Stochastic quantisation of the fractional Φ_3^4 model in the full subcritical regime

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Abstract

We construct the fractional Φ^4 Euclidean quantum field theory on $\mathbb{R}^3$ in the full subcritical regime via parabolic stochastic quantisation. Our approach is based on the use of a truncated flow equation for the effective description of the model at sufficiently small scales and on coercive estimates for the non-linear stochastic partial differential equation describing the interacting field.

Keywords: stochastic quantisation, renormalisation group, fractional Laplacian, singular SPDEs.

A.M.S. subject classification: 60H17, 81T08, 81T17, 35B45, 60H30

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1 Introduction

Take $\varepsilon > 0$ and $M \in \mathbb{N}$, $\mathbb{T}^d_{\varepsilon,M} := (\varepsilon(\mathbb{Z}/M\mathbb{Z}))^d$ and define a probability measure $\nu_{\varepsilon,M}$ on $\Omega_{\varepsilon,N} := \{\varphi : \mathbb{T}^d_{\varepsilon,M} \to \mathbb{R}\}$ by

$$\nu_{\varepsilon,M}(\mathrm{d}\varphi) := \frac{\exp(-S_{\varepsilon,M}(\varphi))}{Z_{\varepsilon,M}} \prod_{x \in \mathbb{T}^d_{\varepsilon,M}} \mathrm{d}\varphi(x) \tag{1}$$

$$S_{\varepsilon,M}(\varphi) := \varepsilon^d \sum_{x \in \mathbb{T}^d_{\varepsilon,M}} \left[\frac{1}{2}\varphi(x)\,(-\Delta_\varepsilon)^s \varphi(x) + \frac{m^2}{2}\,\varphi(x)^2 + \frac{\lambda}{4}\,\varphi(x)^4 - \frac{r_\varepsilon}{2}\,\varphi(x)^2 \right] \tag{2}$$

with $Z_{\varepsilon,M} := \int_{\Omega_{\varepsilon,M}} e^{-S_{\varepsilon,M}(\varphi)} \prod_{x \in \mathbb{T}^d_{\varepsilon,M}} \mathrm{d}\varphi(x)$ and where $m > 0, \lambda > 0, r_\varepsilon$ are constant respectively called *mass*, *coupling constant* and *mass renormalisation* and $(-\Delta_\varepsilon)^s$ is the discrete fractional Laplacian of order $s \in (0, 1]$ defined via functional calculus as the s-th power of the discrete Laplacian Δ_ε.

For the sake of clarity we will restrict our considerations to $d = 3$, in which case $s = 1$ corresponds to the usual Φ^4_3 measure and we will work within the full range of fractional exponents s for which the model is subcritical, i.e. when the non-linear part can be considered as a perturbation of the Gaussian measure in the small scale regime. This corresponds to $s > s_c := 3/4$. Moreover reflection positivity holds for $s \leqslant 1$, so we will restrict to values of $s \in (s_c, 1]$, for more details see the discussion of the measure (1) in [GH19]. We will discuss in detail the case $s \in (3/4, 1)$ but the same proof strategy works with some semplifications for the case $s = 1$ corresponding to the classical Laplacian. The main result of this paper is a proof of the following:

Theorem 1. *Let $d = 3$ and fix $s \in (3/4, 1]$, $m > 0$, $\lambda > 0$. The family $(\nu_{\varepsilon,M})_{\varepsilon,M}$ defines a tight family of probability measure on $\mathcal{S}'(\mathbb{R}^3)$ as $M \to \infty$ and then $\varepsilon \to 0$ and the sequence of mass renormalisations $(r_\varepsilon)_{\varepsilon > 0}$ is chosen suitably with $r_\varepsilon = r_\varepsilon(\lambda) \to +\infty$ as $\varepsilon \to 0$. Any accumulation point ν is non-Gaussian, invariant under translation, reflection positive and satisfies*

$$\int e^{\theta \|(1+|x|)^{-b}(1+(-\Delta)^{1/2})^{-a}\phi\|^4_{L^2}} \nu(\mathrm{d}\phi) < \infty, \tag{3}$$

for sufficiently large $a, b > 0$ and small $\theta > 0$.

Our proof strategy introduces a novel combination of renormalisation group ideas and PDE techniques which we believe can be useful more widely in the context of the theory of subcritical singular SPDEs. We show that the proof also applies to the vector version of the model where the field takes values in $\mathbb{R}^n$ for $n > 1$ and the functional $S_{\varepsilon,M}(\varphi)$ depends on φ in an $O(n)$ symmetric way.

Theorem 1 gives a construction of a model of Euclidean quantum field theory (EQFT) which is referred in the literature as the *fractional* Φ_3^4 model, in reference to the form of the exponential weight (2) and the dimension of the space $\mathbb{R}^3$. In the case $s=1$ this is the well known Φ_3^4 model which is considered a crucial test in constructive quantum field theory since the original results of Glimm and Jaffe [GJ73], Feldman and Osterwalder [Fel74, FO76] and others EQFT constructivists whose goal were to prove the existence of models satisfying the Wightman axioms for local relativistic QFT using probabilistic tools via the concept of Euclidean QFTs [GJ87].

In recent years there has been a renewed interest in EQFTs due to the development of an alternative approach to the proof of theorems like Theorem 1. This new approach is grounded in the basic ideas of stochastic calculus and it is usually called *stochastic quantisation*. This term was introduced by Parisi and Wu [PW81] to describe the quantisation of gauge theories via the construction of a stochastic process evolving in fictious time and whose stationary distribution is the target Euclidean QFT. This stochastic evolution is a non-linear stochastic partial differential equation of a singular kind, for which a particular process of renormalisation is needed to give it a precise meaning. The analysis of such equations require a mix of probabilistic and analytic arguments which escape the usual approach of Ito integration and Ito stochastic differential equations. For this reason it took some time before the SPDE community learned how to handle such singular equation and discovered theories like regularity structures [Hai14] and paracontrolled calculus [GIP15, CC18] or renormalisation group [Kup16] which finally allowed to tackle the problem of the stochastic quantisation of the Φ_3^4 model. Gubinelli and Hofmanová [GH19] obtained the equivalent of Theorem 1 with $s=1$ and a small range of values below that using a mix of paracontrolled calculus for the small scale singularities of the equation and coercive estimates to tame the large scale fluctuations. We refer the reader to the introduction to [GH19] for a deeper review of the literature and the history of constructive QFT and also to contextualise the meaning and consequences of Theorem 1. To out knowledge the construction of the fractional Φ_3^4 model, in the full range of subcritical values of s, is novel.

The probability measure $\nu_{\varepsilon,M}$ in (1) is the equilibrium distribution for the Langevin dynamics given by the stationary solutions to the finite system of SDEs

$$\mathscr{L}_\varepsilon \phi^{(\varepsilon,M)} + \lambda (\phi^{(\varepsilon,M)})^3 - r_\varepsilon \phi^{(\varepsilon,M)} = 2^{1/2}\, \xi^{(\varepsilon,M)}, \tag{4}$$

on $\Lambda_{\varepsilon,M}$ with $\Lambda_{\varepsilon,M} := \mathbb{R}\times\mathbb{T}^d_{\varepsilon,M}$, where $\mathscr{L}_\varepsilon := \partial_t + m^2 + (-\Delta_\varepsilon)^s$ and where $\xi^{(\varepsilon,M)}$ is a (space-time) white noise such that

$$\mathbb{E}[\xi^{(\varepsilon,M)}(t,x)\,\xi^{(\varepsilon,M)}(s,y)] = \delta(t-s)\,\mathbb{1}_{x=y}, \qquad (t,x),(s,y)\in\Lambda_{\varepsilon,M},$$

and $\lambda>0$ and r_ε are constants. By standard stochastic analysis arguments, there exists a unique solution of this equation such that $\mathrm{Law}(\phi^{(\varepsilon,M)}(t)) = \nu_{\varepsilon,M}$ for all $t\in\mathbb{R}$.

The stochastic representation (4) is already quite useful at this point since by PDE estimates in weighted Sobolev spaces and the positivity of the fractional Laplacian one can prove tightness of the family $(\phi^{(\varepsilon,M)})_M$ and therefore the existence of sub-sequential limits as $M\longrightarrow\infty$. This can be done with the arguments in Gubinelli–Hofmanová [GH21]. Any accumulation point $\phi^{(\varepsilon)}$ will be a solution in $\Lambda_\varepsilon := \mathbb{R}\times(\varepsilon\mathbb{Z})^d$ of the infinite system of SDEs

$$\mathscr{L}_\varepsilon \phi^{(\varepsilon)} + \lambda(\phi^{(\varepsilon)})^3 - r_\varepsilon \phi^{(\varepsilon)} = \xi^{(\varepsilon)}, \tag{5}$$

on Λ_ε and where $\xi^{(\varepsilon)}$ is a (space-time) white noise on Λ_ε such that

$$\mathbb{E}[\xi^{(\varepsilon)}(t,x)\,\xi^{(\varepsilon)}(s,y)] = \delta(t-s)\,\mathbb{1}_{x=y}, \qquad (t,x),(s,y)\in\Lambda_\varepsilon. \tag{6}$$

The non-trivial step is now to control the solutions to this equation uniformly as $\varepsilon \longrightarrow 0$. It is expected that in small scales the non-linear term in the dynamics can be considered perturbation of the linear part of the equation with the white-noise source, and therefore also that in the small scale limit, the solutions $\phi^{(\varepsilon)}$ converge locally to distributions in negative Besov spaces of regularity slightly worser than $s-d/2$, which is the kind of regularity allowed by the Gaussian free field. This negative regularity causes problems to control the non-linear term.

Inspired by the work of Wilson and Polchinski [Pol84] on the continuous renormalisation group and by the more recent approach of Duch [Duc21, Duc22], we use a flow equation to effectively describe the solution $\phi^{(\varepsilon)}$ of the SPDE at some spatial scale much larger than ε (see Kupiainen [Kup16] for a discrete counterpart). Let $\phi_\sigma^{(\varepsilon)}$ denote a description of the solution at a scale of order $[\![\sigma]\!] := (1-\sigma) \gg \varepsilon > 0$ for some $\sigma \in (0,1)$, the flow equation approach consists in deriving for $\phi_\sigma^{(\varepsilon)}$ a parabolic equation of the form

$$\mathscr{L}_\varepsilon \phi_\sigma^{(\varepsilon)} = F_\sigma^{(\varepsilon)}(\phi_\sigma^{(\varepsilon)}), \tag{7}$$

where $\psi \longmapsto F_\sigma^{(\varepsilon)}(\psi)$ is an analytic functional of $\xi^{(\varepsilon)}$, the *effective force*, such that for $\sigma = 1$ one recovers the equation (5) with $\phi_1^{(\varepsilon)} = \phi^{(\varepsilon)}$ and

$$F_1^{(\varepsilon)}(\psi) = -\lambda\,\psi^3 + r_\varepsilon\,\psi + \xi^{(\varepsilon)}, \tag{8}$$

The functional $F_\sigma^{(\varepsilon)}$ can be obtained by solving a *flow equation*

$$\partial_\sigma F_\sigma^{(\varepsilon)} = \mathbb{B}_\sigma(F_\sigma^{(\varepsilon)}, F_\sigma^{(\varepsilon)}), \tag{9}$$

backwards for $\sigma \in (\mu, 1]$ with the final condition (8) and where $\mathbb{B}_\sigma$ is an appropriate bilinear operator depending on the specific way the scale-by-scale decomposition $(\phi_\sigma^{(\varepsilon)})_{\sigma\in[0,1]}$ has been introduced. The parameter $\sigma \in [0,1]$ does not have any specific physical meaning and that the spatial scale of the decomposition is fixed conventionally to be of order $[\![\sigma]\!]$, that is $\phi_\sigma^{(\varepsilon)}$ is expected to fluctuate at spatial scales of order $[\![\sigma]\!]$ or larger, and in particular to be a locally bounded function on $\Lambda_0 := \mathbb{R} \times \mathbb{R}^d$ when extended in some reasonable way from the lattice Λ_ε to the continuum. A crucial ingredient is the control of the stochastic process $(F_\sigma^{(\varepsilon)})_\sigma$ given by solutions of the flow equation (9). Following a simple observation of Duch [Duc21, Duc22], this control can be obtained by studying the time evolutions of the cumulants $(\mathscr{F}_\sigma^{(\varepsilon)})_\sigma$ of the random functionals $(F_\sigma^{(\varepsilon)})_\sigma$ which themselves satisfy a kind of higher-order flow equation

$$\partial_\sigma \mathscr{F}_\sigma^{(\varepsilon)} = \mathscr{A}_\sigma(\mathscr{F}_\sigma^{(\varepsilon)}) + \mathscr{B}_\sigma(\mathscr{F}_\sigma^{(\varepsilon)}, \mathscr{F}_\sigma^{(\varepsilon)}),$$

with prescribed initial condition $\mathscr{F}_1^{(\varepsilon)}$. Upon choosing appropriately this initial condition by tuning the parameter r_ε in (8) one can prove uniform in ε estimates for the cumulants $(\mathscr{F}_\sigma^{(\varepsilon)})_\sigma$ and therefore, by a Kolmogorov-type argument, suitable bounds on the effective force $(F_\sigma^{(\varepsilon)})_\sigma$ uniform $\varepsilon \longrightarrow 0$, and with some more work even convergence in law as $\varepsilon \longrightarrow 0$.

Duch's flow equation (9) is bilinear and therefore solvable in general only in a perturbative regime, e.g. in a small interval $I = [\bar{\sigma}, 1]$ near 1 or for small data or small time. Moreover the size of this perturbative region depends crucially on the size of the noise $\xi^{(\varepsilon)}$ and while this dependence can be made uniform in ε there could be large fluctuations in the noise which make the region arbitrarily small and reduce the available proof of existence of solutions to *local* results. A similar difficulty is present in the work of Kupiainen [Kup16] who, instead, uses a discrete renormalisation group (RG) iteration, and in general in all the other approaches which use an expansion of solutions in order to resolve the singular terms and control the limit as $\varepsilon\to 0$, e.g. in regularity structures and also in paracontrolled calculus. This difficulty is the signal of the "large field problem", well known in constructive EQFT. From the point of view of the stochastic quantisation equation, this problem can be solved using the coercivity of the nonlinear term whose sign tends to produce large forces which bring down the solution from infinity. While this observation is standard in PDE theory, it still requires some nontrivial adaptation to be effective for singular SPDEs. The first to solve the problem have been Mourrat and Weber [MW17] in their proof of global existence for the Φ^4_2 dynamics on the full space with the usual Laplacian diffusion term and subsequently Gubinelli and Hofmanová in the context of paracontrolled analysis of Φ^4 models [GH19, GH21] including the parabolic three dimensional setting. Subsequently Chandra, Moinat and Weber [CMW19] provided coercive estimates (the, so called, "coming down from infinity" property) in the full subcritical regime in the framework of regularity structures. In this last paper, the authors consider an SPDE where they modify the covariance of the noise instead of the diffusion term in order to be able to explore the range of regularity arbitrarily near the critical case. The problem with this approach is that the corresponding SPDE in not known to have a reflection positive invariant measure and therefore it cannot be used in the context of stochastic quantisation of Euclidean QFTs, moreover their approach depends heavily on the local nature of the classical Laplacian and therefore seems difficult to adapt to the fractional case or even to the lattice setting.

The main contribution of our work is the individuation of a framework where the flow equation method is coupled to PDE estimates for the dynamics. This combination results in a powerful variation of the RG approach where we are not anymore bound to solve exactly the RG flow equation eq. (9) but we can settle for a suitable approximate solution for which the quantity

$$H_\sigma := \partial_\sigma F^{(\varepsilon)}_\sigma - \mathbb{B}_\sigma(F^{(\varepsilon)}_\sigma, F^{(\varepsilon)}_\sigma), \tag{10}$$

is small enough in an appropriate sense. The price to pay for this approximation is a *remainder* term $R^{(\varepsilon)}_\sigma$ in the SPDE which now reads as a *system* of two equations for the pair of scale-dependent functions $(\phi^{(\varepsilon)}_\sigma, R^{(\varepsilon)}_\sigma)_\sigma$

$$\begin{cases} \mathscr{L}_\varepsilon \phi^{(\varepsilon)}_\sigma = \mathcal{J}_\sigma[F^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma) + R^{(\varepsilon)}_\sigma] \\ \partial_\sigma R^{(\varepsilon)}_\sigma = H^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma) + \mathrm{D}F^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma)(\partial_\sigma G_\sigma) R^{(\varepsilon)}_\sigma, \qquad R^{(\varepsilon)}_1 = 0. \end{cases} \tag{11}$$

where $(\mathcal{J}_\sigma)_\sigma$ is a family of smoothing operators which realises the scale decomposition, $G_\sigma = \mathcal{J}_\sigma \mathscr{L}_\varepsilon^{-1}$ is the localised propagator of the dynamic Gaussian free field (GFF) and $\mathrm{D}F^{(\varepsilon)}_\sigma(\psi)\tilde{\psi}$ denotes the functional derivative of $F^{(\varepsilon)}_\sigma$ at ψ and in the direction of $\tilde{\psi}$ (see below for precise definitions of these objects). Moreover one can prove that the term $F^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma)$ retains the coercive structure of its initial condition, that is,

$$F^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma) = -\lambda(\phi^{(\varepsilon)}_\sigma)^3 + Q^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma),$$

where $Q^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma)$ is smaller than the cubic contribution provided $[\![\sigma]\!] \ll 1$. This together with the linearity in $R^{(\varepsilon)}$ of the second equation of (11) make this system amenable to standard PDE techniques: by choosing $[\![\sigma]\!] \ll 1$ one can control the non-coercive part $Q^{(\varepsilon)}_\sigma(\phi^{(\varepsilon)}_\sigma)$ of the effective force with the coercive part $-\lambda(\phi^{(\varepsilon)}_\sigma)^3$ solving the large field problem. At the same time, since $[\![\sigma]\!] > 0$, we have good estimates for $Q^{(\varepsilon)}_\sigma, \mathrm{D}F^{(\varepsilon)}_\sigma, H^{(\varepsilon)}_\sigma$ which are uniform in $\varepsilon\to 0$ provided the renormalisation constant r_ε is appropriately chosen. This allows the full control of eq. (5) and the proof of tightness of the laws of the processes $(\phi^{(\varepsilon)})_\varepsilon$ and therefore of the family $(\nu^\varepsilon)_\varepsilon$.

Comparison with other approaches. The possibility of working with an approximate flow equation makes it easier to compare the RG approach advocated in this paper with regularity structures and paracontrolled distributions. There is a clear parallel among the various approaches: the flow equation constructs a random object $F_\sigma^{(\varepsilon)}$, the *scale-dependent effective force*, which encodes the effect of the noise and which is a finite polynomial constructed out of the noise term and of the linear part of the equation. It corresponds to the, so-called *model* in the theory of regularity structures or to the *enhanced noise* used in paracontrolled calculus, or even to the *rough path* in rough path analysis. If, on the one hand, $F^{(\varepsilon)}$ is constructed probabilistically, on the other hand, the remainder term R^ε is constructed analytically out of the effective force $F^{(\varepsilon)}$. This deterministic construction clearly corresponds to the analytic machinery in regularity structures and to the paracontrolled operators in the associated calculus, or again to the sewing lemma in controlled rough path theory. When the parameter s is near its critical value of $s_c = 3/4$ the number of terms which have to be accounted for in the approximation $F^{(\varepsilon)}$ of the solution of the flow equation grows in an unbounded manner. An advantage of the flow equation approach, over other methods, is that the analysis is quite insensitive to this growth in complexity of the associated probabilistic object. Indeed the analysis of the flow equation is quite compact and does not really depend strongly on the distance to the critical values of the parameters in play. This "efficiency" of the flow equation to analyse the behaviour of non-linear propagation of randomness was discovered by Polchinski [Pol84] in its proof of perturbative renormalisability of the Euclidean ϕ_4^4 QFT. For a modern account of this approach to perturbation theory of QFTs the reader can consult the book of Salmhofer [Sal07] or Kopper [Kop07]. As we already noted, the application of RG ideas to SPDEs is made very efficient by the observation of Duch [Duc21, Duc22] that the flow equations can be also used to estimate the cumulants and avoid explicit and painful inductive arguments on trees, very much like Polchinski's approach avoids the inductive arguments of BPHZ renormalisation. There is some similarities with recent ideas of Otto, Weber and collaborators of using PDE arguments to obtain systematically the probabilistic estimates for the modes in the theory of regularity structures [OSSW21, LOTT21]. The flow equation approach has however the added advantage that the renormalisation conditions can be analysed in terms of setting the right boundary conditions when solving the flow equation. Recently, the combination of the flow equation approach with stochastic quantisation in the context of the construction of EQFT has been exploited in the Grassmann setting in [DFG22] where the authors developed a new approach to Euclidean Fermionic theories using a forward-backwards stochastic differential equation together with the approximate analysis of the Fermionic analog of Polchinski's flow equation.

Plan of the paper. In Section 2 we introduce the main objects of our analysis: the scale decomposition, the space-time weighted norms which will be used to control the large fields and all the intermediate results which are needed in the proof of Theorem 1. The coercive estimates will be proven in Section 3 while the approximate flow equation for the effective force will be analysed in Section 4 via the flow equation for the cumulants. Finally in Section 5 we explain the modification needed in the proofs to adapt them to the case of the $O(n)$ symmetric vector model. Appendix A contains some technical lemmas while Appendix B contains details on the detailed definition of the various contributions to the flow equations for the cumulants and their analytic estimations.

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1.1 Notations, phase-space decomposition and weights

In this section we shall introduce the main notations we are going to use throughout the paper.

We let $\mathbb{R}_\varepsilon = \varepsilon\mathbb{Z}$ for $\varepsilon > 0$ and $\mathbb{R}_0 = \mathbb{R}$. On $\Lambda_\varepsilon := \mathbb{R} \times \mathbb{R}_\varepsilon^d$ we consider the fractional parabolic distance $|z|_s := |t|^{1/(2s)} + |x|$ where $z = (t, x) \in \Lambda_\varepsilon$ and we denote $\langle z \rangle := (1 + |z|_s^2)^{1/2}$ the so called Japanese bracket.

Denoting with $\mathcal{S}(\Lambda_\varepsilon)$ the space of rapidly decreasing functions over Λ_ε, the Fourier transform is defined, for $f \in \mathcal{S}(\Lambda_\varepsilon)$ as

$$\hat{f}(\omega, k) := \int_{\Lambda_\varepsilon} e^{-i(\omega t + k \cdot x)} f(t, x) \, \mathrm{d}t \, \mathrm{d}x, \qquad (\omega, k) \in \Lambda_\varepsilon^* := \mathbb{R} \times (-\pi/\varepsilon, \pi/\varepsilon)^d,$$

and extended by duality to $\mathcal{S}'(\Lambda_\varepsilon)$ as usual. The inverse Fourier transform is then given by

$$f(t, x) = \int_{\Lambda_\varepsilon^*} \hat{f}(\omega, k) \, e^{i(\omega t + k \cdot x)} \frac{\mathrm{d}\omega \, \mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t, x) \in \Lambda_\varepsilon.$$

For $\varepsilon \geqslant 0$, the (discrete, when $\varepsilon > 0$) Laplacian Δ_ε has symbol given by $\xi \longmapsto -q_\varepsilon(\xi)^2$ where

$$q_\varepsilon(k) := \left[\sum_{i=1}^d \left(\frac{1}{\varepsilon} \sin(\varepsilon k_i) \right)^2 \right]^{1/2}, \tag{12}$$

is the Fourier multiplier corresponding to the operator $(-\Delta_\varepsilon)^{1/2}$. When $\varepsilon = 0$ we have $q_0(k) = |k|$.

The fractional Laplacian

For $s \in (0, 1)$ the (negative) fractional Laplacian $(-\Delta_\varepsilon)^s$ is defined as the Fourier multiplier with symbol $\xi \in (\mathbb{R}_\varepsilon^d)^* \longmapsto q_\varepsilon^{2s}(\xi)$ (cfr. Eq. (12)). In particular it is self-adjoint and positive in $L^2(\mathbb{R}_\varepsilon^d)$ and for $s \in (0, 1]$ it has the (discrete, when $\varepsilon > 0$) heat–kernel representation [Kwa17]

$$(-\Delta_\varepsilon)^s f = C_s \int_{\mathbb{R}_+} (f - e^{\theta \Delta_\varepsilon} f) \, \theta^{-1-s} \mathrm{d}\theta,$$

with the constant $C_s = |\Gamma(-s)|^{-1}$. In the continuum, the fractional Laplacian has, for $s \in (0, 1)$, the integral representation [Kwa17]:

$$(-\Delta_0)^s f(x) = C_{d,s} P.V. \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+2s}} \mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{13}$$

where $C_{d,s}$ is an universal constant. In the discrete setting a similar formula holds [CRS+15]:

$$(-\Delta_\varepsilon)^s f(x) = \varepsilon^d \sum_{y \in \mathbb{R}_\varepsilon^d : y \neq x} K(x - y) (f(x) - f(y)), \qquad x \in \mathbb{R}_\varepsilon^d, \tag{14}$$

where the kernel $K \colon \mathbb{R}_\varepsilon^d \longrightarrow \mathbb{R}$ is positive and such that

$$|K(x)| \leqslant C'_{d,s} |x|^{-d-2s}, \qquad x \in \mathbb{R}_\varepsilon^d,$$

uniformly in ε for some constant $C'_{d,s} > 0$.

We can encode the integral representation, Eq (13) or Eq. (14), of the fractional Laplacian, via a positive measure ν_s on $\Lambda \times \Lambda$ (depending on ε and symmetric for the exchange $z \leftrightarrow z'$) for which

$$\langle f, (-\Delta)^s g \rangle = \int \nu_s(\mathrm{d}z \, \mathrm{d}z') \, f(z) \, (g(z) - g(z')).$$

We also define the kernel $\nu_s(z, \mathrm{d}z')$ arising from the disintegration of ν_s as

$$\nu_s(z, \mathrm{d}z') \mathrm{d}z = \nu_s(\mathrm{d}z \, \mathrm{d}z'). \tag{15}$$

Note that it is a symmetric kernel, i.e. $\nu_s(z, \mathrm{d}z')\mathrm{d}z = \nu_s(z', \mathrm{d}z)\mathrm{d}z'$. With these notations we have also the following Leibnitz rule with a remainder

$$I_s(f,g) := (-\Delta)^s(fg) - f\,(-\Delta)^s g - g\,(-\Delta)^s f = \int_{\mathbb{R}^d} (f(z)-f(z'))\,(g(z)-g(z'))\,\nu_s(z,\mathrm{d}z'). \tag{16}$$

Remark 2. A basic observation is that the fractional Laplacian (whether in the continuum or on the lattice) satisfies an inequality w.r.t. application of convex functions. Let $\Phi: \mathbb{R} \longrightarrow \mathbb{R}$ be a convex C^1 function and Φ' one of its sub-differentials, then, for any $\varepsilon \geqslant 0$ and $u \in C^2(\Lambda_\varepsilon)$ we have

$$(-\Delta_\varepsilon)^s \Phi(u) \leqslant \Phi'(u)\,(-\Delta_\varepsilon)^s u.$$

Indeed, note that $e^{\theta\Delta_\varepsilon}$ has a positive definite kernel. Let $\Phi: \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function, then

$$\Phi(a) - \Phi(b) \leqslant \Phi'(a)\,(a-b), \qquad a, b \in \mathbb{R},$$

so if $u: \Lambda \longrightarrow \mathbb{R}$ is a continuous function we have

$$\Phi(u) - e^{\theta\Delta_\varepsilon}\Phi(u) \leqslant \Phi'(u)\,(u - e^{\theta\Delta_\varepsilon} u),$$

and therefore the claimed inequality. It is clear that the same proof works for $\varepsilon = 0$. We will incorporate this idea in the proof of the key Lemma 16, below.

The operator $G^\varepsilon = \mathscr{L}_\varepsilon^{-1}$ is defined as

$$(\mathscr{L}_\varepsilon^{-1} f)(t) := \int_{-\infty}^{t} e^{-(m^2 + (-\Delta_\varepsilon)^s)(t-u)} f(u)\mathrm{d}u, \qquad t \in \mathbb{R},$$

and will be applied to continuous function of time with at most a limited polynomial growth in space-time. On account of Lemma 5.4 of [Gri03] together with the argument of Section 1 of [GT01], the kernel $G^\varepsilon(t,x)$ of G^ε satisfies

$$G^\varepsilon(t,x) \lesssim e^{-m^2 t} \min\left\{ t^{-\frac{d}{2s}}, \frac{t}{|x|^{d+2s}} \right\} \lesssim \frac{t\, e^{-c m^2 t}}{(|x| + |t|^{1/2s})^{d+2s}} \tag{17}$$

uniformly in $\varepsilon \geqslant 0$.

Scale decomposition

Let us introduce now a scale decomposition of space–time functions parametrized by $\sigma \in [0, 1]$ and where we let $[\![\sigma]\!] := (1-\sigma)$ for convenience. The value $\sigma = 1$ corresponds to allowing fluctuations at all scales while $\sigma < 1$ only at spatial scales $\gtrsim [\![\sigma]\!]$ or equivalently at Fourier scales $\lesssim [\![\sigma]\!]^{-1}$.

Definition 3. *Consider a smooth and compactly supported function $j: \mathbb{R} \longrightarrow \mathbb{R}_+$ such that*

$$j(\eta) = \begin{cases} 1 & \text{if} \quad \eta \leqslant 1, \\ 0 & \text{if} \quad \eta \geqslant 2, \end{cases}$$

and which is chosen in a Gevrey class G^r for $r > 1$, see [GMR21, MW17] for details. For $\ell = 0, 1, 2, \ldots$ denote

$$j_{\sigma,\ell}(\eta) := j(2^{-\ell} \sigma^{-1} [\![\sigma]\!]\, \eta), \qquad \eta \in \mathbb{R},$$

and let $j_\sigma := j_\sigma, 0$. Note that $j_{\sigma,\ell}(\eta) j_{\sigma,\ell'}(\eta) = j_{\sigma,\ell}(\eta)$ for $0 \leqslant \ell < \ell'$.

Definition 4. *We introduce a family $(\mathfrak{J}_\sigma^{(\varepsilon)})_{\sigma \in (0,1)}$ of Fourier multipliers acting on distributions $f \in \mathscr{S}'(\Lambda_\varepsilon)$ as*

$$\mathfrak{J}_\sigma f(t,x) = \mathfrak{J}_\sigma^{(\varepsilon)} f(t,x) := \int_{\Lambda_\varepsilon^*} j_\sigma((|\omega|^{1/2s} + q_\varepsilon(k)))\, \hat{f}(\omega,k)\, e^{i(\omega t + k \cdot x)} \frac{\mathrm{d}\omega \mathrm{d}k}{(2\pi)^{d+1}}, \qquad (t,x) \in \Lambda_\varepsilon. \tag{18}$$

In addition, for $\ell \in \mathbb{N}$, we introduce a family of smoothing operator, $\tilde{\mathfrak{J}}_{\sigma,\ell}^{(\varepsilon)}$ for $\sigma \in (0,1)$ and $\ell = 1, 2, \ldots$ which is as above but constructed out of the function $j_{\sigma,\ell}$. Moreover we let $\tilde{\mathfrak{J}}_\sigma^{(\varepsilon)} := \tilde{\mathfrak{J}}_{\sigma,1}^{(\varepsilon)}$.

In order to lighten the notation, in the following we shall leave understood the dependence on ε of $\mathcal{J}_\sigma^{(\varepsilon)}$ by using $\mathcal{J}_\sigma$.

Note that $\mathcal{J}_\sigma^{(\varepsilon)} f \longrightarrow f$ as $\sigma \nearrow 1$ and that $\mathcal{J}_\sigma^{(\varepsilon)} f \longrightarrow 0$ as $\sigma \searrow 0$ in $\mathcal{S}'(\Lambda_\varepsilon)$. We let $q_0(k) := |k|$ so that $q_\varepsilon(k) \longrightarrow q_0(k)$ point-wise for $k \in \mathbb{R}^d$. We observe that, on account of the above definitions, for any $\ell \in \mathbb{N}$ and $\sigma \in (0,1)$, it holds that

$$\tilde{\mathcal{J}}_{\sigma,\ell} \mathcal{J}_\sigma = \mathcal{J}_\sigma, \qquad \tilde{\mathcal{J}}_{\sigma,\ell+1} \tilde{\mathcal{J}}_{\sigma,\ell} = \tilde{\mathcal{J}}_{\sigma,\ell}. \tag{19}$$

Remark 5. We observe that the operators $\tilde{\mathcal{J}}_{\sigma,\ell}$ and $\mathcal{J}_\sigma$ are contractions in the class of L^p spaces thanks to the Young convolution inequality. This property holds true also in the setting of weighted L^p spaces, see [MW17, GH19] for further details. In particular this is true for polynomial weights as well as for the stretched exponential weights which we will soon introduce.

Remark 6. We shall use the fact that if $\sigma < \mu_i$, where μ_i is as in Definition 8 below, then $\mathcal{J}_\sigma \mathcal{J}_{\mu_{i+1}} = \mathcal{J}_\sigma$. This is a consequence of the very definition of $\mathcal{J}_\sigma$, μ_i and of the fact that, as $\sigma < \mu_i$,

$$2\,\sigma\,[\![\sigma]\!]^{-1} \leqslant 2\,\mu_i\,[\![\mu_i]\!]^{-1} \approx 2\,(2^i - 1) = 2^{i+1} - 2 < 2^{i+1} - 1 \approx \mu_{i+1}\,[\![\mu_{i+1}]\!]^{-1},$$

yielding $j_\sigma(\eta)\, j_{\mu_{i+1}}(\eta) = j_\sigma(\eta)$ for any $\eta \in \mathbb{R}$.

In the tightness argument we will also need a Littlewood–Paley decomposition which acts only on the space variables:

Definition 7. *(Spatial LP blocks) Let $(\hat{\Delta}_i : \mathbb{R} \longrightarrow \mathbb{R}_+)_{i \geqslant -1}$ be a dyadic partition of unity for $\mathbb{R}$, where $\hat{\Delta}_{-1}$ is supported in a ball centred in 0 of radius $\frac{1}{2}$, $\hat{\Delta}_0$ is supported in an annulus and $\hat{\Delta}_i(\cdot) := \hat{\Delta}_0(2^{-i}\cdot)$ such that if $|i-j| > 1$, $\operatorname{supp}(\hat{\Delta}_i) \cap \operatorname{supp}(\hat{\Delta}_j) = \emptyset$. In addition we also require the functions $\hat{\Delta}_{-1}$ and $\hat{\Delta}_0$ to be in a Gevrey class G^r for $r > 1$. Out of this partition of unity we define Littlewood-Paley $(\Delta_i^x)_{i \geqslant -1}$ blocks for $\mathbb{R}_\varepsilon^d$ as the Fourier multipliers associated with $\hat{\Delta}_i(q_\varepsilon(k))$.*

Space-time weights

To analyse the global in space behaviour of the solutions to the SQ equation we shall use several classes of space-time localisation.

Definition 8. *(Weights)*

a) Fix $\nu > 0$ and let

$$\rho(z) := \langle z \rangle^{-\nu},$$

be a polynomial weight where $z = (t,x) \in \Lambda$.

b) Let $(\chi_i : \Lambda \longrightarrow \mathbb{R}_+)_{i \in \mathbb{Z}_+}$ a dyadic partition of unity for Λ with χ_i supported on an annulus of radius $\sim 2^{ai}$ for $i \geqslant 0$, $a > 0$ and $\sum_i \chi_i = 1$.

c) Let $(\tilde{\chi}_i : \Lambda \longrightarrow \mathbb{R}_+)_{i \in \mathbb{Z}_+}$ be a family of smooth and compactly supported functions such that for any $i \in \mathbb{Z}_+$, $\tilde{\chi}_i|_{\operatorname{supp}(\chi_i)} = 1$, where $\operatorname{supp}(\chi_i)$ denotes the support of the function χ_i introduced above.

d) Fix increasing numbers $\{\mu_i\}_{i \geqslant 0} \subseteq (0,1)$ such that $\mu_i \nearrow 1$ and $[\![\mu_i]\!] \approx 2^{-i}$, $i \in \mathbb{Z}_+$.

e) For $i \in \mathbb{N}$, let $\zeta_i : \Lambda \longrightarrow \mathbb{R}$ radial cutoff functions of the form

$$\zeta_i(z) := e^{-c(|z|_s - 2^{ai})_+^\omega}, \qquad z \in \Lambda,$$

with $\omega = 1/r < 1$ where $r > 1$ is the parameter of the Gevrey class of the cutoff function j in eq. (18) and where the constant $c > 0$ will be chosen to be small enough. We observe that $\zeta_i(z)$ is identically equal to one for $|z|_s \leqslant [\![\mu_i]\!]^{-a}$ while it vanishes (stretched-)exponentially fast for larger $|z|_s$.

f) For $j\in\mathbb{N}$, let

$$\rho_j(z):=(2^{2aj}+(|z|_s^2-2^{2aj})_+)^{-\nu/2}.$$

This is a polynomial weight which is $2^{-j\gamma}$ up to scale $|z|_s\approx 2^{aj}$ while it decays as $2^{-k\gamma}$ for $|z|_s\approx 2^{ak}$ with $k>j$.

Remark 9. These weights satisfy the following properties.

a) For any $i\in\mathbb{Z}_+$ it holds that $\chi_i^n\lesssim\zeta_i$ for any $n\geqslant 1$.

b) For any $z,z_1\in\Lambda$,

$$\rho(z)\,\rho^{-1}(z_1)\lesssim\rho^{-1}(z-z_1). \tag{20}$$

Moreover analogous property is also satisfied by the weights ζ_i and ρ_i uniformly in i. Indeed, since $\omega<1$, we have, for any i,

$$\begin{aligned}\zeta_i(z)\,\zeta_i^{-1}(z_1) &\leqslant \exp\{c\,[(|z_1-z|_s+|z|_s-2^{ai})_+^{\omega}-(|z|_s-2^{ai})_+^{\omega}]\}\\ &\leqslant \exp\{c\,|z-z_1|_s^{\omega}\}=:\zeta^{-1}(z-z_1).\end{aligned} \tag{21}$$

and similarly

$$\begin{aligned}\rho_j(z)\,\rho_j^{-1}(z_1) &\lesssim \left(\frac{2^{2aj}+(|z|_s^2-2^{2aj})_++|z_1-z|_s^2}{2^{2aj}+(|z|_s^2-2^{2aj})_+}\right)^{\nu/2}\lesssim\left(1+\frac{|z_1-z|_s^2}{2^{2aj}+(|z|_s^2-2^{2aj})_+}\right)^{\nu/2}\\ &\lesssim (1+|z_1-z|_s^2)^{\nu/2}=\rho^{-1}(z-z_1).\end{aligned} \tag{22}$$

c) The parameter $\gamma>0$ shall be fixed in Section 4. Concerning the parameters ν and a introduced in Definition 8, we shall require that $a\nu=\gamma$. For future convenience we observe that this implies that for any $n\in\mathbb{R}$,

$$\|\zeta_i\rho^{-n}\|_{L^\infty}+\|\chi_i\rho^{-n}\|_{L^\infty}\lesssim[\![\mu_i]\!]^{-n\gamma}.$$

As a consequence, since the only fixed parameter will be γ, this grants some freedom in the choice of ν. In particular, as we shall discuss in some technical lemmas in the following, *e.g.*, Lemma 16, Lemma 35 and 41, we fix ν such that $\nu\in(0,s/3)$.

2 Stochastic quantisation

In this section we lay out the main steps in the proof of Theorem 1, starting from the effective equation at (fractional, parabolic) space–time scale $[\![\sigma]\!]:=(1-\sigma)$, $\sigma\in(0,1)$ obtained via a scale decomposition and the introduction of the notion of approximate effective force.

