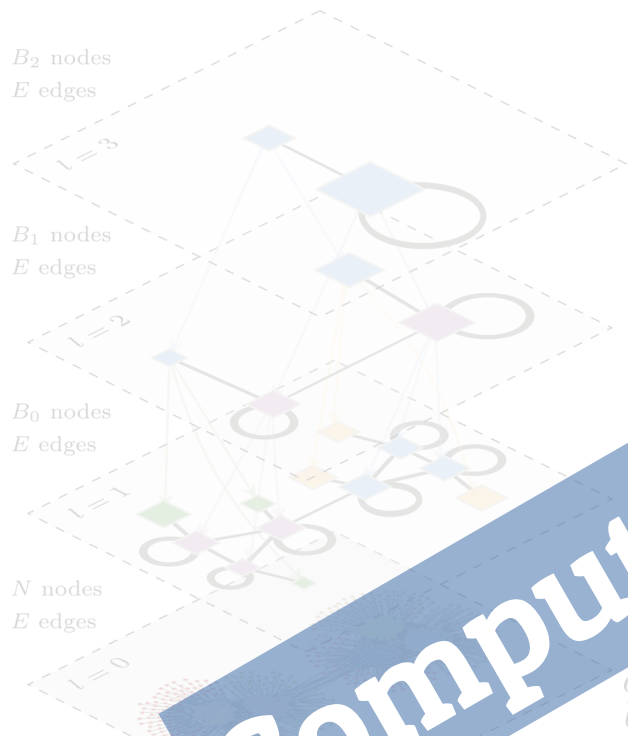
[Clauset, Moore & Newman 2008]



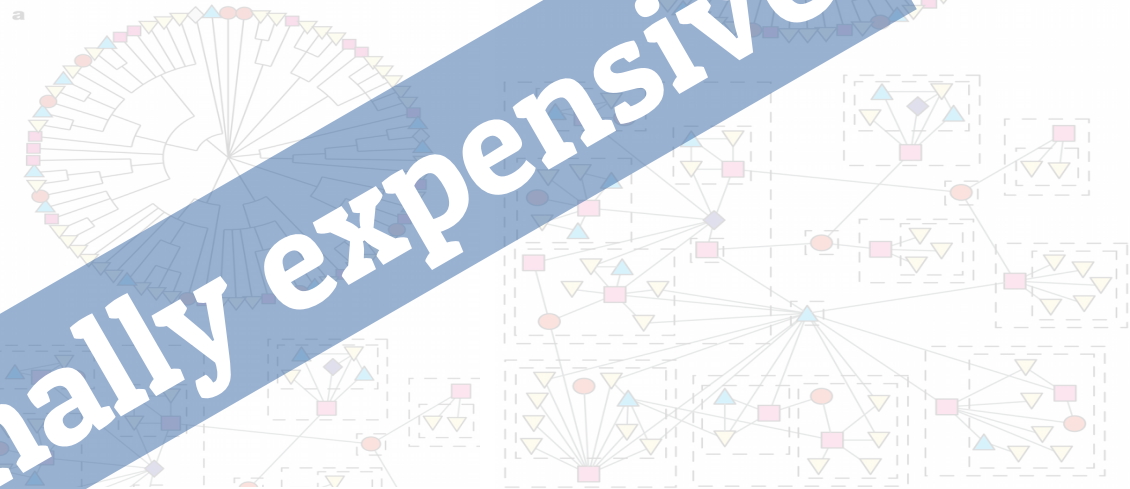
[Blundell & Teh 2013]

Hierarchical block structures

Computationally expensive!



[Peixoto 2014]



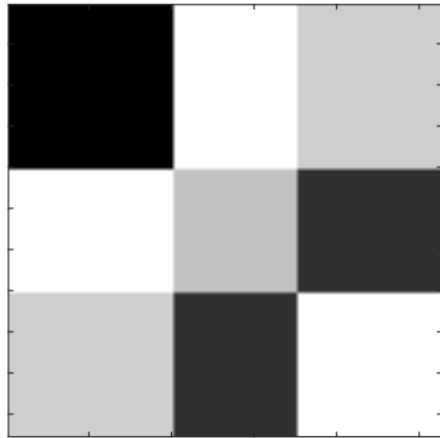
[Clauset, Moore & Newman 2008]

$$\begin{array}{c}
 \phi = abcdef \\
 p(\phi) = \pi
 \end{array}
 \quad
 \begin{array}{c}
 ab|cde|f \\
 (1-\pi)\pi^2
 \end{array}
 \quad
 \begin{array}{c}
 a|b|cde|f \\
 (1-\pi)^2\pi
 \end{array}
 \quad
 \begin{array}{c}
 ab|c|d|e|f \\
 (1-\pi)^2\pi
 \end{array}
 \quad
 \begin{array}{c}
 a|b|c|d|e|f \\
 (1-\pi)^3
 \end{array}$$

