



Science and  
Technology  
Facilities Council



# GALAHAD

# EQP

USER DOCUMENTATION

GALAHAD Optimization Library version 3.3

## 1 SUMMARY

This package uses an iterative method to solve the **equality-constrained quadratic programming problem**

$$\text{minimize } \frac{1}{2} \mathbf{x}^T \mathbf{H} \mathbf{x} + \mathbf{g}^T \mathbf{x} + f \quad (1.1)$$

subject to the linear constraints

$$\mathbf{A} \mathbf{x} + \mathbf{c} = \mathbf{0}, \quad (1.2)$$

where the  $n$  by  $n$  symmetric matrix  $\mathbf{H}$ , the  $m$  by  $n$  matrix  $\mathbf{A}$ , the vectors  $\mathbf{g}$  and  $\mathbf{c}$ , and the scalar  $f$  are given. Full advantage is taken of any zero coefficients in the matrices  $\mathbf{H}$  and  $\mathbf{A}$ .

The package may alternatively be used to minimize the (shifted) squared-least-distance objective

$$\frac{1}{2} \sum_{j=1}^n w_j^2 (x_j - x_j^0)^2 + \mathbf{g}^T \mathbf{x} + f, \quad (1.3)$$

subject to the linear constraints (1.2), for given vectors  $\mathbf{w}$  and  $\mathbf{x}^0$ .

**ATTRIBUTES — Versions:** GALAHAD\_EQP\_single, GALAHAD\_EQP\_double. **Uses:** GALAHAD\_CLOCK, GALAHAD\_SYMBOLS, GALAHAD\_SPACE, GALAHAD\_QPD, GALAHAD\_QPT, GALAHAD\_FDC, GALAHAD\_SBLS, GALAHAD\_GLTR, GALAHAD\_SPECFILE. **Date:** March 2006. **Origin:** N. I. M. Gould, Rutherford Appleton Laboratory. **Language:** Fortran 95 + TR 15581 or Fortran 2003. **Parallelism:** Some options may use OpenMP and its runtime library.

## 2 HOW TO USE THE PACKAGE

Access to the package requires a USE statement such as

*Single precision version*

```
USE GALAHAD_EQP_single
```

*Double precision version*

```
USE GALAHAD_EQP_double
```

If it is required to use both modules at the same time, the derived types QPT\_problem\_type, EQP\_time\_type, EQP\_control\_type, EQP\_inform\_type and EQP\_data\_type (Section 2.4) and the subroutines EQP\_initialize, EQP\_solve, EQP\_resolve, EQP\_terminate, (Section 2.5) and EQP\_read\_specfile (Section 2.7) must be renamed on one of the USE statements.

### 2.1 Matrix storage formats

Both the Hessian matrix  $\mathbf{H}$  and the constraint Jacobian  $\mathbf{A}$  may be stored in a variety of input formats.

#### 2.1.1 Dense storage format

The matrix  $\mathbf{A}$  is stored as a compact dense matrix by rows, that is, the values of the entries of each row in turn are stored in order within an appropriate real one-dimensional array. Component  $n * (i - 1) + j$  of the storage array A%val will hold the value  $a_{ij}$  for  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ . Since  $\mathbf{H}$  is symmetric, only the lower triangular part (that is the part  $h_{ij}$  for  $1 \leq j \leq i \leq n$ ) need be held. In this case the lower triangle will be stored by rows, that is component  $i * (i - 1) / 2 + j$  of the storage array H%val will hold the value  $h_{ij}$  (and, by symmetry,  $h_{ji}$ ) for  $1 \leq j \leq i \leq n$ .

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### 2.1.2 Sparse co-ordinate storage format

Only the nonzero entries of the matrices are stored. For the  $l$ -th entry of  $\mathbf{A}$ , its row index  $i$ , column index  $j$  and value  $a_{ij}$  are stored in the  $l$ -th components of the integer arrays `A%row`, `A%col` and real array `A%val`, respectively. The order is unimportant, but the total number of entries `A%ne` is also required. The same scheme is applicable to  $\mathbf{H}$  (thus requiring integer arrays `H%row`, `H%col`, a real array `H%val` and an integer value `H%ne`), except that only the entries in the lower triangle need be stored.

### 2.1.3 Sparse row-wise storage format

Again only the nonzero entries are stored, but this time they are ordered so that those in row  $i$  appear directly before those in row  $i + 1$ . For the  $i$ -th row of  $\mathbf{A}$ , the  $i$ -th component of an integer array `A%ptr` holds the position of the first entry in this row, while `A%ptr(m + 1)` holds the total number of entries plus one. The column indices  $j$  and values  $a_{ij}$  of the entries in the  $i$ -th row are stored in components  $l = \text{A\%ptr}(i), \dots, \text{A\%ptr}(i + 1) - 1$  of the integer array `A%col`, and real array `A%val`, respectively. The same scheme is applicable to  $\mathbf{H}$  (thus requiring integer arrays `H%ptr`, `H%col`, and a real array `H%val`), except that only the entries in the lower triangle need be stored.

For sparse matrices, this scheme almost always requires less storage than its predecessor.

### 2.1.4 Diagonal storage format

If  $\mathbf{H}$  is diagonal (i.e.,  $h_{ij} = 0$  for all  $1 \leq i \neq j \leq n$ ) only the diagonal entries  $h_{ii}$ ,  $1 \leq i \leq n$ , need be stored, and the first  $n$  components of the array `H%val` may be used for the purpose. There is no sensible equivalent for the non-square  $\mathbf{A}$ .

## 2.2 Integer kinds

We use the term long `INTEGER` to denote `INTEGER(kind=long)`, where `long = selected_int_kind(18)`.

## 2.3 OpenMP

OpenMP may be used by the `GALAHAD_EQP` package to provide parallelism for some solver options in shared memory environments. See the documentation for the `GALAHAD` package `SLS` for more details. To run in parallel, OpenMP must be enabled at compilation time by using the correct compiler flag (usually some variant of `-openmp`). The number of threads may be controlled at runtime by setting the environment variable `OMP_NUM_THREADS`.