2.1 Scale decomposition

Let $\phi^{(\varepsilon)}$ be a stationary solution to the infinite system of SDEs (5) and define the scale-dependent field $\phi_\sigma^\varepsilon:=\mathcal{J}_\sigma\phi^{(\varepsilon)}$ localised at (fractional, parabolic) space-time scales $\gtrsim[\![\sigma]\!]$. It is a solution to

$$\mathscr{L}_\varepsilon\phi_\sigma^\varepsilon=\mathcal{J}_\sigma(F^\varepsilon(\phi^{(\varepsilon)})),$$

where

$$F^\varepsilon(\psi):=-\lambda(\psi)^3-r_\varepsilon\,\psi+\xi^{(\varepsilon)}.$$

Let $\mathscr{E}^0:=C(\Lambda_\varepsilon)$ the space of continuous functions on Λ_ε with at most polynomial growth at infinity wrt. the space-time fractional norm. Consider a family $(F_\sigma^\varepsilon\colon\mathscr{E}^0\longrightarrow\mathscr{S}'(\Lambda_\varepsilon))_{\sigma\in[0,1]}$ of functionals, differentiable in $\sigma\in(0,1)$ and such that $F_1^\varepsilon=F^\varepsilon$. Using that $\phi^{(\varepsilon)}=\phi_1^\varepsilon$ we have

$$F^\varepsilon(\phi^\varepsilon)=F_1^\varepsilon(\phi_1^\varepsilon)=F_\mu^\varepsilon(\phi_\mu^\varepsilon)+R_\mu^\varepsilon,$$

for all $\mu\in[0,1]$, where

$$R^{\varepsilon}_{\mu} := \int_{\mu}^{1}[\partial_{\sigma}F^{\varepsilon}_{\sigma}(\phi^{\varepsilon}_{\sigma}) + \mathrm{D}F^{\varepsilon}_{\sigma}(\phi^{\varepsilon}_{\sigma})(\partial_{\sigma}\phi^{\varepsilon}_{\sigma})]\mathrm{d}\sigma,$$

and where $\mathrm{D}F_{\sigma}(\psi)\psi'$ denotes the Fréchet derivative of F_{σ} in the direction of $\psi'\in\mathscr{C}^0$ at the point $\psi\in\mathscr{C}^0$. Since we also have

$$\partial_{\sigma}\phi^{\varepsilon}_{\sigma} = \dot{G}_{\sigma}(F^{\varepsilon}(\phi^{(\varepsilon)})) = \dot{G}_{\sigma}(F^{\varepsilon}_{\sigma}(\phi^{\varepsilon}_{\sigma}) + R^{\varepsilon}_{\sigma}),$$

with $\dot{G}_{\sigma} := \mathscr{L}_{\varepsilon}^{-1}\dot{\mathcal{J}}_{\sigma}$ and $\dot{\mathcal{J}}_{\sigma} := \partial_{\sigma}\mathcal{J}_{\sigma}$, we deduce that the pair $(\phi^{\varepsilon}_{\mu}, R^{\varepsilon}_{\mu})_{\mu\in(0,1)}$ satisfies the system of equations

$$\begin{cases} \mathscr{L}_{\varepsilon}\phi^{\varepsilon}_{\mu} = \mathcal{J}_{\mu}(F^{\varepsilon}_{\mu}(\phi^{\varepsilon}_{\mu}) + R^{\varepsilon}_{\mu}) \\ R^{\varepsilon}_{\mu} = \int_{\mu}^{1} H^{\varepsilon}_{\sigma}(\phi^{\varepsilon}_{\sigma})\mathrm{d}\sigma + \int_{\mu}^{1}[\mathrm{D}F^{\varepsilon}_{\sigma}(\phi^{\varepsilon}_{\sigma})\,\dot{G}_{\sigma}R^{\varepsilon}_{\sigma}]\mathrm{d}\sigma, \end{cases} \tag{23}$$

where

$$H^{\varepsilon}_{\sigma}(\psi) := \partial_{\sigma}F^{\varepsilon}_{\sigma}(\psi) + \mathrm{D}F^{\varepsilon}_{\sigma}(\psi)\,\dot{G}_{\sigma}F^{\varepsilon}_{\sigma}(\psi),$$

for any choice of $(F^{\varepsilon}_{\sigma})_{\sigma\in[0,1]}$. Our main goal will be that of showing that this system allows for good a-priori estimates for a well chosen trajectory $(F^{\varepsilon}_{\sigma})_{\sigma\in[0,1]}$.

Remark 10. Apart from Section 2.3, in the rest of the paper we keep $\varepsilon>0$ fixed. The analysis which follows is, however, quite insensitive to the fact that the spatial domain is discrete or continuous. The main differences lie in the explicit form of the Fourier multiplier operators $(\mathcal{J}_{\sigma})_{\sigma}$ and $(\dot{G}_{\sigma})_{\sigma}$ together with the different action of the discrete fractional Laplacian $(-\Delta_{\varepsilon})^s$ with respect to its continuous counterpart $(-\Delta_0)^s$. The qualitative behaviour of these objects will be the same and all the estimates we obtain will involve constants which are uniform in the lattice spacing ε unless stated otherwise. In order to lighten the exposition we will suppress the explicit dependence of the domain Λ_{ε} on the lattice spacing ε and writing Λ for either $\Lambda_0 := \mathbb{R}^{d+1}$ or Λ_{ε} with $\varepsilon>0$. Similarly we write $(-\Delta)^s$ for either the discrete or the continuous fractional Laplacian.

2.2 A-priori estimates in weighted spaces

We introduce weighted norms for the function $\psi\in\mathcal{S}'(\Lambda)$ as

$$|||\psi||| = |||\psi|||_{\bar{\mu}} := \inf \sup_{\sigma\in[\bar{\mu},1)} \left\{ \|\rho\,\psi^{<}_{\sigma}\|_{L^{\infty}} + [\![\sigma]\!]^{\gamma}\,\|\psi^{\geqslant}_{\sigma}\|_{L^{\infty}} \right\}, \tag{24}$$

where the infimum is taken over all decompositions $\psi^{<}_{\sigma} + \psi^{\geqslant}_{\sigma} = \mathcal{J}_{\sigma}\psi$, $\sigma\in(0,1)$. Here $\bar{\mu}\in(0,1)$ is a (possibly random) parameter which will be tuned later on depending on the size of the noise.

We denote by $\mathscr{X}\subseteq\mathcal{S}'(\Lambda)$ the space of distributions ψ for which the norm $|||\psi|||$ is finite for some $\bar{\mu}$. The norm (24) is inspired by a similar decomposition in [GH19]: the fields are decomposed in a part $\phi^{<}_{\sigma}$ which is regular but which grows in space and in a part $\phi^{\geqslant}_{\sigma}$ which is irregular (i.e. blows up in L^{∞} as $\sigma\nearrow 1$) but stays bounded at large distances. In Section 3 we will prove the following coercive estimates for fractional parabolic equation with cubic non-linearity which allows us to control the large values of the fields.

Theorem 11. *(Coercive estimate) Assume $\phi\in\mathscr{X}$ and let*

$$f_{\sigma} := \partial_t\phi_{\sigma} + (-\Delta)^s\phi_{\sigma} + \lambda\,\phi^3_{\sigma},$$

then, if $\bar{\mu}$ is large enough (depending only on λ), we have

$$|||\phi|||_{\bar{\mu}} \lesssim \lambda^{-1/2}\,[\![\bar{\mu}]\!]^{\gamma/2} + \lambda^{-1/3}\,|||f|||^{1/3}_{\#,\bar{\mu}} + \lambda^{-1/3}\,[\![\bar{\mu}]\!]^{\gamma/3}\,|||\mathscr{L}\phi|||^{1/3}_{\#,\bar{\mu}},$$

with an universal constant, where the norm on the r.h.s. is defined as (cf. Definition 8)

$$|||f|||_{\#} = |||f|||_{\#,\bar{\mu}} := \sup_{i\geqslant 0} \sup_{\sigma \geqslant \mu_i \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_i^3 f_\sigma\|_{L^\infty}. \tag{25}$$

Proof. For $\bar{\mu}$ larger than some deterministic constant, using successively Lemma 37, Theorem 15, Lemma 39 and Lemma 41 we have

$$\begin{aligned} |||\phi|||_{\bar{\mu}} &\lesssim \sup_{i\geqslant 0} \left[\mathbb{1}_{\sigma\geqslant\mu_i} [\![\sigma]\!]^{\gamma} \|\chi_i \phi_\sigma\|_{L^\infty}\right]. \\ &\lesssim \sup_{i\geqslant 0} \left\{\mathbb{1}_{\sigma\geqslant\mu_i} [\![\sigma]\!]^{\gamma} \left[\lambda^{-1/2} + \lambda^{-1/3}\left(\|\chi_i^3 f_\sigma\| + 2^{\gamma i}\|\rho_i\phi_\sigma\| + 2^{3\gamma i}\|D(\rho_i\phi_\sigma)\|\right)^{1/3}\right]\right\} \\ &\lesssim \lambda^{-1/2}[\![\bar{\mu}]\!]^{\gamma} + \lambda^{-1/3}|||f|||_{\#}^{1/3} + \lambda^{-1/3}[\![\bar{\mu}]\!]^{\gamma/3}|||\phi|||^{1/3} + \lambda^{-1/3}[\![\bar{\mu}]\!]^{\gamma/3}|||\mathscr{L}\phi|||_{\#}^{1/3} \end{aligned}$$

Then, by Young's inequality

$$|||\phi||| \lesssim \lambda^{-1/2}[\![\bar{\mu}]\!]^{\gamma/2} + \lambda^{-1/3}|||f|||_{\#}^{1/3} + \frac{1}{3}|||\phi||| + \lambda^{-1/3}[\![\bar{\mu}]\!]^{\gamma/3}|||\mathscr{L}\phi|||_{\#}^{1/3}$$

and taking the term $\frac{1}{3}|||\phi|||$ on the l.h.s. we conclude. □

The two norms $|||\phi|||, |||f|||_{\#}$ in Theorem 11 fix the analytical setting for the global analysis of the SPDE (23).

Theorem 12. *Let (ϕ, R) be a solution of the equation*

$$\begin{cases} \mathscr{L}\phi_\mu = \mathfrak{J}_\mu(F_\mu(\phi_\mu) + R_\mu) \\ R_\mu = \int_\mu^1 H_\sigma(\phi_\sigma)\mathrm{d}\sigma + \int_\mu^1 [\mathrm{D}F_\sigma(\phi_\sigma)\,\dot{G}_\sigma R_\sigma]\mathrm{d}\sigma, \end{cases} \tag{26}$$

where

$$H_\sigma(\psi) := \partial_\sigma F_\sigma(\psi) + \mathrm{D}F_\sigma(\psi)\,\dot{G}_\sigma F_\sigma(\psi).$$

Assume that $(F_\sigma)_\sigma$ satisfies the estimates, for $\sigma \geqslant \nu \geqslant \mu_i \vee \bar{\mu}$,

$$\begin{aligned} \|\chi_i^3[\mathfrak{J}_\sigma F_\sigma(\psi_\sigma) - (-\lambda\psi_\sigma^3)]\|_{L^\infty} &\leqslant [\![\sigma]\!]^{-3\gamma+\zeta}\left[C_F(1+|||\psi|||)^M + (1+|||\psi|||)^2|||\mathscr{L}\psi|||_{\#}\right], \\ \|\zeta_i\tilde{\mathfrak{J}}_\nu H_\sigma(\psi_\sigma)\|_{L^\infty} &\leqslant C_F[\![\sigma]\!]^{\zeta-1}(1+|||\psi|||)^M, \\ \|\zeta_i\tilde{\mathfrak{J}}_\nu \mathrm{D}F_\sigma(\psi_\sigma)\,\dot{G}_\sigma R_\sigma\|_{L^\infty} &\leqslant C_F[\![\sigma]\!]^{\zeta-1}(1+|||\psi|||)^M\|\zeta_i\tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty}, \end{aligned} \tag{27}$$

for some finite $M, \zeta, C_F > 0$. Then there exists universal constants $\hat{C}, C^{\#}, \Phi > 0$ such that, for all $\bar{\mu}$ satisfying

$$[\![\bar{\mu}]\!]^{\zeta} \leqslant \hat{C}C_F^{-1}, \qquad [\![\bar{\mu}]\!]^{\zeta} \leqslant C^{\#},$$

we have

$$|||\phi|||_{\bar{\mu}} \lesssim 1, \qquad |||\mathscr{L}\phi|||_{\#} \lesssim 1, \qquad \sup_i \sup_{\mu\geqslant\mu_i\vee\bar{\mu}} \|\zeta_i\tilde{\mathfrak{J}}_\mu R_\mu\|_{L^\infty} \lesssim 1.$$

Proof. Consider R first. From the Equation (26) and exploiting Remark 5, we obtain

$$\|\zeta_i\mathfrak{J}_\mu R_\mu\|_{L^\infty} \leqslant \|\zeta_i\tilde{\mathfrak{J}}_\mu R_\mu\|_{L^\infty} \leqslant \int_\mu^1 \|\zeta_i\tilde{\mathfrak{J}}_\mu H_\sigma(\phi_\sigma)\|_{L^\infty}\mathrm{d}\sigma + \int_\mu^1 \|\zeta_i\tilde{\mathfrak{J}}_\mu \mathrm{D}F_\sigma(\phi_\sigma)\,\dot{G}_\sigma R_\sigma\|_{L^\infty}\mathrm{d}\sigma,$$

then using Equation (27) we get

$$\|\zeta_i\tilde{\mathfrak{J}}_\mu R_\mu\|_{L^\infty} \leqslant C_F[\![\mu]\!]^{\zeta}(1+|||\phi|||)^M + C_F(1+|||\phi|||)^M\int_\mu^1 [\![\sigma]\!]^{\zeta-1}\|\zeta_i\tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty}\mathrm{d}\sigma,$$

and by Gronwall inequality we get, for $\mu \geqslant \mu_i \vee \bar{\mu}$,

$$\|\zeta_i \tilde{\mathcal{J}}_\mu R_\mu\|_{L^\infty} \leqslant C_F [\![\bar{\mu}]\!]^\zeta (1 + |||\phi|||)^M \exp(C C_F [\![\bar{\mu}]\!]^\zeta (1 + |||\phi|||)^M),$$

and therefore, by Remark 9 (a),

$$|||\sigma \mapsto \mathcal{J}_\sigma R_\sigma|||_\# \lesssim C_F [\![\bar{\mu}]\!]^{\zeta + 3\gamma} (1 + |||\phi|||)^M \exp(C C_F [\![\bar{\mu}]\!]^\zeta (1 + |||\phi|||)^M). \tag{28}$$

Let's bound now $|||\mathscr{L}\phi|||_\#$. By triangular inequality, it holds that

$$|||\mathscr{L}\phi|||_\# \lesssim |||\sigma \mapsto (\mathcal{J}_\sigma F_\sigma(\phi_\sigma) + \lambda \phi_\sigma^3)|||_\# + |||\sigma \mapsto \lambda \phi_\sigma^3|||_\# + |||\sigma \mapsto \mathcal{J}_\sigma R_\sigma|||_\#.$$

The last term has been just estimated, for the other two we have

$$|||\sigma \mapsto \lambda \phi_\sigma^3|||_\# = \lambda \sup_i \sup_{\sigma \geqslant \mu_i \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_i^3 \phi_\sigma^3\| \lesssim \lambda \sup_i |||\phi|||_{\mu_i \vee \bar{\mu}}^3 \lesssim \lambda |||\phi|||_{\bar{\mu}}^3,$$

and, on account of the first bound in Equation (27),

$$|||\sigma \mapsto (\mathcal{J}_\sigma F_\sigma(\phi_\sigma) + \lambda \phi_\sigma^3)|||_\# \lesssim [\![\bar{\mu}]\!]^\zeta C_F (1 + |||\phi|||)^M + [\![\bar{\mu}]\!]^\zeta (1 + |||\phi|||)^2 |||\mathscr{L}\phi|||_\#.$$

Overall we have

$$|||\mathscr{L}\phi|||_\# \lesssim \lambda |||\phi|||^3 + [\![\bar{\mu}]\!]^\zeta C_F (1 + |||\phi|||)^M + [\![\bar{\mu}]\!]^\zeta (1 + |||\phi|||)^2 |||\mathscr{L}\phi|||_\# + |||\sigma \mapsto \mathcal{J}_\sigma R_\sigma|||_\#. \tag{29}$$

Finally let us put all together using Theorem 11 to get

$$\begin{aligned}
|||\phi|||^3 \;\lesssim\; & \lambda^{-3/2} [\![\bar{\mu}]\!]^{3\gamma/2} + \lambda^{-1} \sup_i \sup_{\sigma \geqslant \mu_i \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_i^3 (\mathcal{J}_\sigma (F_\sigma(\phi_\sigma)) + \lambda \phi_\sigma^3)\| \\
& + \lambda^{-1} \sup_i \sup_{\sigma \geqslant \mu_i \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\zeta_i \mathcal{J}_\sigma R_\sigma\| + \lambda^{-1} [\![\bar{\mu}]\!]^\gamma |||\mathscr{L}\phi|||_\# \\
\lesssim\; & \lambda^{-3/2} [\![\bar{\mu}]\!]^{3\gamma/2} + \lambda^{-1} C_F [\![\bar{\mu}]\!]^\zeta [(1 + |||\phi|||)^M + (1 + |||\phi|||)^2 |||\mathscr{L}\phi|||_\#] + \lambda^{-1} |||\sigma \mapsto \mathcal{J}_\sigma R_\sigma|||_\#,
\end{aligned} \tag{30}$$

where in the second inequality we exploited Equation (27) and $[\![\bar{\mu}]\!] \leqslant 1$. And therefore, by Equations (29) and (30),

$$\begin{aligned}
& |||\phi|||^3 / \Phi^3 + c \lambda^{-1} |||\mathscr{L}\phi|||_\# / \Phi_{\mathscr{L}} \\
\leqslant\; & \lambda^{-3/2} [\![\bar{\mu}]\!]^{3\gamma/2} + \lambda^{-1} [\![\bar{\mu}]\!]^\zeta \left[C_F (1 + |||\phi|||)^M + (1 + |||\phi|||)^2 |||\mathscr{L}\phi|||_\# \right] + \lambda^{-1} |||\sigma \mapsto \mathcal{J}_\sigma R_\sigma|||_\# \\
& + c \lambda^{-1} \left\{ \lambda |||\phi|||^3 + [\![\bar{\mu}]\!]^\zeta \left[C_F (1 + |||\phi|||)^M + (1 + |||\phi|||)^2 |||\mathscr{L}\phi|||_\# \right] + |||\sigma \mapsto \mathcal{J}_\sigma R_\sigma|||_\# \right\} \\
\leqslant\; & \lambda^{-3/2} [\![\bar{\mu}]\!]^{3\gamma/2} + c |||\phi|||^3 + (1 + c) \lambda^{-1} [\![\bar{\mu}]\!]^\zeta \left[C_F (1 + |||\phi|||)^M + (1 + |||\phi|||)^2 |||\mathscr{L}\phi|||_\# \right] \\
& + (1 + c) \lambda^{-1} C_F [\![\bar{\mu}]\!]^{\zeta + 3\gamma} (1 + |||\phi|||)^M \exp\left(C C_F [\![\bar{\mu}]\!]^\zeta (1 + |||\phi|||)^M \right),
\end{aligned}$$

with, possibly large constants $\Phi, \Phi_{\mathscr{L}} \geqslant 1$ and with c chosen small so that $c \leqslant \Phi^{-3}/2$ and $c \leqslant 1$ to obtain

$$\begin{aligned}
|||\phi|||^3 / 2\Phi^3 + c \lambda^{-1} |||\mathscr{L}\phi|||_\# / \Phi_{\mathscr{L}} \;\leqslant\; & \lambda^{-3/2} [\![\bar{\mu}]\!]^{3\gamma/2} + 2 \lambda^{-1} [\![\bar{\mu}]\!]^\zeta \left[C_F (1 + |||\phi|||)^M + (1 + |||\phi|||)^2 |||\mathscr{L}\phi|||_\# \right] \\
& + 2 \lambda^{-1} C_F [\![\bar{\mu}]\!]^{\zeta + 3\gamma} (1 + |||\phi|||)^M \exp\left(C C_F [\![\bar{\mu}]\!]^\zeta (1 + |||\phi|||)^M \right).
\end{aligned} \tag{31}$$

We observe that the constant M appearing in Equation (31) is not the same of Equation (30), nonetheless, as the precise value of M is not important, we shall keep this notation. Let $\bar{\mu}_* \in (0, 1)$ be such that

$$\lambda^{-3/2} [\![\bar{\mu}]\!]^{3\gamma/2} \leqslant 1, \qquad C_F [\![\bar{\mu}_*]\!]^\zeta (3\Phi)^M \leqslant \delta, \qquad [\![\bar{\mu}_*]\!]^\zeta (3\Phi)^2 \leqslant c / 4\Phi_{\mathscr{L}},$$

where $\delta = \delta(\lambda, \Phi)$ is a small constant such that

$$2 \lambda^{1/2} \delta + 2 \lambda^{1/2} \delta \exp(C \delta) \leqslant 2^{-4}.$$

With this choice we have also

$$1+2\,\lambda^{-1}\,C_F\,[\![\bar{\mu}_*]\!]^{\zeta}\,(3\,\Phi)^M+2\,\lambda^{-1}\,C_F\,[\![\bar{\mu}_*]\!]^{3\gamma+\zeta}(3\,\Phi)^M\exp\Big(C\,C_F\,[\![\bar{\mu}_*]\!]^{\zeta}\,(3\,\Phi)^M\Big)\leqslant 1+2^{-4}. \tag{32}$$

Define now $A:=\{\mu\in[\bar{\mu}_*,1):|||\phi|||_\mu\leqslant 2\,\Phi\}$. Then $A\neq\varnothing$ since for $\mu\longrightarrow 1$ we have $|||\phi|||_\mu\longrightarrow 0$. We want to prove that A is both open and closed. Observe that it is open since if $\mu\in A$ then $|||\phi|||_\mu\leqslant 2\,\Phi$ and using Equation (31), Equation (32) and $\mu>\bar{\mu}_*$ we have

$$\begin{aligned}|||\phi|||_\mu^3/2\Phi^3+c\,\lambda^{-1}\,|||\mathscr{L}\phi|||_\#/2\,\Phi_{\mathscr{L}} &\leqslant 1+\lambda^{-1}\,[\![\bar{\mu}_*]\!]^{\zeta}\,(3\,\Phi)^M+\lambda^{-1}\,C_F\,[\![\bar{\mu}_*]\!]^{3\gamma+\zeta}\,(3\,\Phi)^M\exp\Big(C_F\,[\![\bar{\mu}_*]\!]^{\zeta}\,(3\,\Phi)^M\Big)\\ &\leqslant 1+2^{-4},\end{aligned}$$

so $|||\phi|||_\mu\leqslant(2+2^{-3})^{1/3}\,\Phi<2\,\Phi$ and therefore for $\delta>0$ small enough $|||\phi|||_\nu\leqslant\Phi$ for $\nu\in(\mu-\delta,\mu+\delta)$ since $\nu\longmapsto|||\phi|||_\nu$ is continuous. Let us prove that A is also closed. Take $\mu_n\longrightarrow\mu$ such that $\mu_n\in A$ for all $n\geqslant 1$. Then by continuity $|||\phi|||_{\mu_n}\leqslant 2\,\Phi\Longrightarrow|||\phi|||_\mu\leqslant 2\,\Phi$, so $\mu\in A$. As a consequence $A=[\bar{\mu}_*,1)$ and in particular $\bar{\mu}_*\in A$, that is $|||\phi|||_{\bar{\mu}_*}\leqslant 2\,\Phi$ and therefore also

$$|||\mathscr{L}\phi|||_\#\leqslant(2+2^{-3})\,\lambda\,\Phi_{\mathscr{L}}/\,c\lesssim 1,$$

and

$$\sup_i\sup_{\mu\geqslant\mu_i\vee\bar{\mu}_*}\|\zeta_i\,\tilde{\mathfrak{J}}_\mu R_\mu^\varepsilon\|_{L^\infty}\leqslant C_F\,[\![\bar{\mu}_*]\!]^{\zeta}\,(3\,\Phi)^M\exp(C_F\,[\![\bar{\mu}_*]\!]^{\zeta}\,(3\,\Phi)^M)\lesssim 1.$$

□

In order to apply the previous theorem to the analysis of Equation (23) we need suitable approximate solutions of the flow equation for the effective force $(F_\sigma^\varepsilon)_\sigma$

$$\partial_\sigma F_\sigma^\varepsilon+\mathrm{D}F_\sigma^\varepsilon\,\dot{G}_\sigma F_\sigma^\varepsilon=0,$$

for $\sigma\in(0,1]$ with final condition

$$F_1^\varepsilon(\psi)=-\lambda\,(\psi)^3-r_\varepsilon\,\psi+\xi^{(\varepsilon)}. \tag{33}$$

Section-4 will be devoted to prove the following result.

Theorem 13. *There exists a choice of $(r_\varepsilon)_\varepsilon$ for which the random scale-dependent functional $(F_\mu^\varepsilon)_{\mu\in(0,1)}$ with boundary condition (33) satisfies the estimates (27) where $C_F=\|F^{\varepsilon,\mathfrak{A}}\|$ is a finite random constant such that*

$$\sup_{\varepsilon>0}\mathbb{E}[\|F^{\varepsilon,\mathfrak{A}}\|^p]<\infty,$$

for any p (not uniformly).

2.3 Tightness

In order to pass to the limit as $\varepsilon\longrightarrow 0$ we want to embed all the random fields $\varphi^{(\varepsilon)}$ in the same space by extending them from Λ_ε to Λ_0. In order to do so we let

$$\phi^{[\varepsilon]}(t,x):=\theta(\varepsilon\,\mathrm{D})\varphi^{(\varepsilon)}(t,x)=\int_{\Lambda_\varepsilon^*}\theta(\varepsilon\,|k|)\,\hat{\varphi}^{(\varepsilon)}(\omega,k)\,e^{i(\omega t+k\cdot x)}\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^{d+1}},\qquad(t,x)\in\Lambda_0, \tag{34}$$

where $\theta:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a smooth bump function such that $\theta(0)=1$ and $\theta(\eta)=0$ for $|\eta|>1$. The random fields $\phi^{[\varepsilon]}$ all live now in the continuum domain Λ_0 for any $\varepsilon>0$. For convenience we have dealt so far with space-time distributional norms. However in order to obtain informations about the EQFT we need to evaluate the marginal at a fixed time of the solution of the SPDE. The necessary regularity is obtained observing that we have the following Schauder estimate tailored to our norms (cfr. Lemma 36 in Appendix A). For any small $\kappa>0$,

$$\sup_i 2^{-i(3\gamma-2s+\kappa)}\|\rho\,\Delta_i^x\phi^{(\varepsilon)}\|\lesssim[\![\bar{\mu}]\!]^{-\gamma}\,|||\phi^\varepsilon|||+|||\mathscr{L}\phi^\varepsilon|||_\#.$$

Now, we fix $\bar{\mu} \in (0,1)$ such that $[\![\bar{\mu}]\!]^{-\zeta} = 1 + \hat{C}^{-1} \|F^{\varepsilon,\mathfrak{A}}\|$. By Theorem 12 we have that $|||\phi^{\varepsilon}||| \approx |||\mathscr{L}\phi^{\varepsilon}||| \approx 1$ so by Lemma 36 we obtain that ϕ^{ε} can be controlled in a space of continuous functions of time with values in a weighted spatial Besov space of (negative) regularity $-3\gamma + 2s - \kappa$, namely

$$\sup_i \left(2^{-(3\gamma-2s+\kappa)i} \|\rho\, \Delta_i^x \phi^{(\varepsilon)}\|\right) \lesssim [\![\bar{\mu}]\!]^{-\gamma} \lesssim (1 + \|F^{\varepsilon,\mathfrak{A}}\|)^{\gamma/\zeta}. \tag{35}$$

Together with Theorem 13 this allows us to estimate

$$\sup_{\varepsilon>0} \mathbb{E}\left[\left(\sup_i 2^{-(3\gamma-2s+\kappa)i} \|\rho\, \Delta_i^x \phi^{(\varepsilon)}\|\right)^n\right] \lesssim \sup_{\varepsilon>0} \mathbb{E}\left[(1 + \|F^{\varepsilon,\mathfrak{A}}\|)^{\gamma n/\zeta}\right] < +\infty, \tag{36}$$

for any n large enough. The Besov norms behave well through the embedding $\phi^{(\varepsilon)} \longrightarrow \phi^{[\varepsilon]}$, indeed

$$\|\rho\, \Delta_i^x \phi^{[\varepsilon]}\|_{L^\infty(\Lambda_0)} = \|\rho\, \theta(\varepsilon \mathrm{D}) \Delta_i^x \phi^{(\varepsilon)}\|_{L^\infty(\Lambda_0)} \lesssim \|\rho\, \Delta_i^x \phi^{(\varepsilon)}\|_{L^\infty(\Lambda_\varepsilon)}$$

and therefore the bound (36) translates into the tightness of the family $(\phi^{[\varepsilon]}(t))_{\varepsilon>0}$ in $\mathscr{S}'(\Lambda_0)$ for any fixed $t \in \mathbb{R}$. We have then proved the first part of Theorem 1.

2.4 Integrability

In order to complete the proof of Theorem 1 we need to establish the integrability of the measures ν^{ε} uniformly in ε and obtain the bound (3) for any limit points. The main tool for this task is the Hairer–Steele argument [HS22] which gives optimal estimates (as far as growth of the function is concerned). We look to estimate quantities of the form

$$Z_{\varepsilon,\theta} := \int e^{\theta \|h Q_\varepsilon \phi\|_{L^2}^4} \nu^{\varepsilon}(\mathrm{d}\phi),$$

where $Q_\varepsilon := (1 + (-\Delta_\varepsilon)^{1/2})^{-a}$ is a regularising kernel and $h(x) = (1 + |x|)^{-b}$ is a polynomially decaying weight in the space variable alone. The specific value of a and b will be suitable chosen below, respectively in Equations (40) and (39): in particular we shall require them to be large enough. We define a new *tilted* probability measure

$$\nu^{\varepsilon,\theta}(\mathrm{d}\phi) := \frac{e^{\theta \|h Q_\varepsilon \phi\|_{L^2}^4} \nu^{\varepsilon}(\mathrm{d}\phi)}{Z_{\varepsilon,\theta}},$$

and observe that, by Jensen's inequality,

$$1 = \int \nu^{\varepsilon}(\mathrm{d}\phi) = Z_{\varepsilon,\theta} \int e^{-\theta \|h Q_\varepsilon \phi\|_{L^2}^4} \nu^{\varepsilon,\theta}(\mathrm{d}\phi) \geqslant Z_{\varepsilon,\theta} \exp\left[-\theta \int \|h\, Q_\varepsilon \phi\|_{L^2}^4 \nu^{\varepsilon,\theta}(\mathrm{d}\phi)\right]$$

so we conclude that

$$\log Z_{\varepsilon,\theta} \leqslant \theta \int \|h\, Q_\varepsilon \phi\|_{L^2}^4 \nu^{\varepsilon,\theta}(\mathrm{d}\phi). \tag{37}$$

The task of controlling the size of $Z_{\varepsilon,\theta}$ is reduced via (37) to that of estimating some polynomial moments of ϕ under the new tilted measure $\nu^{\varepsilon,\theta}$. By stochastic quantisation this tilted measure can be identified with the marginal of a stationary solution to the SPDE

$$[\partial_t + m^2 + (-\Delta_\varepsilon)^s] \phi^{(\varepsilon)} + \lambda (\phi^{(\varepsilon)})^3 - r_\varepsilon \phi^{(\varepsilon)} = O(\phi^{(\varepsilon)}) + \xi^{(\varepsilon)}, \tag{38}$$

where the additional perturbation $O(\phi)$ is given by

$$O(\phi) = -\theta \frac{\delta}{\delta \phi} \|h\, Q_\varepsilon \phi\|_{L^2}^4 = -2\theta \|h\, Q_\varepsilon \phi\|_{L^2}^2 (Q_\varepsilon h^2 Q_\varepsilon \phi).$$

Estimates for this new equation uniform in $\varepsilon > 0$ come from the flow equation argument that we developed so far. In particular, repeating the arguments at the beginning of this section, we can write

$$\mathscr{L}_\varepsilon \phi_\sigma^{(\varepsilon)} = \mathfrak{J}_\sigma(F(\phi^{(\varepsilon)}) + O(\phi^{(\varepsilon)}))$$

and by including the contributions of O in $R^{(\varepsilon)}$ we have the new system

$$\mathscr{L}_\varepsilon \phi_\sigma^{(\varepsilon)} = \mathfrak{J}_\sigma\big(F_\sigma\big(\phi_\sigma^{(\varepsilon)}\big) + R_\sigma^{(\varepsilon),O}\big)$$

$$R_\mu^{(\varepsilon),O} = \int_\mu^1 \big(\partial_\sigma F_\sigma^{(\varepsilon)}\big(\phi_\sigma^{(\varepsilon)}\big) + \mathrm{D}F_\sigma^{(\varepsilon)}\big(\phi_\sigma^{(\varepsilon)}\big)\, \dot{G}_\sigma F_\sigma^{(\varepsilon)}\big(\phi_\sigma^{(\varepsilon)}\big)\big)\mathrm{d}\sigma + \int_\mu^1 \mathrm{D}F_\sigma^{(\varepsilon)}\big(\phi_\sigma^{(\varepsilon)}\big)\, \dot{G}_\sigma R_\sigma^{(\varepsilon),O}\mathrm{d}\sigma + O(\phi^{(\varepsilon)}).$$

In estimating this equation we have to consider $\|\zeta_j \mathfrak{J}_\sigma R_\sigma^{(\varepsilon),O}\|$, and therefore we need a bound for the space-time norm $\|\zeta_j \mathfrak{J}_\sigma O(\phi^{(\varepsilon)})\|$ of source term:

$$\begin{aligned}
\|\zeta_j \mathfrak{J}_\sigma O(\phi^{(\varepsilon)})\| &\lesssim 2|\theta| \left\| t \longrightarrow \zeta_j(t,\cdot) \|h\, Q_\varepsilon \phi(t,\cdot)\|_{L_x^2(\mathbb{R}^d)}^2 (\mathfrak{J}_\sigma Q_\varepsilon h^2 Q_\varepsilon \phi(t,\cdot)) \right\| \\
&= 2|\theta| \left\| t \longrightarrow \zeta_j(t,\cdot)\rho^{-1}(t,\cdot) \|h\,\rho^{-1}(t,\cdot)\,\rho(t,\cdot) Q_\varepsilon \phi(t,\cdot)\|_{L_x^2(\mathbb{R}^d)}^2 (\rho\, \mathfrak{J}_\sigma Q_\varepsilon h^2 Q_\varepsilon \phi)(t,\cdot) \right\| \\
&\lesssim 2|\theta| \left\| t \longrightarrow \zeta_j(t,\cdot)\rho^{-1}(t,\cdot) \|h\,\rho^{-1}(t,\cdot)\|_{L_x^2(\mathbb{R}_\varepsilon^d)}^2 \right\| \|\rho\, Q_\varepsilon \phi\|^2 \|\rho\, \mathfrak{J}_\sigma Q_\varepsilon h^2 Q_\varepsilon \phi\|
\end{aligned}$$

Now if we assume that $h \in L^2(\mathbb{R}_\varepsilon^d)$ is decaying fast enough at infinity so that $\|h\,\rho(0,\cdot)\|_{L_x^2(\mathbb{R}_\varepsilon^d)}^2 \lesssim 1$, we have, exploiting $\rho^{-1}(t,x) \lesssim \rho^{-1}(t,0)\rho^{-1}(0,x)$, which is a consequence of the property $\rho^{-1}(z_1) \lesssim \rho^{-1}(z - z_1)\rho^{-1}(z_1)$ of the weight ρ,

$$\|h\,\rho^{-1}(t,\cdot)\|_{L_x^2(\mathbb{R}^d)}^2 \lesssim \rho(t,0)^{-2} \|h\,\rho(0,\cdot)\|_{L_x^2(\mathbb{R}_\varepsilon^d)}^2 \lesssim \rho(t,0)^{-2}, \tag{39}$$

so, recalling that $\rho(t,0)^{-2} \leqslant \rho(t,x)^{-2}$ for any $x \in \Lambda$ as well as the localising properties of ζ_j as per Remark 9,

$$\|t \longrightarrow \zeta_j(t,\cdot)\,\rho^{-1}(t,\cdot)\,\rho(t,0)^{-2}\| \leqslant \|\zeta_j \rho^{-3}\| \lesssim 2^{3\gamma j},$$

and since both ρ, h are nice weights we have

$$\|\rho\, \mathfrak{J}_\sigma Q_\varepsilon h^2 Q_\varepsilon \phi\| \lesssim \|\rho\, Q_\varepsilon h^2 Q_\varepsilon \phi\| \lesssim \|\rho\, Q_\varepsilon \phi\| \lesssim \sum_i 2^{-ia} \|\Delta_i^x \rho\, \phi\|.$$

Therefore

$$\|\zeta_j \mathfrak{J}_\sigma O(\phi^{(\varepsilon)})\| \lesssim 2|\theta|\, 2^{3\gamma j} \left(\sum_i 2^{-ia} \|\rho\, \Delta_i^x \phi\|\right)^3.$$

By the support properties of the functions χ_j, Remark 9 as well as the Schauder estimate (98),

$$\begin{aligned}
\|\rho\, \Delta_i^x \phi\| &\lesssim \sup_j \left\| \sum_{e=-1}^{1} \chi_{j+e}^3 \rho\, \Delta_i^x \phi \right\| \lesssim \sup_j 2^{-\gamma j} \|\chi_j^3\, \Delta_i^x \phi\| \\
&\lesssim \sup_j 2^{-\gamma j} [\mathbb{1}_{i \lesssim j} [\![\mu_j \vee \bar{\mu}]\!]^{-\gamma} |||\phi^\varepsilon||| + 2^{i(3\gamma - 2s + \kappa)} |||\mathscr{L}\phi^\varepsilon|||_\#] \\
&\lesssim [\![\bar{\mu}]\!]^{-\gamma} |||\phi^\varepsilon||| + 2^{i(3\gamma - 2s + \kappa)} |||\mathscr{L}\phi^\varepsilon|||_\#,
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_i 2^{-ia} \|\rho\, \Delta_i^x \phi\| &\lesssim \sum_i 2^{-ia} [[\![\bar{\mu}]\!]^{-\gamma} |||\phi^\varepsilon||| + 2^{i(3\gamma - 2s + \kappa)} |||\mathscr{L}\phi^\varepsilon|||_\#] \\
&\lesssim [\![\bar{\mu}]\!]^{-\gamma} |||\phi^\varepsilon||| + |||\mathscr{L}\phi^\varepsilon|||_\#,
\end{aligned} \tag{40}$$

where we used that for a large enough, $3\gamma - 2s + \kappa - a < 0$. Summing up we have

$$\|\zeta_j \mathfrak{J}_\sigma O(\phi^{(\varepsilon)})\| \lesssim 2|\theta|\, 2^{3\gamma j} \big[[\![\bar{\mu}]\!]^{-3\gamma} |||\phi^\varepsilon|||^3 + |||\mathscr{L}\phi^\varepsilon|||_\#^3\big] \tag{41}$$

and now from the Grownall lemma applied to R we get, for $\sigma \geqslant \mu_j \vee \bar{\mu}$,

$$\|\zeta_j \mathfrak{J}_\sigma R_\sigma^{(\varepsilon),O}\| \lesssim \Big[\|F^{\varepsilon,\mathfrak{A}}\| [\![\bar{\mu}]\!]^\zeta (1+|||\phi|||)^M + |\theta|\, 2^{3\gamma j} [\![\bar{\mu}]\!]^{-3\gamma} |||\phi|||^3 + |\theta|\, 2^{3\gamma j} |||\mathscr{L}\phi^\varepsilon|||_\#^3\Big] \times$$
$$\times \exp(C\|F^{\varepsilon,\mathfrak{A}}\| [\![\bar{\mu}]\!]^\zeta (1+|||\phi|||)^M)$$

and therefore, since $[\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} [\![\mu_j]\!]^{-3\gamma} \lesssim 1$,

$$\sup_j \sup_{\sigma \geqslant \mu_j \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\zeta_j \mathfrak{J}_\sigma R_\sigma^{(\varepsilon),O}\| \lesssim \Big[\|F^{\varepsilon,\mathfrak{A}}\| [\![\bar{\mu}]\!]^{3\gamma+\zeta} (1+|||\phi|||)^M + |\theta|\, |||\phi|||^3 + |\theta|\, |||\mathscr{L}\phi^\varepsilon|||_\#^3\Big] \times$$
$$\times \exp(C\|F^{\varepsilon,\mathfrak{A}}\| [\![\bar{\mu}]\!]^\zeta (1+|||\phi|||)^M).$$

By repeating the same continuity argument following (31) we deduce that, provided $\bar{\mu}$ is chosen such that

$$\|F^{\varepsilon,\mathfrak{A}}\| [\![\bar{\mu}]\!]^\zeta (3\Phi)^M \leqslant \delta,$$

for some constant Φ and with a small $\delta = \delta(\lambda, \Phi, \theta)$ satisfying

$$2\lambda^{-1}\delta + 2\{\lambda^{-1}\delta + |\theta|\Phi^3 + |\theta|\Phi_{\mathscr{L}}^3\} \exp(C\delta) \leqslant 2^{-4},$$

we can conclude that $|||\phi||| \leqslant 2\Phi$ and $|||\mathscr{L}\phi^\varepsilon|||_\# \leqslant \Phi_{\mathscr{L}}$. Such a δ exists provided $|\theta| > 0$ is small enough. Taking $\bar{\mu}$ such that $\|F^{\varepsilon,\mathfrak{A}}\| [\![\bar{\mu}]\!]^\zeta (3\Phi)^M = \delta$ we have

$$\|h\, Q_\varepsilon \phi^{(\varepsilon)}(t)\|_{L^2} \lesssim [\![\bar{\mu}]\!]^{-\gamma} \lesssim \|F^{\varepsilon,\mathfrak{A}}\|^{\gamma/\zeta},$$

for any fixed $t \in \mathbb{R}$ (with constants depending on t). As a consequence of Theorem 13 we have also

$$\sup_{\varepsilon>0} \mathbb{E}[\|h\, Q_\varepsilon \phi^{(\varepsilon)}(t)\|_{L^2(\Lambda_\varepsilon)}^n] \lesssim \sup_{\varepsilon>0} \mathbb{E}[\|F^{\varepsilon,\mathfrak{A}}\|^{\gamma n/\zeta}] < \infty,$$

for any large n and fixed $t \in \mathbb{R}$. From this we derive easily that any accumulation point ν of the sequence ν^ε satisfies Eq. (3) provided $\theta > 0$ is small enough. This proves both the exponential integrability required for the Osterwalder–Schrader reconstruction theorem and also proves that the measure is non-Gaussian, since Gaussians measure cannot integrate functions growing so fast, completing the proof of Theorem 1.