$$\begin{array}{c}
 a \quad b \quad c \quad d \quad e \quad f \\
 a \quad \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array} \\
 b \quad \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array} \\
 c \quad \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array} \\
 d \quad \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array} \\
 e \quad \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array} \\
 f \quad \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}
 \end{array}
 =
 \begin{array}{|c|c|c|c|c|c|} \hline f & & & & & \\ \hline \end{array}
 +
 \begin{array}{|c|c|c|c|c|c|} \hline f & & g & & & \\ \hline & f & & & & \\ \hline & & f & & & \\ \hline & & & f & & \\ \hline & & & & f & \\ \hline & & & & & f \\ \hline \end{array}
 +
 \begin{array}{|c|c|c|c|c|c|} \hline f & g & & & & \\ \hline f & f & & & & \\ \hline & & f & & & \\ \hline & & & f & & \\ \hline & & & & f & \\ \hline & & & & & f \\ \hline \end{array}
 +
 \begin{array}{|c|c|c|c|c|c|} \hline f & & g & & & \\ \hline & f & g & & & \\ \hline & & f & f & & \\ \hline & & & f & & \\ \hline & & & & f & \\ \hline & & & & & f \\ \hline \end{array}
 +
 \begin{array}{|c|c|c|c|c|c|} \hline f & g & & & & \\ \hline f & f & & & & \\ \hline & & f & g & & \\ \hline & & & f & f & \\ \hline & & & & f & \\ \hline & & & & & f \\ \hline \end{array}$$

[Blundell & Teh 2013]

The Stochastic Blockmodel (SBM)