The code may be compiled and run in serial mode.

## 2.4 The derived data types

Six derived data types are accessible from the package.

### 2.4.1 The derived data type for holding matrices

The derived data type `SMT_TYPE` is used to hold the matrices  $\mathbf{A}$  and  $\mathbf{H}$ . The components of `SMT_TYPE` used here are:

`m` is a scalar component of type default `INTEGER`, that holds the number of rows in the matrix.

`n` is a scalar component of type default `INTEGER`, that holds the number of columns in the matrix.

`ne` is a scalar variable of type default `INTEGER`, that holds the number of matrix entries.

`type` is a rank-one allocatable array of type default `CHARACTER`, that is used to indicate the matrix storage scheme used. Its precise length and content depends on the type of matrix to be stored (see §2.4.2).

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- `val` is a rank-one allocatable array of type default `REAL` (double precision in `GALAHAD_EQP_double`) and dimension at least `ne`, that holds the values of the entries. Each pair of off-diagonal entries  $h_{ij} = h_{ji}$  of a *symmetric* matrix **H** is represented as a single entry (see §2.1.1–2.1.3). Any duplicated entries that appear in the sparse co-ordinate or row-wise schemes will be summed.
- `row` is a rank-one allocatable array of type default `INTEGER`, and dimension at least `ne`, that may hold the row indices of the entries. (see §2.1.2).
- `col` is a rank-one allocatable array of type default `INTEGER`, and dimension at least `ne`, that may hold the column indices of the entries (see §2.1.2–2.1.3).
- `ptr` is a rank-one allocatable array of type default `INTEGER`, and dimension at least `m + 1`, that may hold the pointers to the first entry in each row (see §2.1.3).

### 2.4.2 The derived data type for holding the problem

The derived data type `QPT_problem_type` is used to hold the problem. The components of `QPT_problem_type` are:

- `n` is a scalar variable of type default `INTEGER`, that holds the number of optimization variables,  $n$ .
- `m` is a scalar variable of type default `INTEGER`, that holds the number of linear constraints,  $m$ .
- `Hessian_kind` is a scalar variable of type default `INTEGER`, that is used to indicate what type of Hessian the problem involves. Possible values for `Hessian_kind` are:

- <0 In this case, a general quadratic program of the form (1.1) is given. The Hessian matrix **H** will be provided in the component `H` (see below).
- 0 In this case, a linear program, that is a problem of the form (1.3) with weights  $\mathbf{w} = 0$ , is given.
- 1 In this case, a least-distance problem of the form (1.3) with weights  $w_j = 1$  for  $j = 1, \dots, n$  is given.
- >1 In this case, a weighted least-distance problem of the form (1.3) with general weights  $\mathbf{w}$  is given. The weights will be provided in the component `WEIGHT` (see below).

By default `Hessian_kind` = - 1.

- `H` is scalar variable of type `SMT_TYPE` that holds the Hessian matrix **H**. The following components are used:

`H$type` is an allocatable array of rank one and type default `CHARACTER`, that is used to indicate the storage scheme used. If the dense storage scheme (see Section 2.1.1) is used, the first five components of `H$type` must contain the string `DENSE`. For the sparse co-ordinate scheme (see Section 2.1.2), the first ten components of `H$type` must contain the string `COORDINATE`, for the sparse row-wise storage scheme (see Section 2.1.3), the first fourteen components of `H$type` must contain the string `SPARSE_BY_ROWS`, and for the diagonal storage scheme (see Section 2.1.4), the first eight components of `H$type` must contain the string `DIAGONAL`.

For convenience, the procedure `SMT_put` may be used to allocate sufficient space and insert the required keyword into `H$type`. For example, if `prob` is of derived type `EQP_problem_type` and involves a Hessian we wish to store using the co-ordinate scheme, we may simply

```
CALL SMT_put( prob%H$type, 'COORDINATE', istat )
```

See the documentation for the GALAHAD package `SMT` for further details on the use of `SMT_put`.

`H$ne` is a scalar variable of type default `INTEGER`, that holds the number of entries in the **lower triangular** part of **H** in the sparse co-ordinate storage scheme (see Section 2.1.2). It need not be set for any of the other three schemes.

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H%val is a rank-one allocatable array of type default REAL (double precision in GALAHAD\_EQP\_double), that holds the values of the entries of the **lower triangular** part of the Hessian matrix **H** in any of the storage schemes discussed in Section 2.1.

H%row is a rank-one allocatable array of type default INTEGER, that holds the row indices of the **lower triangular** part of **H** in the sparse co-ordinate storage scheme (see Section 2.1.2). It need not be allocated for any of the other three schemes.

H%col is a rank-one allocatable array variable of type default INTEGER, that holds the column indices of the **lower triangular** part of **H** in either the sparse co-ordinate (see Section 2.1.2), or the sparse row-wise (see Section 2.1.3) storage scheme. It need not be allocated when the dense or diagonal storage schemes are used.

H%ptr is a rank-one allocatable array of dimension  $n+1$  and type default INTEGER, that holds the starting position of each row of the **lower triangular** part of **H**, as well as the total number of entries plus one, in the sparse row-wise storage scheme (see Section 2.1.3). It need not be allocated when the other schemes are used.

If  $\text{Hessian\_kind} \geq 0$ , the components of **H** need not be set.

WEIGHT is a rank-one allocatable array type default REAL (double precision in GALAHAD\_EQP\_double), that should be allocated to have length  $n$ , and its  $j$ -th component filled with the value  $w_j$  for  $j = 1, \dots, n$ , whenever  $\text{Hessian\_kind} > 1$ . If  $\text{Hessian\_kind} \leq 1$ , WEIGHT need not be allocated.

target\_kind is a scalar variable of type default INTEGER, that is used to indicate whether the components of the targets  $\mathbf{x}^0$  (if they are used) have special or general values. Possible values for target\_kind are:

0 In this case,  $\mathbf{x}^0 = 0$ .

1 In this case,  $x_j^0 = 1$  for  $j = 1, \dots, n$ .

$\neq 0, 1$  In this case, general values of  $\mathbf{x}^0$  will be used, and will be provided in the component X0 (see below).