Remark 14. The choice of the norm to verify the exponential integrability is quite arbitrary. Since we need to determine an SPDE for it we want a differentiable norm. In general we could replace the L^2 norm by any L^{2n} norm, as long as n is finite and similarly use different space weight h and smoothing operator Q_ε, as long as they remain compatible with our Schauder estimates.

3 A-priori estimates

This section is devoted to prove weighted bounds for classical solutions to a fractional parabolic equation with a cubic coercive term. To our knowledge this result is new, even if given the proof follows closely the proof in the case of the Laplacian, see e.g. [GH19].

Theorem 15. *Let u be a classical solution of the fractional parabolic equation*

$$\partial_t u + (-\Delta)^s u + \lambda u^3 = f, \tag{42}$$

where $\lambda > 0$. Then it holds that, for any i,

$$\|\chi_i u\| \lesssim \lambda^{-1/2} + \lambda^{-1/3}[\|\chi_i^3 f\| + 2^{\gamma i}\|\rho_i u\| + 2^{\gamma i}\|D(\rho_i^3 u)\|]^{1/3} \tag{43}$$

where, recalling the kernel $v_s(z, \mathrm{d}z')$ defined in Eq. (15) we let

$$D(f)(z) := \left[\int [f(z') - f(z)]^2 v_s(z, \mathrm{d}z')\right]^{1/2}.$$

Proof. We can assume that $\|\chi_i^3 f\| + \|\rho_i u\| + \|D(\rho_i^3 u)\| < \infty$, otherwise there is nothing to prove. Let $v := \chi_i u$, $\Phi(\xi) = \Phi_L(\xi) := (\xi - L)_+$ and $\Phi'(\xi) = \Phi_L'(\xi) := \mathbb{1}_{\xi \geqslant L}$ so that $\Phi(a) - \Phi(b) \leqslant \Phi'(a)(a - b)$. Consider Equation (42) and test it against $\Phi_L(v)\Phi_L'(v)\chi_i^3$ to get

$$\langle \Phi_L(v)\Phi_L'(v), \chi_i^3 f\rangle = \langle \Phi_L(v)\Phi_L'(v), \chi_i^3 \partial_t u + \chi_i^3(-\Delta)^s u + \lambda v^3\rangle.$$

By Lemma 16 below we have

$$\langle \Phi_L(v)\Phi_L'(v), \chi_i^3(-\Delta)^s u\rangle \gtrsim -\int \Phi_L(v)^2 - [2^{\gamma i}\|\rho_i u\| + 2^{\gamma i}\|D(\rho_i^3 u)\|]\int \Phi_L(v).$$

For the time derivative we observe that

$$\langle \Phi_L(v)\Phi_L'(v), \chi_i^3 \partial_t u\rangle = \langle \Phi_L(v)\Phi_L'(v), \chi_i^2 \partial_t v - \chi_i(\partial_t \chi_i) v\rangle.$$

Then, leaving implicit the space variable, by the convexity of Φ^2 we have

$$\begin{aligned}
\int_\Lambda \chi_i^2(t)\Phi_L(v(t))\Phi_L'(v(t))\partial_t v(t)\mathrm{d}t\mathrm{d}x &= \lim_{h\searrow 0}\int_\Lambda \chi_i^2(t)\Phi_L(v(t))\Phi_L'(v(t))\frac{[v(t) - v(t-h)]}{h}\mathrm{d}t\mathrm{d}x \\
&\geqslant \lim_{h\searrow 0}\int_\Lambda \chi_i^2(t)\frac{[\Phi_L^2(v(t)) - \Phi_L^2(v(t-h))]}{h}\mathrm{d}t\mathrm{d}x \\
&= \lim_{h\searrow 0}\int_\Lambda \frac{\chi_i^2(t) - \chi_i^2(t+h)}{h}\Phi_L^2(v(t))\mathrm{d}t\mathrm{d}x \\
&= -2\int_\Lambda \chi_i(\partial_t \chi_i)\Phi_L^2(v) \\
&\geqslant -2\|\partial_t \chi_i\|\int_\Lambda \Phi_L^2(v),
\end{aligned} \tag{44}$$

therefore the lower bound

$$\langle \Phi(v)\Phi'(v), \chi_i^3 \partial_t u\rangle \geqslant -2\|\partial_t \chi_i\|\int_\Lambda \Phi_L^2(v) - \|v\|\,\|\partial_t \chi_i\|\int_\Lambda \Phi_L(v) \gtrsim -\int_\Lambda \Phi_L^2(v) - 2^{\gamma i}\|\rho_i u\|\int_\Lambda \Phi_L(v),$$

holds. Now we have

$$\begin{aligned}
\langle \Phi_L\Phi_L'(v), \lambda v^3\rangle &= \lambda\langle (v-L)_+, ((v-L)+L)v^2\rangle \geqslant \lambda\langle (v-L)_+^2, v^2\rangle + \lambda L\langle (v-L)_+, v^2\rangle \\
&\geqslant \lambda L^2\int_\Lambda \Phi_L^2(v) + \lambda L^3\int_\Lambda \Phi_L(v).
\end{aligned}$$

but then, also

$$\begin{aligned}
\langle \Phi_L\Phi_L'(v), \lambda v^3\rangle &= \langle \Phi_L(v)\Phi_L'(v), \chi_i^3 f\rangle - \langle \Phi_L(v)\Phi_L'(v), \chi_i^3(\partial_t u + (-\Delta)^s u)\rangle \\
&\leqslant \|\chi_i^3 f\|\int_\Lambda \Phi_L(v) + \int_\Lambda \Phi_L(v)^2 + [2^{\gamma i}\|\rho_i u\| + 2^{\gamma i}\|D(\rho_i^3 u)\|]\int_\Lambda \Phi_L(v).
\end{aligned}$$

Together these two inequalities imply that

$$[\lambda L^2 - C]\int_\Lambda \Phi_L^2(v) + \left[\lambda L^3 - \|\chi_i^3 f\| - C\,2^{\gamma i}\|\rho_i u\| - C\,2^{\gamma i}\|D(\rho_i^3 u)\|\right]\int_\Lambda \Phi_L(v) \leqslant 0,$$

for some $C>0$. Taking

$$L>L_*=\max(\lambda^{-1/2}C,\lambda^{-1/3}[\|\chi_i^3 f\|+C\,2^{\gamma i}\|\rho_i\,u\|+C\,2^{\gamma i}\|D(\rho_i^3 u)\|]^{1/3}),$$

we deduce that

$$\int_\Lambda \Phi_L(v)=\int_\Lambda \Phi_L^2(v)=0,$$

which in turn implies that $v\leq L$ a.e. on Λ and by a similar computation for $v=-\chi_i u$ we obtain that $|v|\leq L$ a.e. on Λ. We conclude that

$$\|v\|\leq \inf_{L>L_*} L=L_*\lesssim \lambda^{-1/2}+\lambda^{-1/3}[\|\chi_i^3 f\|+2^{\gamma i}\|\rho_i\,u\|+2^{\gamma i}\|D(\rho_i^3 u)\|]^{1/3}$$

as claimed. □

The proof is completed by the following lemma.

Lemma 16. *With the notation of Theorem 15, we have*

$$\langle \Phi_L(v)\,\Phi_L'(v),\chi_i^3\,(-\Delta)^s u\rangle\gtrsim-\int \Phi_L(v)^2-[2^{\gamma i}\|\rho_i u\|+2^{3\gamma i}\|D(\rho_i^3\,u)\|]\int \Phi_L(v)\,\Phi_L'(v).$$

Proof. We can write,

$$A:=\langle \Phi_L(v)\,\Phi_L'(v),\chi_i^3\,(-\Delta)^s u\rangle=\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^3(z)\,(u(z)-u(z'))$$

$$=\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,(v(z)-v(z'))+\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\chi_i^2(z)(\chi_i(z')-\chi_i(z))u(z'),$$

and using convexity of Φ_L, namely that $\Phi_L'(a)(a-b)\geq\Phi_L(a)-\Phi_L(b)$, and its positivity, we have

$$\begin{aligned}
A \;&\geq\; \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\chi_i^2(z)\,(\Phi_L(v(z))-\Phi_L(v(z')))\\
&\quad+\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi'(v(z))\,\chi_i^2(z)\,(\chi_i(z')-\chi_i(z))\,u(z')\\
&= \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\chi_i(z)\,\chi_i(z')\,(\Phi_L(v(z))-\Phi_L(v(z'))) && [=:(\mathrm{I})]\\
&\quad+\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,(\chi_i(z')-\chi_i(z))\,u(z') && [=:(\mathrm{II})]\\
&\quad+\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\chi_i(z)\,[\chi_i(z)-\chi_i(z')]\,(\Phi_L(v(z))-\Phi_L(v(z'))) && [=:(\mathrm{III})]
\end{aligned}$$

Symmetrising the integral (I) w.r.t. the exchange $z\leftrightarrow z'$ we have

$$(\mathrm{I})=\frac{1}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z))-\Phi_L(v(z'))]^2>0$$

while, again via algebraic manipulations

$$\begin{aligned}
(\mathrm{II}) \;&=\; \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,(\chi_i(z')-\chi_i(z))\,\rho_i^{-3}(z')\,(\rho_i^3\,u)(z')\\
&= \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,(\chi_i(z')-\chi_i(z))\,(\rho_i^{-3}(z')-\rho_i^{-3}(z))\,(\rho_i^3\,u)(z')\\
&\quad+\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,(\chi_i(z')-\chi_i(z))\,\rho_i^{-3}(z)\,(\rho_i^3\,u)(z')\\
&= \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,u(z)\,(\chi_i(z')-\chi_i(z)) && [=:(\mathrm{II}_1)]\\
&\quad+\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,(\chi_i(z')-\chi_i(z))\,(\rho_i^{-3}(z')-\rho_i^{-3}(z))\,(\rho_i^3\,u)(z') && [=:(\mathrm{II}_2)]\\
&\quad+\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,\rho_i^{-3}(z)\,(\chi_i(z')-\chi_i(z))\,[(\rho_i^3\,u)(z')-(\rho_i^3\,u)(z)] && [=(\mathrm{II}_3)]
\end{aligned}$$

For $(\mathbb{III}_1)$ we have

$$\begin{aligned}(\mathbb{III}_1) &= \int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,u(z)\int (\chi_i(z')-\chi_i(z))\,\nu_s(z,\mathrm{d}z')\\ &\gtrsim -\|\chi_i^2\,u\|\int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z))\\ &\gtrsim -2^{\gamma i}\|\rho_i\,u\|\int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z)),\end{aligned}$$

where we used that

$$\sup_{i,z}\left|\int (\chi_i(z')-\chi_i(z))\,\nu_s(z,\mathrm{d}z')\right| = \sup_i \|(-\Delta)^s\chi_i\| \lesssim 1.$$

Next, using the definition of ρ_i and the inequality (22) for the nice weight ρ_i, we have

$$\begin{aligned}(\mathbb{III}_2) &\geqslant -\|\rho_i^3\,u\|\int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\int |\chi_i(z')-\chi_i(z)|\,|\rho_i^{-3}(z')-\rho_i^{-3}(z)|\,\nu_s(z,\mathrm{d}z')\\ &\gtrsim -\|\rho_i^3\,u\|\int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z))\,\chi_i^2(z)\,\rho_i^{-3}(z)\\ &\gtrsim -2^{3\gamma i}\|\rho_i^3\,u\|\int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z)),\end{aligned}$$

where in the next to the last bound we used the fact that

$$|\rho_i^{-3}(z')-\rho_i^{-3}(z)| \lesssim \rho_i^{-3}(z)\,|\rho^{-3}(z-z')-1| \lesssim \rho_i^{-3}(z)\,(|z-z'|\,\mathbb{1}_{|z-z'|\leqslant 1}+\mathbb{1}_{|z-z'|>1}\,\rho^{-3}(z-z')),$$

which implies that, provided $3\nu<2s$,

$$\sup_i \int |\chi_i(z')-\chi_i(z)|\,|\rho_i^{-3}(z')-\rho_i^{-3}(z)|\,\nu_s(z,\mathrm{d}z') \lesssim \rho_i^{-3}(z).$$

Next, using Cauchy–Schwarz inequality, and the bounds $\|\chi_i^2\,\rho_i^{-3}\| \lesssim 2^{3\gamma i}$ and

$$\sup_{i,z}\int (\chi_i(z')-\chi_i(z))^2\,\nu_s(z,\mathrm{d}z') \lesssim 1,$$

we have

$$\begin{aligned}(\mathbb{III}_3) &\geqslant -2^{3\gamma i}\int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z))\left[\int [(\rho_i^3\,u)(z')-(\rho_i^3\,u)(z)]^2\,\nu_s(z,\mathrm{d}z')\right]^{1/2}\\ &\gtrsim -2^{3\gamma i}\|D(\rho_i^3 u)\|\int \mathrm{d}z \Phi_L(v(z))\,\Phi_L'(v(z)).\end{aligned}$$

Finally we have, for $\alpha>0$ which will be fixed in a moment,

$$\begin{aligned}(\mathbb{IIII}) \geqslant& -\frac{1}{2\alpha}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))^2\,[\chi_i(z')-\chi_i(z)]^2-\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)^2\,[\Phi_L(v(z))-\Phi_L(v(z'))]^2\\ \geqslant& -\frac{1}{2\alpha}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))^2\,[\chi_i(z')-\chi_i(z)]^2+\\ & -\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z))-\Phi_L(v(z'))]^2+\\ & -\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,[\chi_i(z)-\chi_i(z')]\,[\Phi_L(v(z))-\Phi_L(v(z'))]^2\\ \geqslant& -\frac{1}{2\alpha}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))^2\,[\chi_i(z')-\chi_i(z)]^2-\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z))-\Phi_L(v(z'))]^2+\\ & -\frac{\alpha}{4}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')[\chi_i(z)-\chi_i(z')]^2[\Phi_L(v(z))-\Phi_L(v(z'))]^2\\ \geqslant& -\frac{1}{2\alpha}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi_L(v(z))^2\,[\chi_i(z')-\chi_i(z)]^2-\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z))-\Phi_L(v(z'))]^2+\\ & -\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')[\chi_i(z)-\chi_i(z')]^2\,\Phi_L(v(z))^2\\ \gtrsim& -\left[\frac{1}{2\alpha}+\frac{\alpha}{2}\right]\int \Phi_L(v(z))^2\mathrm{d}z-\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z))-\Phi_L(v(z'))]^2\end{aligned}$$

Summing up, for a constant $C>0$,

$$A \geqslant (\mathbb{I}) + (\mathbb{III}_1) + (\mathbb{III}_2) + (\mathbb{III}_3) + (\mathbb{IIII}) \geqslant \frac{1}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z)) - \Phi_L(v(z'))]^2 +$$

$$-C[2^{\gamma i}\|\rho_i u\| + 2^{3\gamma i}\|D(\rho_i^3 u)\|]\int \Phi_L(v(z))\,\Phi_L'(v(z))\mathrm{d}z +$$

$$-C\left[\frac{1}{2\alpha} + \frac{\alpha}{2}\right]\int \Phi_L(v(z))^2\mathrm{d}z - C\frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z)) - \Phi_L(v(z'))]^2$$

$$\geqslant \left[\frac{1}{2} - C\frac{\alpha}{2}\right]\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi_L(v(z)) - \Phi_L(v(z'))]^2 +$$

$$-C\left[\frac{1}{2\alpha} + \frac{\alpha}{2}\right]\int \Phi_L(v(z))^2\mathrm{d}z - C\,[2^{\gamma i}\|\rho_i u\| + 2^{3\gamma i}\|D(\rho_i^3 u)\|]\int \Phi_L(v(z))\,\Phi_L'(v(z))\mathrm{d}z,$$

so for α small enough such that the first term is positive and can be ignored in the lower bound, we have established that

$$\langle \Phi_L(v)\,\Phi_L'(v), \chi_i^3(-\Delta)^s u\rangle \geqslant -C\left[\frac{1}{2\alpha} + \frac{\alpha}{2}\right]\int \Phi_L(v)^2 - C\,[2^{\gamma i}\|\rho_i u\| + 2^{3\gamma i}\|D(\rho_i^3 u)\|]\int \Phi_L(v)\,\Phi_L'(v),$$

as claimed. □

4 Analysis of the flow equation

In this section we prove Theorem 13 stating the existence of an approximate solution to the flow equation with nice bounds. These bounds are possible only because we can "tune" the initial condition of the flow equation by adding an appropriate ε-dependent renormalisation term to its initial condition. Conceptually we are dealing with a random bilinear equation whose solution is analysed by deriving a corresponding evolution equation for its cumulants. The equation for the cumulants has a similar structure and propagate similar bounds *backwards* from the final condition at $\sigma = 1$ apart from a low dimensional (so called, relevant) subspace for which we need to propagate the bounds *forward* from small to large σ. This procedure entails the "tuning" of an appropriate initial condition to lie on a particular trajectory which has good bounds. In order for this tuning to be as simple as possible one need to modify the flow equation in order to reduce the relevant subspace to one dimension. Once the bounds for the cumulants are established, a Kolmogorov-type argument allows to deduce also path-wise bounds on the effective force. The section ends with a technical "post-processing" to extract the coercive term crucial for the global a-priori estimates.

4.1 The random flow equation

We need to set up the appropriate spaces to consider such an equation. We discuss the case $\varepsilon > 0$, the continuum case $\varepsilon = 0$ (which however is not used directly in the proof of the main result) being easier and requiring only minimum changes. Let $\mathscr{E}^0 := C(\Lambda)$ and $\mathscr{E} := C^\infty(\Lambda)$ with polynomial growth at infinity and let $\mathscr{J}\mathscr{E}$ the vector bundle of jets over $\mathscr{E}$, i.e. the (infinite) collection of all the partial derivatives $(\partial^A\psi)_A$ of $\psi \in \mathscr{E}$ where A is a multi-index $A = (A_1 \cdots A_n)$ with $\partial^A = \partial^{A_1} \cdots \partial^{A_n}$ and where $A_i = \{0, 1\pm, \ldots, d\pm\}$ with $\partial^0 = \partial_t$ and $\partial^{k\pm}$ the discrete forward/backward derivative in direction $k = 1, \ldots, d$, defined as $\partial^{k\pm}\psi(z) := \varepsilon^{-1}(\psi(z \pm e_k) - \psi(z))$. Let $[A] = |A_1| + \cdots + |A_n|$ and $|0| = 2s$, $|1| = 1$ denoting the (formal, fractional) parabolic homogeneity of the partial derivatives ∂^α. We denote any $\Psi \in \mathscr{J}\mathscr{E}$ with $\Psi = (\Psi^A)_A$.

Any function $\psi \in \mathscr{E}$ can be lifted to $\Psi = \mathscr{J}\psi \in \mathfrak{J}\mathscr{E}$ by letting $(\mathscr{J}\psi)^A = \partial^A \psi$, and any functional F on $\mathscr{E}$ can be represented by a functional $\mathscr{J}F$ on $\mathscr{J}\mathscr{E}$ such that $F(\psi) = \mathscr{J}F(\mathscr{J}\psi)$. This representation is not unique and we will exploit this freedom to our advantage below. Note that by the chain rule

$$\mathrm{D}F(\psi) \cdot h = \mathrm{D}\mathscr{J}F(\mathfrak{J}\psi) \cdot \mathscr{J}h, \qquad h \in \mathscr{E}.$$

To study approximate solutions $(F_\sigma)_{\sigma \in [0,1]}$ of the flow equation

$$\partial_\sigma F_\sigma + \mathrm{D}F_\sigma \cdot \dot{G}_\sigma F_\sigma = 0, \qquad F_1 = F.$$

we will lift this equation to $(F_\sigma : \mathscr{J}\mathscr{E} \longrightarrow \mathscr{S}')_{\sigma \in [0,1]}$ by the identification above. We then have for $F_\sigma \in C^1(\mathscr{J}\mathscr{E}, \mathscr{S}')$ the lifted equation

$$\partial_\sigma F_\sigma + \mathrm{D}F_\sigma \cdot \mathscr{J}(\dot{G}_\sigma F_\sigma) = 0, \qquad F_1 = F, \tag{45}$$

where by abuse of language we let $F \in C^1(\mathscr{J}\mathscr{E}, \mathscr{S}')$ be a particular representative of our force. This equation can be approximatively solved in the space $\mathscr{P}(\mathscr{J}\mathscr{E})$ of finite polynomials on the jets $\mathscr{J}\mathscr{E}$ with values in $\mathscr{S}'$, this is an algebra with a grading induced by the degree of monomials, let $\mathscr{P}_k(\mathscr{J}\mathscr{E})$ the component of grade k. For $F \in \mathscr{P}(\mathfrak{J}\mathscr{E})$ we denote by $F^{(k)} \in \mathscr{P}_k(\mathscr{J}\mathscr{E})$ the component of degree k and by abuse of language also the associated distributional kernel, so that

$$F^{(k)}(\Psi)(z) = \int_{\Lambda^k} F^{(k)}(z; z_1, \ldots, z_k) \Psi(z_1) \cdots \Psi(z_k) \mathrm{d}z_1 \cdots \mathrm{d}z_k,$$

for $z \in \Lambda$ where $F^{(k)}(z; z_1, \ldots, z_k)$ is a suitable linear map which contracts the various components of $\Psi(z_i)$, since the various components scale differently we will use the notation

$$F^{(k)}(\Psi)(z) = \sum_{A_1, \ldots, A_k} \int_{\Lambda^k} F^{\langle A_1, \ldots, A_k \rangle}(z; z_1, \ldots, z_k) \Psi^{A_1}(z_1) \cdots \Psi^{A_k}(z_k) \mathrm{d}z_1 \cdots \mathrm{d}z_k, \tag{46}$$

to denote the various components of the kernels.

We only care about approximate solution of (45), thus we introduce a formal parameter $\hbar$ and look for solutions $({}^\hbar F_\sigma)_{\sigma \in [0,1]}$,

$${}^\hbar F_\sigma = \sum_{\ell \geqslant 0} F_\sigma^{[\ell]} \hbar^\ell,$$

in the space $\mathscr{P}(\mathscr{J}\mathscr{E})[\![\hbar]\!]$ of formal power series in $\hbar$ with coefficients in $\mathscr{P}(\mathfrak{J}\mathscr{E})$ of the *perturbative flow equation*

$$\partial_\sigma {}^\hbar F_\sigma + \hbar \mathrm{D}^\hbar F_\sigma \cdot \mathscr{J}(\dot{G}_\sigma {}^\hbar F_\sigma) = 0, \qquad {}^\hbar F_1 = F \hbar^0. \tag{47}$$

This setup has the advantage that now the equation has a unique global solution which can be determined by induction on the degree $\hbar$. An approximate solution to (45) is then obtained by fixing an integer $\bar{\ell} \geqslant 0$ and letting

$$F_\sigma := \sum_{\ell=0}^{\bar{\ell}} F_\sigma^{[\ell]}.$$

The precise value for $\bar{\ell}$ will be discussed in Lemma 31. We observe that, thanks to (47), this truncation implies the existence of a maximal polynomial order $\bar{k} = \bar{k}(\bar{\ell})$ in the fields for the kernels.

Let us now introduce a condensed notation to manipulate these kernels. Let

$$\mathfrak{A} := \{(\ell, A_1, \ldots, A_k) : 0 \leqslant \ell \leqslant \bar{\ell}, 0 \leqslant k \leqslant \bar{k}\}. \tag{48}$$

For $\mathfrak{a}\in\mathfrak{A}$ with $\mathfrak{a}=(\ell,A_1,\ldots,A_k)$ we let $k(\mathfrak{a}):=k$, $\ell(\mathfrak{a}):=\ell$, $|A(\mathfrak{a})|:=|A_1|+\cdots+|A_k|$,

$$F^{\mathfrak{a}}:=F^{[\ell],\langle A_1,\ldots,A_k\rangle}, \qquad \text{and} \qquad [\mathfrak{a}]:=-\alpha+\delta\ell(\mathfrak{a})+\beta k(\mathfrak{a})+|A(\mathfrak{a})|, \tag{49}$$

for suitable positive parameters α,δ and β whose value we shall discuss in the following, in particular in Section 4.2 and 4.7. For the kernel $F^{\mathfrak{a}}$ we define a (scale dependent) norm $\|F^{\mathfrak{a}}\|_{\mu,\sigma}$ by

$$\|F^{\mathfrak{a}}\|_{\mu,\sigma}:=\sup_{\nu|\mu\leqslant\nu\leqslant\sigma}\|w(\tilde{\mathfrak{J}}^{\mathfrak{a}}_{\nu}F^{\mathfrak{a}})\,e^{T_{\bar{\nu}}(\mathfrak{a})}\|, \tag{50}$$

where

$$\|F^{\mathfrak{a}}\|:=\sup_{z\in\Lambda}\int_{\Lambda^k}|F^{[\ell],\langle A_1,\ldots,A_k\rangle}(z;z_1,\ldots,z_k)|\mathrm{d}z_1\cdots\mathrm{d}z_k,$$

and where

a) w is a polynomial space-time weight of the form $w(z)=\langle z\rangle^{-\kappa_w}$ for $z\in\Lambda$ and $\kappa_w>0$ a small number to be fixed later in Section 4.7.

b) $\nu\in(0,1)^{1+k(\mathfrak{a})}$ is understood to be a multi-index, namely $\nu=(\nu^0,\nu^1\ldots,\nu^{k(\mathfrak{a})})$,

$$\bar{\nu}:=\min\{\nu^0,\nu^1\ldots,\nu^{k(\mathfrak{a})}\},$$

and where $\mu\leqslant\nu\leqslant\sigma$ means that for any $i\in(0,\ldots,k(\mathfrak{a}))$, $\mu\leqslant\nu^i\leqslant\sigma$, and with

$$\tilde{\mathfrak{J}}^{\mathfrak{a}}_{\nu}:=\bigotimes_{i=0}^{k(\mathfrak{a})}\tilde{\mathfrak{J}}_{\nu_i,\ell(\mathfrak{a})},$$

c) $T_{\bar{\nu}}(\mathfrak{a})$ is a tree weight $T_{\bar{\nu}}(\mathfrak{a})(z,z_1,\ldots,z_{k(\mathfrak{a})})$ that depends on the variables $(z,z_1,\ldots,z_{k(\mathfrak{a})})$ of the kernel $F^{\mathfrak{a}}$ and defined by

$$T_{\bar{\nu}}(\mathfrak{a}):=2^{-\ell(\mathfrak{a})}\tau_{\bar{\nu}}(1+k(\mathfrak{a})), \tag{51}$$

with $\tau_{\bar{\nu}}(1+k(\mathfrak{a}))$ a tree weight defined as

$$\tau_\mu(1+k(\mathfrak{a}))(\boldsymbol{z}):=C([\![\mu]\!]^{-1}\,\mathrm{St}(\boldsymbol{z}))^{\omega},$$

where $\boldsymbol{z}=\{z_i\}_{i=0}^k$ with $z_i\in\Lambda$ for any $i\in\{0,1,\ldots,k(\mathfrak{a})\}$, $\mu\in(0,1)$ is a scale parameter, C is a constant to be fixed below in Remark 17, $\omega=1/r<1$ (where $r>1$ is the parameter of the Gevrey class characterising the cutoff function j in Equation (18)) and where $\mathrm{St}(\boldsymbol{z})$ stands for the Steiner diameter [GMR21] of the set $\boldsymbol{z}$, namely the length of the shortest tree connecting all these points, measured with respect to the fractional space-time distance $|(t,x)|_s:=|x|+|t|^{1/2s}$.

We conclude by introducing a norm for the family $F^{\mathfrak{A}}:=(F^{\mathfrak{a}}_\sigma)_{\mathfrak{a}\in\mathfrak{A},\sigma\in(0,1)}$ as

$$\|F^{\mathfrak{A}}\|:=\sup_{\mathfrak{a}\in\mathfrak{A}}\left[\sup_{\sigma\geqslant\mu\geqslant0}[\![\sigma]\!]^{-[\mathfrak{a}]}\|F^{\mathfrak{a}}_\sigma\|_{\mu,\sigma}\right]\vee\left[\sup_{\sigma\geqslant\mu\geqslant0}[\![\sigma]\!]^{3\gamma-\zeta}\|F^{[0],(0)}_\sigma\|_{\mu,\sigma}\right] \tag{52}$$

where γ,ζ are fixed positive constants to be determined later on.

Remark 17. Thanks to the Gevrey condition required on j in Definition 3, there exist constants $C_1,C_2>0$, uniform with respect to σ and ε, depending on the cut-off function j, such that, for any $z\in\Lambda$ and any multi-index $A=(A_0,\ldots,A_d)$, *cf.* Lemma 34,

$$|\partial^A\dot{G}_\mu(z)|\leqslant C_1(\varepsilon[\![\mu]\!]^{-1}\vee1)^{-d}\,[\![\mu]\!]^{-d-1-|A|}\,e^{-C_2([\![\mu]\!]^{-1}|z|_s)^{\omega}}. \tag{53}$$

For future convenience, we choose the constant C in the definition of the weight function $\tau_\mu(\boldsymbol{z}):=C([\![\mu]\!]^{-1}\mathrm{St}(\boldsymbol{z}))^{\omega}$ to be $C<\frac{C_2}{2}$. This implies, for any $\nu\leqslant\mu$,

$$\|\partial^A\dot{G}_\mu\,e^{2\tau_\nu(2)}\|_{\mathscr{L}(L^\infty,L^\infty)}\leqslant\|\partial^A\dot{G}_\mu\,e^{2\tau_\mu(2)}\|_{\mathscr{L}(L^\infty,L^\infty)}\lesssim(\varepsilon[\![\mu]\!]^{-1}\vee1)^{-d}[\![\mu]\!]^{2s-1-|A|}.$$

In addition, by Equation (53) together with the above condition on the constant C of the tree weight,

$$\|\partial^A \dot{G}_\mu e^{2\tau_\mu(2)}\|_{\mathscr{L}(L^1,L^\infty)} \lesssim (\varepsilon[\![\mu]\!]^{-1} \vee 1)^{-d} [\![\mu]\!]^{-d-1-|A|}.$$

Remark 18. We are now in position to fix the constant $c>0$ in the cutoff functions $\zeta_i(x) := e^{-c(|x|-2^i)_+^\omega}$ and $\zeta(x) := e^{-c|x|^\omega}$we introduced in Definition 8. This constant $c>0$ is chosen small enough so that, given two points x and y it holds that

$$\|e^{-2^{-\bar{\ell}}\tau_0(2)} \zeta^{-1}(x-y)\|_{L^\infty} < \infty,$$

where $\tau_0(2)$ is understood to depend on the variables x and y. Namely we are requiring the divergent behaviour at infinity of ζ^{-1} to be weaker than the vanishing behaviour of the associated tree weight.

In order to prove probabilistic bounds for L^p moments of the norm $\|F^{\mathfrak{A}}\|$ we first prove bounds for the cumulants of the family $(F^{\mathfrak{a}})_{\mathfrak{a}}$, i.e. for the family of deterministic kernels $(\mathscr{F}^{\boldsymbol{a}})_{\boldsymbol{a}}$ for

$$\boldsymbol{a} \in \boldsymbol{A} := \{(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) : \mathfrak{a}_k \in \mathfrak{A}, n \leqslant 2N + \bar{\ell} - L(\boldsymbol{a}), L(\boldsymbol{a}) \leqslant \bar{\ell}\},$$

defined by

$$\mathscr{F}^{\boldsymbol{a}} := \kappa_n(F^{\mathfrak{a}_1}, \cdots, F^{\mathfrak{a}_n}),$$

where $N, \bar{\ell}$ are fixed number, which will be chosen later on, see Equation (83) in Section 4.7 and Lemma 31 respectively, and where we introduce also the notations $n(\boldsymbol{a}) := n$, $L(\boldsymbol{a}) := \ell(\mathfrak{a}_1) + \cdots + \ell(\mathfrak{a}_n)$, $K(\boldsymbol{a}) := k(\mathfrak{a}_1) + \cdots + k(\mathfrak{a}_n)$, $|A(\boldsymbol{a})| := |A(\mathfrak{a}_1)| + \cdots + |A(\mathfrak{a}_n)|$. We define the global homogeneity of these kernels as

$$[\boldsymbol{a}] := -\rho + n(\boldsymbol{a})(\theta + \alpha) + [\mathfrak{a}_1] + \cdots + [\mathfrak{a}_n], \tag{54}$$

for suitable parameters ρ and θ whose value we shall discuss in the following, see Section 4.2. In particular, note that we have

$$[\boldsymbol{a}] = -\rho + \theta n(\boldsymbol{a}) + \delta L(\boldsymbol{a}) + \beta K(\boldsymbol{a}) + |A(\boldsymbol{a})|.$$

The (finite) space of cumulants is endowed with the norm $\|\mathscr{F}^{\boldsymbol{a}}\|$ obtained by taking L^1 norms of all the input variables and L^1 norms of all the $n(\boldsymbol{a})-1$ output variables apart from the first one which is taken in the L^∞ norm. We observe that this is a generalisation of the norm we considered for the kernels. On top of this basic norm we define the family of scale dependent weighted norms

$$\|\mathscr{F}^{\boldsymbol{a}}\|_{\mu,\sigma} := \sup_{\nu|\mu \leqslant \nu \leqslant \sigma} \|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}} \mathscr{F}^{\boldsymbol{a}}) e^{T_{\bar{\nu}}(\boldsymbol{a})}\|, \tag{55}$$

where

$$\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}} := \bigotimes_{i=1}^{n(\boldsymbol{a})} \tilde{\mathfrak{J}}_{\nu_i}^{\mathfrak{a}_i}.$$

With this notation, ν_i is a $(1+k(\mathfrak{a}_i))$-multi-index and thus ν here is a family of $n(\boldsymbol{a})$ multi-index, and where also here $\bar{\nu}$ is the smallest scale involved and

$$T_{\bar{\nu}}(\boldsymbol{a}) = \sum_{i=1}^{n(\boldsymbol{a})} T_{\bar{\nu}}(\mathfrak{a}_i) + 2^{-(\bar{\ell}+1)} \tau_{\bar{\nu}}(n(\boldsymbol{a})), \tag{56}$$

where $\tau_{\bar{\nu}}(n(\boldsymbol{a}))$ is defined as in the above case, with the only difference that in this case the Steiner diameter involves the output variables of the cumulant.

In Section 4.7 we will go from estimates on the norm $\|\mathscr{F}^{\boldsymbol{A}}\|$ for the family $\mathscr{F}^{\boldsymbol{A}} := (\mathscr{F}^{\boldsymbol{a}}_\sigma)_{\boldsymbol{a}\in\boldsymbol{A},\sigma\in(0,1)}$ defined as

$$\|\mathscr{F}^{\boldsymbol{A}}\| := \sup_{\boldsymbol{a}\in\boldsymbol{A}} \left[\sup_{\sigma\geqslant\mu\geqslant 0} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}^{\boldsymbol{a}}_\sigma\|_{\mu,\sigma} \right]^{1/n(\boldsymbol{a})}. \tag{57}$$

to those on $\|F^{\mathfrak{A}}\|$ via a Kolmogorov-type argument. Following Duch [Duc21, Duc22], we introduce a flow equations for cumulants to control the norm $\|\mathscr{F}^{\boldsymbol{A}}\|$.

Lemma 19. *The cumulants satisfy Duch's flow equation:*

$$\partial_\sigma \mathscr{F}^{\boldsymbol{a}}_\sigma = \sum_{\boldsymbol{b}} \mathscr{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma) + \sum_{\boldsymbol{b},\boldsymbol{c}} \mathscr{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma, \mathscr{F}^{\boldsymbol{c}}_\sigma), \tag{58}$$

where the linear operators $\mathscr{A}^{\boldsymbol{a}}_{\boldsymbol{b}}$ and $\mathscr{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}$ are non-zero when

$$\mathscr{A}^{\boldsymbol{a}}_{\boldsymbol{b}} \neq 0 \Longleftrightarrow \begin{cases} n(\boldsymbol{a}) = n(\boldsymbol{b}) - 1, L(\boldsymbol{a}) = L(\boldsymbol{b}) + 1, K(\boldsymbol{a}) = K(\boldsymbol{b}) - 1, \\ [\boldsymbol{a}] = [\boldsymbol{b}] - \theta + \delta - \beta, \end{cases} \tag{59}$$

$$\mathscr{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}} \neq 0 \Longleftrightarrow \begin{cases} n(\boldsymbol{a}) = n(\boldsymbol{b}) + n(\boldsymbol{c}) - 1, L(\boldsymbol{a}) = L(\boldsymbol{b}) + L(\boldsymbol{c}) + 1, K(\boldsymbol{a}) = K(\boldsymbol{b}) + K(\boldsymbol{c}) - 1, \\ [\boldsymbol{a}] = [\boldsymbol{b}] + [\boldsymbol{c}] + \rho - \theta + \delta - \beta. \end{cases} \tag{60}$$

Moreover, they have bounds

$$[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{A}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma)\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} (\varepsilon[\![\sigma]\!]^{-1} \vee 1)^{-d} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma}.$$

and

$$[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{B}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma, \mathscr{F}^{\boldsymbol{c}}_\sigma)\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1} (\varepsilon[\![\sigma]\!]^{-1} \vee 1)^{-d} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma} \|\mathscr{F}^{\boldsymbol{c}}_\sigma\|_{\mu,\sigma},$$

provided the compatibility conditions

$$\theta + \beta - \delta - d = 0, \qquad -\rho + \theta + \beta - \delta + 2s = 0, \tag{61}$$

hold.

Proof. The derivation of the flow equation is a direct consequence of the definition of cumulants, see [Duc21, Duc22]. The detailed form of the operators is not very important in the following discussion and is explicitated in Appendix B where the claims of the Lemma are also proven (see Lemma 44 and Lemma 45). □

This general structure of the flow equation (58) allows us to propagate estimates for the kernels of the form

$$\sup_{\sigma\geqslant\mu} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}^{\boldsymbol{a}}_\sigma\|_{\mu,\sigma} < \infty.$$

However, depending on the sign of $[\boldsymbol{a}]$, we shall handle differently the cumulants: in particular, for $[\boldsymbol{a}] > 0$, the flow equation can be solved *backward* starting from the final condition at $\sigma = 1$. We shall refer to cumulants $\mathscr{F}^{\boldsymbol{a}}_\sigma$ with $[\boldsymbol{a}] > 0$ as *irrelevant cumulants*. On the other hand, this approach does not work for cumulants for which $[\boldsymbol{a}] < 0$ as in this case the flow equation cannot be integrated close to $\sigma = 1$. As we shall see in Section 4.4, we will solve the flow equation for this class of cumulants, called *relevant cumulants*, by integrating it *forward*. Finally, we shall say that a cumulant $\mathscr{F}^{\boldsymbol{a}}_\sigma$ is *marginal* if $[\boldsymbol{a}] = 0$.

Remark 20. Before going on with the analysis of the flow equations for cumulants, some comments about symmetries are in order. First of all we observe that the SPDE we are considering, namely the Φ^4_d model, is invariant under the transformation $\Psi \mapsto -\Psi$ and $\xi \mapsto -\xi$ which also preserves the law of the noise ξ. At the level of cumulants, this entails that if

$$n(\boldsymbol{a}) + K(\boldsymbol{a}) \in 2\mathbb{N} + 1,$$

then $\mathscr{F}_1^{\boldsymbol{a}}=0$. This feature is preserved by the flow equation for cumulants, Equations (58), due to the conditions of Equations (59) and (60). Indeed if we consider a cumulant such that $n(\boldsymbol{a})+K(\boldsymbol{a})\in 2\mathbb{N}+1$, then also on the right hand side of the flow equation there are only terms satisfying the same property: we see from Equation (59) that the only non-vanishing terms $\mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}})$ satisfy

$$n(\boldsymbol{b})+K(\boldsymbol{b})=n(\boldsymbol{a})+K(\boldsymbol{a})+2\in 2\mathbb{N}+1.$$

Similarly, for the terms $\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}})$ the only non-vanishing contributions satisfy

$$n(\boldsymbol{b})+K(\boldsymbol{b})+n(\boldsymbol{c})+K(\boldsymbol{c})=n(\boldsymbol{a})+K(\boldsymbol{a})+2\in 2\mathbb{N}+1,$$

due to Equation (60). As a consequence, we conclude that if $n(\boldsymbol{a})+K(\boldsymbol{a})\in 2\mathbb{N}+1$, then $\mathscr{F}_\mu^{\boldsymbol{a}}=0$ for any $\mu\in[0,1]$. A further symmetry is represented by spatial reflection, *i.e.,* the transformation

$$\Psi(t,x)\mapsto\Psi(t,-x), \qquad \xi(t,x)\mapsto\xi(t,-x),$$

which preserves the law of ξ too. Using an argument as the one above, we conclude that for any $\boldsymbol{a}$, $\mathscr{F}_\mu^{\boldsymbol{a}}$ is symmetric with respect to such transformation.