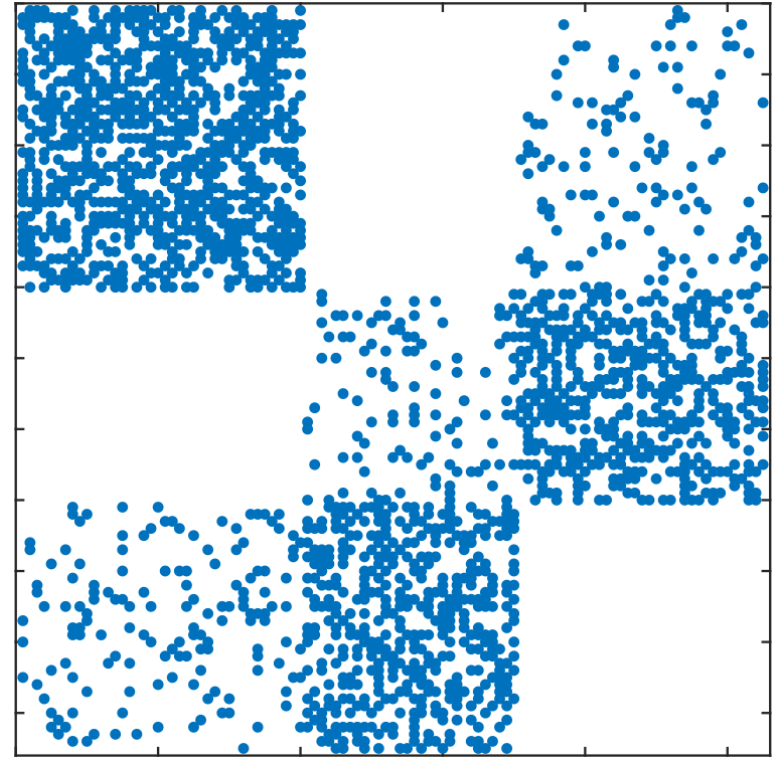


Mixing Matrix

generation

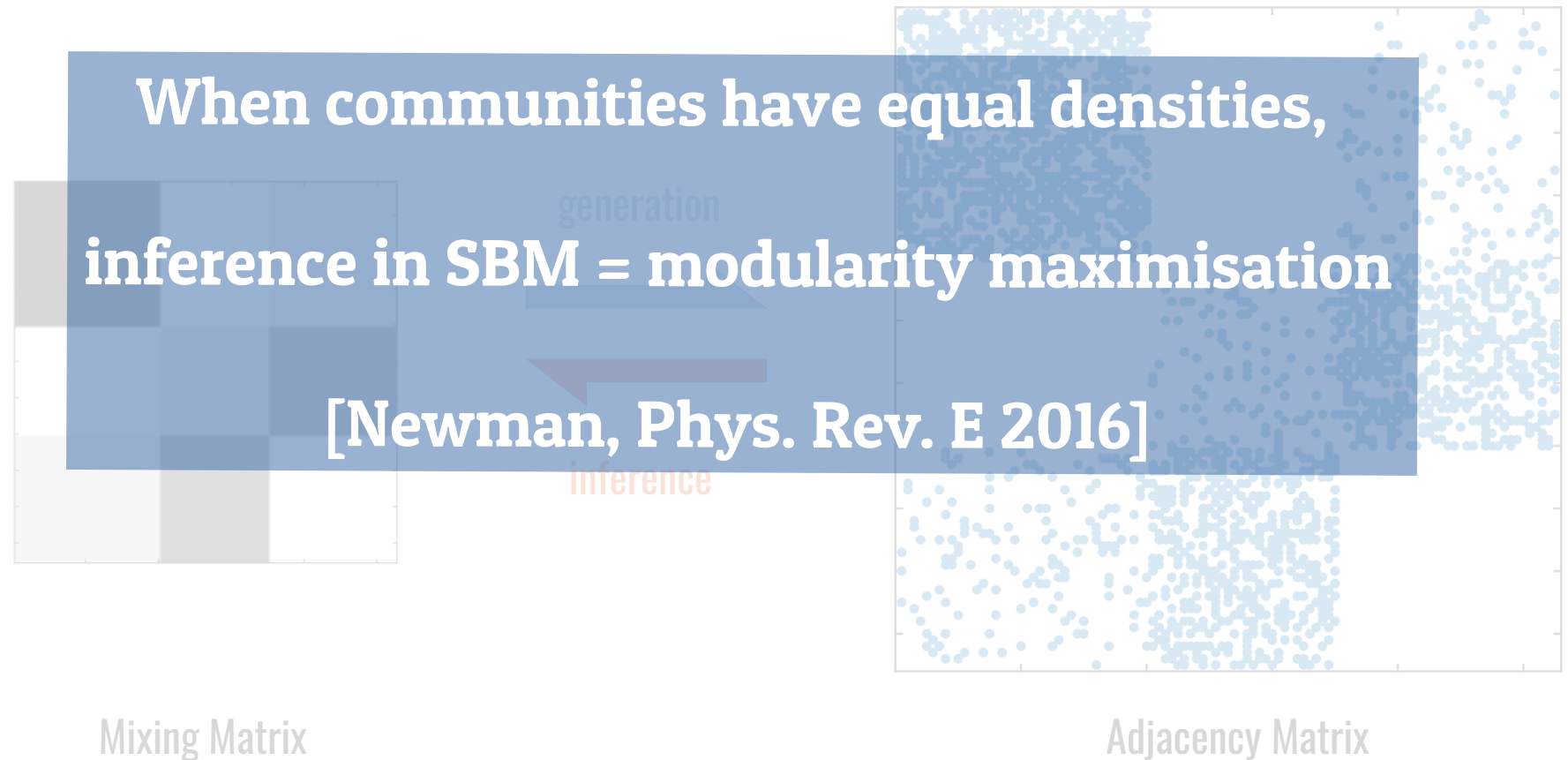


inference



Adjacency Matrix

The Stochastic Blockmodel (SBM)



Spectral methods of inference

Adjacency matrix [Donath and Hoffmann 1972]

Laplacian [Fiedler 1973]

Modularity Matrix -- e.g., [Newman 2006], [Nadakuditi & Newman 2012]

Non-Backtracking matrix --- e.g., [Krzakala et al 2013]

Regularized spectral clustering --- e.g., [Rohe 2011], [Le et al 2016]

Bethe Hessian --- [Saade et al. 2014], [Le & Levina 2015]

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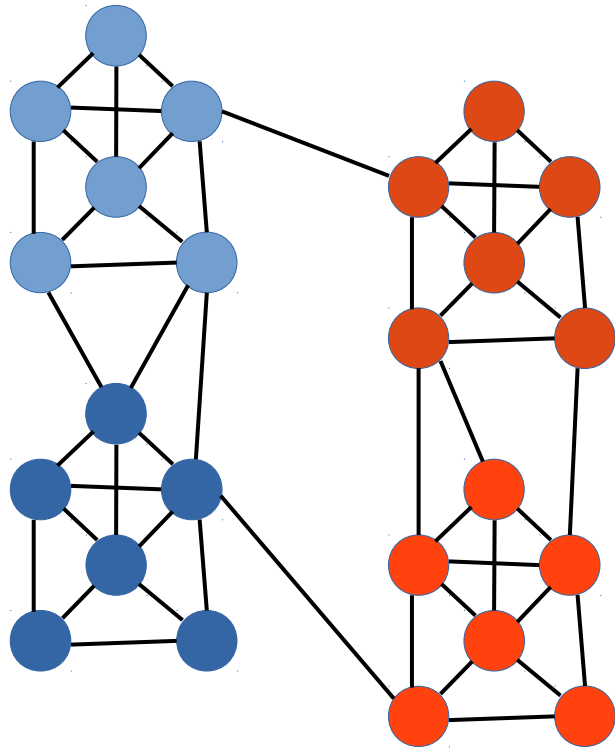
Non-Backtracking matrix --- e.g., [Krzakala et al 2013]