By default target\_kind = - 1.

X0 is a rank-one allocatable array type default REAL (double precision in GALAHAD\_EQP\_double), that should be allocated to have length  $n$ , and its  $j$ -th component filled with the value  $x_j^0$  for  $j = 1, \dots, n$ , whenever  $\text{Hessian\_kind} > 0$  and target\_kind  $\neq 0, 1$ . If  $\text{Hessian\_kind} \leq 0$  or target\_kind = 0, 1, X0 need not be allocated.

gradient\_kind is a scalar variable of type default INTEGER, that is used to indicate whether the components of the gradient **g** have special or general values. Possible values for gradient\_kind are:

0 In this case,  $\mathbf{g} = 0$ .

1 In this case,  $g_j = 1$  for  $j = 1, \dots, n$ .

$\neq 0, 1$  In this case, general values of **g** will be used, and will be provided in the component G (see below).

By default gradient\_kind = - 1.

G is a rank-one allocatable array of dimension  $n$  and type default REAL (double precision in GALAHAD\_EQP\_double), that holds the gradient **g** of the linear term of the quadratic objective function. The  $j$ -th component of **G**,  $j = 1, \dots, n$ , contains  $g_j$ . If gradient\_kind = 0, 1, G need not be allocated.

f is a scalar variable of type default REAL (double precision in GALAHAD\_EQP\_double), that holds the constant term,  $f$ , in the objective function.

A is scalar variable of type SMT\_TYPE that holds the Jacobian matrix **A**. The following components are used:

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`A%type` is an allocatable array of rank one and type default `CHARACTER`, that is used to indicate the storage scheme used. If the dense storage scheme (see Section 2.1.1) is used, the first five components of `A%type` must contain the string `DENSE`. For the sparse co-ordinate scheme (see Section 2.1.2), the first ten components of `A%type` must contain the string `COORDINATE`, while for the sparse row-wise storage scheme (see Section 2.1.3), the first fourteen components of `A%type` must contain the string `SPARSE_BY_ROWS`.

Just as for `H%type` above, the procedure `SMT_put` may be used to allocate sufficient space and insert the required keyword into `A%type`. Once again, if `prob` is of derived type `EQP_problem_type` and involves a Jacobian we wish to store using the sparse row-wise storage scheme, we may simply

```
CALL SMT_put( prob%A%type, 'SPARSE_BY_ROWS', istat )
```

`A%ne` is a scalar variable of type default `INTEGER`, that holds the number of entries in **A** in the sparse co-ordinate storage scheme (see Section 2.1.2). It need not be set for either of the other two schemes.

`A%val` is a rank-one allocatable array of type default `REAL` (double precision in `GALAHAD_EQP_double`), that holds the values of the entries of the Jacobian matrix **A** in any of the storage schemes discussed in Section 2.1.

`A%row` is a rank-one allocatable array of type default `INTEGER`, that holds the row indices of **A** in the sparse co-ordinate storage scheme (see Section 2.1.2). It need not be allocated for either of the other two schemes.

`A%col` is a rank-one allocatable array variable of type default `INTEGER`, that holds the column indices of **A** in either the sparse co-ordinate (see Section 2.1.2), or the sparse row-wise (see Section 2.1.3) storage scheme. It need not be allocated when the dense storage scheme is used.

`A%ptr` is a rank-one allocatable array of dimension `m+1` and type default `INTEGER`, that holds the starting position of each row of **A**, as well as the total number of entries plus one, in the sparse row-wise storage scheme (see Section 2.1.3). It need not be allocated when the other schemes are used.

**C** is a rank-one allocatable array of dimension `m` and type default `REAL` (double precision in `GALAHAD_EQP_double`), that holds the values of the vector of constant terms for the constraints. The  $i$ -th component of **C**,  $i = 1, \dots, m$ , contains  $c_i$ .

**X** is a rank-one allocatable array of dimension `n` and type default `REAL` (double precision in `GALAHAD_EQP_double`), that holds the values **x** of the optimization variables. The  $j$ -th component of **X**,  $j = 1, \dots, n$ , contains  $x_j$ .

**Y** is a rank-one allocatable array of dimension `m` and type default `REAL` (double precision in `GALAHAD_EQP_double`), that holds the values **y** of estimates of the Lagrange multipliers corresponding to the linear constraints (see Section 4). The  $i$ -th component of **Y**,  $i = 1, \dots, m$ , contains  $y_i$ .

### 2.4.3 The derived data type for holding control parameters

The derived data type `EQP_control_type` is used to hold controlling data. Default values may be obtained by calling `EQP_initialize` (see Section 2.5.1), while components may also be changed by calling `GALAHAD_EQP_read_spec` (see Section 2.7.1). The components of `EQP_control_type` are:

`error` is a scalar variable of type default `INTEGER`, that holds the stream number for error messages. Printing of error messages in `EQP_solve` and `EQP_terminate` is suppressed if `error`  $\leq 0$ . The default is `error` = 6.

`out` is a scalar variable of type default `INTEGER`, that holds the stream number for informational messages. Printing of informational messages in `EQP_solve` is suppressed if `out`  $< 0$ . The default is `out` = 6.

`print_level` is a scalar variable of type default `INTEGER`, that is used to control the amount of informational output which is required. No informational output will occur if `print_level`  $\leq 0$ . If `print_level` = 1, a single line of output will be produced for each iteration of the process. If `print_level`  $\geq 2$ , this output will be increased to provide significant detail of each iteration. The default is `print_level` = 0.

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`new_h` is a scalar variable of type default `INTEGER`, that is used to indicate how **H** has changed (if at all) since the previous call to `EQP_solve`. Possible values are:

- 0 **H** is unchanged
- 1 the values in **H** have changed, but its nonzero structure is as before.
- 2 both the values and structure of **H** have changed.