4.2 Bounds on parameters

We shall now bound the parameters ρ,θ,β,δ and s introduced before. To start, we need to take into account the compatibility conditions as per Equation (61). Another constraint comes from the scaling of the initial condition $(\mathscr{F}_1^{\boldsymbol{a}})_{\boldsymbol{a}}$. Due to the Gaussian nature of the noise, it has nonzero contributions only for $n(\boldsymbol{a})\in\{1,2\}$ and $L(\boldsymbol{a})=0$. When $n(\boldsymbol{a})=2$ only the two point function of the noise contributes so we must have $k(\boldsymbol{a})=0$, $|A(\boldsymbol{a})|=0$, in which case, at the level of integral kernel,

$$\mathscr{F}_1^{\boldsymbol{a}}(t,x;t',x')=\kappa(F^{[0],(0)}(t,x),F^{[0],(0)}(t',x'))=\mathbb{E}[\xi^{(\varepsilon)}(t,x)\,\xi^{(\varepsilon)}(t',x')]=\delta(t-t')\mathbb{1}_{x=x'},$$

from Equation (6). So

$$\|\mathscr{F}_1^{\boldsymbol{a}}\|_{\mu,\sigma}=\sup_{\nu|\mu\leq\nu_0,\nu_1\leq\sigma}\|\check{j}_{\nu_0}*\check{j}_{\nu_1}e^{T_{\min\{\nu_0,\nu_1\}}(2)}\|\approx 1,$$

uniformly in $\varepsilon,\mu,\sigma>0$. As a consequence we have to require it to be a marginal cumulant, namely

$$[\boldsymbol{a}]=-\rho+2\theta\leq 0, \tag{62}$$

Moreover note that for $n(\boldsymbol{a})=1$ we have contributions $\boldsymbol{a}$ with $k(\boldsymbol{a})=1,3$ and when $k(\boldsymbol{a})=3$, $\mathscr{F}_1^{\boldsymbol{a}}=\kappa(F^{[0],(3)})$, with, at the level of its integral kernel

$$(\tilde{J}_\nu\mathscr{F}_1^{\boldsymbol{a}})(z;z_1,z_2,z_3)=-\lambda\int \mathrm{d}z'\,\check{j}_{\nu_0}(z-z')\prod_{i=1}^{3}\check{j}_{\nu_i}(z'-z_i),$$

and thus $\|\mathscr{F}_1^{\boldsymbol{a}}\|_{\mu,\sigma}=\lambda$, uniformly in $\varepsilon,\mu,\sigma>0$, which requires

$$[\boldsymbol{a}]=-\rho+\theta+3\beta=0. \tag{63}$$

Now, from Equation (63) and Equation (61), we have that

$$\beta=s-\frac{\delta}{2}, \qquad \rho=d+2s, \qquad \theta=d+2s-3\beta. \tag{64}$$

These equations are compatible with Eq. (62) provided

$$\delta\leq\frac{4s-d}{3}=:\delta_*, \tag{65}$$

which plays the role of an upper bound on the value of $\delta > 0$. An additional bound on δ will come from the regularity of the white noise, see (86) below and will imply that the inequality in (65) must be strict. We further observe that Equation (63) entails

$$\theta = \rho - 3\beta = 3\beta - 3[\delta_* - \delta].$$

In the following we will fix δ as large as possible, namely such that $\delta_* - \delta > 0$ is as small as we like.

4.3 Classification of cumulants

Given these bounds on the parameters, we can investigate the class of cumulants being relevant or marginal that is when it holds $[\boldsymbol{a}] \leqslant 0$. Observing that $-\rho + 2\theta = d - 4s + 3\delta = 0$, the condition $[\boldsymbol{a}] \leqslant 0$ can be written as

$$[\boldsymbol{a}] = \theta(n(\boldsymbol{a}) - 2) + \beta K(\boldsymbol{a}) + \delta L(\boldsymbol{a}) = \beta(3n(\boldsymbol{a}) - 6 + K(\boldsymbol{a})) + \delta L(\boldsymbol{a}) - 3[\delta_* - \delta]n(\boldsymbol{a}) \leqslant 0.$$

We see that, provided $\delta_* - \delta$ is chosen sufficiently small we have

a) if $n(\boldsymbol{a}) > 2$, there are no relevant/marginal cumulants;

b) if $n(\boldsymbol{a}) = 2$, the only relevant/marginal cumulant is the one with $L(\boldsymbol{a}) = K(\boldsymbol{a}) = 0$, that is the two point function of the noise: $\mathscr{F}^{\boldsymbol{a}} = \kappa_2\big(F_\sigma^{[0](0)}, F_\sigma^{[0](0)}\big)$.

c) if $n(\boldsymbol{a}) = 1$, the only relevant/marginal cumulants are (at most) those with $K(\boldsymbol{a}) \leqslant 3$.

Summarising, the only relevant/marginal cumulants are

$$\kappa_2\big(F_\sigma^{[0](0)}, F_\sigma^{[0](0)}\big), \qquad \kappa_1\big(F_\sigma^{[\ell](k)}\big), \quad k = 0, 1, 2, 3.$$

We can further restrict the set of cumulants to be further analyzed. Indeed, the flow equation for the cumulants with $L(\boldsymbol{a}) = 0$ is trivial and there is no evolution, so they coincide with their initial values. This applies to $\kappa_2\big(F_\sigma^{[0](0)}, F_\sigma^{[0](0)}\big)$ and $\kappa_1\big(F_\sigma^{[0](3)}\big)$. Moreover $\kappa_1\big(F_\sigma^{[\ell](3)}\big)$ for $\ell \geqslant 1$ is irrelevant. As for the others, by Remark 20, we know that the cumulants $\kappa_1\big(F_\sigma^{[\ell](0)}\big)$ and $\kappa_1\big(F_\sigma^{[\ell](2)}\big)$ are vanishing due to symmetry arguments. Thus the only remaining cumulant that we have to consider more in detail is $\kappa_1\big(F_\sigma^{[\ell](1)}\big)$ for $\ell \geqslant 1$. This will be the topic of the next section.

4.4 Localisation

The handling of the relevant cumulants $\big(\kappa_1\big(F_\sigma^{[\ell](1)}\big)\big)_{\ell \geqslant 1}$, for which the flow equation cannot propagate information backwards along the flow, relies on exploiting the ambiguity in the lift of the flow equation to the jet space. Note first that our model does not show derivative fields in the flow equation, so the kernels $F_\sigma^{[\ell]\langle A_1, \ldots, A_k\rangle}$ have components only for $A_1 = \cdots = A_k = 0$, i.e. in the fields space. We have

$$F_\sigma^{[\ell](1)}(\mathscr{J}\psi)(z) = \int_\Lambda F_\sigma^{[\ell]\langle 0\rangle}(z; z_1)\psi(z_1)\mathrm{d}z_1,$$

and we can rewrite this by performing a Taylor expansion of the field ψ around $\psi(z)$. At first order this reads

$$\psi(z_1) \;=\; \psi(z) + \sum_i \int_0^1 \partial^i \psi(z + \rho_{z_1 - z}(t))[\mathrm{d}\rho_{z_1 - z}(t)]^i,$$

where $i \in \{0, 1\pm, 2\pm, \ldots, d\pm\}$ with ∂^0 the time derivative and $\partial^{k\pm}\psi(z') := \pm\varepsilon^{-1}[\psi(z \pm e_k) - \psi(z)]$ denote resp. the discrete forward $(k+)$ and backward $(k-)$ derivatives in the k-th spatial direction, for any $h \in \Lambda$ the function $\rho_h\colon [0,1] \longrightarrow \Lambda$ is a bounded variation path such that $\rho_h(0) = 0$ and $\rho_h(1) = h$ and the notation $[\mathrm{d}\rho_{z_1-z}(t)]^{k\pm} := \pm(\mathrm{d}\rho^k_{z_1-z}(t))_\pm$ denotes resp. the positive or negative component of the measure $\mathrm{d}\rho^k_{z_1-z}(t)$ multiplied with the corresponding sign, while we let $[\mathrm{d}\rho_{z_1-z}(t)]^0 := \mathrm{d}\rho^0_{z_1-z}(t)$. Expanding once more we have

$$
\begin{aligned}
\psi(z_1) &= \psi(z) + \sum_{i \neq 0} \partial^i\psi(z)[z_1 - z]^i + \int_0^1 \partial^0\psi(z + \rho_{z_1-z}(t))\mathrm{d}\rho^0_{z_1-z}(t)\\
&\quad + \sum_{i \neq 0, j} \int_0^1\int_0^t \partial^i\partial^j\psi(z + \rho_{z_1-z}(u))[\mathrm{d}\rho_{z_1-z}(u)]^j[\mathrm{d}\rho_{z_1-z}(t)]^i.
\end{aligned}
$$

Note that the path ρ_h is piecewise constant in space so that the signed measure $\mathrm{d}\rho^i_h$ is well defined and given by a sum of delta functions times increments. We choose it so that its total mass is bounded by $|h|$ and $\int_0^1 \mathrm{d}\rho_h(t) = h$ and such that $\rho^0_{z_1-z}(u) = (z_1 - z)_0 u$. As a consequence

$$
\begin{aligned}
F^{[\ell](1)}_\sigma(\mathscr{J}\psi)(z) &= \psi(z)\int_\Lambda \mathrm{d}z_1 F^{[\ell]\langle 0\rangle}_\sigma(z, z_1) + \sum_{i \neq 0} \mathbb{1}_{i \neq 0}\int_\Lambda \mathrm{d}z_1 [z_1 - z]^i F^{[\ell]\langle 0\rangle}_\sigma(z, z_1)\,\partial^i\psi(z) +\\
&\quad + \int_\Lambda \mathrm{d}z_1 \int_0^1 \mathrm{d}\rho^0_{z_1-z}(t) F^{[\ell]\langle 0\rangle}_\sigma(z, z_1)\partial^0\psi(z + \rho_{z_1-z}(t)) +\\
&\quad + \sum_{i \neq 0, j} \mathbb{1}_{i \neq 0}\int_\Lambda \mathrm{d}z_1 \int_0^1 [\mathrm{d}\rho_{z_1-z}(t)]^i \int_0^t [\mathrm{d}\rho_{z_1-z}(u)]^j F^{[\ell]\langle 0\rangle}_\sigma(z, z_1)\partial^i\partial^j\psi(z + \rho_{z_1-z}(u)),
\end{aligned}
$$

Now observe that $\partial^{k-}\psi(z) - \partial^{k+}\psi(z) = \varepsilon(\partial^{k+}\partial^{k-}\psi)(z)$ so,

$$
\begin{aligned}
\sum_{i \neq 0}\int_\Lambda \mathrm{d}z_1 [z_1 - z]^i F^{[\ell]\langle 0\rangle}_\sigma(z, z_1)\,\partial^i\psi(z) &= \sum_k \Big[\int_\Lambda (z_1 - z)^k_+ F^{[\ell]\langle 0\rangle}_\sigma(z, z_1)\mathrm{d}z_1\,\partial^{k+}\psi(z) +\\
&\quad - \int_\Lambda (z_1 - z)^k_- \mathrm{d}z_1\, F^{[\ell]\langle 0\rangle}_\sigma(z, z_1)\,\partial^{k-}\psi(z)\Big]\\
= \sum_k \frac{\partial^k_+\psi(z) + \partial^{k-}\psi(z)}{2}\int_\Lambda (z_1 - z)^k \mathrm{d}z_1\, F^{[\ell]\langle 0\rangle}_\sigma(z, z_1) &- \sum_k \varepsilon(\partial^{k+}\partial^{k-}\psi)(z)\int (z_1 - z)^k F^{[\ell]\langle 0\rangle}_\sigma(z, z_1)\mathrm{d}z_1.
\end{aligned}
$$

So when computed on jets of smooth functions $\mathscr{J}\psi$ we can express them as a localised term and a remainder. In particular we define two operations $\boldsymbol{L}$ and $\boldsymbol{R}$ so that

$$
\begin{aligned}
(\boldsymbol{L}^\hbar F_\sigma)(\Psi)(z) &:= \sum_{\mathfrak{b}: k(\mathfrak{b}) = 1} \Psi^{\varnothing}(z)\hbar^{\ell(\mathfrak{b})}\int_\Lambda (\tilde{\mathfrak{J}}^{\otimes 2}_{\sigma,\ell(\mathfrak{b})} F^{[\ell(\mathfrak{b})]\langle 0\rangle}_\sigma)(z; z_1)\mathrm{d}z_1\\
&\quad + \sum_{\mathfrak{b}: k(\mathfrak{b}) = 1}\ \sum_{j \in \{1, \ldots, d\}} \Psi^{(j\pm)}(z)\hbar^{\ell(\mathfrak{b})}\int_\Lambda \frac{(z_1 - z)^j}{2}(\tilde{\mathfrak{J}}^{\otimes 2}_{\sigma,\ell(\mathfrak{b})} F^{[\ell(\mathfrak{b})]\langle 0\rangle}_\sigma)(z; z_1)\mathrm{d}z_1\\
&\quad + \sum_{\mathfrak{b}: k(\mathfrak{b}) \neq 1} \hbar^{\ell(\mathfrak{b})} F^{\mathfrak{b}}_\sigma(\Psi),\\
(\boldsymbol{R}^\hbar F_\sigma)(\Psi)(z) &:= \sum_{\mathfrak{b}: k(\mathfrak{b}) = 1} \hbar^{\ell(\mathfrak{b})}\Big[\int_\Lambda \mathrm{d}z_1 \int_0^1 \mathrm{d}\rho^0_{z_1-z}(t)\,(\tilde{\mathfrak{J}}^{\otimes 2}_{\sigma,\ell(\mathfrak{b})} F^{[\ell(\mathfrak{b})]\langle 0\rangle}_\sigma)(z, z_1)\,\Psi^{(0)}(z + \rho_{z_1-z}(t)) +\\
&\quad - \sum_{j \in \{1, \ldots, d\}} \varepsilon\,(\Psi^{(j+j-)}(z) + \Psi^{(j-j+)}(z))\int_\Lambda \frac{(z_1 - z)^j}{2}(\tilde{\mathfrak{J}}^{\otimes 2}_{\sigma,\ell(\mathfrak{b})} F^{[\ell(\mathfrak{b})]\langle 0\rangle}_\sigma)(z; z_1)\mathrm{d}z_1\\
&\quad + \sum_{A = (i,j), i \neq 0} \int_\Lambda \mathrm{d}z_1 \int_0^1 [\mathrm{d}\rho_{z_1-z}(t)]^i \int_0^t [\mathrm{d}\rho_{z_1-z}(u)]^j\,(\tilde{\mathfrak{J}}^{\otimes 2}_{\sigma,\ell(\mathfrak{b})} F^{[\ell(\mathfrak{b})]\langle 0\rangle}_\sigma)(z; z_1)\,\Psi^A(z + \rho_{z_1-z}(u))\Big],
\end{aligned}
\tag{66}
$$

and which satisfy

$$
\tilde{\mathfrak{J}}_\sigma{}^\hbar F_\sigma(\mathscr{J}\tilde{\mathfrak{J}}_\sigma\psi) = \tilde{\mathfrak{J}}_\sigma \boldsymbol{L}^\hbar F_\sigma(\mathscr{J}\tilde{\mathfrak{J}}_\sigma\psi) + \tilde{\mathfrak{J}}_\sigma \boldsymbol{R}^\hbar F_\sigma(\mathscr{J}\tilde{\mathfrak{J}}_\sigma\psi).
$$

Remark 21. We observe that in Equation (66) we inserted the smoothing operator $\tilde{\mathfrak{J}}^{\otimes 2}_{\sigma,\ell}$ acting on the variables $z, z_1 \in \Lambda$. This is legitimate since we shall only be interested in controlling the kernels with respect to the family of seminorms we introduced in Equation (50) and we stress that the difference $F^{[\ell]\langle 0\rangle}_\sigma - \tilde{\mathfrak{J}}^{\otimes 2}_{\sigma,\ell} F^{[\ell]\langle 0\rangle}_\sigma$ is vanishing w.r.t. this family of seminorms.

We can then modify the perturbative flow equation (47) to read

$$\partial_\sigma {}^\hbar F_\sigma + \hbar(\boldsymbol{L} + \boldsymbol{R})\big[\mathrm{D}^\hbar F_\sigma \cdot \mathscr{J}(\dot{G}_\sigma {}^\hbar F_\sigma)\big] = 0, \qquad {}^\hbar F_1 = F\hbar^0, \tag{67}$$

since its flow projects down to the previous flow on smooth sections. Note that the $\boldsymbol{R}$ operation increases the total number of derivatives of the fields of two, but no other operation introduces new derivatives. Moreover the initial condition does not contain derivatives of fields, so we can deduce that at any perturbative order only finitely many terms have derivatives of order at most two and that no higher order derivatives are created by solving the flow equations.

The localisation is reflected easily on the flow equation for the cumulants, producing a structurally similar equation

$$\partial_\sigma \mathscr{F}^{\boldsymbol{a}}_\sigma = \sum_{\boldsymbol{b}} \tilde{\mathscr{A}}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma) + \sum_{\boldsymbol{b},\boldsymbol{c}} \tilde{\mathscr{B}}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma, \mathscr{F}^{\boldsymbol{c}}_\sigma), \tag{68}$$

with modified operators $\tilde{\mathscr{A}}, \tilde{\mathscr{B}}$ whose specific form is discussed in Appendix B.2. The most important feature of these operators is that they satisfy the same estimates of the operators $\mathscr{A}$ and $\mathscr{B}$ we discussed before. In particular we have the following lemma (see Appendix B.2 for the proof, in particular Lemma 47 and Lemma 48).

Lemma 22. *Under the compatibility conditions (61), it holds that, for $\sigma \geq \mu$,*

$$[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\tilde{\mathscr{A}}^{\boldsymbol{a}}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma)\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma},$$

$$[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\tilde{\mathscr{B}}^{\boldsymbol{a}}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma, \mathscr{F}^{\boldsymbol{c}}_\sigma)\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma} \|\mathscr{F}^{\boldsymbol{c}}_\sigma\|_{\mu,\sigma}.$$

Remark 23. With reference to Section 4.3, we saw that the family of relevant cumulants consists of those $\mathscr{F}^{\boldsymbol{a}}$ with $n(\boldsymbol{a}) = 1$ and $K(\boldsymbol{a}) = 1$. On account of the above argument, we have that they are of the form $\mathscr{F}^{\boldsymbol{a}}_\sigma = \kappa_1\big(F^{[\ell]\langle A(\boldsymbol{a})\rangle}_\sigma\big)$. More into the detail, we observe that the only relevant cumulant is $\mathscr{F}^{\boldsymbol{a}}_\sigma = \kappa_1(F^{[\ell]\langle 0\rangle}_\sigma)$ as the other ones are either irrelevant or vanishing. Indeed,

- if $A(\boldsymbol{a})$ identifies a first order derivative with respect to time, then $|A(\boldsymbol{a})| = 2s$ and thus, on account of Equation (61),

  $$[\boldsymbol{a}] = -\rho + \theta + \beta + \delta L(\boldsymbol{a}) + 2s = \delta\ell,$$

  which is positive for $L(\boldsymbol{a}) > 0$, while we observe that for $L(\boldsymbol{a}) = 0$ this cumulant is marginal and since $\kappa_1\big(F^{[0]\langle A(\boldsymbol{a})\rangle}_1\big) = 0$ and $\partial_\sigma \kappa_1\big(F^{[0]\langle A(\boldsymbol{a})\rangle}_\sigma\big) = 0$, we have $\kappa_1\big(F^{[0]\langle A(\boldsymbol{a})\rangle}_\sigma\big) = 0$ for any σ;
- if $A(\boldsymbol{a})$ identifies a second order derivative with respect to space, then $|A(\boldsymbol{a})| = 2$ and thus

  $$[\boldsymbol{a}] = -\rho + \theta + \beta + \delta L(\boldsymbol{a}) + 2 > 0,$$

  for any $L(\boldsymbol{a}) \geq 0$ and thus this cumulant is irrelevant;
- if $A(\boldsymbol{a})$ identifies a first order derivative with respect to space, then $\kappa_1\big(F^{[L(\boldsymbol{a})]\langle A(\boldsymbol{a})\rangle}_\sigma\big)$ is vanishing since the cumulants are symmetric under spatial reflections as per Remark 20 and the coefficient of the monomial $\Psi^{(k\pm)}$ in Eq. (66) vanishes;
- if $A(\boldsymbol{a})$ identifies higher order derivatives with respect to space and/or time, then *a fortiori*, the associated cumulant is irrelevant.

As a consequence, the only relevant cumulants are $\mathscr{F}_\sigma^{\boldsymbol{a}} = \kappa_1(F_\sigma^{[L(\boldsymbol{a})]\langle 0\rangle})$ for $L(\boldsymbol{a}) \in (0, 2\beta/\delta]$. We further observe that these cumulants are local by construction and the stationarity in law of the cumulants implies that there exist constants $(r_\sigma^\ell)_{\ell \geqslant 0}$ such that

$$\kappa_1(F_\sigma^{[\ell]\langle 0\rangle}(z, z_1)) = \mathbb{E}(F_\sigma^{[\ell]\langle 0\rangle}(z, z_1)) = r_\sigma^\ell \delta(z - z_1). \tag{69}$$

4.5 Inductive procedure

The aim of this section is to prove that we can solve the flow equation for cumulants, Equation (68), through an induction procedure over the order ℓ. This is due to the fact that, by Lemma 19, it is a triangular system with respect to the parameter $L(\boldsymbol{a})$.

Lemma 24. *For any $\bar{r} \in \mathbb{R}$ there exists (non-unique) constants $(r_1^{\ell,\varepsilon})_{\ell=1,\ldots,\bar{\ell}}$ such that the solution of the approximate flow equation with initial condition*

$$F_1(\Psi) = \lambda(\Psi^0)^3 + r_\varepsilon \Psi^0 + \xi^{(\varepsilon)}$$

where $r_\varepsilon := \bar{r} + \sum_{\ell \geqslant 1}^{\bar{\ell}} r_1^{\ell,\varepsilon}$, satisfies

$$\|\mathscr{F}^{\boldsymbol{A}}\| = \sup_{\boldsymbol{a} \in \boldsymbol{A}} \left[\sup_{\sigma \geqslant \mu} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma}\right]^{1/n(\boldsymbol{a})} < \infty.$$

Proof. We split the initial condition on the levels ℓ as follows

$$F_1^{[0]}(\Psi) = \lambda(\Psi^0)^3 + \bar{r}\,\Psi^0 + \xi^{(\varepsilon)}, \qquad F_1^{[\ell]}(\Psi) = r_1^{\ell,\varepsilon}\Psi^0, \qquad \ell = 1, \ldots, \bar{\ell},$$

with $(r_1^{\ell,\varepsilon})_{\ell=1,\ldots,\bar{\ell}}$ quantities to be determined later. For a fixed ℓ let $N_\ell := 2N + \bar{\ell} - \ell$ and

$$M_\ell := \sup_{\substack{\boldsymbol{a}: L(\boldsymbol{a}) \leqslant \ell, \\ n(\boldsymbol{a}) \leqslant N_\ell}} \left[\sup_{\sigma \geqslant \mu} [\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma}\right]^{1/n(\boldsymbol{a})}.$$

We will prove by induction that $M_\ell < \infty$ for all $\ell = 0, \ldots, \bar{\ell}$. First of all we discuss the case $\ell = L(\boldsymbol{a}) = 0$, for which we have $\partial_\sigma \mathscr{F}_\sigma^{\boldsymbol{a}} = 0$ by definition and thus $\mathscr{F}_\sigma^{\boldsymbol{a}} = \mathscr{F}_1^{\boldsymbol{a}}$ for all $\sigma \in [0, 1]$. For irrelevant cumulants it holds that $\mathscr{F}_1^{\boldsymbol{a}} = 0$, yielding $\mathscr{F}_\sigma^{\boldsymbol{a}} = 0$. Considering instead the relevant cases, as we discussed in Section 4.2, the norms $[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}_1^{\boldsymbol{a}}\|_{\mu,\sigma}$ are uniformly bounded in μ and σ. The only relevant case we have not discussed in Section 4.2 is the case of $\mathscr{F}_1^{\boldsymbol{a}}$ with $n(\boldsymbol{a}) = 1$, $L(\boldsymbol{a}) = 0$ and $K(\boldsymbol{a}) = 1$: we observe that in this case $\mathscr{F}_1^{\boldsymbol{a}} = \bar{r} \in \mathbb{R}$ and thus, being $[\boldsymbol{a}] < 0$, $[\![\sigma]\!]^{-[\boldsymbol{a}]} \|\mathscr{F}_1^{\boldsymbol{a}}\|_{\mu,\sigma}$ is uniformly bounded in μ and σ. As a consequence, it holds that $M_0 < \infty$.

Let us now consider the induction step. Assume that $M_\ell < \infty$, for a fixed ℓ and we shall prove that also $M_{\ell+1}$ is finite. From Lemma 22, we have that

$$\begin{aligned} \|\tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}})\|_{\mu,\sigma} &\lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1} M_\ell^{n(\boldsymbol{a})+1}, \\ \|\tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}})\|_{\mu,\sigma} &\lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1} M_\ell^{n(\boldsymbol{b})} M_\ell^{n(\boldsymbol{c})} \lesssim [\![\sigma]\!]^{[\boldsymbol{a}]-1} M_\ell^{n(\boldsymbol{a})+1}. \end{aligned} \tag{70}$$

where recall that $n(\boldsymbol{b}) = n(\boldsymbol{a}) + 1$ in the first line and $n(\boldsymbol{b}) + n(\boldsymbol{c}) = n(\boldsymbol{a}) + 1$ in the second, so this inequalities hold for any $\boldsymbol{a}$ with $L(\boldsymbol{a}) = \ell + 1$ and $n(\boldsymbol{a}) \leqslant N_\ell - 1 = N_{\ell+1}$.

We shall distinguish between two cases, whether $\mathscr{F}_\sigma^{\boldsymbol{a}}$ is irrelevant, *i.e.* $[\boldsymbol{a}] > 0$, or relevant, *i.e.*, $[\boldsymbol{a}] < 0$. For what concerns *marginal* cumulants, namely $[\boldsymbol{a}] = 0$, we observe that on account of the discussion of Section 4.3 this is the case of $\kappa_1(F_\sigma^{[0](3)})$ which is accounted for by M_0 above and thus it does not enter into play for $L(\boldsymbol{a}) = \ell + 1 > 0$.

If we assume that $\mathscr{F}^{\boldsymbol{a}}$ is irrelevant, *i.e.* $[\boldsymbol{a}] > 0$, then we can solve the flow equation inductively *backward* from the final condition at $\sigma = 1$. First of all we observe that for $[\boldsymbol{a}] > 0$, it holds that $\mathscr{F}_1^{\boldsymbol{a}} = 0$ and thus, from Equation (68),

$$\mathscr{F}_\eta^{\boldsymbol{a}} = \int_\eta^1 \left[\sum_{\boldsymbol{b}} \tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}) + \sum_{\boldsymbol{b},\boldsymbol{c}} \tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}})\right] \mathrm{d}\sigma.$$

As a consequence, by triangular inequality and Equation (70), there exists a constant $C>0$ such that

$$\|\mathscr{F}_\eta^{\boldsymbol{a}}\|_{\mu,\eta} \leqslant C M_\ell^{n(\boldsymbol{a})+1} \int_\eta^1 [\![\sigma]\!]^{[\boldsymbol{a}]-1}\,\mathrm{d}\sigma \lesssim C\,[\![\eta]\!]^{[\boldsymbol{a}]}\,M_\ell^{n(\boldsymbol{a})+1}.$$

On the other hand, if we assume $\mathscr{F}^{\boldsymbol{a}}$ to be relevant, *i.e.* $[\boldsymbol{a}]<0$, then we can solve the flow equation *forward* starting from an arbitrary intermediate condition at an arbitrary but fixed scale $\mu_0 \in (0,1)$.

In particular, recalling Equation (69), we can then fix $r_{\mu_0}^{\ell+1}$ to be some arbitrary value $\mathfrak{c}^{\ell+1} \in \mathbb{R}$ at some reference scale $\mu_0<1$ and then using the flow equations *forward* to determine it for all the higher scales $\mu \in (\mu_0, 1)$ with the equation

$$r_\mu^{\ell+1,\varepsilon} := \mathfrak{c}^{\ell+1} + \int_{\mu_0}^{\mu} \partial_\sigma \kappa_1\big(F_\sigma^{[\ell+1],(1)}\big)\mathrm{d}\sigma. \tag{71}$$

As a consequence, denoting the relevant cumulant $\kappa_1\big(F_\sigma^{[\ell+1],(1)}\big)$ at hand by $\mathscr{F}_\sigma^{\boldsymbol{a}}$ with $N(\boldsymbol{a})=1$, $L(\boldsymbol{a})=\ell+1, K(\mathfrak{a}_1)=1, |A(\mathfrak{a}_1)|=0$ and observing that, on account of Equation (69), $|r_\sigma^{\ell+1,\varepsilon}| \lesssim \|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma}$, by the estimates above, we have, using

$$0>[\boldsymbol{a}] = -\rho+\theta+\delta(\ell+1)+\beta = -2s+\delta(\ell+2),$$

and Equations (70) but taking into account the more refined ε-dependent estimate for $\dot{G}$ in Lemma 34,

$$\begin{aligned}
\|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma} &\leqslant |\mathfrak{c}^{\ell+1}| + \int_{\mu_0}^{\sigma} \|\partial_{\sigma'}\mathscr{F}_{\sigma'}^{\boldsymbol{a}}\|_{\mu,\sigma'}\mathrm{d}\sigma' \\
&\lesssim |\mathfrak{c}^{\ell+1}| + \left[\int_{\mu_0}^{\sigma} \|\tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_{\sigma'}, \mathscr{F}_{\sigma'}^{\boldsymbol{b}})\|_{\mu,\sigma'}\mathrm{d}\sigma' + \int_{\mu_0}^{\sigma} \|\tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{G}_{\sigma'}, \mathscr{F}_{\sigma'}^{\boldsymbol{b}}, \mathscr{F}_{\sigma'}^{\boldsymbol{c}})\|_{\mu,\sigma'}\mathrm{d}\sigma'\right] \\
&\lesssim |\mathfrak{c}^{\ell+1}| + C M_\ell^{n(\boldsymbol{a})+1} \int_{\mu_0}^{\sigma} [\![\sigma']\!]^{[\boldsymbol{a}]-1}(\varepsilon[\![\sigma']\!]^{-1}\vee 1)^{-d}\mathrm{d}\sigma' \\
&\lesssim |\mathfrak{c}^{\ell+1}| + C M_\ell^{n(\boldsymbol{a})+1}\left[\int_{[\![\sigma]\!]\vee\varepsilon}^{[\![\mu_0]\!]} \mu^{[\boldsymbol{a}]-1}(\varepsilon^{-d}\mu^d\wedge 1)\mathrm{d}\mu + \int_{[\![\sigma]\!]}^{[\![\sigma]\!]\vee\varepsilon} \mu^{[\boldsymbol{a}]-1}(\varepsilon^{-d}\mu^d\wedge 1)\mathrm{d}\mu\right] \\
&\lesssim |\mathfrak{c}^{\ell+1}| + C M_\ell^{n(\boldsymbol{a})+1}\left[\int_{[\![\sigma]\!]\vee\varepsilon}^{[\![\mu_0]\!]} \mu^{[\boldsymbol{a}]-1}\mathrm{d}\mu + \varepsilon^{-d}\int_{[\![\sigma]\!]}^{[\![\sigma]\!]\vee\varepsilon} \mu^{[\boldsymbol{a}]-1+d}\mathrm{d}\mu\right] \\
&\lesssim |\mathfrak{c}^{\ell+1}| + C M_\ell^{n(\boldsymbol{a})+1}([\![\sigma]\!]\vee\varepsilon)^{[\boldsymbol{a}]}\big[1+\varepsilon^{-d}([\![\sigma]\!]\vee\varepsilon)^d \mathbb{1}_{[\![\sigma]\!]>\varepsilon}\big] \\
&\lesssim |\mathfrak{c}^{\ell+1}| + C M_\ell^{n(\boldsymbol{a})+1}([\![\sigma]\!]\vee\varepsilon)^{[\boldsymbol{a}]}.
\end{aligned} \tag{72}$$

As a first consequence, we see that for any cumulant $\mathscr{F}_\mu^{\boldsymbol{a}}$ with $L(\boldsymbol{a})=\ell+1$ and $n(\boldsymbol{a}) \leqslant N_{\ell+1}$, it holds that, for a suitable constant $C>0$,

$$\tilde{M}_{\ell+1} := \sup_{\substack{\boldsymbol{a}: L(\boldsymbol{a})=\ell+1,\\ n(\boldsymbol{a})\leqslant N_{\ell+1}}} \left[\sup_{\sigma\geqslant\mu} [\![\sigma]\!]^{-[\boldsymbol{a}]}\,\|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma}\right]^{1/n(\boldsymbol{a})} \leqslant C \sup_{1\leqslant n\leqslant N} M_\ell^{1+1/n(\boldsymbol{a})} \leqslant C\,(1+M_\ell)^2 < \infty,$$

yielding $M_{\ell+1} \leqslant \sup\{M_\ell, \tilde{M}_{\ell+1}\} \leqslant C(1+M_\ell)^2 < \infty$, which closes the induction step and we conclude that

$$\|\mathscr{F}^{\boldsymbol{A}}\| = \sup_{\substack{\boldsymbol{a}: L(\boldsymbol{a})\leqslant\bar{\ell},\\ n(\boldsymbol{a})+L(\boldsymbol{a})\leqslant 2N+\bar{\ell}}} \left[\sup_{\sigma\geqslant\mu} [\![\sigma]\!]^{-[\boldsymbol{a}]}\,\|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma}\right]^{1/n(\boldsymbol{a})} = M_{\bar{\ell}} \lesssim (1+M_0)^{2^{\bar{\ell}}} < \infty.$$

The main consequence of this procedure to fix a suitable solution of the perturbative flow equation is that now we have modified its boundary value at $\sigma=1$. Indeed Eq. (71) implies that at level $\ell+1$ the initial condition of the flow equation reads $F_1^{[\ell+1]}(\Psi) = r_1^{\ell+1,\varepsilon}\Psi^0$ and Eq. (72) implies that $r_1^{\ell+1,\varepsilon}$ is indeed finite and diverges in ε as

$$|r_1^{\ell+1,\varepsilon}| \lesssim ([\![1]\!]\vee\varepsilon)^{[\boldsymbol{a}]} \lesssim \varepsilon^{-2s+\delta(\ell+2)}.$$

□

Remark 25. Using more precisely the induction procedure we have actually the more general estimate

$$\sup_{\substack{\boldsymbol{a}:L(\boldsymbol{a})\leqslant\bar{\ell},\\ n(\boldsymbol{a})\leqslant 2N+\bar{\ell}-L(\boldsymbol{a})}}\left[\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\boldsymbol{a}]}\|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma}\right]^{1/n(\boldsymbol{a})}\leqslant C(1+M_0)^{2^{\bar{\ell}}}.$$

4.6 Control of derivatives

In the forthcoming analysis, in particular in the Kolmogorov-type argument of Section 4.7, we shall also be interested in cumulants of kernels derived with respect to the parameter σ, namely cumulants of the form $\kappa_{n(\boldsymbol{a})}(\partial_\sigma^{p_1}F_\sigma^{\mathfrak{a}_1},\ldots,\partial_\sigma^{p_{n(\boldsymbol{a})}}F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}})$ where $p_i\in\{0,1\}$ for any $i\in\{1,\ldots,n(\boldsymbol{a})\}$, as we shall only be interested in up to first order derivatives.

To this end we shall slightly modify the index encoding the information about cumulants. We let $N_{\ell,p}:=N+(\bar{\ell}-\ell)+(N-p)$ and introduce extended cumulants $(\mathscr{F}^{\tilde{\boldsymbol{a}}})_{\tilde{\boldsymbol{a}}\in\tilde{\boldsymbol{A}}}$ with

$$\tilde{\boldsymbol{a}}\in\tilde{\boldsymbol{A}}:=\{(\mathfrak{a}_1,p_1,\ldots,\mathfrak{a}_n,p_n):\mathfrak{a}_k\in\mathfrak{A},p_k\in\{0,1\},n\leqslant N_{L(\boldsymbol{a}),|p(\boldsymbol{a})|},|p(\tilde{\boldsymbol{a}})|\leqslant N,L(\tilde{\boldsymbol{a}})\leqslant\bar{\ell}\},$$

defined by

$$\mathscr{F}_\sigma^{\tilde{\boldsymbol{a}}}:=\kappa_{n(\tilde{\boldsymbol{a}})}(\partial_\sigma^{p_1}F_\sigma^{\mathfrak{a}_1},\ldots,\partial_\sigma^{p_{n(\boldsymbol{a})}}F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}),\qquad[\tilde{\boldsymbol{a}}]:=[\boldsymbol{a}]-|p(\boldsymbol{a})|,$$

where $\boldsymbol{a}=(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)\in\boldsymbol{A}$, $p(\boldsymbol{a})=(p_1,\ldots,p_{n(\boldsymbol{a})})$ is a multi-index with $p_i\in\{0,1\}$ for any $i\in\{1,\ldots,n(\boldsymbol{a})\}$ and where $|p(\boldsymbol{a})|:=\sum_{i=1}^{n(\boldsymbol{a})}p_i$. We observe that $|p(\boldsymbol{a})|\leqslant n(\boldsymbol{a})\wedge N$ and that for $|p(\boldsymbol{a})|=0$, we have $\mathscr{F}_\sigma^{\tilde{\boldsymbol{a}}}=\mathscr{F}_\sigma^{\boldsymbol{a}}$ and $[\tilde{\boldsymbol{a}}]=[\boldsymbol{a}]$. We recall that N is a fixed number which will be chosen later on.

Lemma 26. *Under the conditions of Lemma 24 we have*

$$\|\mathscr{F}^{\boldsymbol{A}}\|\leqslant\|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|:=\sup_{\tilde{\boldsymbol{a}}\in\tilde{\boldsymbol{A}}}\left[\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{a}}]}\|\mathscr{F}_\sigma^{\tilde{\boldsymbol{a}}}\|_{\mu,\sigma}\right]^{1/n(\tilde{\boldsymbol{a}})}\lesssim(1+\|\mathscr{F}^{\boldsymbol{A}}\|)^N<\infty.\tag{73}$$

Proof. We shall proceed as in Section 4.5 by an induction that now will be over $L(\boldsymbol{a})+|p(\boldsymbol{a})|\in\{0,\ldots,\bar{\ell}+N\}$. For $\ell=0,\ldots,\bar{\ell}$ and $p=0,\ldots,N$ let

$$M_{\ell,p}:=\sup_{\tilde{\boldsymbol{a}}\in\tilde{\boldsymbol{A}}_{\ell,p}}\left[\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{a}}]}\|\mathscr{F}_\sigma^{\tilde{\boldsymbol{a}}}\|_{\mu,\sigma}\right]^{1/n(\tilde{\boldsymbol{a}})},\qquad M_k:=\sup_{\substack{\ell,p:\ell+p\leqslant k\\ p\leqslant N,\ell\leqslant\bar{\ell}}}M_{\ell,p},$$

where $\tilde{\boldsymbol{A}}_{\ell,p}:=\{\tilde{\boldsymbol{a}}\in\tilde{\boldsymbol{A}}:L(\tilde{\boldsymbol{a}})=\ell,|p(\tilde{\boldsymbol{a}})|=p,n(\tilde{\boldsymbol{a}})\leqslant N_{\ell,p}\}$. As before we have $M_0=M_{0,0}<\infty$. For the induction step assume that M_k is finite and let us prove that M_{k+1} is also finite, that is we need to bound both $M_{\leqslant\ell+1,\leqslant p}$ and $M_{\leqslant\ell,\leqslant p+1}$ for $\ell+p=k$.