Regularized spectral clustering --- e.g., [Rohe 2011], [Le et al 2016]

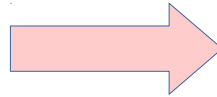
Bethe Hessian --- [Saade et al. 2014], [Le & Levina 2015]

The logo consists of the letters "SBM" in a bold, white, sans-serif font, centered within a solid blue rectangular background.

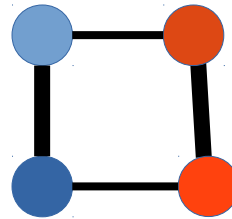
Building the hierarchy



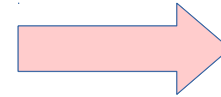
Observed network



infer blocks



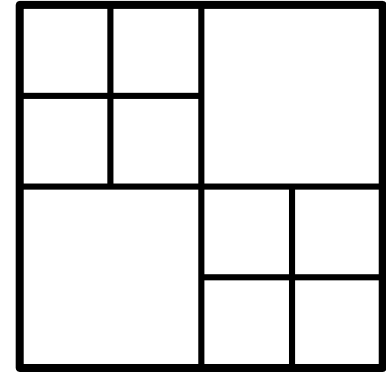
Multigraph



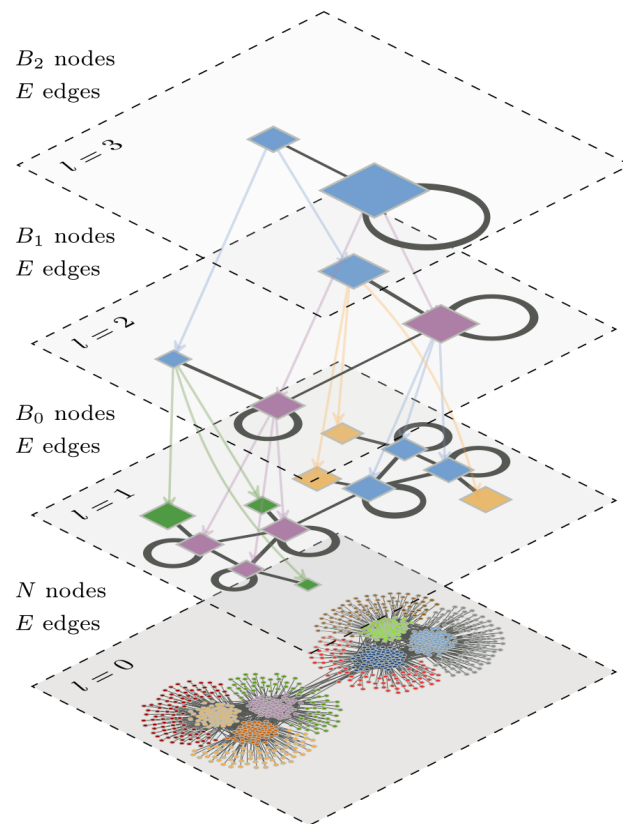
infer blocks



Multigraph



**Similar to something
we've seen before...**



[Peixoto 2014]

Detecting hierarchies(?)