The default is `new_h = 2`.

`new_a` is a scalar variable of type default `INTEGER`, that is used to indicate how **A** has changed (if at all) since the previous call to `EQP_solve`. Possible values are:

- 0 **A** is unchanged
- 1 the values in **A** have changed, but its nonzero structure is as before.
- 2 both the values and structure of **A** have changed.

The default is `new_a = 2`.

`cg_maxit` is a scalar variable of type default `INTEGER`, that is used to limit the number of conjugate-gradient iterations performed in the optimality phase. If `cg_maxit` is negative, no limit will be imposed. The default is `cg_maxit = 200`.

`radius` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that may be used to specify an upper bound on the norm of the allowed solution (a “trust-region” constraint) during the iterative solution of the optimality phase of the problem. This is particularly useful if the problem is unbounded from below. If `radius` is set too small, there is a possibility that this will preclude the package from finding the actual solution. If `initial_radius` is not positive, it will be reset to the default value, `initial_radius = SQRT(0.1*HUGE(1.0)) (SQRT(0.1*HUGE(1.0D0)) in GALAHAD_EQP_double)`.

`inner_stop_relative` and `inner_stop_absolute` are scalar variables of type default `REAL` (double precision in `GALAHAD_EQP_double`), that hold the relative and absolute convergence tolerances for the iterative solution of the optimality phase of the problem using the package `GALAHAD_GLTR`, and correspond to the values `control%stop_relative` and `control%stop_absolute` in that package. The defaults are `inner_stop_relative = 0.01` and `inner_stop_absolute =  $\sqrt{u}$` , where  $u$  is `EPSILON(1.0) (EPSILON(1.0D0) in GALAHAD_EQP_double)`.

`max_infeasibility_relative` and `max_infeasibility_absolute` are scalar variables of type default `REAL` (double precision in `GALAHAD_EQP_double`), that hold the relative and absolute tolerances for assessing infeasibility in the feasibility phase. If the constraints are believed to be rank deficient and the norm of the residual  $\mathbf{Ax}_T + \mathbf{c}$  at a “typical” feasible point is larger than `max(max_infeasibility_relative*||A||, max_infeasibility_absolute)`, the problem will be marked as infeasible. The defaults are `max_infeasibility_relative = max_infeasibility_absolute =  $u^{0.75}$`  where  $u$  is `EPSILON(1.0) (EPSILON(1.0D0) in GALAHAD_EQP_double)`.

`remove_dependencies` is a scalar variable of type default `LOGICAL`, that must be set `.TRUE.` if linear dependent constraints  $\mathbf{Ax} + \mathbf{c} = \mathbf{0}$  should be removed and `.FALSE.` otherwise. The default is `remove_dependencies = .TRUE..`

`space_critical` is a scalar variable of type default `LOGICAL`, that must be set `.TRUE.` if space is critical when allocating arrays and `.FALSE.` otherwise. The package may run faster if `space_critical` is `.FALSE.` but at the possible expense of a larger storage requirement. The default is `space_critical = .FALSE..`

`deallocate_error_fatal` is a scalar variable of type default `LOGICAL`, that must be set `.TRUE.` if the user wishes to terminate execution if a deallocation fails, and `.FALSE.` if an attempt to continue will be made. The default is `deallocate_error_fatal = .FALSE..`

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`FDC_control` is a scalar variable of type `FDC_control_type` whose components are used to control any detection of linear dependencies performed by the package `GALAHAD_FDC`. See the specification sheet for the package `GALAHAD_FDC` for details, and appropriate default values.

`GLTR_control` is a scalar variable argument of type `GLTR_control_type` that is used to pass control options to the conjugate-gradient solver used to solve linear systems that arise. See the documentation for the `GALAHAD` package `GLTR` for further details. In particular, default values are as for `GLTR`.

`SBLS_control` is a scalar variable argument of type `SBLS_control_type` that is used to pass control options to the symmetric block linear equation preconditioner used to help solve linear systems that arise. See the documentation for the `GALAHAD` package `SBLS` for further details. In particular, default values are as for `SBLS`.

`prefix` is a scalar variable of type default `CHARACTER` and length 30, that may be used to provide a user-selected character string to preface every line of printed output. Specifically, each line of output will be prefaced by the string `prefix(2:LEN(TRIM( prefix ))-1)`, thus ignoring the first and last non-null components of the supplied string. If the user does not want to preface lines by such a string, they may use the default `prefix = ""`.

#### 2.4.4 The derived data type for holding timing information

The derived data type `EQP_time_type` is used to hold elapsed CPU and system clock times for the various parts of the calculation. The components of `EQP_time_type` are:

`total` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the total CPU time spent in the package.

`find_dependent` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the CPU time spent detecting and removing dependent constraints prior to solution.

`factorize` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the CPU time spent factorizing the required matrices.

`solve` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the CPU time spent computing the solution given the factorization(s).

`clock_total` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the total elapsed system clock time spent in the package.

`clock_find_dependent` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the elapsed system clock time spent detecting and removing dependent constraints prior to solution.

`clock_factorize` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the elapsed system clock time spent factorizing the required matrices.

`clock_solve` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that gives the elapsed system clock time spent computing the search direction.

#### 2.4.5 The derived data type for holding informational parameters

The derived data type `EQP_inform_type` is used to hold parameters that give information about the progress and needs of the algorithm. The components of `EQP_inform_type` are:

`status` is a scalar variable of type default `INTEGER`, that gives the exit status of the algorithm. See Section 2.6 for details.

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`alloc_status` is a scalar variable of type default `INTEGER`, that gives the status of the last attempted array allocation or deallocation. This will be 0 if `status = 0`.

`bad_alloc` is a scalar variable of type default `CHARACTER` and length 80, that gives the name of the last internal array for which there were allocation or deallocation errors. This will be the null string if `status = 0`.