Assume that $\mathscr{F}_\sigma^{\tilde{\boldsymbol{a}}}$ is a cumulant with $|p(\tilde{\boldsymbol{a}})|=p+1$ and $L(\tilde{\boldsymbol{a}})\leqslant\ell$. Using the (differential) flow equation for one of the kernels $\partial_\sigma^{p_i}F_\sigma^{\mathfrak{a}_i}$ with $p_i=1$ for some $i\in\{1,\ldots,n(\tilde{\boldsymbol{a}})\}$ we have

$$\mathscr{F}_\sigma^{\tilde{\boldsymbol{a}}}=\sum_{\tilde{\boldsymbol{b}}}\tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\tilde{\boldsymbol{b}}})+\sum_{\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}}}\tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\tilde{\boldsymbol{b}}},\mathscr{F}_\sigma^{\tilde{\boldsymbol{c}}}),\tag{74}$$

where on the r.h.s. the indexes $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ without tildas are obtained from $\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}},\tilde{\boldsymbol{c}}$ ignoring the additional information on the derivatives. As a consequence, the contributions on the r.h.s. of (74) come only form cumulants $\mathscr{F}^{\tilde{\boldsymbol{d}}}$ with one derivative less, with cumulant order increased by one and with perturbative order decreased by one, i.e. we have $|p(\tilde{\boldsymbol{d}})|=p$, $L(\tilde{\boldsymbol{d}})\leqslant\ell-1$ and $n(\tilde{\boldsymbol{d}})\leqslant n(\tilde{\boldsymbol{a}})+1=N_{\ell,p+1}+1=N_{\ell,p}\leqslant N_{\ell-1,p}$. By Lemma 47 and Lemma 48 these contributions are estimated by

$$\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{a}}]}\|\tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\tilde{\boldsymbol{b}}})\|_{\mu,\sigma}\leqslant K\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{b}}]}\|\mathscr{F}_\sigma^{\tilde{\boldsymbol{b}}}\|_{\mu,\sigma}\leqslant KM_{\leqslant\ell-1,\leqslant p}^{n(\tilde{\boldsymbol{a}})},$$

and

$$\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{a}}]}\|\tilde{\mathscr{B}}^{\boldsymbol{a},(i)}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma,\mathscr{F}^{\tilde{\boldsymbol{b}}}_\sigma,\mathscr{F}^{\tilde{\boldsymbol{c}}}_\sigma)\|_{\mu,\sigma}\leqslant K\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{b}}]}\|\mathscr{F}^{\tilde{\boldsymbol{b}}}_\sigma\|_{\mu,\sigma}[\![\sigma]\!]^{-[\tilde{\boldsymbol{c}}]}\|\mathscr{F}^{\tilde{\boldsymbol{c}}}_\sigma\|_{\mu,\sigma}\leqslant KM^{n(\tilde{\boldsymbol{b}})}_{\leqslant\ell-1,\leqslant p}M^{n(\tilde{\boldsymbol{c}})}_{\leqslant\ell-1,\leqslant p}.$$

If $\mathscr{F}^{\tilde{\boldsymbol{a}}}_\sigma$ is a cumulant with $|p(\tilde{\boldsymbol{a}})|=p$ and $L(\tilde{\boldsymbol{a}})\leqslant\ell+1$, instead, we look for $i\in\{1,\ldots,n(\tilde{\boldsymbol{a}})\}$ such that $p_i=1$. If there is at least one derivative we proceed as before to bring down by one the value of $L(\tilde{\boldsymbol{a}})$ and of $|p(\tilde{\boldsymbol{a}})|$ and in this case on the r.h.s. we have only cumulants $\mathscr{F}^{\tilde{\boldsymbol{d}}}$ such that $n(\tilde{\boldsymbol{d}})\leqslant n(\tilde{\boldsymbol{a}})+1=N_{\ell+1,p}+1=N_{\ell,p}$. If there are no derivatives of kernels, then we can argue as in the previous section via the integrated flow equation involving also cumulants $\mathscr{F}^{\tilde{\boldsymbol{d}}}$ such that $n(\tilde{\boldsymbol{d}})\leqslant n(\tilde{\boldsymbol{a}})+1=N_{\ell,p}$. Therefore we can still obtain that

$$\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{a}}]}\|\mathscr{F}^{\tilde{\boldsymbol{a}}}_\sigma\|_{\mu,\sigma}\leqslant K(1+M_{\leqslant\ell,\leqslant p})^2.$$

Overall, we proved then that $M_{k+1}\leqslant K(1+M_k)^2$ for all $k\in\{0,\ldots,\bar{\ell}+N\}$. Iterating this relation down from $k=\bar{\ell}+N$ to $k=0$ we conclude

$$\|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|=\sup_{\tilde{\boldsymbol{a}}\in\tilde{\boldsymbol{A}}}\left[\sup_{\sigma\geqslant\mu}[\![\sigma]\!]^{-[\tilde{\boldsymbol{a}}]}\|\mathscr{F}^{\tilde{\boldsymbol{a}}}_\sigma\|_{\mu,\sigma}\right]^{1/n(\tilde{\boldsymbol{a}})}=M_{\bar{\ell}+N}\lesssim(1+M_0)^{2^{\bar{\ell}+N}}<\infty.$$

Note that in particular we have $\|\mathscr{F}^{\boldsymbol{A}}\|\leqslant\|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|<\infty$. □

4.7 Conclusion of the cumulant analysis

The proof of Theorem 13 is obtained by combining Lemma 26 together with the following lemma which shows how to go from estimates on $\mathscr{F}$ to those on F. Recall that the weight w we introduced in Equation (50) is of the form $w(z)=\langle z\rangle^{-\kappa_w}$ for $z\in\Lambda$ and some $\kappa_w>0$.

Lemma 27. *Let the parameter α introduced in Equation (49) be given by*

$$\alpha=3\beta+\kappa_2,$$

for some $\kappa_2>0$ and assume that

$$\gamma>\frac{d+2s}{6}. \tag{75}$$

Then for κ_w and κ_2 small enough there exists N large enough such that

$$\left\{\mathbb{E}[\|F^{\mathfrak{A}}\|^N]\right\}^{1/N}\lesssim(1+\|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|). \tag{76}$$

The rest of this section is devoted to its proof. Recall the definition of $\|F^{\mathfrak{A}}\|$ in Eq. (52). The term $F^{[0],(0)}$ by construction is the noise $\xi^{(\varepsilon)}$ and there exists a random constant Ξ^ε uniformly bounded in all the $L^p(\mathbb{P})$ spaces such that

$$\|\mathfrak{J}_\sigma F^{[0],(0)}\|_{L^\infty}\lesssim\Xi^\varepsilon[\![\sigma]\!]^{-\frac{d+2s}{2}-\kappa_3},$$

for an arbitrary small $\kappa_3>0$, therefore its contribution to the norm $\|F^{\mathfrak{A}}\|$ in Eq. (52) satisfies the bound (76) provided Eq. (75) holds. We can then focus on the estimation of the other kernels. Moreover, since the sets $\mathfrak{A}$ is finite (cfr. (48)) it is enough to prove

$$\sup_{\mathfrak{a}\in\mathfrak{A}}\left\{\mathbb{E}\left[\sup_{\mu\leqslant\sigma}\sup_{\nu|\mu\leqslant\nu\leqslant\sigma}\left([\![\sigma]\!]^{-[\mathfrak{a}]}\|w(\tilde{\mathfrak{J}}^{\mathfrak{a}}_\nu F^{\mathfrak{a}}_\sigma)e^{T_\nu(\mathfrak{a})}\|\right)^N\right]\right\}\lesssim\|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|. \tag{77}$$

For notational simplicity we shall denote with $Y=(y_1,\ldots,y_{k(\mathfrak{a})})\in\Lambda^{k(\mathfrak{a})}$ the set of input variables of $F^{\mathfrak{a}}$. We require N to be large enough so that

$$\kappa_w>(d+2s)/N. \tag{78}$$

For N even, we have

$$
\begin{aligned}
\mathbb{E}\|w(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}})\, e^{T_{\bar{\nu}}(\mathfrak{a})}\|^N &= \mathbb{E}\left[\sup_{z\in\Lambda} w(z)\int_{\Lambda^{k(\mathfrak{a})}} |(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}})|(z;Y)\, e^{T_{\bar{\nu}}(z,Y)} \mathrm{d}Y\right]\\
&= \mathbb{E}\left[\sup_{z\in\Lambda} w(z)\int_{\Lambda^{k(\mathfrak{a})}} \left|\int_\Lambda \check{j}_{\nu,\ell(\mathfrak{a})+1}(z-z')\,(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F^{\mathfrak{a}})(z';Y)\mathrm{d}z'\right| e^{T_{\bar{\nu}}(z,Y)}\mathrm{d}Y\right]^N\\
&\lesssim \mathbb{E}\left[\sup_{z\in\Lambda} w(z)\int_\Lambda |\check{j}_{\nu,\ell(\mathfrak{a})+1}(z-z')|\, e^{\tau_{\bar{\nu}}(z,z')}\left(\int_{\Lambda^{k(\mathfrak{a})}} |(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F^{\mathfrak{a}})(z';Y)|\, e^{T_{\bar{\nu}}(z,Y)}\mathrm{d}Y\right)\mathrm{d}z'\right]^N\\
&\lesssim \mathbb{E}\left[\sup_z \int\left(|\check{j}_{\nu,\ell(\mathfrak{a})+1}(z-z')|\, e^{\tau_{\bar{\nu}}(z,z')}\,\frac{w(z)}{w(z')}\right)\times\right.\\
&\qquad\qquad \left.\times\left(w(z')\int |(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F^{\mathfrak{a}})(z';Y)|\, e^{T_{\bar{\nu}}(z',Y)}\mathrm{d}Y\right)\mathrm{d}z'\right]^N,
\end{aligned}
$$

where in the second inequality we exploited the second identity in Equation (19) while in the first inequality we used $e^{T_{\bar{\nu}}(z,Y)} \leqslant e^{T_{\bar{\nu}}(z',Y)}\, e^{\tau_{\bar{\nu}}(z,z')}$. Observing now that $w(z)\, w^{-1}(z') \lesssim w^{-1}(z-z')$ and by exploiting the Young inequality for convolutions we get, with $p=N/(N-1)$,

$$
\begin{aligned}
\mathbb{E}\|w(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}}) e^{T_{\bar{\nu}}(\mathfrak{a})}\|^N &\lesssim \left[\int_\Lambda (|\check{j}_{\nu,\ell(\mathfrak{a})+1}(z-z')|\, e^{T_\nu(z-z')}\, w^{-1}(z-z'))^p \mathrm{d}z'\right]^{N/p}\times\\
&\qquad \times\mathbb{E}\left[\int_\Lambda\left(w(z)\int_{\Lambda^{k(\mathfrak{a})}} |(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F^{\mathfrak{a}})(z;Y)|\, e^{T_{\bar{\nu}}(z,Y)}\mathrm{d}Y\right)^N \mathrm{d}z\right]\\
&\lesssim [\![\nu]\!]^{-(d+2s)}\int_\Lambda\int_{(\Lambda^{k(\mathfrak{a})})^N} w^N(z)\,\mathbb{E}\prod_{k=1}^N |(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F^{\mathfrak{a}})(z;Y_k)|\, e^{T_{\bar{\nu}}(z,Y_k)}\mathrm{d}Y_1\cdots\mathrm{d}Y_N\mathrm{d}z\\
&\lesssim [\![\nu]\!]^{-(d+2s)}\left(\int_\Lambda w^N(z)\mathrm{d}z\right)\|\mathbb{E}[(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}}\, e^{T_{\bar{\nu}}(\mathfrak{a})})^{\otimes N}]\|_{(L^\infty)^N},
\end{aligned}
$$

where in the last expression the L^1 norms on the input variables are implicit and where $\int_\Lambda w^N(z)\mathrm{d}z<\infty$ thanks to condition (78). Using the relation between expectation and cumulants and its homogeneity we can bound this by

$$
\mathbb{E}\|w(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}})\, e^{T_{\bar{\nu}}(\mathfrak{a})}\|^N \lesssim_N [\![\nu]\!]^{-(d+2s)}\sup_{m\leqslant N}\left\{\|(\tilde{\mathcal{J}}_\nu^{\boldsymbol{a}}\mathscr{F}_\sigma^{\boldsymbol{a}})\,(e^{T_{\bar{\nu}}(\mathfrak{a})})^{\otimes m}\|_{(L^\infty)^m}\right\}^{N/m},
$$

with $\boldsymbol{a}=m\mathfrak{a}:=(\mathfrak{a},\ldots,\mathfrak{a})$ (m times). By Bernstein inequality we can bound the $(L^\infty)^m$ norm in the r.h.s. by the $L^\infty(L^1)^{m-1}$ norm, where we mean that the first output variable is estimated in L^∞ and the others in L^1. Since all the variables involved are associated to smoothing operators $\tilde{\mathcal{J}}_\nu$ with $\nu\in[\mu,\sigma]$ we have an overall loss of a factor $[\![\nu]\!]^{-(d+2s)}$ for each variable. Still leaving implicit the L^1 norms on the input variables we can then write

$$
\begin{aligned}
\sup_{\nu|\mu\leqslant\nu\leqslant\sigma}\mathbb{E}\|w(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}})\, e^{T_{\bar{\nu}}(\mathfrak{a})}\|^N &\lesssim_N \sup_{m\leqslant N}\left\{[\![\nu]\!]^{-(d+2s)(m/N+m-1)}\sup_{\nu|\mu\leqslant\nu\leqslant\sigma}\|(\tilde{\mathcal{J}}_\nu^{\boldsymbol{a}}\mathscr{F}_\sigma^{\boldsymbol{a}})\,(e^{T_{\bar{\nu}}(\mathfrak{a})})^{\otimes m}\|_{L^\infty(L^1)^{m-1}}\right\}^{N/m}\\
&\lesssim \sup_{m\leqslant N}\left\{[\![\nu]\!]^{-(d+2s)(m/N+m-1)}\sup_{\nu|\mu\leqslant\nu\leqslant\sigma}\|(\tilde{\mathcal{J}}_\nu^{\boldsymbol{a}}\mathscr{F}_\sigma^{\boldsymbol{a}})\, e^{T_{\bar{\nu}}(\boldsymbol{a})}\|_{L^\infty(L^1)^{m-1}}\right\}^{N/m}\\
&= \sup_{m\leqslant N}\left\{[\![\sigma]\!]^{-(d+2s)(m/N+m-1)}\|\mathscr{F}_\sigma^{\boldsymbol{a}}\|_{\mu,\sigma}\right\}^{N/m},
\end{aligned}
\tag{79}
$$

where in the second inequality we used $(e^{T_{\bar{\nu}}(\mathfrak{a})})^{\otimes m}\lesssim e^{T_{\bar{\nu}}(\boldsymbol{a})}$ which holds true by construction, *cf.* Equations (51) and (56) and in the last that $[\![\nu]\!]\geqslant[\![\sigma]\!]$. This analysis allows us to deduce that, for $\boldsymbol{a}=m\mathfrak{a}$,

$$
\sup_{\mu\leqslant\sigma}\sup_{\nu|\mu\leqslant\nu\leqslant\sigma}\mathbb{E}\left[\left([\![\sigma]\!]^{-[\mathfrak{a}]}\|w(\tilde{\mathcal{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}})\, e^{T_{\bar{\nu}}(\mathfrak{a})}\|\right)^N\right]\lesssim\|\mathscr{F}^{\boldsymbol{A}}\|^N\sup_{\mu\leqslant\sigma}\sup_{1\leqslant m\leqslant N}\left\{[\![\sigma]\!]^{[m\mathfrak{a}]-(d+2s)(m/N+m-1)-m[\mathfrak{a}]}\right\}^{N/m},
\tag{80}
$$

where on the r.h.s. we used that, due to Lemma 24, for $\sigma \geqslant \mu$, it holds that $\|\mathscr{F}_\sigma^{m\mathfrak{a}}\|_{\mu,\sigma} \leqslant \|\mathscr{F}^{\boldsymbol{A}}\|^m [\![\sigma]\!]^{[m\mathfrak{a}]}$. We will see below that a good choice of parameter renders the constant in the r.h.s. finite.

Eq. (80) is still not good enough since we need to control the random variable on the r.h.s. of Eq. (77) where $\sup_{\mu \leqslant \sigma} \sup_{\nu | \mu \leqslant \nu \leqslant \sigma}$ are inside the expectation. In order to achieve the definitive estimate we use a standard argument (essentially via a Sobolev embedding). This is a bit tedious but straighforward and it is the reason for the estimates in Section 4.6. Let $H_\nu^{\mathfrak{a}} := \tilde{\mathfrak{J}}_\nu^{\mathfrak{a}} F^{\mathfrak{a}}$ and observe that, for $\sigma \geqslant \max\{\nu\}$,

$$H_\nu^{\mathfrak{a}} = H_{(\sigma)}^{\mathfrak{a}} - \int_{\nu_0}^{\sigma} \ldots \int_{\nu_{k(\mathfrak{a})}}^{\sigma} (\partial_{\nu'} H_{\nu'}^{\mathfrak{a}}) \mathrm{d}\nu',$$

where for notational simplicity we denote, recalling that ν is a multi-index, $\partial_{\nu'} := \partial_{\nu_0'} \otimes \ldots \otimes \partial_{\nu_{k(\mathfrak{a})}'}$ and $\mathrm{d}\nu' = \mathrm{d}\nu_0' \ldots \mathrm{d}\nu_{k(\mathfrak{a})}'$. We also underline that in the above equation, in $H_{(\sigma)}^{\mathfrak{a}} = \tilde{\mathfrak{J}}_\sigma^{\mathfrak{a}} F^{\mathfrak{a}}$ all the scales are at σ, this is specified by the bracket in the notation $H_{(\sigma)}^{\mathfrak{a}}$. We thus observe that, by triangular inequality,

$$\begin{aligned}
\|w H_\nu^{\mathfrak{a}} e^{T_{\bar{\nu}}(\mathfrak{a})}\| &\leqslant \|w H_\sigma^{\mathfrak{a}} e^{T_{\bar{\nu}}(\mathfrak{a})}\| + \int_{\nu_0}^{\sigma} \ldots \int_{\nu_{k(\mathfrak{a})}}^{\sigma} \|w (\partial_{\nu'} H_{\nu'}^{\mathfrak{a}}) e^{T_{\bar{\nu}}(\mathfrak{a})}\| \mathrm{d}\nu' \\
&\leqslant \|w H_\sigma^{\mathfrak{a}} e^{T_\sigma(\mathfrak{a})}\| + \int_{\nu_0}^{\sigma} \ldots \int_{\nu_{k(\mathfrak{a})}}^{\sigma} \|w (\partial_{\nu'} H_{\nu'}^{\mathfrak{a}}) e^{T_{\bar{\nu}'}(\mathfrak{a})}\| \mathrm{d}\nu',
\end{aligned}$$

where in the second inequality we used $e^{T_{\bar{\nu}}(\mathfrak{a})} \leqslant e^{T_\sigma(\mathfrak{a})}$ as well as $e^{T_{\bar{\nu}}(\mathfrak{a})} \leqslant e^{T_{\bar{\nu}'}(\mathfrak{a})}$. As a consequence, by taking $\kappa' > 0$ small and using Jensen's inequality,

$$\mathbb{E}\left[\sup_{\nu | \mu \leqslant \nu \leqslant \sigma} \|w H_\nu^{\mathfrak{a}} e^{T_{\bar{\nu}}(\mathfrak{a})}\|^N\right] \lesssim \mathbb{E}\|w H_\sigma^{\mathfrak{a}} e^{T_\sigma(\mathfrak{a})}\|^N + \int_{[\mu,\sigma]^{1+k(\mathfrak{a})}} \mathbb{E}[[\![\nu']\!]^{1-\kappa'} \|w (\partial_{\nu'} H_{\nu'}^{\mathfrak{a}}) e^{T_{\bar{\nu}'}(\mathfrak{a})}\|]^N \frac{\mathrm{d}\nu'}{[\![\nu']\!]^{1-\kappa'}},$$

where again, with a slight abuse of notation, we denote $[\![\nu']\!]^{1-\kappa'} := \prod_{i=0}^{k(\mathfrak{a})} [\![\nu_i']\!]^{1-\kappa'}$. It follows that

$$\begin{aligned}
\mathbb{E}\|F^{\mathfrak{a}}\|_{\mu,\sigma}^N &= \mathbb{E}\left[\sup_{\nu | \mu \leqslant \nu \leqslant \sigma} \|w H_\nu^{\mathfrak{a}} e^{T_{\bar{\nu}}(\mathfrak{a})}\|^N\right] \\
&\lesssim \mathbb{E}\|w H_\sigma^{\mathfrak{a}} e^{T_\sigma(\mathfrak{a})}\|^N + \sup_{\nu | \mu \leqslant \nu \leqslant \sigma} \mathbb{E}[[\![\nu]\!]^{1-\kappa'} \|w (\partial_\nu H_\nu^{\mathfrak{a}}) e^{T_{\bar{\nu}}(\mathfrak{a})}\|]^N \\
&\lesssim \|\mathscr{F}^{\boldsymbol{A}}\|^N \sup_{\mu \leqslant \sigma} \sup_{1 \leqslant m \leqslant n} \left\{[\![\sigma]\!]^{[m\mathfrak{a}] - (d+2s)(m/N + m - 1) - m[\mathfrak{a}] - m(k(\mathfrak{a})+1)\kappa'}\right\}^{N/m},
\end{aligned} \tag{81}$$

where in the last inequality we exploited Equation (80), together with

$$\|w (\partial_\nu H_\nu^{\mathfrak{a}}) e^{T_{\bar{\nu}}(\mathfrak{a})}\| \lesssim [\![\nu]\!]^{-(k(\mathfrak{a})+1)} \|w H_\nu^{\mathfrak{a}} e^{T_{\bar{\nu}}(\mathfrak{a})}\|,$$

and with the argument we used to get Equation (79) and (80).

The argument to handle the supremum over $\sigma \geqslant \mu$ is quite similar, to this end we let

$$H_\sigma^{\mathfrak{a}} := [\![\sigma]\!]^{-[\mathfrak{a}]} F_\sigma^{\mathfrak{a}},$$

and observe that $H_\sigma^{\mathfrak{a}} = H_{\mu_0}^{\mathfrak{a}} + \int_{\mu_0}^{\sigma} (\partial_{\sigma'} H_{\sigma'}^{\mathfrak{a}}) \mathrm{d}\sigma'$, for some fixed scale μ_0, so by triangular inequality, taking $\kappa'' > 0$ small and using Jensen's inequality,

$$\mathbb{E}\left[\sup_{\sigma \geqslant \mu} \|H_\sigma^{\mathfrak{a}}\|_{\mu,\sigma}^N\right] \lesssim_n \mathbb{E}\left[\|H_{\mu_0}^{\mathfrak{a}}\|_{\mu,\sigma}^N\right] + \int_{\mu_0}^{1} \mathbb{E}\left[\|[\![\sigma']\!]^{1-\kappa''} (\partial_{\sigma'} H_{\sigma'}^{\mathfrak{a}})\|_{\mu,\sigma}^N\right] \frac{\mathrm{d}\sigma'}{[\![\sigma]\!]^{1-\kappa''}},$$

so, as above

$$\mathbb{E}\left[\sup_{\sigma \geqslant \mu} \|H_\sigma^{\mathfrak{a}}\|_{\mu,\sigma}^N\right] \lesssim \sup_{\mu} \mathbb{E}\left[\|H_{\mu_0}^{\mathfrak{a}}\|_{\mu,\sigma}^N\right] + \sup_{\sigma \geqslant \mu} \mathbb{E}\left[\left\|[\![\sigma]\!]^{1-\kappa''} (\partial_\sigma H_\sigma^{\mathfrak{a}})\right\|_{\mu,\sigma}^N\right].$$

The first terms is bounded by Equation (81) while, to bound the second term, we observe that

$$\partial_\sigma H_\sigma^{\mathfrak{a}} = \partial_\sigma([\![\sigma]\!]^{-[\mathfrak{a}]} F_\sigma^{\mathfrak{a}}) = -[\mathfrak{a}] [\![\sigma]\!]^{-[\mathfrak{a}]-1} F_\sigma^{\mathfrak{a}} + [\![\sigma]\!]^{-[\mathfrak{a}]} \partial_\sigma F_\sigma^{\mathfrak{a}},$$

and thus

$$\sup_{\sigma\geq\mu}\mathbb{E}\left[\left\|[\![\sigma]\!]^{1-\kappa''}(\partial_\sigma H_\sigma^{\mathfrak{a}})\right\|_{\mu,\sigma}^N\right]\lesssim\sup_{\sigma\geq\mu}\mathbb{E}\left[\left\|[\![\sigma]\!]^{-[\mathfrak{a}]-\kappa''}F_\sigma^{\mathfrak{a}}\right\|_{\mu,\sigma}^N\right]+\sup_{\sigma\geq\mu}\mathbb{E}\left[\left\|[\![\sigma]\!]^{-[\mathfrak{a}]+1-\kappa''}(\partial_\sigma F_\sigma^{\mathfrak{a}})\right\|_{\mu,\sigma}^N\right].$$

For the first term we can use the above estimates, in particular Equation (81), to conclude that

$$\begin{aligned}\sup_{\sigma\geq\mu}\mathbb{E}\left[\left\|[\![\sigma]\!]^{-[\mathfrak{a}]-\kappa''}F_\sigma^{\mathfrak{a}}\right\|_{\mu,\sigma}^N\right] &\lesssim \|\mathscr{F}^{\boldsymbol{A}}\|^N\sup_{\sigma\geq\mu}\sup_{m\leq n}\left\{[\![\sigma]\!]^{[m\mathfrak{a}]-(d+2s)(m/N+m-1)-m[\mathfrak{a}]-m(k(\mathfrak{a})+1)\kappa'-m\kappa''}\right\}^{N/m}\\ &\lesssim \|\mathscr{F}^{\boldsymbol{A}}\|^N\sup_{\sigma\geq\mu}[\![\sigma]\!]^{N(\alpha-3\beta-m(k(\mathfrak{a})+1)\kappa'-m\kappa'')-\rho},\end{aligned}$$

where in the second inequality we exploited $\rho=d+2s$, $\theta-\rho=-3\beta$ and the definition of $[m\mathfrak{a}]$, *cf.* Equation (54).

For what concerns the term with $\partial_\sigma F_\sigma^{\mathfrak{a}}$, we can repeat the argument we used to derive Equations (79) and (80) with the only difference that in this case we refer to the analysis discussed in Section (4.6), in particular Equation (73), yielding

$$\begin{aligned}\sup_{\sigma\geq\mu}\mathbb{E}\left[\left\|[\![\sigma]\!]^{-[\mathfrak{a}]+1-\kappa'}(\partial_\sigma F_\sigma^{\mathfrak{a}})\right\|_{\mu,\sigma}^N\right] &\lesssim \|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|^N\sup_{\sigma\geq\mu}\sup_{m\leq n}\left\{[\![\sigma]\!]^{[m\mathfrak{a}]-m-(d+2s)(m/N+m-1)-m[\mathfrak{a}]-m(k(\mathfrak{a})+1)\kappa'-m\kappa''}\right\}^{N/m}\\ &\lesssim \|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|^N\sup_{\sigma\geq\mu}[\![\sigma]\!]^{N(\alpha-3\beta-(k(\mathfrak{a})+1)\kappa'-\kappa'')-\rho},\end{aligned}$$

where again we exploited $\rho=d+2s$, $\theta-\rho=-3\beta$ and the definition of $[m\mathfrak{a}]$. On account of the result of Section 4.6 and $\|\mathscr{F}^{\boldsymbol{A}}\|\leq\|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|$, we have that

$$\sup_{\sigma\geq\mu}\mathbb{E}\left[\left\|[\![\sigma]\!]^{1-\kappa'}(\partial_\sigma H_\sigma^{\mathfrak{a}})\right\|_{\mu,\sigma}^N\right]\lesssim\|\mathscr{F}^{\tilde{\boldsymbol{A}}}\|^N\sup_{\sigma\geq\mu}[\![\sigma]\!]^{N(\alpha-3\beta-m(k(\mathfrak{a})+1)\kappa'-m\kappa'')-\rho},$$

and we observe that in order to conclude that $\mathbb{E}[\sup_{\sigma\geq\mu}\|H_\sigma^{\mathfrak{a}}\|_{\mu,\sigma}^N]$ is finite, we need the right hand side above to be finite, namely

$$\alpha\geq 3\beta+(k(\mathfrak{a})+1)\kappa'+\kappa''+\frac{d+2s}{N}.$$

We are not in position to choose all the parameters. For any small $\kappa_w>0$ and $\kappa_2>0$ we can choose $\kappa',\kappa''>0$ small enough and N large enough so that

$$\alpha=3\beta+\kappa_2,\qquad \kappa_2>(K+1)\kappa'+\kappa''+\frac{d+2s}{N},\tag{82}$$

and this concludes the proof of Lemma 27.

In particular, as we will see in Section 4.8, we shall choose κ_w and κ_2 small so that

$$\delta>\kappa_w+\kappa_2,\tag{83}$$

and this condition provides the rationale to choose the maximum order N of the cumulants we consider in the flow equation analysis.

4.8 Post-processing

The aim of this section is to prove the estimates in Equation (27) starting from the analysis of the flow equation we performed so far in Section 4. These estimates will be established in Lemma 28 and in Lemma 31. To begin with, we extract the coercive term from the effective force by letting

$$Q_\sigma(\psi):=\mathcal{J}_\sigma F_\sigma(\psi_\sigma)-(-\lambda\psi_\sigma^3),$$

for a generic field ψ, where $\psi_\sigma := \mathfrak{J}_\sigma \psi$. Observe that

$$\begin{aligned} Q_\sigma(\psi) &= \mathfrak{J}_\sigma F_\sigma(\psi_\sigma) - (-\lambda\,\psi_\sigma^3) \\ &= \mathfrak{J}_\sigma F_\sigma(\psi_\sigma) - \mathfrak{J}_\sigma(-\lambda\,\psi_\sigma^3) + (1-\mathfrak{J}_\sigma)(\lambda\,\psi_\sigma^3) \\ &= \sum_{\mathfrak{a}|\ell(\mathfrak{a})>0} \mathfrak{J}_\sigma F_\sigma^{[\ell(\mathfrak{a})]}(\psi_\sigma) + \mathfrak{J}_\sigma F^{[0],(1)}\psi_\sigma + \mathfrak{J}_\sigma F^{[0],(0)} + (1-\mathfrak{J}_\sigma)(\lambda\,\psi_\sigma^3) \\ &=: \mathfrak{J}_\sigma F_\sigma^{[>0]}(\psi_\sigma) + \mathfrak{J}_\sigma F^{[0],(1)}\psi_\sigma + \mathfrak{J}_\sigma F^{[0],(0)} + (1-\mathfrak{J}_\sigma)(\lambda\,\psi_\sigma^3), \end{aligned} \tag{84}$$

where we recall that $F^{[0],1}(\psi) = r\,\psi$ and $F^{[0],0} = \xi^{(\varepsilon)}$ and where in the last equality we introduced the notation $\mathfrak{J}_\sigma F_\sigma^{[>0]}(\psi_\sigma) := \sum_{\mathfrak{a}|\ell(\mathfrak{a})>0} \mathfrak{J}_\sigma F_\sigma^{[\ell(\mathfrak{a})]}(\psi_\sigma)$.

In the forthcoming analysis we shall explicitly use the value of γ. In particular we fix $\gamma > 0$ so that

$$\beta - \frac{\delta - \kappa_2 - \kappa_w}{3} < \gamma \leqslant \beta, \tag{85}$$

where $\kappa_w > 0$ is the polynomial decay of the weight w, see Equation (50) and Section 4.7. These constraints are imposed, in particular, by the computations in Lemma 29 below and are necessary to prove that the irrelevant terms generated by the flow equation give sub-dominant contributions wrt. the coercive cubic term. Recall also that we have an additional constraint on γ resulting from the regularity of the white-noise ξ, cfr. (75), namely

$$\gamma > \frac{d+2s}{6}.$$

We observe that the conditions in (85) are compatible provided we choose κ_w and κ_2 small so that $\delta > \kappa_w + \kappa_2$, while $\beta \geqslant \gamma$ together with Eq. (75) implies an upper bound on δ, so overall we have

$$\kappa_w + \kappa_2 < \delta < \delta_* = \frac{4s-d}{3}. \tag{86}$$

In the rest of the section we assume that these conditions holds.

Lemma 28. *For $\sigma \geqslant \bar{\mu} \vee \mu_i$, it holds that*

$$\|\chi_i^3\, Q_\sigma(\psi)\|_{L^\infty} \leqslant [\![\sigma]\!]^{-3\gamma+\zeta} [\|F^{\mathfrak{A}}\|(1 + |||\psi|||)^M + (1 + |||\psi|||)^2 |||\mathscr{L}\psi|||_\#],$$

for some non-random constants $M, \zeta > 0$. In particular

$$|||Q(\psi)|||_\# \lesssim [\![\bar{\mu}]\!]^\zeta \|F^{\mathfrak{A}}\|(1 + |||\psi|||)^M + (1 + |||\psi|||)^2 |||\mathscr{L}\psi|||_\#,$$

where recall the norm $|||Q(\psi)|||_\#$ defined in Equation (25).

Proof. This is a direct consequence of the triangular inequality applied to Equation (84) together with the results of Lemma 29, 30 below and of Lemma 42 in Appendix A. □

We start by a localised version of $\mathfrak{J}_\sigma F_\sigma^{[>0]}$.

Lemma 29. *For $\sigma \geqslant \bar{\mu} \vee \mu_i$, it holds that,*

$$\|\chi_i^3\, \mathfrak{J}_\sigma F_\sigma^{[>0]}(\mathscr{J}\psi_\sigma)\|_{L^\infty} \lesssim [\![\sigma]\!]^{-3\gamma+\zeta} \|F^{\mathfrak{A}}\|(1 + |||\psi|||)^M,$$

for some non-random constants $M > 0$ and $\zeta := 3\gamma - \alpha + \delta - \kappa_w > 0$.

Proof. First of all we recall that

$$\chi_i^3\, \mathfrak{J}_\sigma F_\sigma^{[>0]}(\mathscr{J}\psi_\sigma) = \sum_{\mathfrak{a}|\ell(\mathfrak{a})>0} \chi_i^3\, \mathfrak{J}_\sigma F_\sigma^{\mathfrak{a}}(\mathscr{J}\psi_\sigma).$$

For the moment being we shall only consider one specific term $\chi_i^3\,\mathfrak{J}_\sigma F_\sigma^{\mathfrak{a}}(\mathscr{J}\psi_\sigma)$ for $\mathfrak{a}$ such that $\ell(\mathfrak{a})>0$, coming back to the sum at the end of the proof. It holds that,

$$\begin{aligned}
\|\chi_i^3\,\mathfrak{J}_\sigma F_\sigma^{\mathfrak{a}}(\mathscr{J}\psi_\sigma)\|_{L^\infty} &= \left\|\chi_i^3\,\mathfrak{J}_\sigma F_\sigma^{\mathfrak{a}} \bigotimes_{m=1}^{k(\mathfrak{a})} (\mathscr{J}\psi_\sigma^< + \mathscr{J}\psi_\sigma^\geq)^{A_m(\mathfrak{a})}\right\|_{L^\infty}\\
&\lesssim \sum_{k_1+k_2=k(\mathfrak{a})} [\![\sigma]\!]^{-|A(\mathfrak{a})|}\,\|w\,(\tilde{\mathfrak{J}}_\sigma^{\mathfrak{a}} F_\sigma^{\mathfrak{a}}) e^{T_{\bar{\nu}}(\mathfrak{a})}\|\,\|\chi_i^3 w^{-1} e^{-T_{\bar{\nu}}(\mathfrak{a})}(\rho^{-1})^{\otimes k_1}\otimes 1^{\otimes k_2}\|_{L^\infty}\times\\
&\qquad\qquad \times\|\rho\,\mathscr{J}\psi_\sigma^<\|_{L^\infty}^{k_1}\|\mathscr{J}\psi_\sigma^\geq\|_{L^\infty}^{k_2}\\
&\lesssim \sum_{k_1+k_2=k(\mathfrak{a})} [\![\sigma]\!]^{-|A(\mathfrak{a})|} \sup_{\nu|\mu_i\vee\bar{\mu}\leq\nu\leq\sigma} \|w\,(\tilde{\mathfrak{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}})\,e^{T_{\bar{\nu}}(\mathfrak{a})}\|\times\\
&\qquad\qquad \times\|\chi_i^3\,w^{-1}\,e^{-T_{\bar{\nu}}(\mathfrak{a})}\,(\rho^{-1})^{\otimes k_1}\otimes 1^{\otimes k_2}\|_{L^\infty}\|\rho\,\mathscr{J}\psi_\sigma^<\|_{L^\infty}^{k_1}\|\mathscr{J}\psi_\sigma^\geq\|_{L^\infty}^{k_2},
\end{aligned}$$

where in the first inequality we used Equation (19) while in the second one we exploited Remark 5. We now observe that

$$\|\rho\,\mathscr{J}\psi_\sigma^<\|_{L^\infty}^{k_1}\|\mathscr{J}\psi_\sigma^\geq\|_{L^\infty}^{k_2}\lesssim [\![\sigma]\!]^{-\gamma k_2}\,|\!|\!|\psi|\!|\!|^{k(\mathfrak{a})},$$

as well as

$$\|\chi_i^3\,w^{-1}\,e^{-T_\sigma(\mathfrak{a})}\,(\rho^{-1})^{\otimes k_1}\otimes 1^{\otimes k_2}\|_{L^\infty}\lesssim [\![\mu_i]\!]^{-\gamma k_1-\kappa_w},$$

and, on account of the above analysis of the flow equation,

$$\sup_{\nu|\mu_i\vee\bar{\mu}\leq\nu\leq\sigma} \|w\,(\tilde{\mathfrak{J}}_\nu^{\mathfrak{a}} F_\sigma^{\mathfrak{a}})\,e^{T_{\bar{\nu}}(\mathfrak{a})}\|\lesssim [\![\sigma]\!]^{-\alpha+\beta k(\mathfrak{a})+\delta\ell(\mathfrak{a})+|A(\mathfrak{a})|}\,\|F^{\mathfrak{a}}\|_{\mu_i\vee\bar{\mu},\sigma}.$$

As a consequence, it holds that

$$\|\chi_i^3\,\mathfrak{J}_\sigma F_\sigma^{\mathfrak{a}}(\mathscr{J}\psi_\sigma)\|_{L^\infty}\lesssim [\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-\kappa_w}\,\|F^{\mathfrak{a}}\|_{\mu,\sigma}\,|\!|\!|\psi|\!|\!|^{k(\mathfrak{a})},$$

and thus

$$\begin{aligned}
\|\chi_i^3\,\mathfrak{J}_\sigma F_\sigma^{[>0]}(\psi)\|_{L^\infty} &\leq \sum_{\mathfrak{a}|\ell(\mathfrak{a})>0,k(\mathfrak{a})\leq\bar{k}} \|\chi_i^3\,\mathfrak{J}_\sigma F_\sigma^{\mathfrak{a}}(\mathscr{J}\psi_\sigma)\|_{L^\infty}\\
&\lesssim \sum_{\mathfrak{a}|k(\mathfrak{a})\leq\bar{k}} [\![\sigma]\!]^{-\alpha+(\beta-\gamma)k(\mathfrak{a})+\delta\ell(\mathfrak{a})-\kappa_w}\,\|F^{\mathfrak{a}}\|_{\mu,\sigma}\,|\!|\!|\psi|\!|\!|^{k(\mathfrak{a})}\\
&\lesssim [\![\sigma]\!]^{-\alpha+\delta-\kappa_w}\,\|F^{\mathfrak{A}}\|\,(1+|\!|\!|\psi|\!|\!|)^{\bar{k}},
\end{aligned}$$

where we used that $\beta\geq\gamma$ and $\ell(\mathfrak{a})\geq 1$. Finally we get the sought bound setting

$$\zeta := 3\gamma-\alpha+\delta-\kappa_w = 3(\gamma-\beta)+\delta-\kappa_w-\kappa_2>0. \qquad \square$$

Lemma 30. *It holds that*

$$\|\chi_i^3\,\mathfrak{J}_\sigma(F^{[0],(1)}\psi_\sigma+F^{[0],(0)})\|_{L^\infty}\lesssim [\![\sigma]\!]^{-3\gamma+\zeta}\,\|F^{\mathfrak{A}}\|\,(1+|\!|\!|\psi|\!|\!|),$$

for some $\zeta>0$.

Proof. We consider separately the two contributions. We recall that, due to the flow equation (67) $F^{[0],(1)}$ is independent of σ and thus, for some $\zeta>0$,

$$\|\chi_i^3\,\mathfrak{J}_\sigma F^{[0],(1)}\psi_\sigma\|_{L^\infty}\lesssim [\![\sigma]\!]^{-\gamma}\,|\!|\!|\psi|\!|\!|\lesssim [\![\sigma]\!]^{-3\gamma+\zeta}\,|\!|\!|\psi|\!|\!|.$$

Moreover, thanks to the definition of $\|F^{\mathfrak{A}}\|$ (cfr. Eq. (52)) it follows directly that

$$\|\mathfrak{J}_\sigma F^{[0],(0)}\|_{L^\infty}\lesssim \|F^{\mathfrak{A}}\|\,[\![\sigma]\!]^{-3\gamma+\zeta}. \qquad \square$$

We now move to the analysis of the terms $\mathrm{D}F_\sigma$ and H_σ appearing in Eq. (26). Recall that

$$H_\sigma(\psi) := \partial_\sigma F_\sigma(\psi)+\mathrm{D}F_\sigma(\psi)\,\dot{G}_\sigma F_\sigma(\psi).$$

Lemma 31. *Fix $\bar{\ell}$ large enough so that*

$$\pi := \delta\bar{\ell} - \left(3s + \frac{\delta}{2} + 2(\kappa_w + \kappa_2)\right) > 0. \tag{87}$$

Then for $\sigma \geqslant \nu \geqslant \mu_i \vee \bar{\mu}$,

$$\|\zeta_i \tilde{\mathfrak{J}}_\nu H_\sigma(\psi_\sigma)\|_{L^\infty} \lesssim [\![\sigma]\!]^{\pi-1} \|F^{\mathfrak{A}}\| (1 + |||\psi|||)^{\bar{k}},$$

$$\|\zeta_i \tilde{\mathfrak{J}}_\nu [\mathrm{D}F_\sigma(\phi_\sigma) \dot{G}_\sigma R_\sigma(\psi)]\|_{L^\infty} \lesssim [\![\sigma]\!]^{\delta-\kappa_2-1} \|F^{\mathfrak{A}}\| (1 + |||\psi|||)^{\bar{k}} \|\zeta_i \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty}.$$

Note also that $\delta - \kappa_2 > 0$.