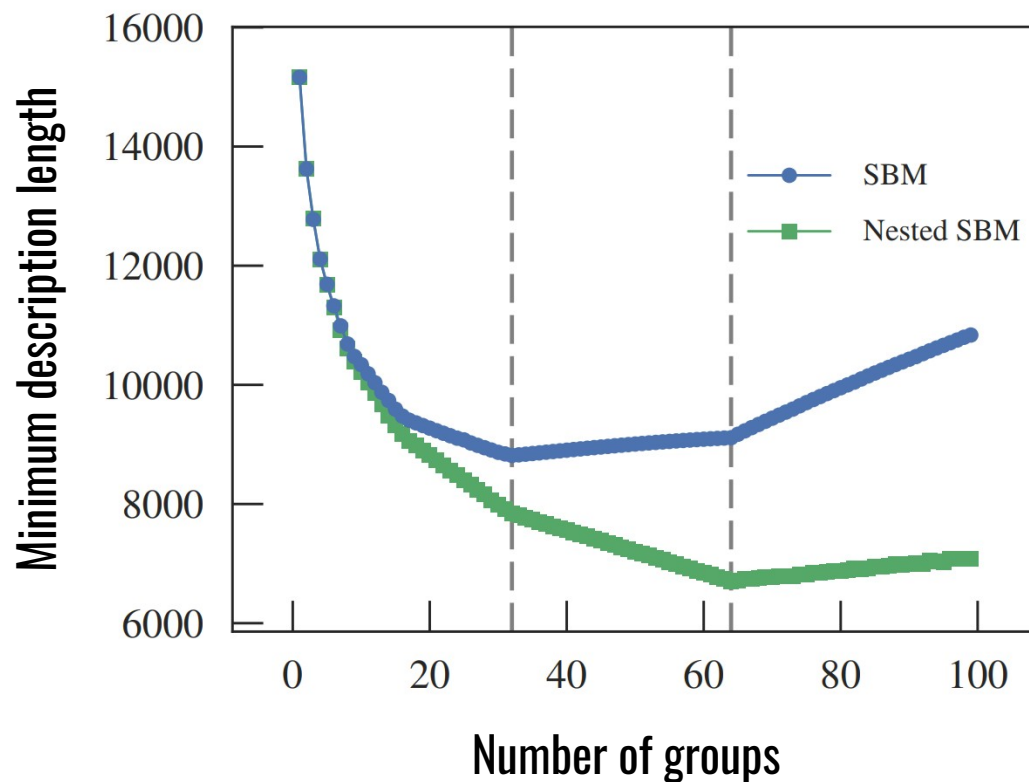
64 cliques of 10 nodes

[Peixoto 2017]

Detecting hierarchies(?)



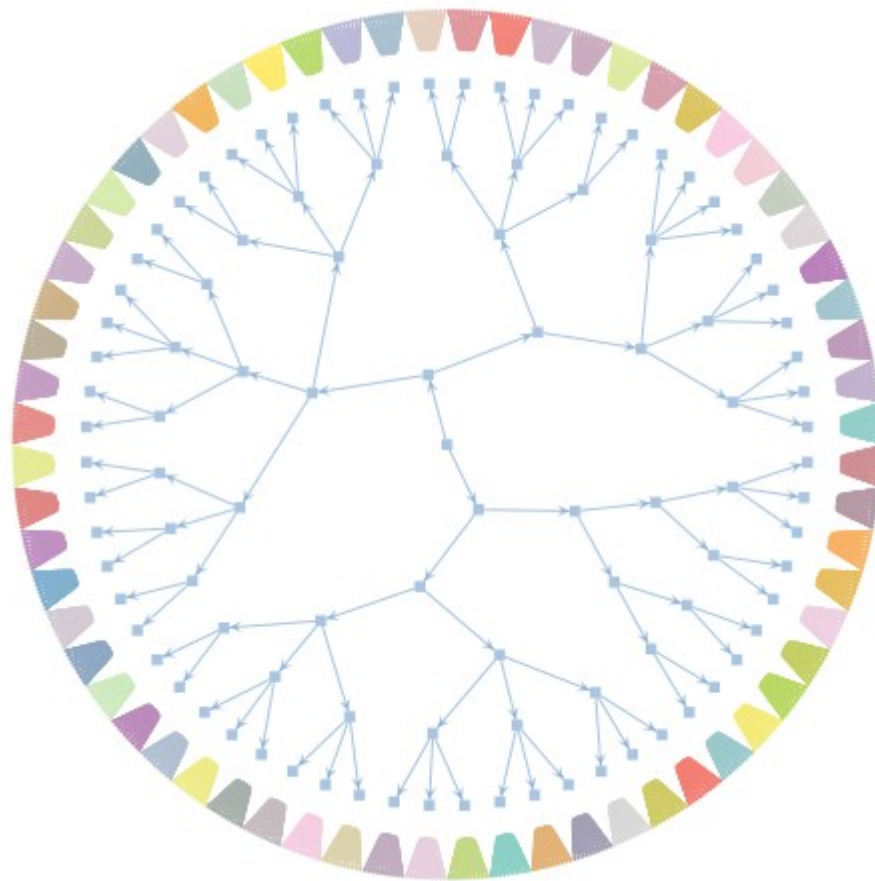
[Peixoto 2017]



Detecting hierarchies(?)