`cg_iter` is a scalar variable of type default `INTEGER`, that gives the total number of conjugate-gradient iterations required.

`factorization_integer` is a scalar variable of type long `INTEGER`, that gives the amount of integer storage used for the matrix factorization.

`factorization_real` is a scalar variable of type long `INTEGER`, that gives the amount of real storage used for the matrix factorization.

`obj` is a scalar variable of type default `REAL` (double precision in `GALAHAD_EQP_double`), that holds the value of the objective function at the best estimate of the solution found.

`time` is a scalar variable of type `EQP_time_type` whose components are used to hold elapsed CPU and system clock times for the various parts of the calculation (see Section 2.4.4).

`FDC_inform` is a scalar variable of type `FDC_inform_type` whose components are used to provide information about any detection of linear dependencies performed by the package `GALAHAD_FDC`. See the specification sheet for the package `GALAHAD_FDC` for details.

`SBLS_inform` is a scalar variable of type `SBLS_inform_type` whose components are used to hold information relating to the formation and factorization of the preconditioner. See the documentation for the `GALAHAD` package `SBLS` for further details.

`GLTR_inform` is a scalar variable of type `GLTR_inform_type` whose components are used to hold information relating to the computation of the solution via the conjugate-gradient method. See the documentation for the `GALAHAD` package `GLTR` for further details.

#### 2.4.6 The derived data type for holding problem data

The derived data type `EQP_data_type` is used to hold all the data for a particular problem, or sequences of problems with the same structure, between calls of `EQP` procedures. This data should be preserved, untouched, from the initial call to `EQP_initialize` to the final call to `EQP_terminate`.

### 2.5 Argument lists and calling sequences

There are three procedures for user calls (see Section 2.7 for further features):

1. The subroutine `EQP_initialize` is used to set default values, and initialize private data, before solving one or more problems with the same sparsity and bound structure.
2. The subroutine `EQP_solve` is called to solve the problem.
3. The subroutine `EQP_terminate` is provided to allow the user to automatically deallocate array components of the private data, allocated by `EQP_solve`, at the end of the solution process.

---

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### 2.5.1 The initialization subroutine

Default values are provided as follows:

```
CALL EQP_initialize( data, control, inform )
```

`data` is a scalar `INTENT(INOUT)` argument of type `EQP_data_type` (see Section 2.4.6). It is used to hold data about the problem being solved.

`control` is a scalar `INTENT(OUT)` argument of type `EQP_control_type` (see Section 2.4.3). On exit, `control` contains default values for the components as described in Section 2.4.3. These values should only be changed after calling `EQP_initialize`.

`inform` is a scalar `INTENT(INOUT)` argument of type `EQP_inform_type` (see Section 2.4.5). A successful call to `EQP_initialize` is indicated when the component status has the value 0. For other return values of status, see Section 2.6.

### 2.5.2 The equality-constrained-quadratic programming subroutine

The equality-constrained quadratic programming algorithm is called as follows:

```
CALL EQP_solve( p, data, control, inform )
```

`p` is a scalar `INTENT(INOUT)` argument of type `QPT_problem_type` (see Section 2.4.2). It is used to hold data about the problem being solved. The user must have allocated all array components, and set appropriate values for all components. Users are free to choose whichever of the matrix formats described in Section 2.1 is appropriate for **A** and **H** for their application—different formats may be used for the two matrices.

The components `p%X` and `p%Y` must be set to initial estimates,  $\mathbf{x}^0$ , of the solution variables, **x**, and Lagrange multipliers for the constraints, **y**. Inappropriate initial values will be altered, so the user should not be overly concerned if suitable values are not apparent, and may be content with merely setting `p%X=0.0` and `p%Y=0.0`.

On exit, the components `p%X` and `p%Y` will contain the best estimates of the solution variables **x**, and Lagrange multipliers for the constraints **y**. **Restrictions:** `p%n > 0` and, `p%m ≥ 0`, and `prob%H_type` and `prob%A_type` ∈ {`'DENSE'`, `'COORDINATE'`, `'SPARSE_BY_ROWS'`, `'DIAGONAL'` }.

`data` is a scalar `INTENT(INOUT)` argument of type `EQP_data_type` (see Section 2.4.6). It is used to hold data about the problem being solved. It must not have been altered **by the user** since the last call to `EQP_initialize`.

`control` is a scalar `INTENT(INOUT)` argument of type `EQP_control_type` (see Section 2.4.3). Default values may be assigned by calling `EQP_initialize` prior to the first call to `EQP_solve`.

`inform` is a scalar `INTENT(OUT)` argument of type `EQP_inform_type` (see Section 2.4.5). A successful call to `EQP_solve` is indicated when the component status has the value 0. For other return values of status, see Section 2.6.

### 2.5.3 The resolve subroutine

Once `EQP_solve` has been called, further quadratic programs, for which the data **g**, **c** and **f** may have been altered but **A** and **H** are unchanged, may be solved more efficiently as follows:

```
CALL EQP_resolve( p, data, control, inform )
```

`p` is a scalar `INTENT(INOUT)` argument of type `QPT_problem_type` as described for `EQP_solve` but for which only the components `%G`, `%f` and `%C` may have been altered since the last call to `EQP_solve`. As before, on exit, the components `p%X` and `p%Y` will contain the best estimates of the solution variables **x**, and Lagrange multipliers for the constraints **y**.

`data`, `control` and `inform` are precisely as described for `EQP_solve`.

---

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### 2.5.4 The termination subroutine

All previously allocated arrays are deallocated as follows:

```
CALL EQP_terminate( data, control, info )
```

`data` is a scalar `INTENT(INOUT)` argument of type `EQP_data_type` exactly as for `EQP_solve`, which must not have been altered **by the user** since the last call to `EQP_initialize`. On exit, array components will have been deallocated.

`control` is a scalar `INTENT(IN)` argument of type `EQP_control_type` exactly as for `EQP_solve`.