Proof. We observe that on account of the perturbative flow equation, Equation (47), it holds

$$H_\sigma(\psi_\sigma) := \sum_{m=0}^{\bar{\ell}} \sum_{\mathfrak{a}|\ell(\mathfrak{a})=0}^{m} \mathrm{D}F_\sigma^{[\bar{\ell}-\ell(\mathfrak{a})]}(\psi_\sigma)\, \dot{G}_\sigma F_\sigma^{[m]}(\psi_\sigma).$$

Then,

$$\begin{aligned}
\|\zeta_i \tilde{\mathfrak{J}}_\nu H_\sigma(\psi_\sigma)\|_{L^\infty} &= \left\|\zeta_i \sum_{m=0}^{\bar{\ell}} \sum_{\mathfrak{a}|\ell(\mathfrak{a})=0}^{m} \tilde{\mathfrak{J}}_\nu \mathrm{D}F_\sigma^{[\bar{\ell}-\ell(\mathfrak{a})]}(\psi_\sigma)\, \dot{G}_\sigma F_\sigma^{[m]}(\psi_\sigma)\right\|_{L^\infty} \\
&\leqslant \sum_{m=0}^{\bar{\ell}} \sum_{\mathfrak{a}|\ell(\mathfrak{a})=0}^{m} \sum_{k'=0}^{k(\mathfrak{a})} \|\zeta_i \tilde{\mathfrak{J}}_\nu F_\sigma^{[\bar{\ell}-\ell(\mathfrak{a})],(k'+1)}(\mathscr{J}\psi_\sigma)^{\otimes k'}\, \dot{G}_\sigma F_\sigma^{[m],(k(\mathfrak{a})-k')}(\mathscr{J}\psi_\sigma)^{\otimes(k(\mathfrak{a})-k')}\|_{L^\infty}.
\end{aligned}$$

Working as in the proof of Lemma 29, it holds that, for $\sigma \geqslant \nu \geqslant \mu_i \vee \bar{\mu}$,

$$\begin{aligned}
&\|\zeta_i \tilde{\mathfrak{J}}_\nu F_\sigma^{[\bar{\ell}-\ell(\mathfrak{a})],(k'+1)}(\mathscr{J}\psi_\sigma)^{\otimes k'}\, \dot{G}_\sigma F_\sigma^{[m],(k(\mathfrak{a})-k')}(\mathscr{J}\psi_\sigma)^{\otimes(k(\mathfrak{a})-k')}\|_{L^\infty} \lesssim \\
&\quad\lesssim \sum_{k_1+k_2=k(\mathfrak{a})} \|w \tilde{\mathfrak{J}}_\nu \tilde{\mathfrak{J}}_\sigma^{\otimes(k'+1)} F_\sigma^{[\bar{\ell}-\ell(\mathfrak{a})],(k'+1)}\, e^{T_\nu(k'+1)}\|_{L^\infty(L^1)^{k'+1}} \|\dot{G}_\sigma e^{\tau_\sigma(2)}\|_{\mathscr{L}(L^\infty,L^\infty)} \\
&\qquad \times \|w \tilde{\mathfrak{J}}_\sigma \tilde{\mathfrak{J}}_\sigma^{\otimes(k(\mathfrak{a})-k')} F_\sigma^{[m],(k(\mathfrak{a})-k')}\, e^{T_\sigma(k(\mathfrak{a})-k')}\|_{L^\infty(L^1)^{k(\mathfrak{a})-k'}} \times \\
&\qquad \times |||\psi|||^{k(\mathfrak{a})} \|\zeta_i w^{-2} e^{-T_\sigma(k(\mathfrak{a}))} (\rho^{-1})^{\otimes k_1} \otimes 1^{\otimes k_2}\|_{L^\infty} [\![\sigma]\!]^{-k_2\gamma} \\
&\quad\lesssim \|F^{\mathfrak{A}}\|\, |||\psi|||^{k(\mathfrak{a})} \sum_{k_1+k_2=k(\mathfrak{a})} [\![\sigma]\!]^{-\alpha+\beta k'+\beta+\delta(\bar{\ell}-\ell(\mathfrak{a}))} [\![\sigma]\!]^{2s-1} [\![\sigma]\!]^{-\alpha+\beta(k(\mathfrak{a})-k')+\delta m} \times \\
&\qquad \times [\![\mu_i]\!]^{-\gamma k_1-2\kappa_w} [\![\sigma]\!]^{-k_2\gamma} \\
&\quad\lesssim \|F^{\mathfrak{A}}\|\, |||\psi|||^{k(\mathfrak{a})} \sum_{k_1+k_2=k(\mathfrak{a})} [\![\sigma]\!]^{2s-1-2\alpha+\beta k(\mathfrak{a})+\beta+\delta(\bar{\ell}-\ell(\mathfrak{a})+m)-k_2\gamma} [\![\mu_i]\!]^{-\gamma k_1-2\kappa_w} \\
&\quad\lesssim \|F^{\mathfrak{A}}\|\, |||\psi|||^{k(\mathfrak{a})} [\![\sigma]\!]^{2s-1-2\alpha+(\beta-\gamma)k(\mathfrak{a})+\beta+\delta(\bar{\ell}-\ell(\mathfrak{a})+m)-2\kappa_w}.
\end{aligned}$$

As a consequence

$$\begin{aligned}
\|\zeta_i \tilde{\mathfrak{J}}_\nu H_\sigma(\psi_\sigma)\|_{L^\infty} &\lesssim \|F^{\mathfrak{A}}\| (1 + |||\psi|||)^{\bar{k}} \sum_{m=0}^{\bar{\ell}} \sum_{\mathfrak{a}|\ell(\mathfrak{a})=0}^{m} \sum_{k'=0}^{k(\mathfrak{a})} [\![\sigma]\!]^{2s-1-2\alpha+(\beta-\gamma)k(\mathfrak{a})+\beta+\delta(\bar{\ell}-\ell(\mathfrak{a})+m)-2\kappa_w} \\
&\lesssim \|F^{\mathfrak{A}}\| (1 + |||\psi|||)^{\bar{k}} \sum_{m=0}^{\bar{\ell}} \sum_{\mathfrak{a}|\ell(\mathfrak{a})=0}^{m} \sum_{k'=0}^{k(\mathfrak{a})} [\![\sigma]\!]^{-1+(\beta-\gamma)k(\mathfrak{a})+\delta(\bar{\ell}-\ell(\mathfrak{a})+m)-2(\kappa_w+\kappa_2)-3s-\delta/2},
\end{aligned}$$

where we used that $\beta - 2\alpha + 2s = -3s - \frac{\delta}{2} - 2\kappa_2$. Due to the constraints on $\ell(\mathfrak{a})$ and m,

$$(\beta - \gamma)\, k(\mathfrak{a}) + \delta\, (\bar{\ell} - \ell(\mathfrak{a}) + m) - 2(\kappa_w + \kappa_2) - 3s - \delta/2 \geqslant \pi > 0,$$

where we exploited Equation (87). As a consequence, we get

$$\|\zeta_i \tilde{\mathfrak{J}}_\nu S_\sigma(\psi_\sigma)\|_{L^\infty} \lesssim [\![\sigma]\!]^{\pi-1} \|F^{\mathfrak{A}}\| (1 + |||\psi|||)^{\bar{k}}.$$

Moving to the second bound we have, by using Equation (19),

$$\|\zeta_i \tilde{\mathfrak{J}}_v \mathrm{D}F_\sigma(\psi_\sigma)\,\dot{G}_\sigma R_\sigma\|_{L^\infty} \leqslant \sum_{\mathfrak{a}|k(\mathfrak{a})=0}^{\bar{k}-1} \|\zeta_i \tilde{\mathfrak{J}}_v F_\sigma^{(k(\mathfrak{a})+1)}(\mathscr{J}\psi_\sigma)^{\otimes k(\mathfrak{a})}\,\dot{G}_\sigma \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty}.$$

As before, we have for $\sigma \geqslant \mu_i \vee \bar{\mu} \vee \nu$,

$$\begin{aligned}\|\zeta_i \tilde{\mathfrak{J}}_v F_\sigma^{(k(\mathfrak{a})+1)}(\mathscr{J}\psi_\sigma)^{\otimes k(\mathfrak{a})}\,\dot{G}_\sigma \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty} \lesssim & \sum_{k_1+k_2=k(\mathfrak{a})} \|w\,\tilde{\mathfrak{J}}_v \tilde{\mathfrak{J}}_\sigma^{\otimes(k(\mathfrak{a})+1)} F_\sigma^{(k(\mathfrak{a})+1)}\,e^{T_v(k(\mathfrak{a})+1)}\|_{L^\infty(L^1)^{k'+1}} \times \\ & \times \|\dot{G}_\sigma e^{\tau_\sigma(2)}\|_{\mathscr{L}(L^\infty,L^\infty)} \|\zeta_i\, w^{-1}\,e^{-T_v(k(\mathfrak{a})+1)}\,(\rho^{-1})^{\otimes k_1} \otimes 1^{\otimes k_2}\,\zeta_i^{-1}\|_{L^\infty} \times \\ & \times \|\zeta_i \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty}\, [\![\sigma]\!]^{-k_2\gamma}\, |||\psi|||^{k(\mathfrak{a})}.\end{aligned}$$

We now proceed as above with the only observation that

$$\|\zeta_i\, w^{-1}\,e^{-T_v(k(\mathfrak{a})+1)}\,(\rho^{-1})^{\otimes k_1} \otimes 1^{\otimes k_2}\,\zeta_i^{-1}\|_{L^\infty} \lesssim [\![\mu_i]\!]^{-k_1\gamma-\kappa_w},$$

since the contribution ζ_i^{-1} is controlled by the tree weight $e^{-T_v(k(\mathfrak{a})+1)}$ thanks to Remark 18. As a consequence, we get

$$\begin{aligned}\|\zeta_i \tilde{\mathfrak{J}}_v F_\sigma^{(k(\mathfrak{a})+1)}(\mathscr{J}\psi_\sigma)^{\otimes k(\mathfrak{a})}\,\dot{G}_\sigma R_\sigma\|_{L^\infty} &\lesssim \|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar{k}} \sum_{\mathfrak{a}|k(\mathfrak{a})=0}^{\bar{k}-1} \sum_{k_1+k_2=k(\mathfrak{a})} [\![\sigma]\!]^{-\alpha+\beta+2s-1+\beta k(\mathfrak{a})-k_2\gamma} \times \\ &\quad \times [\![\mu_i]\!]^{-k_1\gamma-\kappa_w} \|\zeta_i \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty} \\ &\lesssim \|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar{k}} \sum_{\mathfrak{a}|k(\mathfrak{a})=0}^{\bar{k}-1} [\![\sigma]\!]^{-\alpha+\beta+2s-1+(\beta-\gamma)k(\mathfrak{a})-\kappa_w} \|\zeta_i \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty}. \\ &\lesssim \|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar{k}}\,[\![\sigma]\!]^{-\alpha+\beta+2s-1} \|\zeta_i \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty} \\ &\lesssim \|F^{\mathfrak{A}}\|\,(1+|||\psi|||)^{\bar{k}}\,[\![\sigma]\!]^{\delta-\kappa_w-\kappa_2-1} \|\zeta_i \tilde{\mathfrak{J}}_\sigma R_\sigma\|_{L^\infty},\end{aligned}$$

where in the third inequality we used $\beta \geqslant \gamma$ while in the last we used $\beta-\alpha+2s=\delta-\kappa_2>0$. □

5 The vector model

In this last section we discuss the modifications to implement in order to extend our results to the vector model where the field $\phi^{(\varepsilon)}$ takes values in the Euclidean space $\mathbb{R}^n$ with some $n>1$. We denote with $(\phi^{(\varepsilon),a})_{a=1,\ldots,n}$ the components of the field in the canonical basis. The dynamics reads

$$\mathscr{L}_\varepsilon \phi^{(\varepsilon),a} + \lambda\,|\phi^{(\varepsilon)}|^2\,\phi^{(\varepsilon),a} - r_\varepsilon \phi^{(\varepsilon),a} = \xi^{(\varepsilon),a}, \qquad a=1,\ldots,n, \tag{88}$$

on Λ_ε and where $\xi^{(\varepsilon)}$ is a vector-valued, space-time white noise on Λ_ε such that

$$\mathbb{E}[\xi^{(\varepsilon),a}(t,x)\,\xi^{(\varepsilon),b}(s,y)] = \delta(t-s)\,\delta_{a,b}\,\mathbb{1}_{x=y}, \qquad (t,x),(s,y)\in\Lambda_\varepsilon,\ a,b=1,\ldots,n. \tag{89}$$

The main difference is how to establish appropriate a-priori estimates in this case. The rest of the analysis does not depend much on the scalar nature of the equation, until the classification of the relevant cumulants in Section 4.3 and Section 4.4. There one need to use the $O(n)$ symmetry of the noise (89) to conclude that also the flow cumulants are symmetric and therefore the only contribution to the first order kernel is diagonal in vector indexes and can be reabsorbed in a redefinition of the renormalisation constant r_ε.

Let us discuss the changes w.r.t. the proof of the scalar a-priori estimates given in Theorem 11.

Theorem 32. *Consider a classical solution of the vector equation*

$$\partial_t u^a + (-\Delta)^s u^a + \lambda |u|^2 u^a = f^a, \qquad a = 1, \ldots, n. \tag{90}$$

It holds that

$$\|\chi_i u\| \lesssim \lambda^{-1/2} + \lambda^{-1/3} [\|\chi_i^3 f\| + 2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|]^{1/3}. \tag{91}$$

Proof. The proof is similar to the one of Theorem 15 and thus we shall not give all the details and refer to it for the notations. We test Eq. (90) against $\Phi(|v|^2)\,\Phi'(|v|^2)\, v^a \chi_i^3$ where $v := \chi_i u$ and we implicitly sum over the repeated index $a = 1, \ldots, n$,

$$\int \Phi(|v|^2)\,\Phi'(|v|^2)\,(\chi_i^3 f \cdot v) = \int \Phi(|v|^2)\,\Phi'(|v|^2)\,[\chi_i^3\, v \cdot \partial_t u + \chi_i^3\, v^a (-\Delta)^s u^a + \lambda (|v|^2)^2].$$

We have, by Lemma 33

$$\langle \Phi(|v|^2)\,\Phi'(|v|^2)\, v^a, \chi_i^3 (-\Delta)^s u^a \rangle \gtrsim -\int \Phi(|v|^2)^2 - \|v\| \,[2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|] \int \Phi(|v|^2).$$

For the time derivative we observe that

$$\int \chi_i^2\, \Phi(|v|^2)\,\Phi'(|v|^2)\,\chi_i^3\, v \cdot \partial_t u = \frac{1}{2} \int \chi_i^2\, \Phi(|v|^2)\,\Phi'(|v|^2)\,[\partial_t |v|^2 - (\chi_i \partial_t \chi_i)\,|v|^2].$$

Then, leaving implicit the space variable, by the convexity of Φ^2 and working as in Equation (44) we have

$$\frac{1}{2} \int \chi_i^2\, \Phi(|v|^2)\,\Phi'(|v|^2)\,\partial_t |v|^2 \;\geqslant\; -\|\partial_t \chi_i\| \int_\Lambda \Phi_L^2(|v|^2),$$

therefore the lower bound

$$\begin{aligned} \langle \Phi(|v|^2)\,\Phi'(|v|^2), \chi_i^3\, v \cdot \partial_t u \rangle \;&\geqslant\; -\|\partial_t \chi_i\| \int_\Lambda \Phi_L^2(|v|^2) - \|v\|^2 \,\|\partial_t \chi_i\| \int_\Lambda \Phi_L(|v|^2) \\ &\gtrsim\; -\int_\Lambda \Phi_L^2(|v|^2) - 2^{\gamma i} \|\rho_i u\|\,\|v\| \int_\Lambda \Phi_L(|v|^2), \end{aligned}$$

holds. If we let $\tilde{f} := \chi_i^3 f$ and

$$H_L := \langle \Phi_L(|v|^2)\,\Phi_L'(|v|^2), \lambda (|v|^2)^2 - \tilde{f} \cdot v \rangle = -\langle \Phi_L(v)\,\Phi_L'(v), \chi_i^3\, v \cdot (\partial_t u + (-\Delta)^s u) \rangle,$$

we have obtained the uniform bound

$$H_L \lesssim \int_\Lambda \Phi_L(|v|^2)^2 + \|v\| \,[2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|] \int_\Lambda \Phi_L(|v|^2). \tag{92}$$

Now, we have

$$\langle \Phi_L(|v|^2)\,\Phi_L'(|v|^2), \lambda (|v|^2)^2 \rangle \;\geqslant\; \lambda L \int_\Lambda \Phi_L^2(|v|^2) + \lambda L^2 \int_\Lambda \Phi_L(|v|^2).$$

We can estimate, via the Cauchy-Schwartz inequality,

$$\langle \Phi_L(|v|^2)\,\Phi_L'(|v|^2), \tilde{f} \cdot v \rangle \lesssim \int_\Lambda |\tilde{f}|\,|v|\,\Phi_L(|v|^2) \lesssim \|\tilde{f}\|\,\|v\| \int_\Lambda \Phi_L(|v|^2),$$

this gives

$$[\lambda L - C] \int_\Lambda \Phi_L^2(|v|^2) + \left[\lambda L^2 - \|v\|\,\|\tilde{f}\| - C\,\|v\|\,(2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|)\right] \int_\Lambda \Phi_L(|v|^2) \leqslant 0,$$

Taking

$$L > L_* = \max(\lambda^{-1} C, \lambda^{-1/2} \|v\|^{1/2} [\|\tilde{f}\| + C\,(2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|)]^{1/2}),$$

we deduce that

$$\int_\Lambda \Phi_L(|v|^2) = \int_\Lambda \Phi_L^2(|v|^2) = 0,$$

which in turn implies that $|v|^2 \leqslant L$ a.e. on Λ. We conclude that

$$\|v\|^2 \leqslant \inf_{L > L_*} L = L_* \lesssim \lambda^{-1} + \lambda^{-1/2} \|v\|^{1/2} [\|\tilde{f}\| + C(2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|)]^{1/2}.$$

This implies that

$$\|v\| \leqslant \lambda^{-1/2} + \lambda^{-1/4} \|v\|^{1/4} [\|\tilde{f}\| + C(2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|)]^{1/4}.$$

By Young inequality we have

$$\lambda^{-1/4} \|v\|^{1/4} [\|\tilde{f}\| + C(2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|)]^{1/4} \leqslant \frac{\|v\|}{4} + C\lambda^{-1/3} [\|\tilde{f}\| + 2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|]^{1/3},$$

and thus

$$\|v\| \lesssim \lambda^{-1/2} + \lambda^{-1/3} [\|\tilde{f}\| + 2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|]^{1/3}.$$

□

For convenience of the reader we provide the detailed argument that replaces Lemma 16 albeit the proof is a small variation on the same theme.

Lemma 33. *Let $v := \chi_i u$, then we have*

$$\langle \Phi(|v|^2) \Phi'(|v|^2) v^a, \chi_i^3 (-\Delta)^s u^a \rangle \gtrsim -\int \Phi(|v|^2)^2 - \|v\| [2^{\gamma i} \|\rho_i u\| + 2^{\gamma i} \|D(\rho_i u)\|] \int \Phi(|v|^2).$$

Proof. First of all, observe that

$$\partial_\tau \Phi(|\tau v(z') + (1-\tau) v(z)|^2) = 2\Phi'(|\tau v(z') + (1-\tau) v(z)|^2) \sum_a (\tau v^a(z') + (1-\tau) v^a(z))(v^a(z') - v^a(z)),$$

and

$$\partial_\tau^2 \Phi(|\tau v(z') + (1-\tau) v(z)|^2) = 4\Phi''(|\tau v(z') + (1-\tau) v(z)|^2) \left[\sum_a (\tau v^a(z') + (1-\tau) v^a(z))(v^a(z') - v^a(z))\right]^2$$
$$+ 2\Phi'(|\tau v(z') + (1-\tau) v(z)|^2) \sum_a (v^a(z') - v^a(z))^2 \geqslant 0,$$

so regularising and then taking limits we have

$$2\Phi'(|v(z)|^2) \sum_a v^a(z)(v^a(z) - v^a(z')) \geqslant \Phi(|v(z)|^2) - \Phi(|v(z')|^2). \tag{93}$$

Then we can write, by algebraic manipulations and leaving the sum over a implicit,

$$\begin{aligned}
A &= \langle \Phi(|v|^2) \Phi'(|v|^2) v^a, \chi_i^3 (-\Delta)^s u^a \rangle \\
&= \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \Phi'(|v(z)|^2) v^a(z) \chi_i^3(z) (u^a(z) - u^a(z')) \\
&= \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \Phi'(|v(z)|^2) v^a(z) \chi_i^2(z) (v^a(z) - v^a(z')) \\
&\quad + \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \Phi'(|v(z)|^2) v^a(z) \chi_i^2(z) (\chi_i(z') - \chi_i(z)) u^a(z'),
\end{aligned}$$

and using the inequality (93) we have

$$\begin{aligned}
A &\geqslant \frac{1}{2} \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \chi_i^2(z) [\Phi(|v(z)|^2) - \Phi(|v(z')|^2)] \\
&\quad + \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \Phi'(|v(z)|^2) v^a(z) \chi_i^2(z) (\chi_i(z') - \chi_i(z)) u^a(z') \\
&= \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \chi_i(z) \chi_i(z') [\Phi(|v(z)|^2) - \Phi(|v(z')|^2)] && [=:(\mathrm{I})] \\
&\quad + \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \Phi'(|v(z)|^2) v^a(z) \chi_i^2(z) (\chi_i(z') - \chi_i(z)) u^a(z') && [=:(\mathrm{II})] \\
&\quad + \int \nu_s(\mathrm{d}z\,\mathrm{d}z') \Phi(|v(z)|^2) \chi_i(z) [\chi_i(z) - \chi_i(z')] [\Phi(|v(z)|^2) - \Phi(|v(z')|^2)] && [=:(\mathrm{III})]
\end{aligned}$$

Symmetrising the integral ($\mathbb{I}$) w.r.t the exchange $z \leftrightarrow z'$ we have

$$(\mathbb{I}) \geqslant \frac{1}{4}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi(|v(z)|^2) - \Phi(|v(z')|^2)]^2 > 0,$$

while, again via the same algebraic manipulations as in the scalar case we have

$$\begin{aligned}(\mathbb{II}) \;=\; & \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi(|v(z)|^2)\,\Phi'(|v(z)|^2)\,|v(z)|^2\,\chi_i(z)\,(\chi_i(z') - \chi_i(z)) && [=:(\mathbb{II}_1)]\\ & + \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi(|v(z)|^2)\Phi'(|v(z)|^2)v^a(z)\chi_i^2(z)(\chi_i(z') - \chi_i(z))(\rho_i^{-1}(z') - \rho_i^{-1}(z))(\rho_i u^a)(z') && [=:(\mathbb{II}_2)]\\ & + \int \nu_s(\mathrm{d}z\,\mathrm{d}z')\Phi(|v(z)|^2)\Phi'(|v(z)|^2)v^a(z)\chi_i^2(z)\rho_i^{-1}(z)(\chi_i(z') - \chi_i(z))[(\rho_i u^a)(z') - (\rho_i u^a)(z)] && [=(\mathbb{II}_3)]\end{aligned}$$

and

$$(\mathbb{II}_1) + (\mathbb{II}_2) \gtrsim -2^{2\gamma i}\|\rho_i u\|\,\|v\|\int \mathrm{d}z\Phi(|v(z)|^2),$$

$$(\mathbb{II}_3) \gtrsim -2^{2\gamma i}\|D(\rho_i u)\|\,\|v\|\int \mathrm{d}z\Phi(|v(z)|^2),$$

and finally, again with computation analogous to the scalar case, we have

$$(\mathbb{III}) \gtrsim -\left[\frac{1}{2\alpha} + \frac{\alpha}{2}\right]\int \Phi(|v(z)|^2)^2\mathrm{d}z - \frac{\alpha}{2}\int \nu_s(\mathrm{d}z\,\mathrm{d}z')\chi_i(z)\,\chi_i(z')\,[\Phi(|v(z)|^2) - \Phi(|v(z')|^2)]^2,$$

from which we conclude the claim. □

Appendix A Auxiliary estimates

We collect in this appendix various technical estimates of general character.

Lemma 34. *Let $\dot{G}_\sigma(z)$ be the kernel of $\dot{G}_\sigma$. There exist constants $C_1, C_2 \in \mathbb{R}$ independent of σ and ε such that for any $z \in \Lambda_\varepsilon$*

$$|\partial^A \dot{G}_\sigma(z)| \leqslant C_1\,(\varepsilon[\![\sigma]\!]^{-1} \vee 1)^{-d}[\![\sigma]\!]^{-d-1-|A|}\,e^{-C_2([\![\sigma]\!]^{-1}|z|_s)^\omega}, \tag{94}$$

where $\omega := 1/r < 1$, with $r > 1$ the Gevrey regularity of the function j, cf. Definition 3. Moreover

$$|\check{\dot{j}}_{\sigma,1}(z_1) - \check{\dot{j}}_{\sigma,1}(z_1 + h)| \lesssim (2\sigma[\![\sigma]\!]^{-1})^{d+2s+1}((2\sigma[\![\sigma]\!]^{-1}|h|) \wedge 1)\,e^{-C_2([\![\sigma]\!]^{-1}|z|_s)^\omega}, \tag{95}$$

where $\check{\dot{j}}_{\sigma,1}$ is the kernel of $\dot{\mathcal{J}}_{\sigma,1}$.

Proof. First of all we observe that the symbol of $\dot{\mathcal{J}}_\sigma$ is

$$\partial_\sigma j_\sigma(|k_0|^{1/2s} + q_\varepsilon(k)) = -\frac{1}{\sigma^2}(|k_0|^{1/2s} + q_\varepsilon(k))\,j'(\sigma^{-1}[\![\sigma]\!]\,(|k_0|^{1/2s} + q_\varepsilon(k))),$$

where j' denotes the derivative of the function j. Thus we have

$$\dot{G}_\sigma(t,x) = -\int_{\Lambda_\varepsilon^*} \frac{|k_0'|^{1/2s} + q_\varepsilon(k')}{ik_0' + m^2 + q_\varepsilon^{2s}(k')}\,\frac{1}{\sigma^2}\,j'(\sigma^{-1}[\![\sigma]\!]\,(|k_0'|^{1/2s} + q_\varepsilon(k')))\,e^{i(k'x + k_0' t)}\mathrm{d}k_0'\mathrm{d}k'. \tag{96}$$

We start by considering the regime $[\![\sigma]\!] \gtrsim \varepsilon$. Let $\varepsilon_\sigma := \varepsilon\sigma[\![\sigma]\!]^{-1}$ and $m_\sigma^2 := m^2(\sigma^{-1}[\![\sigma]\!])^{2s}$. By the change of variables $k = \sigma^{-1}[\![\sigma]\!]\, k'$ and $k_0 = (\sigma^{-1}[\![\sigma]\!])^{2s} k_0'$, we get,

$$\dot{G}_\sigma(t,x) = -\frac{(\sigma[\![\sigma]\!]^{-1})^{d+1}}{\sigma^2}\int_{\Lambda_{\varepsilon_\sigma}^*} \frac{|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)}{i k_0 + m_\sigma^2 + q_{\varepsilon_\sigma}^{2s}(k)} \times$$
$$\times\, j'((|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)))\, e^{i(\sigma[\![\sigma]\!]^{-1}kx + (\sigma[\![\sigma]\!]^{-1})^{2s}k_0 t)}\mathrm{d}k_0\mathrm{d}k,$$

where we used the fact that $\sigma^{-1}[\![\sigma]\!]\, q_\varepsilon(k) = q_{\varepsilon_\sigma}(\sigma^{-1}[\![\sigma]\!]\, k)$. This gives, for $\tilde{\alpha} \in \mathbb{N}^d$, $\alpha_0 \in \mathbb{N}$ and denoting $\alpha = (\alpha_0, \tilde{\alpha}) \in \mathbb{N}^{1+d}$,

$$\begin{aligned}(\sigma[\![\sigma]\!]^{-1}x)^{\tilde{\alpha}}((\sigma[\![\sigma]\!]^{-1})^{2s}t)^{\alpha_0}\dot{G}_\sigma(t,x) &= (-1)^{1+|\alpha|}\frac{(\sigma[\![\sigma]\!]^{-1})^{d+1}}{\sigma^2}\int_{\Lambda_{\varepsilon_\sigma}^*} e^{i(\sigma[\![\sigma]\!]^{-1}kx + (\sigma[\![\sigma]\!]^{-1})^{2s}k_0 t)} \times \\ &\quad \times \partial_{k_0}^{\alpha_0}\partial_k^{\tilde{\alpha}}\left(\frac{|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)}{i k_0 + m_\sigma^2 + q_{\varepsilon_\sigma}^{2s}(k)}\, j'((|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)))\right)\mathrm{d}k_0\mathrm{d}k.\end{aligned}$$

We observe that $\frac{|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)}{i k_0 + m_\sigma^2 + q_{\varepsilon_\sigma}^{2s}(k)}$ and $|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)$ are analytic on the support of j'. Since j' is a function in a Gevrey class of regularity $r > 1$, recalling that the composition of a Gevrey function of regularity $r > 1$ with an analytic one is Gevrey of regularity r, *cf.* Prop. 1.4.6. [Rod93], we conclude that

$$\left|\partial_{k_0}^{\alpha_0}\partial_k^{\tilde{\alpha}}\left(\frac{|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)}{i k_0 + m_\sigma^2 + q_{\varepsilon_\sigma}^{2s}(k)}\, j'((|k_0|^{1/2s} + q_{\varepsilon_\sigma}(k)))\right)\right| \leqslant 2^{|\alpha|} C^{|\alpha|} |\alpha|^{|\alpha| r}.$$

We observe that the bound is uniform with respect to σ since $m_\sigma^2 = m^2(\sigma^{-1}[\![\sigma]\!])^{2s}$ is a decreasing function of σ and the origin is outside the support of j'. For what concerns the uniformity of the above constants with respect to ε, we observe that this is due to the fact that $q_\varepsilon(k)$ is well defined for any $\varepsilon \geqslant 0$ while its derivatives with respect to k are not singular with respect to ε. This implies that for any α (the constants C, C_1, C_2 in the following formulas may change from one equation to the next and are independent of σ, ε and α)

$$\sup_{(t,x)\in\Lambda_\varepsilon} |(\sigma[\![\sigma]\!]^{-1}x)^{\tilde{\alpha}}((\sigma[\![\sigma]\!]^{-1})^{2s}t)^{\alpha_0}\dot{G}_\sigma(t,x)| \leqslant C_1\frac{(\sigma[\![\sigma]\!]^{-1})^{d+1}}{\sigma^2}C^{|\alpha|}|\alpha|^{|\alpha| r}.$$

This gives, for any $(t,x) \in \Lambda_\varepsilon$ and $n = |\alpha|$,

$$\begin{aligned}|\dot{G}_\sigma(t,x)| &\leqslant (1 + (\sigma[\![\sigma]\!]^{-1}|x|)^2 + ((\sigma[\![\sigma]\!]^{-1})^{2s}|t|)^2)^{-n/2}\frac{(\sigma[\![\sigma]\!]^{-1})^{d+1}}{\sigma^2}C_1 C^n n^{nr} \\ &= \frac{(\sigma[\![\sigma]\!]^{-1})^{d+1}}{\sigma^2}C_1 u^{-n} n^{nr} = \frac{(\sigma[\![\sigma]\!]^{-1})^{d+1}}{\sigma^2}C_1 e^{nr\ln(nu^{-1/r})},\end{aligned}$$

where we introduced $u := C^{-1}(1 + (\sigma[\![\sigma]\!]^{-1}|x|)^2 + ((\sigma[\![\sigma]\!]^{-1})^{2s}|t|)^2)^{1/2}$. Setting $\omega = 1/r$ and $n = \lfloor u^\omega e^{-1}\rfloor$, we get

$$\begin{aligned}|\dot{G}_\sigma(t,x)| &\leqslant C_1[\![\sigma]\!]^{-d-1}e^{-C_2(1 + ([\![\sigma]\!]^{-2s}t)^2 + |[\![\sigma]\!]^{-1}x|^2)^{\omega/2}} \\ &\leqslant C_1[\![\sigma]\!]^{-d-1}e^{-C_2[\![\sigma]\!]^{-\omega}(|t|^{1/s} + |x|^2)^{\omega/2}} \\ &\leqslant C_1[\![\sigma]\!]^{-d-1}e^{-C_2([\![\sigma]\!]^{-1}|z|_s)^\omega},\end{aligned}$$

where we concluded using that $(|t|^{1/s} + |x|^2)^{1/2} \geqslant 2^{-1/2}|z|_s$ for $z = (t,x)$, with $|z|_s = |t|^{1/2s} + |x|$. Let us now consider the spatial discrete derivative

$$\begin{aligned}\nabla_\varepsilon^i\dot{G}_\sigma(z) &= \frac{1}{\varepsilon}[\dot{G}_\sigma(z + \varepsilon e_i) - \dot{G}_\sigma(z)] \\ &= -\int_{\Lambda_\varepsilon^*}\frac{|k_0|^{1/2s} + q_\varepsilon(k)}{ik_0 + m^2 + q_\varepsilon^{2s}(k)}\frac{1}{\sigma^2}j'(\sigma^{-1}[\![\sigma]\!](|k_0|^{1/2s} + q_\varepsilon(k)))\,e^{i(kx + k_0 t)}\frac{(e^{i\varepsilon k\cdot e_i} - 1)}{\varepsilon}\mathrm{d}k_0\mathrm{d}k.\end{aligned}$$

As before we can rescale the integral and observe that

$$\left|\partial_{k_0}^{\alpha_0}\partial_k^{\tilde{\alpha}}\left(\frac{|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k)}{i k_0+m_\mu^2+q_{\varepsilon_\sigma}^{2s}(k)}\,j'((|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k)))\,\frac{(e^{i\varepsilon\sigma[\![\sigma]\!]^{-1}k\cdot e_i}-1)}{\varepsilon}\right)\right|$$

$$\leqslant \sigma[\![\sigma]\!]^{-1}\int_0^1 \mathrm{d}h\left|\partial_{k_0}^{\alpha_0}\partial_k^{\tilde{\alpha}}\left(\frac{|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k)}{i k_0+m_\sigma^2+q_{\varepsilon_\sigma}^{2s}(k)}\,j'((|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k)))\,(k\cdot e_i)\,e^{i\varepsilon_\sigma hk\cdot e_i}\right)\right|$$

$$\leqslant \sigma[\![\sigma]\!]^{-1}\,2^{|\alpha|}\,C^{|\alpha|}\,|\alpha|^{|\alpha|r},$$

to conclude that

$$|\nabla_\varepsilon^i\dot{G}_\sigma(z)|\leqslant C_1\,[\![\sigma]\!]^{-d-2}\,e^{-C_2([\![\sigma]\!]^{-1}|z|_s)^\omega}.$$

Analogously one can discuss higher order derivatives (and also time derivatives) to get

$$|\partial^A\dot{G}_\sigma(z)|\leqslant C_1\,[\![\sigma]\!]^{-d-1-|A|}\,e^{-C_2([\![\sigma]\!]^{-1}|z|_s)^\omega}. \tag{97}$$

Considering the other regime, namely $[\![\sigma]\!]\lesssim\varepsilon$. When $|z|_\varepsilon\lesssim\varepsilon$ we estimate directly $|\dot{G}_\sigma(z)|$ to have

$$|\dot{G}_\sigma(z)|\leqslant\int_{\Lambda_\varepsilon^*}\left|\frac{|k_0'|^{1/2s}+q_\varepsilon(k')}{ik_0'+m^2+q_\varepsilon^{2s}(k')}\right|\frac{1}{\sigma^2}|j'(\sigma^{-1}[\![\sigma]\!]\,(|k_0'|^{1/2s}+q_\varepsilon(k')))|\,\mathrm{d}k_0'\mathrm{d}k'\lesssim\varepsilon^{-d}[\![\sigma]\!]^{2s-1}[\![\sigma]\!]^{-2s}\lesssim\varepsilon^{-d}[\![\sigma]\!]^{-1},$$

and more generally

$$|\partial^A\dot{G}_\sigma(z)|\lesssim\varepsilon^{-d}[\![\sigma]\!]^{-1-|A|}.$$

When $|z|_\varepsilon\gtrsim\varepsilon$ we have, exploiting the general bound (97)

$$|\partial^A\dot{G}_\sigma(z)|\leqslant C_1\,[\![\sigma]\!]^{-d-1-|A|}\,e^{-\frac{1}{2}C_2([\![\sigma]\!]^{-1}\varepsilon)^\omega}e^{-\frac{1}{2}C_2([\![\sigma]\!]^{-1}|z|_s)^\omega}$$

$$\lesssim C_1\varepsilon^{-d-|A_x|}[\![\sigma]\!]^{-1-|A_t|}e^{-\frac{1}{2}C_2([\![\sigma]\!]^{-1}|z|_s)^\omega}\lesssim C_1\varepsilon^{-d}[\![\sigma]\!]^{-1-|A|}e^{-\frac{1}{2}C_2([\![\sigma]\!]^{-1}|z|_s)^\omega},$$

so this estimate holds actually for all z provided the constants C_1 and C_2 are adjusted accordingly and this allows to conclude (94). Finally consider

$$\begin{aligned}\partial_\sigma\check{j}_{\sigma,1}(z_1)-\partial_\sigma\check{j}_{\sigma,1}(z_1+h) \;=\; & -(2\sigma[\![\sigma]\!]^{-1})^{d+2s+1}\int_{\Lambda_{\varepsilon_{\eta'}}^*}\mathrm{d}k_0\mathrm{d}k\frac{(|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k))}{\eta'^2}\times\\ & \times j'(|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k))(1-e^{i2\sigma[\![\sigma]\!]^{-1}kh})\,e^{i(2\sigma[\![\sigma]\!]^{-1}kx_1+(2\sigma[\![\sigma]\!]^{-1})^{2s}k_0t_1)}.\end{aligned}$$

for which we have

$$\left|\partial_k^{\tilde{\alpha}}\partial_{k_0}^{\alpha_0}(j'(|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k))(|k_0|^{1/2s}+q_{\varepsilon_\sigma}(k))(1-e^{i2\sigma[\![\sigma]\!]^{-1}kh}))\right|\leqslant 2^{|\alpha|}C^{|\alpha|}\,|\alpha|^{|\alpha|r}((2\sigma[\![\sigma]\!]^{-1}|h|)\wedge 1).$$

Reasoning as before we deduce also (95). □

Lemma 35. *Fix $\alpha\in(2s,4s)$ and let h be a polynomial space-time weight $h(z):=\langle z\rangle^{-\nu}$ such that $\nu\in(0,2s)$ and $(\Delta_i^x)_{i\geqslant-1}$ a Littlewood-Paley decomposition in the space variable alone as per Definition 7. There exists a universal constant for which for any $\varphi\in\mathcal{S}'(\Lambda)$ we have*

$$\|h\,\Delta_i^x\varphi\|\lesssim(2+i)\,2^{(\alpha-2s)i}\sup_\sigma[\![\sigma]\!]^\alpha\,\|h\,\mathcal{J}_\sigma\mathscr{L}\varphi\|,\qquad i\geqslant-1.$$

that is $\varphi\in C(\mathbb{R};B_{\infty,\infty}^\theta)$ for any $\theta<2s-\alpha$.

Proof. Let $f:=\mathscr{L}\varphi\in\mathcal{S}'(\Lambda)$ and assume that $\sup_\sigma[\![\sigma]\!]^\alpha\|h\,\mathcal{J}_\sigma f\|<\infty$, otherwise we do not have anything to prove. Choose $(\sigma_i)_i\subseteq(0,1)$ such that $[\![\sigma_i]\!]\approx 2^{-i}$ and define $Q_i:=(\tilde{\mathcal{J}}_{\sigma_i,1}-\tilde{\mathcal{J}}_{\sigma_{i-1,1}})$, so that $\|h\,Q_jf\|\lesssim 2^{\alpha j}$, and

$$\varphi(t)=\sum_j(\mathscr{L}^{-1}Q_jf)(t)=\sum_j\int_{-\infty}^t e^{-(t-u)(m^2+(-\Delta_\varepsilon)^s)}Q_jf(u)\mathrm{d}u.$$

Let $(\Delta_i^t)_{i\geqslant -1}$ be a Littlewood–Paley decomposition in the temporal variable constructed out of the dyadic partition of unity $(\hat{\Delta}_i)_{i\geqslant -1}$ as per Definition 7, for simplicity however we parametrise Δ_i^t according to spatial scales, i.e. when $i\geqslant 0$, Δ_i^t filters an annulus in the Fourier transform of the time variable of size 2^{2si}. Now write

$$\Delta_i^x \varphi(t) = \sum_j (\mathscr{L}^{-1}\Delta_i^x Q_j f)(t) = \sum_{j,\ell} \int_{-\infty}^t e^{-(t-u)(m^2+(-\Delta_\varepsilon)^s)} \Delta_i^x \Delta_\ell^t Q_j f(u) \mathrm{d}u.$$

Now due to the structures of the various decompositions, $\Delta_i^x \Delta_\ell^t Q_j$ is different from zero only when $i\approx j\gtrsim \ell$ or $\ell\approx j\gtrsim i$ since remember that the support $\Delta_i^x\Delta_\ell^t Q_j$ in the dual space should satisfy the bounds $q_\varepsilon(k)\approx 2^i$, $|\omega|^{1/2s}\approx 2^\ell$, $q_\varepsilon(k)+|\omega|^{1/2s}\approx 2^j$. We keep i fixed and only let ℓ, j vary and we denote $\mathbb{I}$ and $\mathbb{II}$ these two regions.