[Peixoto 2017]



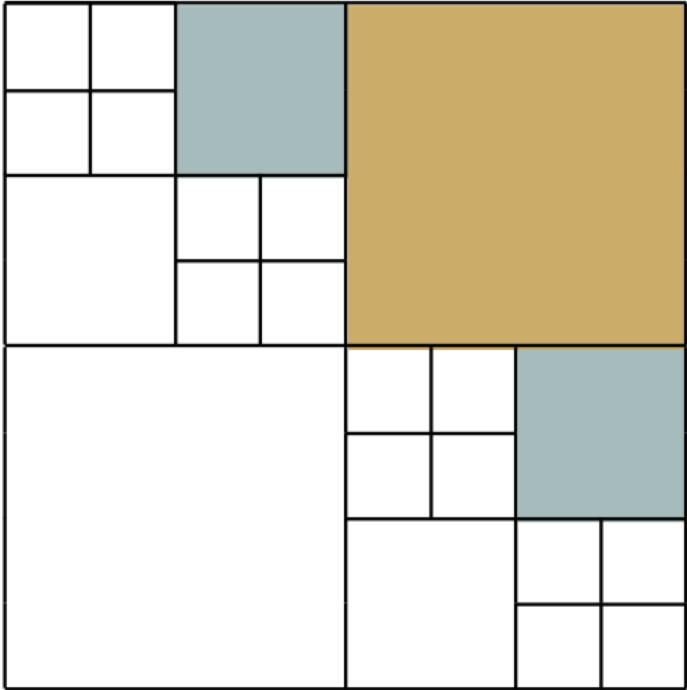
Detecting hierarchies(?)

Hierarchical model \neq Hierarchical structure

[Peixoto 2017]

Generate a simple hierarchy...

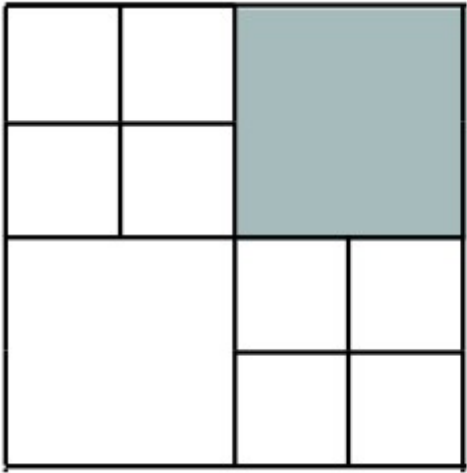
$\mathbb{E}[A] =$



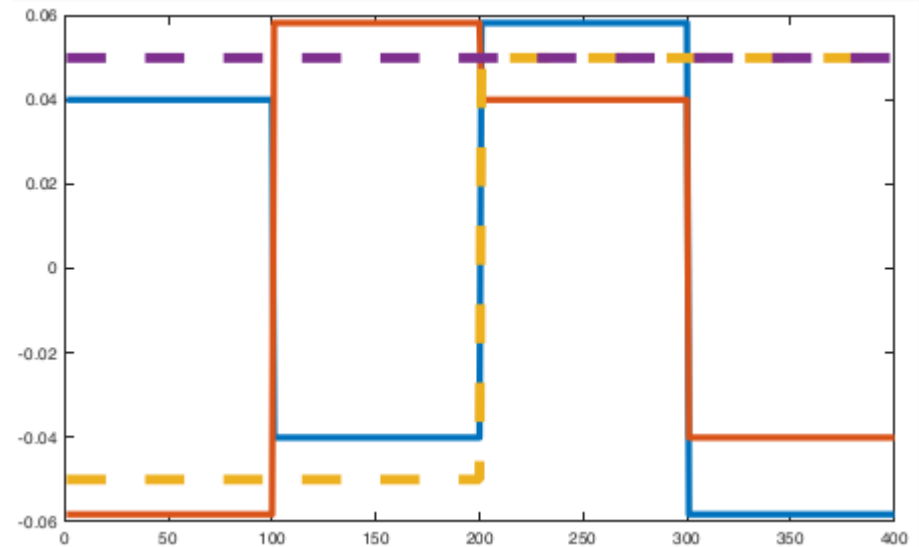
Constant probability between groups at each hierarchical level