`inform` is a scalar `INTENT(OUT)` argument of type `EQP_inform_type` exactly as for `EQP_solve`. Only the component `status` will be set on exit, and a successful call to `EQP_terminate` is indicated when this component `status` has the value 0. For other return values of `status`, see Section 2.6.

### 2.6 Warning and error messages

A negative value of `inform%status` on exit from `EQP_solve` or `EQP_terminate` indicates that an error has occurred. No further calls should be made until the error has been corrected. Possible values are:

- 1. An allocation error occurred. A message indicating the offending array is written on unit `control%error`, and the returned allocation status and a string containing the name of the offending array are held in `inform%alloc_status` and `inform%bad_alloc` respectively.
- 2. A deallocation error occurred. A message indicating the offending array is written on unit `control%error` and the returned allocation status and a string containing the name of the offending array are held in `inform%alloc_status` and `inform%bad_alloc` respectively.
- 3. One of the restrictions `prob%n > 0` or `prob%m ≥ 0` or requirements that `prob%A_type` and `prob%H_type` contain its relevant string 'DENSE', 'COORDINATE', 'SPARSE\_BY\_ROWS' or 'DIAGONAL' has been violated.
- 5. The constraints appear to be inconsistent.
- 9. An error was reported by the subroutine `SILS_analyse` called by `SBLS`. The return status from `SILS_analyse` is given in one of the components of `inform%SBLS_inform%SLS_inform`. See the documentation for the GALAHAD package `SILS` for further details.
- 10. An error was reported by the subroutine `SILS_factorize` called by `SBLS`. The return status from `SILS_factorize` is given in one of the components of `inform%SBLS_inform%SLS_inform`. See the documentation for the GALAHAD package `SILS` for further details.
- 11. An error was reported by the subroutine `SILS_solve` called by `SBLS`. The return status from `SILS_solve` is given in one of the components of `inform%SBLS_inform%SLS_inform`. See the documentation for the GALAHAD package `SILS` for further details.
- 12. An error was reported by the subroutine `ULS_analyse` called by `SBLS`. The return status from `ULS_analyse` is given in one of the components of `inform%SBLS_inform%ULS_inform`. See the documentation for the GALAHAD package `ULS` for further details.
- 14. An error was reported by the subroutine `ULS_solve` called by `SBLS`. The return status from `ULS_solve` is given in one of the components of `inform%SBLS_inform%ULS_inform`. See the documentation for the GALAHAD package `ULS` for further details.
- 15. The computed preconditioner has the wrong inertia and is thus unsuitable.
- 16. The residuals from the preconditioning step are large, indicating that the factorization may be unsatisfactory.
- 25. `EQP_resolve` has been called before `EQP_solve`.

---

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## 2.7 Further features

In this section, we describe an alternative means of setting control parameters, that is components of the variable `control` of type `EQP_control_type` (see Section 2.4.3), by reading an appropriate data specification file using the subroutine `EQP_read_specfile`. This facility is useful as it allows a user to change EQP control parameters without editing and recompiling programs that call EQP.

A specification file, or *specfile*, is a data file containing a number of "specification commands". Each command occurs on a separate line, and comprises a "keyword", which is a string (in a close-to-natural language) used to identify a control parameter, and an (optional) "value", which defines the value to be assigned to the given control parameter. All keywords and values are case insensitive, keywords may be preceded by one or more blanks but values must not contain blanks, and each value must be separated from its keyword by at least one blank. Values must not contain more than 30 characters, and each line of the specfile is limited to 80 characters, including the blanks separating keyword and value.

The portion of the specification file used by `EQP_read_specfile` must start with a "BEGIN EQP" command and end with an "END" command. The syntax of the specfile is thus defined as follows:

```
( .. lines ignored by EQP_read_specfile .. )
  BEGIN EQP
    keyword      value
    .....
    keyword      value
  END
( .. lines ignored by EQP_read_specfile .. )
```

where keyword and value are two strings separated by (at least) one blank. The "BEGIN EQP" and "END" delimiter command lines may contain additional (trailing) strings so long as such strings are separated by one or more blanks, so that lines such as

```
BEGIN EQP SPECIFICATION
```

and

```
END EQP SPECIFICATION
```

are acceptable. Furthermore, between the "BEGIN EQP" and "END" delimiters, specification commands may occur in any order. Blank lines and lines whose first non-blank character is ! or \* are ignored. The content of a line after a ! or \* character is also ignored (as is the ! or \* character itself). This provides an easy manner to "comment out" some specification commands, or to comment specific values of certain control parameters.

The value of a control parameters may be of three different types, namely integer, logical or real. Integer and real values may be expressed in any relevant Fortran integer and floating-point formats (respectively). Permitted values for logical parameters are "ON", "TRUE", ".TRUE.", "T", "YES", "Y", or "OFF", "NO", "N", "FALSE", ".FALSE." and "F". Empty values are also allowed for logical control parameters, and are interpreted as "TRUE".

The specification file must be open for input when `EQP_read_specfile` is called, and the associated device number passed to the routine in `device` (see below). Note that the corresponding file is `REWINDED`, which makes it possible to combine the specifications for more than one program/routine. For the same reason, the file is not closed by `EQP_read_specfile`.

Control parameters corresponding to the components `SBLs_control` and `GLTR_control` may be changed by including additional sections enclosed by "BEGIN SBLs" and "END SBLs", and "BEGIN GLTR" and "END GLTR", respectively. See the specification sheets for the packages `GALAHAD_SBLs` and `GALAHAD_GLTR` for further details.