In the region $\mathbb{II}$ where $\ell\approx j\gtrsim i$ we let, omitting for simplicity the space variable,

$$F_t(u) := [(1+\partial_u)^{-2}(\Delta_i^x\Delta_\ell^t Q_j f)](u), \qquad F_t(t) = \partial_u F_t(u)|_{u=t} = 0.$$

We observe that this function exists, since it suffices to take

$$F_t(u) := \int_u^t \mathrm{d}u_1 e^{-(u_1-u)} \int_{u_1}^t \mathrm{d}u_2 e^{-(u_2-u_1)} (\Delta_i^x \Delta_\ell^t Q_j f)(u_2).$$

We observe that, by integrating over $u\in(-\infty,t)$ the following equality,

$$\begin{aligned}\partial_u^2(e^{-(t-u)(m^2+(-\Delta_\varepsilon)^s)}F_t(u)) &= e^{-(t-u)(m^2+(-\Delta_\varepsilon)^s)}[(1+m^2+(-\Delta_\varepsilon)^s)^2F_t(u) + \\ &\quad +2(1+m^2+(-\Delta_\varepsilon)^s)(1+\partial_u)F_t(u) + \Delta_i^x\Delta_\ell^t Q_j f(u)],\end{aligned}$$

and by observing that

$$\int_{-\infty}^t e^{-(t-u)(m^2+(-\Delta_\varepsilon)^s)}\Delta_i^x\Delta_\ell^t Q_j f(u)\mathrm{d}u = -\int_{-\infty}^t (1+m^2+(-\Delta_\varepsilon)^s) e^{-(t-u)(m^2+(-\Delta_\varepsilon)^s)}(1+\partial_u)F_t(u)\mathrm{d}u,$$

it holds that (by renaming $1+m^2$ with m^2)

$$A(t) := \int_{-\infty}^t e^{-(t-u)(m^2+(-\Delta)^s)}\Delta_i^x\Delta_\ell^t Q_j f(u)\mathrm{d}u = \int_{-\infty}^t e^{-(t-u)(m^2+(-\Delta)^s)}(m^2+(-\Delta_\varepsilon)^s)^2 F(u)\mathrm{d}u,$$

This gives that

$$A(t) = (G(m^2+(-\Delta_\varepsilon)^s)^2 F)(t).$$

To estimate this term we separately deal with the cases $i=-1$ and $i\geqslant 0$.

Considering the case $i=-1$, namely on the ball around the origin in the Fourier space, introducing $\tilde{\Delta}_{-1}^x$ so that $\tilde{\Delta}_{-1}^x\Delta_{-1}^x = \Delta_{-1}^x$,

$$\begin{aligned}\|h\,G(m^2+(-\Delta_\varepsilon)^s)^2\partial_t^{-2}(\Delta_{-1}^x\Delta_\ell^t Q_j f)\| &= \|h\,G(m^2+(-\Delta_\varepsilon)^s)^2\tilde{\Delta}_{-1}^x\partial_t^{-2}(\Delta_{-1}^x\Delta_\ell^t Q_j f)\| \\ &\lesssim \|(h^{-1}G)h\partial_t^{-2}(\Delta_{-1}^x\Delta_\ell^t Q_j f)\| \\ &\lesssim \|h^{-1}G\|_{\mathscr{L}(L^\infty,L^\infty)}\|hF\|.\end{aligned}$$

We now observe that $\|h^{-1}G\|_{\mathscr{L}(L^\infty,L^\infty)}\lesssim 1$ provided the polynomial weight h is not too fast decreasing. Indeed, from (17) we see that it is enough to require $\nu < 2s$. This implies

$$\|h\,G(m^2+(-\Delta_\varepsilon)^s)^2\partial_t^{-2}(\Delta_{-1}^x\Delta_\ell^t Q_j f)\| \lesssim \|hF\| \lesssim 2^{-4s\ell}2^{\alpha j}.$$

For $i\geqslant 0$ instead,

$$h\,G(m^2+(-\Delta_\varepsilon)^s)^2 F = h\,G(m^2+(-\Delta_\varepsilon)^s)^2\tilde{\Delta}_i^x\tilde{\Delta}_\ell^t F,$$

and thus

$$\|h\,G(m^2+(-\Delta_\varepsilon)^s)^2 F\| \lesssim \|h^{-1}G(m^2+(-\Delta_\varepsilon)^s)^2\tilde{\Delta}_i^x\tilde{\Delta}_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}\|hF\|.$$

We want to control $\|h^{-1}G(m^2+(-\Delta_\varepsilon)^s)^2\tilde{\Delta}_i^x\tilde{\Delta}_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}$. The kernel of $G(m^2+(-\Delta_\varepsilon)^s)^2$ can be written as

$$\int_{\Lambda_\varepsilon^*}\mathrm{d}k'\mathrm{d}k_0'\frac{(m^2+q_\varepsilon^{2s}(k'))^2}{i k_0'+m^2+q_\varepsilon^{2s}(k')}\hat{\tilde{\Delta}}(2^{-i}q_\varepsilon(k))\hat{\tilde{\Delta}}(2^{-2\ell s}k_0')e^{i(k'x+k_0't)}=$$

and we can rescale the integral to get

$$=2^{id}2^{2\ell s}\int_{\Lambda_{\varepsilon_i}^*}\mathrm{d}k\mathrm{d}k_0\frac{(m^2+2^{2is}q_{\varepsilon_i}^{2s}(k))^2}{i\,2^{2\ell s}k_0+m^2+2^{2is}q_{\varepsilon_i}^{2s}(k)}\hat{\tilde{\Delta}}(q_{\varepsilon_i}(k))\hat{\tilde{\Delta}}(k_0)e^{i(k2^ix+2^{2\ell s}k_0t)}=$$

$$=2^{id}2^{4is}\int_{\Lambda_{\varepsilon_i}^*}\mathrm{d}k\mathrm{d}k_0\frac{(m^22^{-2is}+q_{\varepsilon_i}^{2s}(k))^2}{i\,k_0+m^22^{-2\ell s}+2^{2(i-\ell)s}q_{\varepsilon_i}^{2s}(k)}\hat{\tilde{\Delta}}(q_{\varepsilon_i}(k))\hat{\tilde{\Delta}}(k_0)e^{i(k2^ix+2^{2\ell s}k_0t)},$$

where we observe that $2^{2(i-\ell)s}\lesssim 1$ since we are in the regime $\ell\gtrsim i$. The stretched exponential decay of our kernel now follows from similar arguments as in Lemma 34 recalling that the Littlewood-Paley blocks have symbols of Gevrey regularity $r>0$, and gives the integrability against h^{-1} as well as

$$\|h^{-1}G(m^2+(-\Delta_\varepsilon)^s)^2\tilde{\Delta}_i^x\tilde{\Delta}_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}\lesssim 2^{id}2^{4is}2^{-id}2^{-2\ell s}=2^{4is}2^{-2\ell s}.$$

This implies that

$$\|h^{-1}G(m^2+(-\Delta_\varepsilon)^s)^2F\|\lesssim 2^{4is}2^{-2\ell s}\|hF\|\lesssim 2^{4is}2^{-2\ell s}2^{-4s\ell}2^{\alpha j}.$$

As a consequence, for $i\geqslant 0$,

$$\left\|h\sum_{j\approx\ell\gtrsim i}A\right\|\lesssim\sum_{j\approx\ell\gtrsim i}2^{4is}2^{-2\ell s}2^{-4s\ell}2^{\alpha j}\lesssim 2^{4is}\sum_{j\gtrsim i}2^{(\alpha-6s)j}\lesssim 2^{(\alpha-2s)i}.$$

For what concerns the second regime, when $i\approx j\gtrsim\ell$. This is simpler. We consider directly

$$A:=G\Delta_i^x\Delta_\ell^tQ_jf,$$

and we have

$$\|hA\|\lesssim\|h^{-1}G\Delta_i^x\Delta_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}\|hQ_jf\|\lesssim\|h^{-1}G\Delta_i^x\Delta_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}2^{\alpha j}\lesssim\|h^{-1}G\Delta_i^x\Delta_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}2^{\alpha i}.$$

For $\|h^{-1}G\Delta_i^x\Delta_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}$ we observe that for $i=-1$, we can use the above argument on the polynomial decay to say that

$$\|h^{-1}G\Delta_{-1}^x\Delta_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}\lesssim 1.$$

For $i\geqslant 0$, with the above argument on the stretched exponential decay, we get

$$\|h^{-1}G\Delta_i^x\Delta_\ell^t\|_{\mathscr{L}(L^\infty,L^\infty)}\lesssim 2^{-2is}.$$

As a consequence, we get $\|hA\|\lesssim 2^{(\alpha-2s)i}$, and thus $\|h\sum_{\ell\lesssim i\approx j}A\|\lesssim i2^{(\alpha-2s)i}$. This concludes the proof. □

Exploiting the previous lemma we obtain a regularity estimate tailored to our norms.

Lemma 36. *Fix any small $\kappa>0$. Then we have, for all $j\geqslant 0$, $i\geqslant -1$,*

$$\|\chi_j^3\,\Delta_i^x\phi^\varepsilon\|\lesssim \mathbb{1}_{i\lesssim j}[\![\mu_j\vee\bar{\mu}]\!]^{-\gamma}|||\phi^\varepsilon|||+2^{i(3\gamma-2s+\kappa)}|||\mathscr{L}\phi^\varepsilon|||_{\#}. \tag{98}$$

In particular,

$$\sup_i 2^{-i(3\gamma-2s+\kappa)}\|\rho\,\Delta_i^x\phi^\varepsilon\|\lesssim[\![\bar{\mu}]\!]^{-\gamma}|||\phi^\varepsilon|||+|||\mathscr{L}\phi^\varepsilon|||_{\#}.$$

Proof. Let us first assume that Equation (98) is satisfied. Then we have

$$\begin{aligned}\|\rho\,\Delta_i^x\phi^\varepsilon\| &\lesssim \sup_j 2^{-j\gamma}\|\chi_j^3\,\Delta_i^x\phi^\varepsilon\|\\ &\lesssim \sup_j\Big[\mathbb{1}_{i\lesssim j}2^{-j\gamma}[\![\mu_j\vee\bar{\mu}]\!]^{-\gamma}|||\phi^\varepsilon|||+2^{i(3\gamma-2s+\kappa)}|||\mathscr{L}\phi^\varepsilon|||_\#\Big]\\ &\lesssim 2^{-i\gamma}[\![\mu_i\vee\bar{\mu}]\!]^{-\gamma}|||\phi^\varepsilon|||+2^{i(3\gamma-2s+\kappa)}|||\mathscr{L}\phi^\varepsilon|||_\#,\end{aligned}$$

where we used the fact that $j\mapsto 2^{-j\gamma}[\![\mu_j\vee\bar{\mu}]\!]^{-\gamma}$ is non-increasing, and therefore we proved that

$$\sup_i 2^{-i(3\gamma-2s+\kappa)}\|\rho\,\Delta_i^x\phi^\varepsilon\|\lesssim[\![\bar{\mu}]\!]^{-\gamma}|||\phi^\varepsilon|||+|||\mathscr{L}\phi^\varepsilon|||_\#,$$

as required. To prove Equation (98), define η_j such that $\eta_j/(2-\eta_j)=\mu_j\vee\bar{\mu}$, and consider the associated operators $\mathcal{J}_{\eta_j}$ and $\mathcal{J}_{>\eta_j}:=1-\mathcal{J}_{\eta_j}$. Then by decomposing ϕ^ε at the scale η_j we have

$$\|\chi_j^3\,\Delta_i^x\phi^\varepsilon\|\lesssim\|\chi_j^3\,\Delta_i^x\phi_{\eta_j}^\varepsilon\|+\|\chi_j^3\,\Delta_i^x\mathcal{J}_{>\eta_j}\phi^\varepsilon\|,$$

so we would conclude provided we prove that

$$\|\chi_j^3\,\Delta_i^x\phi_{\eta_j}^\varepsilon\|\lesssim\mathbb{1}_{i\lesssim j}[\![\mu_j\vee\bar{\mu}]\!]^{-\gamma}|||\phi^\varepsilon|||, \tag{99}$$

and that

$$\|\chi_j^3\,\Delta_i^x\mathcal{J}_{>\eta_j}\phi^\varepsilon\|\lesssim 2^{i(3\gamma-2s+\kappa)}|||\mathscr{L}\phi^\varepsilon|||_\#. \tag{100}$$

We start by estimating $\|\chi_j^3\,\Delta_i^x\phi_{\eta_j}^\varepsilon\|$. Thanks to the definition of η_j we have $\mathcal{J}_{\eta_j}=\mathcal{J}_{\eta_j}\mathcal{J}_{\mu_j\vee\bar{\mu}}$ and therefore using some splitting $\phi^\varepsilon=\phi^{\varepsilon,<}+\phi^{\varepsilon,\geqslant}$ and the definition of the norm $|||\phi^\varepsilon|||$, *cf.* Equation (24), we can estimate

$$\begin{aligned}\|\chi_j^3\,\Delta_i^x\phi_{\eta_j}^\varepsilon\| &\lesssim \mathbb{1}_{i\lesssim j}\|\chi_j^3\,\mathcal{J}_{\eta_j}\Delta_i^x\phi_{\mu_j\vee\bar{\mu}}^\varepsilon\|\\ &\lesssim \mathbb{1}_{i\lesssim j}[\|\chi_j^3\,\mathcal{J}_{\eta_j}\Delta_i^x\phi_{\mu_j\vee\bar{\mu}}^{\varepsilon,<}\|+\|\chi_j^3\,\mathcal{J}_{\eta_j}\Delta_i^x\phi_{\mu_j\vee\bar{\mu}}^{\varepsilon,\geqslant}\|]\\ &\lesssim \mathbb{1}_{i\lesssim j}[2^{j\gamma}\|\rho\,\mathcal{J}_{\eta_j}\Delta_i^x\phi_{\mu_j\vee\bar{\mu}}^{\varepsilon,<}\|+\|\mathcal{J}_{\eta_j}\Delta_i^x\phi_{\mu_j\vee\bar{\mu}}^{\varepsilon,\geqslant}\|]\\ &\lesssim \mathbb{1}_{i\lesssim j}[2^{j\gamma}\|\rho\,\phi_{\mu_j\vee\bar{\mu}}^{\varepsilon,<}\|+\|\phi_{\mu_j\vee\bar{\mu}}^{\varepsilon,\geqslant}\|]\\ &\lesssim \mathbb{1}_{i\lesssim j}[2^{j\gamma}+[\![\mu_j\vee\bar{\mu}]\!]^{-\gamma}]|||\phi^\varepsilon|||\lesssim\mathbb{1}_{i\lesssim j}[\![\mu_j\vee\bar{\mu}]\!]^{-\gamma}|||\phi^\varepsilon|||,\end{aligned}$$

leading to (99). Let now consider (100). We introduce the weight ρ_j of Definition 8 and we observe that by construction, point-wise,

$$2^{-3(k\vee j)\nu}\chi_k^3\lesssim\rho_j^3\lesssim\sum_k 2^{-3(k\vee j)\nu}\chi_k^3,$$

Noting that, for all $j\geqslant 0$, $i\geqslant -1$, we have,

$$2^{-i(3\gamma-2s+\kappa)}2^{-3j\gamma}\|\chi_j^3\,\Delta_i^x\mathcal{J}_{>\eta_j}\phi^\varepsilon\|\lesssim 2^{-i(3\gamma-2s+\kappa)}\|\rho_j^3\,\Delta_i^x\mathcal{J}_{>\eta_j}\phi^\varepsilon\|\lesssim\sup_\sigma[\![\sigma]\!]^{3\gamma}\|\rho_j^3\,\mathcal{J}_\sigma\mathcal{J}_{>\eta_j}\mathscr{L}\phi^\varepsilon\|$$

$$\lesssim\sup_{\sigma\geqslant\eta_j/(2-\eta_j)}[\![\sigma]\!]^{3\gamma}\|\rho_j^3\,\mathscr{L}\phi_\sigma^\varepsilon\|,\lesssim\sup_{\sigma\geqslant\mu_j\vee\bar{\mu}}[\![\sigma]\!]^{3\gamma}\|\rho_j^3\,\mathscr{L}\phi_\sigma^\varepsilon\|,$$

where in the first bound we used the support of χ_j and the decay of ρ_j, in the second we used the Schauder estimate in Lemma 35 with $\alpha=3\gamma$ (with reference to the notation of the lemma, we observe that this is uniform in j thanks to Remark 9 (b) and to the fact that Lemma 42 is stated for a generic polynomial weight h) and finally, in the third, we used that we have chosen η_j such that $\eta_j/(2-\eta_j)=\mu_j\vee\bar{\mu}$. By Lemma 39 below, we have

$$\sup_{\sigma\geqslant\mu_j\vee\bar{\mu}}[\![\sigma]\!]^{3\gamma}\|\rho_j^3\,\mathscr{L}\phi_\sigma^\varepsilon\|\lesssim[\![\mu_j]\!]^{3\gamma}|||\mathscr{L}\phi^\varepsilon|||_\#,$$

and we deduce that

$$2^{-i(3\gamma-2s+\kappa)}\|\chi_j^3\,\Delta_i^x\mathcal{J}_{>\eta_j}\phi^\varepsilon\| \lesssim 2^{3j\gamma}\sup_{\sigma\geqslant\mu_j\vee\bar{\mu}}[\![\sigma]\!]^{3\gamma}\|\rho_j^3\,\mathscr{L}\phi_\sigma^\varepsilon\| \lesssim |\!|\!|\mathscr{L}\phi^\varepsilon|\!|\!|_\#,$$

which is what we needed to establish (100). The proof of Equation (98) is now complete. □

Lemma 37. *It holds*

$$|\!|\!|\phi|\!|\!|_{\bar{\mu}} \lesssim \sup_{i\geqslant 0}\left[\mathbb{1}_{\sigma\geqslant\mu_i}[\![\sigma]\!]^\gamma\|\chi_i\,\phi_\sigma\|_{L^\infty}\right].$$

Proof. Consider the decomposition

$$\phi_\sigma = \sum_{i\geqslant 0}\mathbb{1}_{\sigma<\mu_i}\chi_i\,\phi_\sigma + \sum_{i\geqslant 0}\mathbb{1}_{\sigma\geqslant\mu_i}\chi_i\,\phi_\sigma = \phi_\sigma^< + \phi_\sigma^\geqslant. \tag{101}$$

We observe that on account of Equations (24) and (101), we shall only consider $\sigma\geqslant\bar{\mu}$ and thus we shall only distinguish two regimes here $\sigma\geqslant\mu_i$ and $\bar{\mu}\leqslant\sigma<\mu_i$. Assume that $\bar{\mu}\leqslant\sigma<\mu_i$ and use $\mathcal{J}_\sigma\mathcal{J}_{\mu_{i+1}} = \mathcal{J}_\sigma$ (*cf.* Remark 6) to write

$$\chi_i\,\phi_\sigma = \tilde{\chi}_i\chi_i\,\mathcal{J}_\sigma\phi_{\mu_{i+1}} = \tilde{\chi}_i\,\mathcal{J}_\sigma\chi_i\,\phi_{\mu_{i+1}} + \tilde{\chi}_i\,[\chi_i,\mathcal{J}_\sigma]\,\phi_{\mu_{i+1}},$$

and estimate

$$\|\tilde{\chi}_i\,[\chi_i,\mathcal{J}_\sigma]\,\phi_{\mu_{i+1}}\| \;\leqslant\; \|\tilde{\chi}_i\,[\chi_i,\mathcal{J}_\sigma]\,\phi_{\mu_{i+1}}^<\| + \|\tilde{\chi}_i\,[\chi_i,\mathcal{J}_\sigma]\,\phi_{\mu_{i+1}}^\geqslant\|.$$

Then, by Lemma 38 below, Remark 9 and noting that $L(\chi_i)\lesssim\|\nabla\chi_i\|_{L^\infty}\leqslant[\![\mu_i]\!]^a\|\nabla\chi\|_{L^\infty}\lesssim[\![\mu_i]\!]^a$, we have

$$\begin{aligned}\|\tilde{\chi}_i\,[\chi_i,\mathcal{J}_\sigma]\,\phi_{\mu_{i+1}}\| &\leqslant \|\tilde{\chi}_i\,[\chi_i,\mathcal{J}_\sigma]\,\phi_{\mu_{i+1}}^<\| + \|\tilde{\chi}_i\,[\chi_i,\mathcal{J}_\sigma]\,\phi_{\mu_{i+1}}^\geqslant\|.\\ &\lesssim ([\![\mu_i]\!]^{a-\gamma}[\![\sigma]\!] + [\![\mu_i]\!]^a[\![\sigma]\!]^{1-\gamma})\,|\!|\!|\phi|\!|\!|_{\bar{\mu}}\\ &\leqslant [\![\bar{\mu}]\!]^{1+a-\gamma}\,|\!|\!|\phi|\!|\!|_{\bar{\mu}}.\end{aligned}$$

Thus for $\bar{\mu}\leqslant\sigma<\mu_i$,

$$\|\chi_i\,\phi_\sigma\|_{L^\infty} \leqslant \|\chi_i\,\phi_{\mu_{i+1}}\|_{L^\infty} + ([\![\mu_i]\!]^a[\![\sigma]\!]^{1-\gamma} + [\![\mu_i]\!]^{a-\gamma}[\![\sigma]\!])\,|\!|\!|\phi|\!|\!|_{\bar{\mu}},$$

and, using (101),

$$\begin{aligned}\|\rho\phi_\sigma^<\|_{L^\infty} + [\![\sigma]\!]^\gamma\|\phi_\sigma^\geqslant\|_{L^\infty} &\lesssim \sup_{i\geqslant 0}\left[(\mathbb{1}_{\sigma<\mu_i}[\![\mu_i]\!]^\gamma + \mathbb{1}_{\sigma\geqslant\mu_i}[\![\sigma]\!]^\gamma)\|\chi_i\,\phi_\sigma\|_{L^\infty}\right]\\ &\lesssim \sup_{i\geqslant 0}\left[(\mathbb{1}_{\sigma\geqslant\mu_i}[\![\sigma]\!]^\gamma)\|\chi_i\,\phi_\sigma\|_{L^\infty} + (\mathbb{1}_{\sigma<\mu_i}[\![\mu_i]\!]^{a+\gamma}[\![\sigma]\!]^{1-\gamma} + [\![\mu_i]\!]^a[\![\sigma]\!])\,|\!|\!|\phi|\!|\!|_{\bar{\mu}}\right]\\ &\lesssim \sup_{i\geqslant 0}\left[\mathbb{1}_{\sigma\geqslant\mu_i}[\![\sigma]\!]^\gamma\|\chi_i\,\phi_\sigma\|_{L^\infty}\right] + [\![\bar{\mu}]\!]^{1+a}\,|\!|\!|\phi|\!|\!|_{\bar{\mu}},\end{aligned}$$

Therefore

$$|\!|\!|\phi|\!|\!|_{\bar{\mu}} \leqslant C\sup_{i\geqslant 0}\left[\mathbb{1}_{\sigma\geqslant\mu_i}[\![\sigma]\!]^\gamma\|\chi_i\,\phi_\sigma\|_{L^\infty}\right] + C\,[\![\bar{\mu}]\!]^{1+a}\,|\!|\!|\phi|\!|\!|_{\bar{\mu}},$$

and choosing $\bar{\mu}$ large enough we have $C\,[\![\bar{\mu}]\!]^{1+a}\leqslant 1/2$ and we can conclude. □

Lemma 38. *For any $n\geqslant 0$, it holds that*

$$\|f\,[g,\mathcal{J}_\sigma]\,h\|_{L^\infty} \lesssim [\![\sigma]\!]\,\|\rho^n\,h\|_{L^\infty}\,\|f\rho^{-n}\|_{L^\infty}\,L(g),$$

where

$$L(g) := \sup_{z,z_1\in\Lambda}\frac{|g(z)-g(z_1)|}{|z-z_1|}.$$

Proof. As a first step, we observe that

$$\begin{aligned}
\|f\,[g,\mathcal{J}_\sigma]\,h\|_{L^\infty} &\leqslant \sup_z \int |f(z)\,\check{j}_\sigma(z-z_1)\,(g(z)-g(z_1))\,h(z_1)|\mathrm{d}z_1\\
&= \sup_z \int |f(z)\,\check{j}_\sigma(z-z_1)\,(g(z)-g(z_1))\,\rho^{-n}(z)\,\rho^{n}(z)\,\rho^{-n}(z_1)\,\rho^{n}(z_1)\,h(z_1)|\mathrm{d}z_1\\
&\leqslant \|\rho^n\,h\|_{L^\infty}\sup_z \int |f(z)\,\rho^{-n}(z)\,\check{j}_\sigma(z-z_1)\,(g(z)-g(z_1))\,\rho^{n}(z)\,\rho^{-n}(z_1)|\mathrm{d}z_1\\
&\leqslant \|\rho^n\,h\|_{L^\infty}\|f\rho^{-n}\|_{L^\infty}\sup_z \int |\,\check{j}_\sigma(z-z_1)\,(g(z)-g(z_1))\,\rho^{-n}(z-z_1)|\mathrm{d}z_1\\
&\leqslant L(g)\,\|\rho^n h\|_{L^\infty}\|f\rho^{-n}\|_{L^\infty}\sup_z \int |\,\check{j}_\sigma(z-z_1)||z-z_1|\,|\rho^{-n}(z-z_1)|\mathrm{d}z_1\\
&\lesssim L(g)\,[\![\sigma]\!]\,\|\rho^n\,h\|_{L^\infty}\|f\rho^{-n}\|_{L^\infty}\|z\longmapsto \check{j}(z)\,z\,\rho^{-n}(z)\|_{L^1}\\
&\lesssim L(g)\,[\![\sigma]\!]\,\|\rho^n\,h\|_{L^\infty}\|f\rho^{-n}\|_{L^\infty},
\end{aligned}$$

where in the third second inequality we exploited both $\rho^n(z)\,\rho^{-n}(z_1)\lesssim\rho^{-n}(z-z_1)$. □

Lemma 39. *Provided $\bar{\mu}$ is large enough, we have*

$$\sup_{\sigma\geqslant\mu_j\vee\bar{\mu}} [\![\sigma]\!]^\gamma\,\|\rho_j\,\psi_\sigma\|\lesssim[\![\mu_j]\!]^\gamma\,|||\psi|||,$$

and

$$\sup_{\sigma\geqslant\mu_j\vee\bar{\mu}} [\![\sigma]\!]^{3\gamma}\,\|\rho_j^3\,\mathscr{L}\psi_\sigma\|\lesssim[\![\mu_j]\!]^{3\gamma}\,|||\mathscr{L}\psi|||_\#.$$

Proof. Let us handle first the first inequality. Take $\sigma\geqslant\mu_j\vee\bar{\mu}$, and $k\geqslant j$ and consider two cases. If $\sigma<\mu_k\vee\bar{\mu}$ then we can choose $\sigma_k\geqslant\mu_k\vee\bar{\mu}$ such that $[\![\mu_k\vee\bar{\mu}]\!]\approx[\![\sigma_k]\!]$ and $\mathcal{J}_\sigma\mathcal{J}_{\sigma_k}=\mathcal{J}_\sigma$ so we have

$$\|\rho_k\,\psi_\sigma\|\lesssim\|\rho_k\,\mathcal{J}_\sigma\psi_{\sigma_k}\|\lesssim\|\rho_k\,\psi_{\sigma_k}\|.$$

When $\sigma\geqslant\mu_k\vee\bar{\mu}$ we can take $\sigma_k=\sigma$. In both cases we have $[\![\sigma_k]\!]\approx[\![\mu_k\vee\bar{\mu}\vee\sigma]\!]$ and $\|\rho_k\psi_\sigma\|\lesssim\|\rho_k\psi_{\sigma_k}\|$. Now

$$\begin{aligned}
\|\rho_k\,\psi_{\sigma_k}\| &\lesssim [\![\mu_k]\!]^\gamma\sup_{m|m<k}\|\chi_m\,\psi_{\sigma_k}\|+\sup_{m|m\geqslant k}[\![\mu_m]\!]^\gamma\,\|\chi_m\,\psi_{\sigma_k}\|\\
&\lesssim [\![\mu_k]\!]^\gamma\sup_{m|m<k}[[\![\mu_m]\!]^{-\gamma}+[\![\sigma_k]\!]^{-\gamma}]\,|||\psi|||+[\![\mu_k]\!]^\gamma\sup_{m|m\geqslant k}\|\rho_m\,\psi_{\sigma_k}\|\\
&\lesssim [1+[\![\mu_k]\!]^\gamma\,[\![\sigma_k]\!]^{-\gamma}]\,|||\psi|||+[\![\mu_k]\!]^\gamma\sup_{m|m\geqslant k}\|\rho_m\,\psi_{\sigma_k}\|.
\end{aligned}$$

Using that $[\![\mu_k]\!]^\gamma\,[\![\sigma_k]\!]^{-\gamma}\lesssim 1$ if $\sigma<\mu_k\vee\bar{\mu}$ and $[\![\mu_k]\!]^\gamma\,[\![\sigma_k]\!]^{-\gamma}\lesssim[\![\mu_k]\!]^\gamma\,[\![\sigma]\!]^{-\gamma}$ if $\sigma>\mu_k\vee\bar{\mu}$ we have

$$\sup_{k|k\geqslant j}[1+[\![\mu_k]\!]^\gamma\,[\![\sigma_k]\!]^{-\gamma}]\lesssim[\![\mu_j]\!]^\gamma\,[\![\sigma]\!]^{-\gamma},$$

and then

$$\sup_{k|k\geqslant j}\|\rho_k\,\psi_{\sigma_k}\|\lesssim[\![\mu_j]\!]^\gamma\,[\![\sigma]\!]^{-\gamma}\,|||\psi|||+[\![\mu_j]\!]^\gamma\sup_{k|k\geqslant j}\|\rho_k\,\psi_{\sigma_k}\|.$$

If j is large enough we conclude that

$$\|\rho_j\,\psi_\sigma\|\leqslant\sup_{k|k\geqslant j}\|\rho_k\,\psi_\sigma\|\lesssim\sup_{k|k\geqslant j}\|\rho_k\,\psi_{\sigma_k}\|\lesssim[\![\mu_j]\!]^\gamma\,[\![\sigma]\!]^{-\gamma}\,|||\psi|||.$$

While if μ_j is smaller than $\bar{\mu}$ (chosen large enough) then take j' such that $\mu_{j'} > \bar{\mu} > \mu_j$ and bound

$$\|\rho_j \psi_\sigma\| \lesssim 2^{\gamma(j'-j)} \|\rho_{j'} \psi_\sigma\| \lesssim 2^{\gamma(j'-j)} [\![\mu_{j'}]\!]^\gamma [\![\sigma]\!]^{-\gamma} |||\psi||| \lesssim [\![\mu_j]\!]^\gamma [\![\sigma]\!]^{-\gamma} |||\psi|||.$$

This concludes the proof of the first inequality.

Consider now the second inequality to prove and use the definition of ρ_j to write

$$\sup_{\sigma \geqslant \mu_j \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\rho_j^3 \mathscr{L} \psi_\sigma\| \leqslant \sup_k 2^{-3(j \vee k)\gamma} \sup_{\sigma \geqslant \mu_j \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_k^3 \mathscr{L} \psi_\sigma\| =: A_j.$$

Let us now decompose the supremums over k and σ in the definition of A_j to get three contributions:

$$\begin{aligned} A_j \;\lesssim\; & \sup_{k \leqslant j} 2^{-3j\gamma} \sup_{\sigma \geqslant \mu_j \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_k^3 \mathscr{L} \psi_\sigma\| && [=: A_j^{(1)}] \\ & + \sup_{k > j} 2^{-3k\gamma} \sup_{\sigma \geqslant \mu_k \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_k^3 \mathscr{L} \psi_\sigma\| && [=: A_j^{(2)}] \\ & + \sup_{k | k > j} 2^{-3k\gamma} \sup_{\mu_j \vee \bar{\mu} \leqslant \sigma \leqslant \mu_k \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_k^3 \mathscr{L} \psi_\sigma\|. && [=: A_j^{(3)}] \end{aligned}$$

We estimate each of them separately. For $A_j^{(1)}$ we use that $\mu_j \vee \bar{\mu} \geqslant \mu_k \vee \bar{\mu}$ when $k \leqslant j$:

$$A_j^{(1)} \lesssim \sup_{k \leqslant j} 2^{-3j\gamma} \sup_{\sigma \geqslant \mu_k \vee \bar{\mu}} [\![\sigma]\!]^{3\gamma} \|\chi_k^3 \mathscr{L} \psi_\sigma\| \lesssim \sup_{k \leqslant j} 2^{-3j\gamma} |||\mathscr{L}\psi|||_\# \lesssim 2^{-3j\gamma} |||\mathscr{L}\psi|||_\#.$$

For $A_j^{(2)}$ we can directly estimate

$$A_j^{(2)} \lesssim \sup_{k > j} 2^{-3k\gamma} |||\mathscr{L}\psi|||_\# \lesssim 2^{-3j\gamma} |||\mathscr{L}\psi|||_\#.$$

For $A_j^{(3)}$ we first observe that the contributions to the sup are non-vanishing only when $\mu_k > \bar{\mu}$, so

$$A_j^{(3)} = \sup_{k | k > j, \mu_k > \bar{\mu}} 2^{-3k\gamma} \sup_{\mu_j \vee \bar{\mu} \leqslant \sigma \leqslant \mu_k} [\![\sigma]\!]^{3\gamma} \|\chi_k^3 \mathscr{L} \psi_\sigma\|.$$

Using that $\mathcal{J}_{\mu_{k+1}} \mathcal{J}_\sigma = \mathcal{J}_\sigma$ for $\sigma < \mu_k$ we can write write $\phi_\sigma^\varepsilon = \mathcal{J}_\sigma \phi_{\mu_{k+1}}^\varepsilon$ and therefore

$$A_j^{(3)} \lesssim \sup_{k | k > j, \mu_k > \bar{\mu}} 2^{-3k\gamma} \sup_{\mu_j \vee \bar{\mu} \leqslant \sigma \leqslant \mu_k} [\![\sigma]\!]^{3\gamma} \|\chi_k^3 \mathcal{J}_\sigma \mathscr{L} \psi_{\mu_{k+1}}\|.$$

The estimation of the r.h.s. in this last expression requires some work. By using the fact that $\chi_k \lesssim \zeta_k$ and that ζ_k is a nice weight for which $\mathcal{J}_\sigma$ is a contraction, see Remarks 5 and 9, we have

$$\|\chi_k^3 \mathcal{J}_\sigma \mathscr{L} \psi_{\mu_{k+1}}\| \lesssim \|\zeta_k^3 \mathcal{J}_\sigma \mathscr{L} \psi_{\mu_{k+1}}\| \lesssim \|\zeta_k^3 \mathscr{L} \psi_{\mu_{k+1}}\| \lesssim \sup_m \|\zeta_k^3 \chi_m^3 \mathscr{L} \psi_{\mu_{k+1}}\|,$$

where in the last inequality we again introduced a partition of unity. Now we split the sup in two parts, when $m \leqslant k$ and when $m > k$ and we observe that in this second case $\zeta_k^3 \chi_m^3 \lesssim e^{-c 2^{m\omega}} \chi_m^3$ for some constant $c > 0$, then we have

$$\|\chi_k^3 \mathscr{L} \psi_{\mu_{k+1}}\| \lesssim \sup_{m \leqslant k} \|\chi_m^3 \mathscr{L} \psi_{\mu_{k+1}}\| + \sup_{m > k} e^{-c 2^{m\omega}} \|\chi_m^3 \mathscr{L} \psi_{\mu_{k+1}}\|$$

$$\lesssim \sup_{m \leqslant k} \|\chi_m^3 \mathscr{L} \psi_{\mu_{k+1}}\| + \sup_{m > k} e^{-c 2^{m\omega}} [\![\mu_{k+1}]\!]^{-3\gamma} \sup_{\mu_k \vee \bar{\mu} \leqslant \sigma \leqslant \mu_m} [\![\sigma]\!]^{3\gamma} \|\chi_m^3 \mathscr{L} \psi_\sigma\|.$$

At this point the quantity

$$H_{k,m} := \sup_{\mu_k \vee \bar{\mu} \leqslant \sigma \leqslant \mu_m} [\![\sigma]\!]^{3\gamma} \|\chi_m^3 \mathscr{L} \psi_\sigma\|,$$

has the same form as the contribution in the estimate for $A_j^{(3)}$, and we can write

$$\begin{aligned}A_j^{(3)} \lesssim Q_j &:= \sup_{k|k>j,\mu_k>\bar{\mu}} 2^{-3k\gamma} H_{j,k} \lesssim [\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} \left[\sup_{k|k>j,\mu_k>\bar{\mu}} 2^{-3k\gamma} [\![\mu_{k+1}]\!]^{-3\gamma}\right] |||\mathscr{L}\psi|||_{\#} \\ &+ [\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} \sup_{k|k>j,\mu_k>\bar{\mu}} 2^{-3k\gamma} [\![\mu_{k+1}]\!]^{-3\gamma} \sup_{m>k} \underbrace{e^{-c2^{m\omega}} 2^{\vartheta m}}_{\lesssim e^{-c2^{k\omega}/2}} 2^{-\vartheta m} H_{k,m},\end{aligned}$$

from which, since $\sup_{k>j} 2^{-3k\gamma} k [\![\mu_{k+1}]\!]^{-3\gamma} < +\infty$, we obtain

$$\begin{aligned}Q_j &\lesssim [\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} |||\mathscr{L}\phi^{\varepsilon}|||_{\#} + [\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} \sup_{k|k>j,\mu_k>\bar{\mu}} 2^{-3k\gamma} [\![\mu_{k+1}]\!]^{-3\gamma} e^{-c2^{k\omega}/2} Q_k \\ &\lesssim [\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} |||\mathscr{L}\phi^{\varepsilon}|||_{\#} + [\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} \sup_k Q_k,\end{aligned}$$

and in particular

$$\sup_j Q_j \lesssim [\![\bar{\mu}]\!]^{3\gamma} \left[|||\mathscr{L}\phi^{\varepsilon}|||_{\#} + \sup_k Q_k\right].$$

Therefore, provided $\bar{\mu}$ is large enough, $\sup_j Q_j \lesssim [\![\bar{\mu}]\!]^{3\gamma} |||\mathscr{L}\psi|||_{\#}$, and

$$A_j^{(3)} \lesssim [\![\mu_j \vee \bar{\mu}]\!]^{3\gamma} |||\mathscr{L}\psi|||_{\#} \lesssim 2^{-3\gamma j} |||\mathscr{L}\psi|||_{\#}.$$

□

Lemma 40. *Provided $\sigma \geqslant \mu_j \vee \bar{\mu}$ and $\bar{\mu}$ is large enough, we have, for any $\eta \in (0,1)$,*

$$\|\rho_j^3 \mathcal{J}_{>\eta} \varphi_\sigma\| \lesssim [\![\mu_j]\!]^{3\gamma} [\![\eta]\!]^{2s} [\![\sigma]\!]^{-3\gamma} |||\mathscr{L}\varphi|||_{\#}.$$

Proof. First of all we observe that we can assume $\eta \in (0,1)$ to be such that $\eta [\![\eta]\!]^{-1} < 2\sigma [\![\sigma]\!]^{-1}$ otherwise there is nothing to prove since in the opposite case $\mathcal{J}_\sigma \mathcal{J}_{>\eta} = 0$.