Spectral properties

$$\mathbb{E}[A]$$

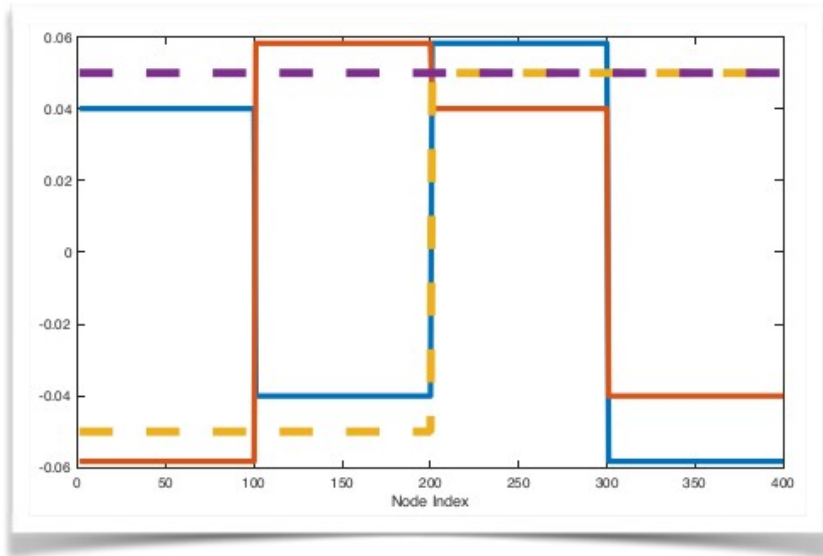


First 4 Eigenvectors of the Laplacian

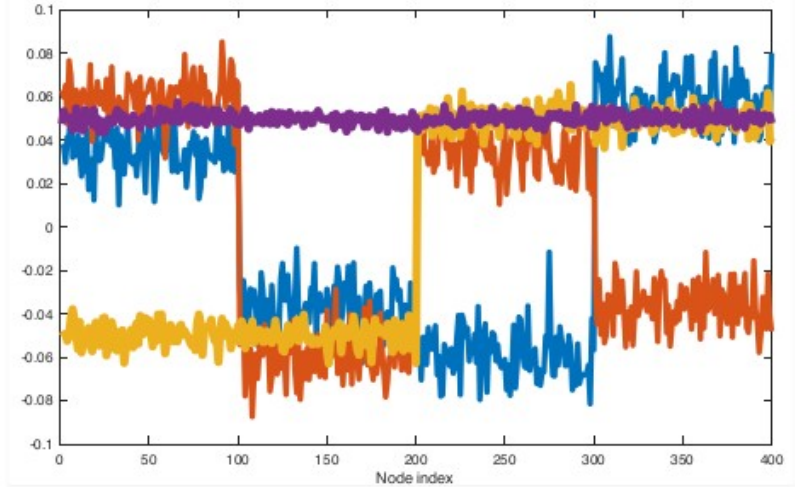
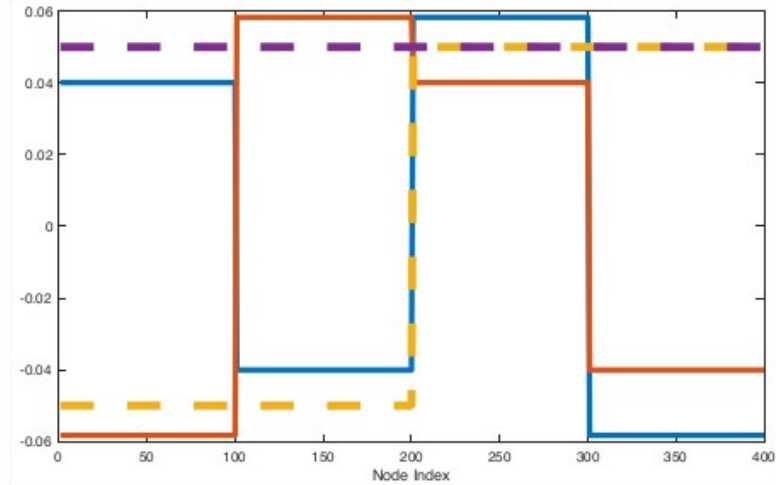


Node index

If we could “see” the expected adjacency matrix,
then we could just look for constant eigenvectors



If we could “see” the expected adjacency matrix,
then we could just look for constant eigenvectors



Theorem : The regularized Laplacian concentrates around its expectation with high probability

[Rohe et al 2011, Le et al 2016...]