---

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### 2.7.1 To read control parameters from a specification file

Control parameters may be read from a file as follows:

```
CALL EQP_read_specfile( control, device )
```

`control` is a scalar `INTENT(INOUT)` argument of type `EQP_control_type` (see Section 2.4.3). Default values should have already been set, perhaps by calling `EQP_initialize`. On exit, individual components of `control` may have been changed according to the commands found in the specfile. Specfile commands and the component (see Section 2.4.3) of `control` that each affects are given in Table 2.1.

command	component of control	value type
error-printout-device	%error	integer
printout-device	%out	integer
print-level	%print_level	integer
maximum-number-of-cg-iterations	%cg_maxit	integer
trust-region-radius	%radius	real
max-relative-infeasibility-allowed	%max_infeasibility_relative	real
max-absolute-infeasibility-allowed	%max_infeasibility_absolute	real
inner-iteration-relative-accuracy-required	%inner_stop_relative	real
inner-iteration-absolute-accuracy-required	%inner_stop_absolute	real
remove-linear-dependencies	%remove_dependencies	logical
space-critical	%space_critical	logical
deallocate-error-fatal	%deallocate_error_fatal	logical

Table 2.1: Specfile commands and associated components of `control`.

`device` is a scalar `INTENT(IN)` argument of type `default INTEGER`, that must be set to the unit number on which the specfile has been opened. If `device` is not open, `control` will not be altered and execution will continue, but an error message will be printed on unit `control%error`.

### 2.8 Information printed

If `control%print_level` is positive, information about the progress of the algorithm will be printed on unit `control%out`. If `control%print_level = 1`, the norm of the constraint violation and the value of the objective function for both the feasibility and optimality phases are reported. Additionally, if `control%print_level = 2`, `print_level = 1` output from both SBLS and GLTR occurs, summarising the factorization and iteration phases, as well as timing statistics from the two phases. If `control%print_level ≥ 3` detailed output from SBLS and GLTR occurs which is unlikely to be useful to general users.

## 3 GENERAL INFORMATION

**Use of common:** None.

**Workspace:** Provided automatically by the module.

**Other routines called directly:** None.

**Other modules used directly:** `EQP_solve` calls the GALAHAD packages `GALAHAD_CLOCK`, `GALAHAD_SYMBOLS`, `GALAHAD_SPACE`, `GALAHAD_QPT`, `GALAHAD_FDC`, `GALAHAD_SBPS`, `GALAHAD_GLTR` and `GALAHAD_SPECFILE`.

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**Input/output:** Output is under control of the arguments `control%error`, `control%out` and `control%print_level`.

**Restrictions:** `prob%n` > 0, `prob%m` ≥ 0, `prob%A_type` and `prob%H_type` ∈ {`'DENSE'`, `'COORDINATE'`, `'SPARSE_BY_ROWS'`, `'DIAGONAL'` }.

**Portability:** ISO Fortran 95 + TR 15581 or Fortran 2003. The package is thread-safe.

## 4 METHOD

Any finite solution  $\mathbf{x}$  to the problem necessarily satisfies the primal optimality conditions

$$\mathbf{Ax} + \mathbf{c} = \mathbf{0} \quad (4.1)$$

and the dual optimality conditions

$$\mathbf{Hx} + \mathbf{g} - \mathbf{A}^T \mathbf{y} = \mathbf{0}, \quad (4.2)$$

where the components of the vector  $\mathbf{y}$  are known as the Lagrange multipliers for the constraints.

A solution to the problem is found in two phases. In the first, a point  $\mathbf{x}_F$  satisfying (4.1) is found. In the second, the required solution  $\mathbf{x} = \mathbf{x}_F + \mathbf{s}$  is determined by finding  $\mathbf{s}$  to minimize  $q(\mathbf{s}) = \frac{1}{2} \mathbf{s}^T \mathbf{H} \mathbf{s} + \mathbf{g}_F^T \mathbf{s} + f_F$  subject to the homogeneous constraints  $\mathbf{As} = \mathbf{0}$ , where  $\mathbf{g}_F = \mathbf{Hx}_F + \mathbf{g}$  and  $f_F = \frac{1}{2} \mathbf{x}_F^T \mathbf{H} \mathbf{x}_F + \mathbf{g}^T \mathbf{x}_F + f$ . The required constrained minimizer of  $q(\mathbf{s})$  is obtained by implicitly applying the preconditioned conjugate-gradient method in the null space of  $\mathbf{A}$ . Any preconditioner of the form

$$\mathbf{K}_G = \begin{pmatrix} \mathbf{G} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{pmatrix}$$

is suitable, and the GALAHAD package `SBLS` provides a number of possibilities. In order to ensure that the minimizer obtained is finite, an additional, precautionary trust-region constraint  $\|\mathbf{s}\| \leq \Delta$  for some suitable positive radius  $\Delta$  is imposed, and the GALAHAD package `GLTR` is used to solve this additionally-constrained problem.

## References:

The preconditioning aspects are described in detail in

H. S. Dollar, N. I. M. Gould and A. J. Wathen. “On implicit-factorization constraint preconditioners”. In *Large Scale Nonlinear Optimization* (G. Di Pillo and M. Roma, eds.) Springer Series on Nonconvex Optimization and Its Applications, Vol. 83, Springer Verlag (2006) 61–82

and

H. S. Dollar, N. I. M. Gould, W. H. A. Schilders and A. J. Wathen “On iterative methods and implicit-factorization preconditioners for regularized saddle-point systems”. *SIAM Journal on Matrix Analysis and Applications*, **28(1)** (2006) 170–189,

while the constrained conjugate-gradient method is discussed in

N. I. M. Gould, S. Lucidi, M. Roma and Ph. L. Toint, Solving the trust-region subproblem using the Lanczos method. *SIAM Journal on Optimization* **9:2** (1999), 504–525.