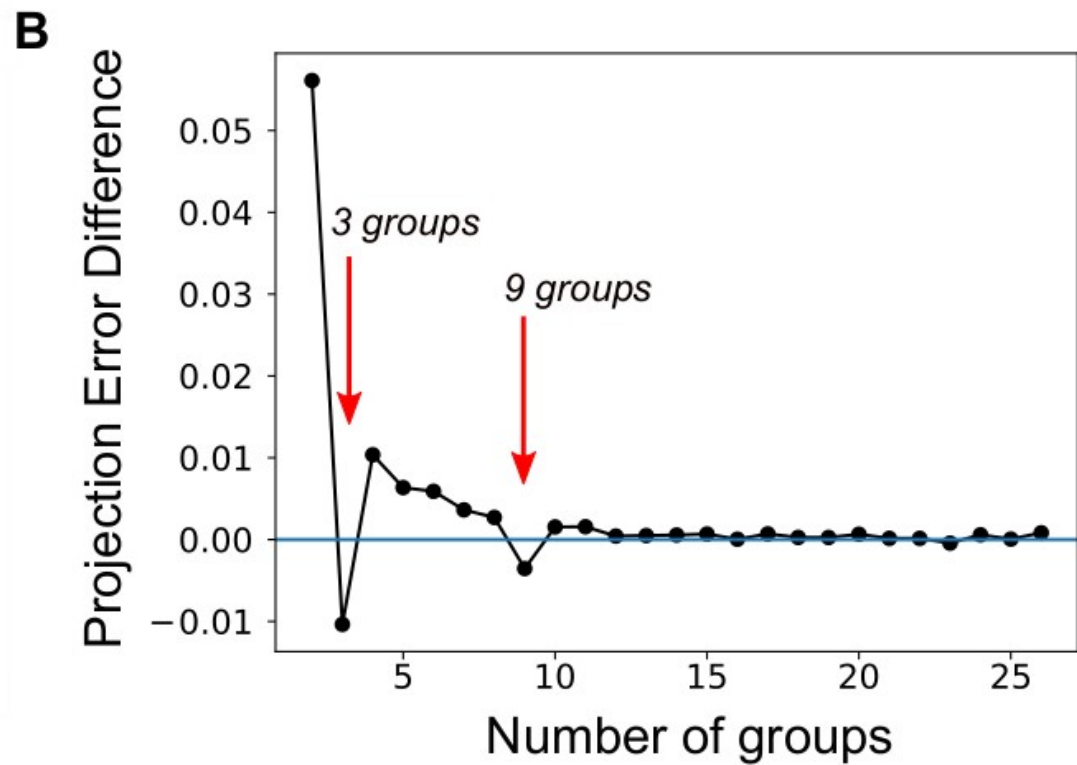
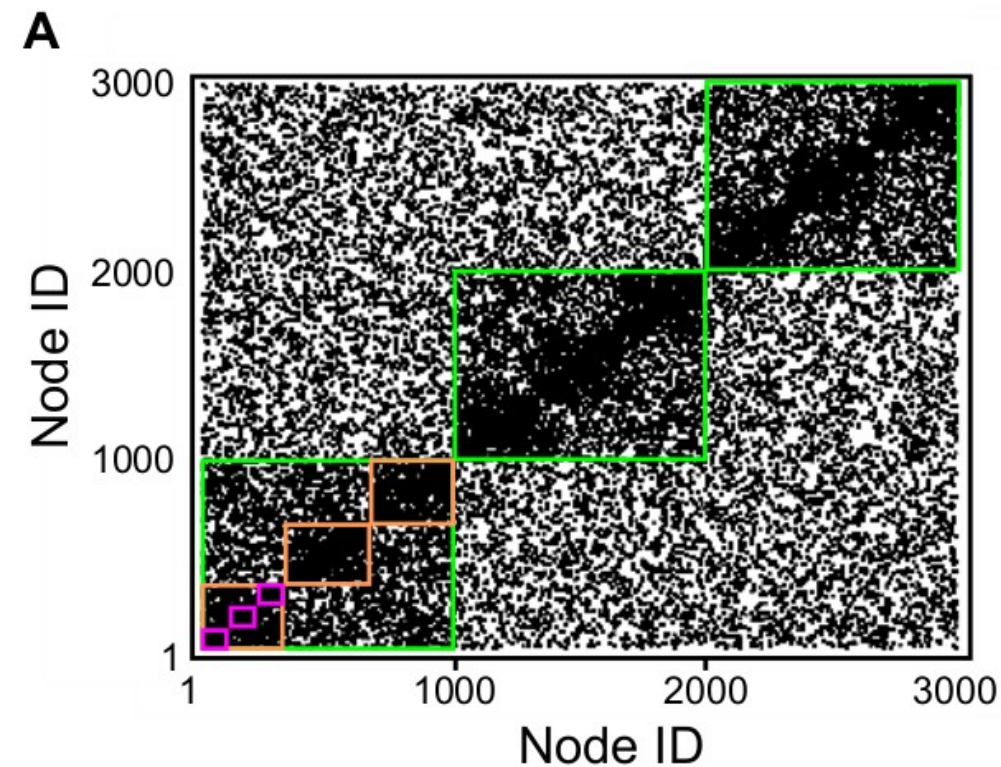
Detecting hierarchies, spectrally

$$\|\mathcal{L}(A) - \mathbb{E}[\mathcal{L}(A)]\| \leq \delta(\theta)$$

For each value of k (starting from k_{\max}):

- 1) Perform spectral clustering for k
- 2) Check if eigenvectors are \sim constant on each group
- 3) Small error = good candidate for agglomeration

Example: 3 x 3 hierarchy



Take home messages...

- The spectral approach
 - fast (scalable) algorithm
- Framed within a generative model – hier. SBM
 - interpretable, explicit assumptions
- **Hierarchical model \neq Hierarchical structure**

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Tue 12:30 Room 1

“Graph-based semi-supervised learning for complex networks”

Wed 16:30 Room 10

“Multiscale mixing patterns in networks”

Thur 12:10 Room 3

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