## 5 EXAMPLE OF USE

Suppose we wish to minimize  $\frac{1}{2}x_1^2 + x_2^2 + \frac{3}{2}x_3^2 + 4x_1x_3 + 2x_2 + 1$  subject to the the general linear constraints  $2x_1 + x_2 - 2 = 0$  and  $x_2 + x_3 - 2 = 0$ . Then, on writing the data for this problem as

$$\mathbf{H} = \begin{pmatrix} 1 & & 4 \\ & 2 & \\ 4 & & 3 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 1 & \\ & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

---

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in sparse co-ordinate format, we may use the following code:

```
! THIS VERSION: GALAHAD 2.1 - 22/03/2007 AT 09:00 GMT.
PROGRAM GALAHAD_EQP_EXAMPLE
USE GALAHAD_EQP_double      ! double precision version
IMPLICIT NONE
INTEGER, PARAMETER :: wp = KIND( 1.0D+0 ) ! set precision
TYPE ( QPT_problem_type ) :: p
TYPE ( EQP_data_type ) :: data
TYPE ( EQP_control_type ) :: control
TYPE ( EQP_inform_type ) :: inform
INTEGER :: s
INTEGER, PARAMETER :: n = 3, m = 2, h_ne = 4, a_ne = 4
! start problem data
ALLOCATE( p%G( n ), p%C( m ), p%X( n ), p%Y( m ) )
p%new_problem_structure = .TRUE.      ! new structure
p%n = n ; p%m = m ; p%f = 1.0_wp      ! dimensions & objective constant
p%G = (/ 0.0_wp, 2.0_wp, 0.0_wp /)    ! objective gradient
p%C = (/ - 2.0_wp, - 2.0_wp /)        ! constraint constants
p%X = 0.0_wp ; p%Y = 0.0_wp          ! start from zero
! sparse co-ordinate storage format
CALL SMT_put( p%H%type, 'COORDINATE', s ) ! Specify co-ordinate
CALL SMT_put( p%A%type, 'COORDINATE', s ) ! storage for H and A
ALLOCATE( p%H%val( h_ne ), p%H%row( h_ne ), p%H%col( h_ne ) )
ALLOCATE( p%A%val( a_ne ), p%A%row( a_ne ), p%A%col( a_ne ) )
p%H%val = (/ 1.0_wp, 2.0_wp, 3.0_wp, 4.0_wp /) ! Hessian H
p%H%row = (/ 1, 2, 3, 3 /) ! NB lower triangle
p%H%col = (/ 1, 2, 3, 1 /) ; p%H%ne = h_ne
p%A%val = (/ 2.0_wp, 1.0_wp, 1.0_wp, 1.0_wp /) ! Jacobian A
p%A%row = (/ 1, 1, 2, 2 /)
p%A%col = (/ 1, 2, 2, 3 /) ; p%A%ne = a_ne
! problem data complete
CALL EQP_initialize( data, control, inform ) ! Initialize control parameters
CALL EQP_solve( p, data, control, inform ) ! Solve problem
IF ( inform%status == 0 ) THEN ! Successful return
  WRITE( 6, "( ' EQP: ', I0, ' CG iteration(s). Optimal objective value =', &
    & ES12.4, '/', ' Optimal solution = ', ( 5ES12.4 ) )" ) &
    inform%cg_iter, inform%obj, p%X
ELSE ! Error returns
  WRITE( 6, "( ' EQP_solve exit status = ', I6 ) " ) inform%status
END IF
CALL EQP_terminate( data, control, inform ) ! delete internal workspace
DEALLOCATE( p%G, p%C, p%X, p%Y ) ! deallocate problem arrays
DEALLOCATE( p%H%val, p%H%row, p%H%col, p%A%val, p%A%row, p%A%col )
DEALLOCATE( p%H%type, p%A%type )
END PROGRAM GALAHAD_EQP_EXAMPLE
```

This produces the following output:

```
EQP: 1 CG iteration(s). Optimal objective value = 7.0541E+00
Optimal solution = 3.2432E-01 1.3514E+00 6.4865E-01
```

The same problem may be solved holding the data in a sparse row-wise storage format by replacing the lines

```
! sparse co-ordinate storage format
...
! problem data complete
```

---

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by

```
! sparse row-wise storage format
CALL SMT_put( p%H$type, 'SPARSE_BY_ROWS', s ) ! Specify sparse-by-row
CALL SMT_put( p%A$type, 'SPARSE_BY_ROWS', s ) ! storage for H and A
ALLOCATE( p%H%val( h_ne ), p%H%col( h_ne ), p%H%ptr( n + 1 ) )
ALLOCATE( p%A%val( a_ne ), p%A%col( a_ne ), p%A%ptr( m + 1 ) )
p%H%val = (/ 1.0_wp, 2.0_wp, 3.0_wp, 4.0_wp /) ! Hessian H
p%H%col = (/ 1, 2, 3, 1 /) ! NB lower triangular
p%H%ptr = (/ 1, 2, 3, 5 /) ! Set row pointers
p%A%val = (/ 2.0_wp, 1.0_wp, 1.0_wp, 1.0_wp /) ! Jacobian A
p%A%col = (/ 1, 2, 2, 3 /)
p%A%ptr = (/ 1, 3, 5 /) ! Set row pointers
! problem data complete
```

or using a dense storage format with the replacement lines

```
! dense storage format
CALL SMT_put( p%H$type, 'DENSE', s ) ! Specify dense
CALL SMT_put( p%A$type, 'DENSE', s ) ! storage for H and A
ALLOCATE( p%H%val( n * ( n + 1 ) / 2 ) )
ALLOCATE( p%A%val( n * m ) )
p%H%val = (/ 1.0_wp, 0.0_wp, 2.0_wp, 4.0_wp, 0.0_wp, 3.0_wp /) ! Hessian
p%A%val = (/ 2.0_wp, 1.0_wp, 0.0_wp, 0.0_wp, 1.0_wp, 1.0_wp /) ! Jacobian
! problem data complete
```

respectively.

If instead **H** had been the diagonal matrix

$$\mathbf{H} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 3 \end{pmatrix}$$

but the other data is as before, the diagonal storage scheme might be used for **H**, and in this case we would instead

```
CALL SMT_put( prob%H$type, 'DIAGONAL', s ) ! Specify dense storage for H
ALLOCATE( p%H%val( n ) )
p%H%val = (/ 1.0_wp, 0.0_wp, 3.0_wp /) ! Hessian values
```

Notice here that zero diagonal entries are stored.