We claim that

$$\|\rho_j^3 G \mathcal{J}_{>\eta} \mathscr{L} \varphi_\sigma\| \lesssim [\![\eta]\!]^{2s} \|\rho_j^3 \mathscr{L} \varphi_\sigma\|, \tag{102}$$

and we observe that this implies

$$\begin{aligned}\|\rho_j^3 \mathcal{J}_{>\eta} \varphi_\sigma\| &= \|\rho_j^3 G \mathcal{J}_{>\eta} \mathscr{L} \varphi_\sigma\| \lesssim [\![\eta]\!]^{2s} \|\rho_j^3 \mathscr{L} \mathcal{J}_{>\eta} \varphi_\sigma\| \\ &\lesssim [\![\eta]\!]^{2s} [\![\sigma]\!]^{-3\gamma} \sup_{\sigma' \geqslant \sigma} [\![\sigma']\!]^{3\gamma} \|\rho_j^3 \mathscr{L} \varphi_{\sigma'}\| \\ &\lesssim [\![\eta]\!]^{2s} [\![\sigma]\!]^{-3\gamma} \sup_{\sigma' \geqslant \mu_j \vee \bar{\mu}} [\![\sigma']\!]^{3\gamma} \|\rho_j^3 \mathscr{L} \varphi_{\sigma'}\|,\end{aligned}$$

and we use the Lemma 39 to have,

$$\sup_{\sigma' \geqslant \mu_j \vee \bar{\mu}} [\![\sigma']\!]^{3\gamma} \|\rho_j^3 \mathscr{L} \varphi_{\sigma'}\| \lesssim [\![\mu_j]\!]^{3\gamma} |||\mathscr{L}\varphi|||_{\#},$$

which concludes the proof. To prove the claim, we first observe that

$$\rho_j^3 G \mathcal{J}_{>\eta} \mathscr{L} \varphi_\sigma = \rho_j^3 G \mathcal{J}_{>\eta} \mathcal{J}_{\sigma,2} \mathscr{L} \varphi_\sigma,$$

and using $\rho_j(z) \lesssim \rho^{-1}(z - z') \rho_j(z')$, we have

$$\|\rho_j^3 G \mathcal{J}_{>\eta} \mathscr{L} \varphi_\sigma\| \lesssim \|\rho^{-3} G \mathcal{J}_{>\eta} \mathcal{J}_{\sigma,2}\|_{\mathscr{L}(L^\infty, L^\infty)} \|\rho_j^3 \mathscr{L} \varphi_\sigma\|,$$

and thus we reduced ourselves to proving that

$$\|\rho^{-3} G \mathcal{J}_{>\eta} \mathcal{J}_{\sigma,2}\|_{\mathscr{L}(L^\infty, L^\infty)} \lesssim [\![\eta]\!]^{2s}.$$

Observe that

$$G\mathfrak{J}_{>\eta}\mathfrak{J}_{\sigma,2}=G\mathfrak{J}_{\sigma,2}(1-\mathfrak{J}_{\eta})=G(\mathfrak{J}_{\sigma,2}-\mathfrak{J}_{\eta})=G(\mathfrak{J}_{\mu}-\mathfrak{J}_{\eta})=\int_{\eta}^{\mu}G\dot{\mathfrak{J}}_{\eta'}\mathrm{d}\eta'=\int_{\eta}^{\mu}\dot{G}_{\eta'}\mathrm{d}\eta',$$

where we introduced the scale $\mu=\frac{4\sigma}{1+3\sigma}$ for which $\mathfrak{J}_{\sigma,2}=\mathfrak{J}_{\mu}$. As a consequence, via Lemma 34 and recalling that ρ^{-3} grows polynomially,

$$\|\rho^{-3}\,G\mathfrak{J}_{>\eta}\mathfrak{J}_{\sigma,2}\|_{\mathscr{L}(L^{\infty},L^{\infty})}\leqslant\int_{\eta}^{\mu}\|\rho^{-3}\,\dot{G}_{\eta'}\|_{\mathscr{L}(L^{\infty},L^{\infty})}\mathrm{d}\eta'\lesssim\int_{\eta}^{\mu}[\![\eta']\!]^{2s-1}\mathrm{d}\eta'\lesssim[\![\eta']\!]^{2s}.$$

□

Lemma 41. *For $\sigma\geqslant\mu_i\vee\bar{\mu}$,*

$$\|D(\rho_i^3\,\varphi_\sigma)\|\lesssim[\![\mu_i]\!]^{3\gamma}\,[\![\sigma]\!]^{-2\gamma}\,(|\!|\!|\varphi|\!|\!|+|\!|\!|\mathscr{L}\varphi|\!|\!|_{\#}).$$

Proof. First of all, we recall that

$$D(\rho_i^3\,\varphi_\sigma)(z)^2=\int\nu_s(z,\mathrm{d}z')\,[(\rho_i^3\,\varphi_\sigma)(z)-(\rho_i^3\,\varphi_\sigma)(z')]^2. \tag{103}$$

Write for any $\eta\in(0,1)$,

$$\|D(\rho_i^3\,\varphi_\sigma)\|\leqslant\|D(\rho_i^3\,\tilde{\mathfrak{J}}_{\eta,1}\varphi_\sigma)\|+\|D(\rho_i^3\,\tilde{\mathfrak{J}}_{>\eta,1}\varphi_\sigma)\|\leqslant\|D(\rho_i^3\,\tilde{\mathfrak{J}}_{\eta,1}\varphi_\sigma)\|+\int_{\eta}^{\sigma}\|D(\rho_i^3\,\dot{\tilde{\mathfrak{J}}}_{\eta',1}\varphi_\sigma)\|\mathrm{d}\eta'.$$

We claim that

$$\|D(\rho_i^3\,\dot{\tilde{\mathfrak{J}}}_{\eta',1}\varphi_\sigma)\|=\|D(\rho_i^3\,\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)\|\lesssim[\![\eta']\!]^{-1-s}\|\rho_i^3\,\mathfrak{J}_{>\eta'}\varphi_\sigma\|, \tag{104}$$

and that

$$\|D(\rho_i^3\,\tilde{\mathfrak{J}}_{\eta,1}\varphi_\sigma)\|\lesssim[\![\eta']\!]^{-s}\|\rho_i^3\,\varphi_\sigma\|. \tag{105}$$

This implies that

$$\|D(\rho_i^3\,\varphi_\sigma)\|\lesssim[\![\eta']\!]^{-s}\|\rho_i^3\,\varphi_\sigma\|+\int_{\eta}^{\sigma}[\![\eta']\!]^{-1-s}\|\rho_i^3\,\mathfrak{J}_{>\eta'}\varphi_\sigma\|\mathrm{d}\eta',$$

and we use Lemmas 39 and 40 to get

$$\begin{aligned}\|D(\rho_i^3\,\varphi_\sigma)\| &\lesssim [\![\eta]\!]^{-s}\|\rho_i^3\,\varphi_\eta\|+\int_{\eta}^{\sigma}[\![\eta']\!]^{s-1}\mathrm{d}\eta'\,[\![\mu_i]\!]^{3\gamma}\,[\![\sigma]\!]^{-3\gamma}\,|\!|\!|\mathscr{L}\varphi|\!|\!|_{\#}\\ &\lesssim [\![\mu_i]\!]^{3\gamma}\,[\![\eta]\!]^{-s}\,[\![\sigma]\!]^{-\gamma}\,|\!|\!|\varphi|\!|\!|+[\![\mu_i]\!]^{3\gamma}\,[\![\eta]\!]^{s}\,[\![\sigma]\!]^{-3\gamma}\,|\!|\!|\mathscr{L}\varphi|\!|\!|_{\#},\end{aligned} \tag{106}$$

and choose η so that $[\![\eta]\!]=[\![\sigma]\!]^{\gamma/s}$, recalling that $\gamma<s$, to conclude

$$\|D(\rho_i^3\,\varphi_\sigma)\|\lesssim[\![\mu_i]\!]^{3\gamma}\,[\![\sigma]\!]^{-2\gamma}\,(|\!|\!|\varphi|\!|\!|+|\!|\!|\mathscr{L}\varphi|\!|\!|_{\#}).$$

To prove the claim (104), we first observe that, by manipulating Equation (103),

$$\|D(\rho_i^3\,\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)\|\leqslant\|\rho_i^3\,D\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)\|+\sup_z\Big(\int\nu_s(z,\mathrm{d}z')\,(\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)^2(z')\,[\rho_i^3(z)-\rho_i^3(z')]^2\Big)^{1/2}. \tag{107}$$

For the second term, we split the integral over a ball $B(z,r)$ for a small $r>0$ and on its complementary set B_c,

$$\begin{aligned}\int\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)^2(z')[\rho_i^3(z)-\rho_i^3(z')]^2 &\lesssim \int_B\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)^2(z')[\rho_i^3(z)-\rho_i^3(z')]^2\\ &\quad+\int_{B_c}\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)^2(z')[\rho_i^3(z)-\rho_i^3(z')]^2.\end{aligned}$$

Outside the ball we have

$$\begin{aligned}\int_{B_c}\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)^2(z')\,[\rho_i^3(z)-\rho_i^3(z')]^2 &\lesssim \int_{B_c}\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)^2(z')\,\rho_i^6(z) &&:=(\mathbb{I})\\ &\quad+\int_{B_c}\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathfrak{J}}}_{\eta',1}\mathfrak{J}_{>\eta'}\varphi_\sigma)^2(z')\,\rho_i^6(z')\Big) &&:=(\mathbb{II})\end{aligned}$$

and we observe that

$$(\mathrm{III})\lesssim\|\rho_i^3\,\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|^2\int_{B_c}\nu_s(z,\mathrm{d}z')\lesssim\|\rho_i^3\,\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|^2,$$

as well as

$$\begin{aligned}(\mathrm{I})\lesssim\int_{B_c}\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma)^2(z')\,\rho_i^6(z)\lesssim\int_{B_c}\nu_s(z,\mathrm{d}z')\,\rho_i^6(z')\,(\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma)^2(z')\,\rho^{-6}(z-z')\lesssim\\ \lesssim\|\rho_i^3\,\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|^2\int_{B_c}\nu_s(z,\mathrm{d}z')\rho^{-6}(z-z')\lesssim\|\rho_i^3\,\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|^2,\end{aligned}$$

where the last bound holds true provided the weight ρ^{-6} is not growing too fast, namely $6\nu<2s$. Concerning the contribution over the ball, we first observe that, for $r>0$ small,

$$|\rho_i^3(z)-\rho_i^3(z')|\lesssim\rho_i^3(z')\,|\rho_i^3(z)\,\rho_i^{-3}(z')-1|\lesssim\rho_i^3(z')\,|\rho^{-3}(z-z')-1|\lesssim\rho_i^3(z')\,|z-z'|_s^2.$$

This implies that

$$\begin{aligned}\int_B\nu_s(z,\mathrm{d}z')\,(\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma)^2(z')\,[\rho_i^3(z)-\rho_i^3(z')]^2\lesssim\int_B\nu_s(z,\mathrm{d}z')\,(\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma)^2(z')\,\rho_i^6(z')\,|z-z'|_s^4\lesssim\\ \lesssim\|\rho_i^3\,\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|^2\int_B\nu_s(z,\mathrm{d}z')|z-z'|_s^4\lesssim\|\rho_i^3\,\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|^2.\end{aligned}$$

As a consequence, provided $6\nu<2s$,

$$\sup_z\left(\int\nu_s(z,\mathrm{d}z')(\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma)^2(z')[\rho_i^3(z)-\rho_i^3(z')]^2\right)^{1/2}\lesssim\|\rho_i^3\,\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|\lesssim[\![\eta']\!]^{-1}\|\rho_i^3\mathcal{J}_{>\eta'}\varphi_\sigma\|. \tag{108}$$

Considering now the first term in Equation (107), we first note that

$$\|\rho_i^3\,D\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|\lesssim\|\rho_i^3\mathcal{J}_{>\eta'}\varphi_\sigma\|\sup_z\left(\int\nu_s(z,\mathrm{d}z')\rho^{-3}(z-z')\left[\int\mathrm{d}\tilde{z}|\partial_{\eta'}\check{j}_{\eta',1}(z-\tilde{z})-\partial_{\eta'}\check{j}_{\eta',1}(z'-\tilde{z})|\right]^2\right)^{1/2}, \tag{109}$$

and thus we can restrict to the analysis of the integral in the r.h.s. Using (95) we have

$$\int\nu_s(z,\mathrm{d}z')\rho^{-3}(z-z')\left[\int\mathrm{d}\tilde{z}|\partial_{\eta'}\check{j}_{\eta',1}(z-\tilde{z})-\partial_{\eta'}\check{j}_{\eta',1}(z'-\tilde{z})|\right]^2\lesssim(2\eta'[\![\eta']\!]^{-1})^2\int\frac{\rho^{-3}(z')}{|z'|^{d+2s-2}}((2\eta'[\![\eta']\!]^{-1}|z'|)\wedge1)^2.$$

Now considering the ball $B=B(z,a)$ of radius $a=2^{-1}\eta'^{-1}[\![\eta']\!]$ we see that, provided $3\nu<2s$,

$$\begin{aligned}\int\frac{\rho^{-3}(z')}{|z'|^{d+2s-2}}((2\eta'[\![\eta']\!]^{-1}|z'|)\wedge1)^2\leqslant\int_B\frac{\rho^{-3}(z')}{|z'|^{d+2s-2}}(2\eta'[\![\eta']\!]^{-1}|z'|)^2+\int_{B^c}\frac{\rho^{-3}(z')}{|z'|^{d+2s-2}}\\ \leqslant(2\eta'[\![\eta']\!]^{-1})^2(2^{-1}\eta'^{-1}[\![\eta']\!])^{2-2s}+(2^{-1}\eta'^{-1}[\![\eta']\!])^{3\nu-2s}\lesssim[\![\eta']\!]^{-2s}.\end{aligned}$$

It follows that, provided $3\nu<2s$,

$$\left(\int\nu_s(z,\mathrm{d}z')\rho^{-3}(z-z')\left[\int\mathrm{d}\tilde{z}|\partial_{\eta'}\check{j}_{\eta',1}(z-\tilde{z})-\partial_{\eta'}\check{j}_{\eta',1}(z'-\tilde{z})|\right]^2\right)^{1/2}\lesssim[\![\eta']\!]^{-1-s},$$

and thus, on account of Equation (109),

$$\|\rho_i^3\,D\dot{\tilde{\mathcal{J}}}_{\eta',1}\mathcal{J}_{>\eta'}\varphi_\sigma\|\lesssim[\![\eta']\!]^{-1-s}\|\rho_i^3\mathcal{J}_{>\eta'}\varphi_\sigma\|.$$

This, together with Equations (107) and (108), yields Equation (104). Equation (105) can be proven analogously. □

Lemma 42. *For $\sigma\geqslant\bar{\mu}\vee\mu_i$, it holds that*

$$\|\chi_i^3\,(1-\mathcal{J}_\sigma)(\psi_\sigma^3)\|\lesssim[\![\sigma]\!]^{-3\gamma+\zeta}\,|||\psi|||^2\,|||\mathscr{L}\psi|||_\#.$$

for some $\zeta := 2s - 2\gamma > 0$.

Proof. Write

$$(1-\mathcal{J}_\sigma)(\psi_\sigma^3) = (1-\mathcal{J}_\sigma)\big[(\mathcal{J}_\eta\psi_\sigma)^3 + (\mathcal{J}_{>\eta}\psi_\sigma)^3 + 3(\mathcal{J}_\eta\psi_\sigma)^2(\mathcal{J}_{>\eta}\psi_\sigma) + 3(\mathcal{J}_\eta\psi_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^2\big]$$

and observe that, provided we choose $\eta = \frac{\sigma}{6-5\sigma}$, which implies $6\eta(1-\eta)^{-1} = \sigma(1-\sigma)^{-1}$, the first contribution $(1-\mathcal{J}_\sigma)(\mathcal{J}_\eta\psi_\sigma)^3$ is vanishing by the Fourier space support properties of the product. As for the other contributions we have for example

$$\|\chi_i^3(1-\mathcal{J}_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^3\| \lesssim [\![\mu_i]\!]^{-5\gamma}\|\rho_i^5(1-\mathcal{J}_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^3\| \lesssim [\![\mu_i]\!]^{-5\gamma}\|\rho_i^5(\mathcal{J}_{>\eta}\psi_\sigma)^3\| \lesssim [\![\mu_i]\!]^{-5\gamma}\|\rho_i\psi_\sigma\|^2\|\rho_i^3(\mathcal{J}_{>\eta}\psi_\sigma)\|.$$

Now by Lemma 39,

$$\|\rho_i\psi_\sigma\| \lesssim [\![\mu_i]\!]^{\gamma}[\![\sigma]\!]^{-\gamma}|\!|\!|\psi|\!|\!|,$$

while by Lemma 40

$$\|\rho_i^3(\mathcal{J}_{>\eta}\psi_\sigma)\| \lesssim [\![\mu_i]\!]^{3\gamma}[\![\eta]\!]^{2s}[\![\sigma]\!]^{-3\gamma}|\!|\!|\mathscr{L}\psi|\!|\!|_{\#},$$

and we have that (since $[\![\eta]\!] \approx [\![\sigma]\!]$)

$$\|\chi_i^3(1-\mathcal{J}_\sigma)(\mathcal{J}_{>\eta}\psi_\sigma)^3\| \lesssim [\![\eta]\!]^{2s}[\![\sigma]\!]^{-5\gamma}|\!|\!|\psi|\!|\!|^2|\!|\!|\mathscr{L}\psi|\!|\!|_{\#} \lesssim [\![\sigma]\!]^{2s-5\gamma}|\!|\!|\psi|\!|\!|^2|\!|\!|\mathscr{L}\psi|\!|\!|_{\#}.$$

Similar estimates hold for the other terms, therefore

$$\|\chi_i^3(1-\mathcal{J}_\sigma)(\psi_\sigma^3)\| \lesssim [\![\sigma]\!]^{2s-5\gamma}|\!|\!|\psi|\!|\!|^2|\!|\!|\mathscr{L}\psi|\!|\!|_{\#},$$

and we can set $\zeta := 2s - 2\gamma > 0$ to prove our claim. □

Appendix B Details on the flow equations

B.1 Flow equations for cumulants

First of all we observe that, on account of the flow equation (47) as well as of the properties of joint cumulants, it holds that, for $\boldsymbol{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in \boldsymbol{A}$ and with the notation of Equations (58) and (46),

$$\sum_{\boldsymbol{b}} \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}) = \sum_{i=1}^{n(\boldsymbol{a})}\sum_{\boldsymbol{b}} \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}), \qquad \sum_{\boldsymbol{b},\boldsymbol{c}} \mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}}) = \sum_{i=1}^{n(\boldsymbol{a})}\sum_{\boldsymbol{b},\boldsymbol{c}} \mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}}),$$

where

$$\sum_{\boldsymbol{b}} \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}) := \sum_{\ell'=0}^{\ell(\mathfrak{a}_i)-1}\sum_{k'=0}^{k(\mathfrak{a}_i)} (k'+1)\kappa_{n(\boldsymbol{a})+1}\big(F_\sigma^{\mathfrak{a}_1}, \ldots, F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(k'+1)}, \mathscr{J}\dot{G}_\sigma F_\sigma^{[\ell'],(k(\mathfrak{a}_i)-k')}, \ldots, F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}\big), \tag{110}$$

where the range of the sum over $\boldsymbol{b}$ is understood to be given by the constraints on (ℓ', k') on the right hand side of the above equation. Concerning the other term,

$$\begin{aligned}\sum_{\boldsymbol{b},\boldsymbol{c}} \mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}, \mathscr{F}_\sigma^{\boldsymbol{c}}) \;:=\; &\sum_{I_1,I_2}\sum_{\ell'=0}^{\ell(\mathfrak{a}_i)-1}\sum_{k'=0}^{k(\mathfrak{a}_i)} (k'+1)\kappa_{|I_1|+1}\big(F_\sigma^{\mathfrak{a}_1}, \ldots, F_\sigma^{\mathfrak{a}_{i-1}}, F_\sigma^{(\ell(\mathfrak{a}_i)-1-\ell', k'+1)}\big) \times \\ &\times \mathscr{J}\dot{G}_\sigma\kappa_{|I_2|+1}\big(F_\sigma^{(\ell', k(\mathfrak{a}_i)-k')}, F_\sigma^{\mathfrak{a}_{i+1}}, \ldots, F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}\big),\end{aligned} \tag{111}$$

where now, for a fixed $i \in \{1, \ldots, n(\boldsymbol{a})\}$, the sum $\sum_{I_1, I_2}$ runs over all the partitions of the set $I_1 \sqcup I_2 = \{1, \ldots, i-1, i+1, \ldots, n(\boldsymbol{a})\}$ and where now the range of the sum over $\boldsymbol{b}$ and $\boldsymbol{c}$ is given by the constraints over (I_1, I_2, ℓ', k') on the right hand side.

The aim of this appendix is to prove some of the results concerning the flow equation we discussed in Section 4.

Remark 43. As a premise, we introduce some notation: we let ${}^iT_{\bar{\nu}}(\boldsymbol{a})$ and ${}^i\tilde{T}_{\bar{\nu}}(\boldsymbol{a})$ be tree weights which are derived from the definition of $T_{\bar{\nu}}(\boldsymbol{a})$ by replacing the i-th tree $T_{\bar{\nu}}^{(i)}(\mathfrak{a}_i) = \tau_{\bar{\nu}}(1 + k(\mathfrak{a}_i))2^{-\ell(\mathfrak{a}_i)}$ respectively with

$$
{}^iT_{\bar{\nu}}^{(i)}(\mathfrak{a}_i) = \tau_{\bar{\nu}}(1 + k(\mathfrak{a}_i))\, 2^{-\ell(\mathfrak{a}_i)+1}, \qquad {}^i\tilde{T}_{\bar{\nu}}^{(i)}(\mathfrak{a}_i) = \tau_{\bar{\nu}}(1 + k(\mathfrak{a}_i))\, 2^{-\ell(\mathfrak{a}_i)+1/2}.
$$

Lemma 44. *It holds that, for $\sigma \geqslant \mu$, $\boldsymbol{a} \in \boldsymbol{A}$ and $i \leqslant n(\boldsymbol{a})$,*

$$
[\![\sigma]\!]^{-[\boldsymbol{a}]} \sup_{\nu | \mu \leqslant \nu \leqslant \sigma} \| (\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}} \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}))\, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})} \| \lesssim_{\boldsymbol{a}} [\![\sigma]\!]^{-[\boldsymbol{b}]-1} \| \mathscr{F}_\sigma^{\boldsymbol{b}} \|_{\mu,\sigma},
$$

and, in particular,

$$
[\![\sigma]\!]^{-[\boldsymbol{a}]} \| \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}) \|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} \| \mathscr{F}_\sigma^{\boldsymbol{b}} \|_{\mu,\sigma}.
$$

Proof. We start from the first bound. We have that

$$
\| (\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}} \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}))\, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})} \| = \| (\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}} \mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma, \mathscr{F}_\sigma^{\boldsymbol{b}}))\, e^{T_{\bar{\nu}}(\boldsymbol{b}) + \tau_{\bar{\nu}}(2)}\, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a}) - T_{\bar{\nu}}(\boldsymbol{b}) - \tau_{\bar{\nu}}(2)} \|,
$$

where $\tau_{\bar{\nu}}(2)$ depends on the variables on which $\dot{G}_\sigma$ depends. In addition,

$$
\begin{aligned}
{}^iT_{\bar{\nu}}(\boldsymbol{a}) - T_{\bar{\nu}}(\boldsymbol{b}) - \tau_{\bar{\nu}}(2) &= \sum_{m=1}^{n(\boldsymbol{a})} {}^iT_{\bar{\nu}}(\mathfrak{a}_m) + 2^{-(\bar{\ell}+1)}\tau_{\bar{\nu}}(n(\boldsymbol{a})) - \sum_{j=1}^{n(\boldsymbol{b})} T_{\bar{\nu}}^{(j)}(\mathfrak{b}_j) - 2^{-(\bar{\ell}+1)}\tau_{\bar{\nu}}(n(\boldsymbol{b})) - \tau_{\bar{\nu}}(2) \\
&\leqslant \sum_{m=1}^{n(\boldsymbol{a})} {}^iT_{\bar{\nu}}(\mathfrak{a}_m) - \sum_{j=1}^{n(\boldsymbol{b})} T_{\bar{\nu}}^{(j)}(\mathfrak{b}_j) - \tau_{\bar{\nu}}(2) \\
&= {}^iT_{\bar{\nu}}^{(i)}(\mathfrak{a}_i) - T_{\bar{\nu}}^{(i)}(\mathfrak{b}_i) - T_{\bar{\nu}}^{(i+1)}(\mathfrak{b}_{i+1}) - \tau_{\bar{\nu}}(2) \\
&= \tau_{\bar{\nu}}(1 + k(\mathfrak{a}_i))2^{-\ell(\mathfrak{a}_i)+1} - \tau_{\bar{\nu}}(1 + k(\mathfrak{b}_i))2^{-\ell(\mathfrak{b}_i)} - \tau_{\bar{\nu}}(1 + k(\mathfrak{b}_{i+1}))2^{-\ell(\mathfrak{b}_{i+1})} - \tau_{\bar{\nu}}(2),
\end{aligned}
$$

where in the second step we exploited $n(\boldsymbol{a}) < n(\boldsymbol{b})$ as well as the fact that the points appearing in the tree associated with $n(\boldsymbol{a})$ are contained also in the one associated with $n(\boldsymbol{b})$, while in the third step we used that, due to Equation (110), it holds that

$$
\sum_{m=1}^{n(\boldsymbol{a})} {}^iT_{\bar{\nu}}(\mathfrak{a}_m) - \sum_{j=1}^{n(\boldsymbol{b})} T_{\bar{\nu}}^{(j)}(\mathfrak{b}_j) = {}^iT_{\bar{\nu}}^{(i)}(\mathfrak{a}_i) - T_{\bar{\nu}}^{(i)}(\mathfrak{b}_i) - T_{\bar{\nu}}^{(i+1)}(\mathfrak{b}_{i+1}).
$$

Again from Equation (110), we see that

$$
\ell(\mathfrak{b}_i) = \ell(\mathfrak{a}_i) - 1 - \ell', \qquad \ell(\mathfrak{b}_{i+1}) = \ell',
$$

for some $0 \leqslant \ell' \leqslant \ell(\mathfrak{a}_i) - 1$, which implies that $2^{-\ell(\mathfrak{a}_i)+1} \leqslant 2^{-\ell(\mathfrak{b}_i)}$ and $2^{-\ell(\mathfrak{a}_i)+1} \leqslant 2^{-\ell(\mathfrak{b}_{i+1})}$. Furthermore, we observe that, from the definition of Steiner diameter it follows that, by triangular inequality,

$$
\tau_{\bar{\nu}}(1 + k(\mathfrak{a}_i)) \leqslant \tau_{\bar{\nu}}(1 + k(\mathfrak{b}_i)) + \tau_{\bar{\nu}}(1 + k(\mathfrak{b}_{i+1})) + \tau_{\bar{\nu}}(2),
$$

and thus, overall

$$
{}^iT_{\bar{\nu}}(\boldsymbol{a}) - T_{\bar{\nu}}(\boldsymbol{b}) - \tau_{\bar{\nu}}(2) \leqslant 0.
$$

This implies that

$$\begin{aligned}\sup_{\nu|\mu\leq\nu\leq\sigma}\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}\mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))\,e^{{}^iT_{\bar\nu}(\boldsymbol{a})}\| &\lesssim \sup_{\nu|\mu\leq\nu\leq\sigma}\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}\mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a}}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))\,e^{T_{\bar\nu}(\boldsymbol{b})+\tau_{\bar\nu}(2)}\|\\ &\lesssim \sup_A\|\partial^A\dot G_\sigma e^{\tau_{\bar\nu}(2)}\|_{\mathscr{L}(L^1,L^\infty)}\sup_{\nu|\mu\leq\nu\leq\sigma}\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}\mathscr{F}_\sigma^{\boldsymbol{b}})\,e^{T_{\bar\nu}(\boldsymbol{b})}\|\\ &= \sup_A\|\partial^A\dot G_\sigma e^{\tau_\sigma(2)}\|_{\mathscr{L}(L^1,L^\infty)}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma},\end{aligned}$$

where the sup is over A such that $|A(\boldsymbol{b})|=|A(\boldsymbol{a})|+|A|$. In the second inequality above, resorting to Equation (110), in order to have the norm $\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}$ on the right hand side, we need two more smoothing operators, with respect to $\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}$, as it holds that $n(\boldsymbol{b})+K(\boldsymbol{b})=n(\boldsymbol{a})+K(\boldsymbol{a})+2$. In particular we observe that these two smoothing operators are missing on the input and output variables of $\dot G_\sigma$. To this end, we can exploit the relation $\dot G_\sigma\tilde{\mathfrak{J}}_{\sigma,\ell}=\dot G_\sigma$, which is true for any ℓ as consequence of Equation (19). This justifies the above inequality. Recalling that, since $\sigma\geq\bar\nu$, $\|\partial^A\dot G_\sigma e^{\tau_{\bar\nu}(2)}\|_{\mathscr{L}(L^1,L^\infty)}\lesssim [\![\sigma]\!]^{-|A|-1-d}$, we get the thesis by observing that

$$-[\boldsymbol{a}]-1-d-|A|=-[\boldsymbol{b}]-1+\theta+\beta-\delta-d=-[\boldsymbol{b}]-1,$$

where in the last equality we used the compatibility condition $\theta+\beta-\delta-d=0$. For what concerns the last bound, it follows from the first bound together with the inequality $e^{T_{\bar\nu}(\boldsymbol{a})}\leq e^{{}^iT_{\bar\nu}(\boldsymbol{a})}$. □

Lemma 45. *It holds that, for $\sigma\geq\mu$, $\boldsymbol{a}\in\boldsymbol{A}$ and $i\leq n(\boldsymbol{a})$,*

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\sup_{\nu|\mu\leq\nu\leq\sigma}\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))\,e^{{}^i\tilde T_{\bar\nu}(\boldsymbol{a})}\|\lesssim_{\boldsymbol{a}}[\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma},$$

and, in particular,

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\|\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}})\|_{\mu,\sigma}\lesssim[\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma}.$$

Proof. Also in this case, we start from the first bound. We have that

$$\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))\,e^{{}^i\tilde T_{\bar\nu}(\boldsymbol{a})}\|=\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))\,e^{T_{\bar\nu}(\boldsymbol{b})+T_{\bar\nu}(\boldsymbol{c})+2\tau_{\bar\nu}(2)}\,e^{{}^i\tilde T_{\bar\nu}(\boldsymbol{a})-T_{\bar\nu}(\boldsymbol{b})-T_{\bar\nu}(\boldsymbol{c})-2\tau_{\bar\nu}(2)}\|,\tag{112}$$

where $\tau_{\bar\nu}(2)$ depends on the variables on which $\dot G_\sigma$ depends. In addition

$$\begin{aligned}{}^i\tilde T_{\bar\nu}(\boldsymbol{a})-T_{\bar\nu}(\boldsymbol{b})-T_{\bar\nu}(\boldsymbol{c})-2\tau_{\bar\nu}(2) &= \sum_{m=1}^{n(\boldsymbol{a})}{}^i\tilde T_{\bar\nu}(\mathfrak{a}_m)+2^{-(\bar\ell+1)}\tau_{\bar\nu}(n(\boldsymbol{a}))-\sum_{j=1}^{n(\boldsymbol{b})}T_{\bar\nu}^{(j)}(\mathfrak{b}_j)-2^{-(\bar\ell+1)}\tau_{\bar\nu}(n(\boldsymbol{b}))+\\ &\quad-\sum_{u=1}^{n(\boldsymbol{c})}T_{\bar\nu}^{(u)}(\mathfrak{c}_u)-2^{-(\bar\ell+1)}\tau_{\bar\nu}(n(\boldsymbol{c}))-2\tau_{\bar\nu}(2)\\ &= \sum_{m=1}^{n(\boldsymbol{a})}{}^i\tilde T_{\bar\nu}(\mathfrak{a}_m)-\sum_{j=1}^{n(\boldsymbol{b})}T_{\bar\nu}^{(j)}(\mathfrak{b}_j)-\sum_{u=1}^{n(\boldsymbol{c})}T_{\bar\nu}^{(u)}(\mathfrak{c}_u)-2\tau_{\bar\nu}(2)+2^{-(\bar\ell+1)}\tau_{\bar\nu}^{\mathrm{out}}(2)+\\ &\quad+2^{-(\bar\ell+1)}(\tau_{\bar\nu}(n(\boldsymbol{a}))-\tau_{\bar\nu}(n(\boldsymbol{b}))-\tau_{\bar\nu}(n(\boldsymbol{c}))-\tau_{\bar\nu}^{\mathrm{out}}(2)),\end{aligned}$$

where in the last step we added and subtracted $2^{-(\bar\ell+1)}\tau_{\bar\nu}^{\mathrm{out}}(2)$ where $\tau_{\bar\nu}^{\mathrm{out}}(2)$ is the Steiner diameter connecting the last output variable in the cumulant $\mathscr{F}_\sigma^{\boldsymbol{b}}$ with the first output variable of $\mathscr{F}_\sigma^{\boldsymbol{c}}$. First of all, by definition of Steiner diameter,

$$\tau_{\bar\nu}(n(\boldsymbol{a}))-\tau_{\bar\nu}(n(\boldsymbol{b}))-\tau_{\bar\nu}(n(\boldsymbol{c}))-\tau_{\bar\nu}^{\mathrm{out}}(2)\leq 0,$$

and thus

$$ {}^{i}\tilde{T}_{\bar{v}}(\boldsymbol{a})-T_{\bar{v}}(\boldsymbol{b})-T_{\bar{v}}(\boldsymbol{c})-2\tau_{\bar{v}}(2)\leqslant\sum_{m=1}^{n(\boldsymbol{a})}{}^{i}\tilde{T}_{\bar{v}}^{(m)}(\mathfrak{a}_m)-\sum_{j=1}^{n(\boldsymbol{b})}T_{\bar{v}}^{(j)}(\mathfrak{b}_j)-\sum_{u=1}^{n(\boldsymbol{c})}T_{\bar{v}}^{(u)}(\mathfrak{c}_u)-2\tau_{\bar{v}}(2)+2^{-(\bar{\ell}+1)}\tau_{\bar{v}}^{\mathrm{out}}(2). $$

Using an argument analogous to the one we exploited in the proof of Lemma 44, we get

$$ \begin{aligned} {}^{i}\tilde{T}_{\bar{v}}(\boldsymbol{a})-T_{\bar{v}}(\boldsymbol{b})-T_{\bar{v}}(\boldsymbol{c})-2\tau_{\bar{v}}(2) &\leqslant {}^{i}\tilde{T}_{\bar{v}}^{(i)}(\mathfrak{a}_i)-T_{\bar{v}}^{(n(\boldsymbol{b}))}(\mathfrak{b}_{n(\boldsymbol{b})})-T_{\bar{v}}^{(1)}(\mathfrak{c}_1)-2\tau_{\bar{v}}(2)+2^{-(\bar{\ell}+1)}\tau_{\bar{v}}^{\mathrm{out}}(2)\\ &= \tau_{\bar{v}}(1+k(\mathfrak{a}_i))2^{-\ell(\mathfrak{a}_i)+1/2}-\tau_{\bar{v}}(1+k(\mathfrak{b}_{n(\boldsymbol{b})}))2^{-\ell(\mathfrak{b}_{n(\boldsymbol{b})})}-\tau_{\bar{v}}(1+k(\mathfrak{c}_1))2^{-\ell(\mathfrak{c}_1)}+\\ &\qquad -2\tau_{\bar{v}}(2)+2^{-(\bar{\ell}+1)}\tau_{\bar{v}}^{\mathrm{out}}(2), \end{aligned} $$

But now we recall that

$$ \ell(\mathfrak{b}_{n(\boldsymbol{b})})=\ell(\mathfrak{a}_i)-1-\ell',\qquad \ell(\mathfrak{c}_1)=\ell', $$

for some $0\leqslant\ell'\leqslant\ell(\mathfrak{a}_i)-1$, which implies that $2^{-\ell(\mathfrak{a}_i)+1}\leqslant 2^{-\ell(\mathfrak{b}_{n(\boldsymbol{b})})}$ and $2^{-\ell(\mathfrak{a}_i)+1}\leqslant 2^{-\ell(\mathfrak{c}_1)}$. It follows that

$$ \begin{aligned} {}^{i}\tilde{T}_{\bar{v}}(\boldsymbol{a})-T_{\bar{v}}(\boldsymbol{b})-T_{\bar{v}}(\boldsymbol{c})-2\tau_{\bar{v}}(2) &\lesssim 2^{-\ell(\mathfrak{a}_i)+1/2}(\tau_{\bar{v}}(1+k(\mathfrak{a}_i))-\sqrt{2}\,\tau_{\bar{v}}(1+k(\mathfrak{b}_{n(\boldsymbol{b})}))-\sqrt{2}\,\tau_{\bar{v}}(1+k(\mathfrak{c}_1))-\tau_{\bar{v}}(2))\\ &\qquad -\tau_{\bar{v}}(2)+2^{-(\bar{\ell}+1)}\tau_{\bar{v}}^{\mathrm{out}}(2)\\ &\leqslant -2^{-\ell(\mathfrak{a}_i)+1/2}(\sqrt{2}-1)(\tau_{\bar{v}}(1+k(\mathfrak{b}_{n(\boldsymbol{b})}))+\tau_{\bar{v}}(1+k(\mathfrak{c}_1)))-\tau_{\bar{v}}(2)\\ &\qquad +2^{-(\bar{\ell}+1)}\tau_{\bar{v}}^{\mathrm{out}}(2), \end{aligned} $$

where in the last inequality we used that, by definition of Steiner diameter,

$$ \tau_{\bar{v}}(1+k(\mathfrak{a}_i))\leqslant\tau_{\bar{v}}(1+k(\mathfrak{b}_{n(\boldsymbol{b})}))+\tau_{\bar{v}}(1+k(\mathfrak{c}_1))+\tau_{\bar{v}}(2). $$

Finally, since $2^{-(\bar{\ell}+1)}\leqslant 2^{-\ell(\mathfrak{a}_i)+1/2}(\sqrt{2}-1)$ for any i by definition of $\bar{\ell}$ and since by triangular inequality

$$ -\tau_{\bar{v}}(1+k(\mathfrak{b}_{n(\boldsymbol{b})}))-\tau_{\bar{v}}(1+k(\mathfrak{c}_1))-\tau_{\bar{v}}(2)+\tau_{\bar{v}}^{\mathrm{out}}(2)\leqslant 0, $$

we conclude that

$$ {}^{i}\tilde{T}_{\bar{v}}(\boldsymbol{a})-T_{\bar{v}}(\boldsymbol{b})-T_{\bar{v}}(\boldsymbol{c})-2\tau_{\bar{v}}(2)\leqslant 0. $$

On account of Equation (112), this yields

$$ \|(\tilde{\mathfrak{J}}_{v}^{\boldsymbol{a}}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))\,e^{{}^{i}\tilde{T}_{\bar{v}}(\boldsymbol{a})}\|\lesssim\|(\tilde{\mathfrak{J}}_{v}^{\boldsymbol{a}}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))\,e^{T_{\bar{v}}(\boldsymbol{b})+T_{\bar{v}}(\boldsymbol{c})+2\tau_{\bar{v}}(2)}\|, $$

which in turn, by resorting to Equation (111) and exploiting as in the previous lemma $\dot{G}_\sigma\tilde{\mathfrak{J}}_\sigma=\dot{G}_\sigma$, gives

$$ \begin{aligned} \sup_{v|\mu\leqslant v\leqslant\sigma}\|(\tilde{\mathfrak{J}}_{v}^{\boldsymbol{a}}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))e^{{}^{i}\tilde{T}_{\bar{v}}(\boldsymbol{a})}\| &\lesssim \sup_{v|\mu\leqslant v\leqslant\sigma}\|(\tilde{\mathfrak{J}}_{v}^{\boldsymbol{a}}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))e^{T_{\bar{v}}(\boldsymbol{b})+T_{\bar{v}}(\boldsymbol{c})+2\tau_{\bar{v}}(2)}\|\\ &\lesssim \sup_{A}\|\partial^A\dot{G}_\sigma e^{2\tau_\sigma(2)}\|_{\mathscr{L}(L^\infty,L^\infty)}\sup_{v|\mu\leqslant v\leqslant\sigma}\|(\tilde{\mathfrak{J}}_{v}^{\boldsymbol{a}}\mathscr{F}_\sigma^{\boldsymbol{b}})e^{T_{\bar{v}}(\boldsymbol{b})}\|\|(\tilde{\mathfrak{J}}_{v}^{\boldsymbol{a}}\mathscr{F}_\sigma^{\boldsymbol{c}})e^{T_{\bar{v}}(\boldsymbol{c})}\|\\ &= \sup_{A}\|\partial^A\dot{G}_\sigma e^{2\tau_\sigma(2)}\|_{\mathscr{L}(L^\infty,L^\infty)}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma}, \end{aligned} $$

where the sup is over A such that $|A(\boldsymbol{b})|+|A(\boldsymbol{c})|=|A(\boldsymbol{a})|+|A|$. Recalling now that

$$ \|\partial^A\dot{G}_\sigma e^{2\tau_\sigma(2)}\|_{\mathscr{L}(L^\infty,L^\infty)}\lesssim[\![\sigma]\!]^{2s-1-|A|}, $$

we get the thesis by observing that

$$ -[\boldsymbol{a}]+2s-1-|A|=-[\boldsymbol{b}]-[\boldsymbol{c}]-\rho+\theta+\beta-\delta+2s-1=-[\boldsymbol{b}]-[\boldsymbol{c}]-1, $$

where we used the compatibility condition $-\rho+\theta+\beta-\delta+2s=0$.

Concerning the last bound, it follows from the first one and the inequality $e^{T_{\bar{v}}(\boldsymbol{a})}\leqslant e^{{}^{i}\tilde{T}_{\bar{v}}(\boldsymbol{a})}$. □

B.2 Localised flow equation

Taking into account the localised flow equation (67) and the modified flow equation for cumulants (68), we can decompose the operators $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ in contributions involving the operator $\boldsymbol{R}$ and $\boldsymbol{L}$ respectively, as

$$\tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}})=(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}+(\mathfrak{L}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}, \tag{113}$$

and

$$\tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}})=(\mathfrak{R}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}+(\mathfrak{L}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}. \tag{114}$$

Starting from the first contribution in Equation (113), we observe that this is non-vanishing only if $k(\mathfrak{a}_i)=1$ and $1<|A_1(\mathfrak{a}_i)|\leqslant 2$ and, in such cases it holds that, as a consequence of the definition of the operator $\boldsymbol{R}$ as per Equation (66), denoting with the letter Z the set of space-time variables on which $(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}$ depends, apart from z and z_1,

$$\begin{aligned}(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}(Z,z,z_1)&=(k'+1)\int_{\Lambda^2}\mathrm{d}z_2\mathrm{d}z_3\mathscr{J}\dot{G}_\sigma(z_2,z_3)\times\\ &\times\ [\mathfrak{Z}^{\{z,z_1\}}\kappa_{n(\boldsymbol{a})+1}(F_\sigma^{\mathfrak{a}_1},\ldots,F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(1)}(z;z_2),F_\sigma^{[\ell'],(1)}(z_3,z_1),\ldots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}})+\\ &+\ \mathfrak{Z}^{\{z,z_1\}}\kappa_{n(\boldsymbol{a})+1}(F_\sigma^{\mathfrak{a}_1},\ldots,F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(2)}(z;z_1,z_2),F_\sigma^{[\ell'],(0)}(z_3),\ldots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}})],\end{aligned} \tag{115}$$

where $\mathfrak{Z}^{\{z,z_1\}}$ is the extension to distributions of the map which on smooth functions acts as, for the case of two derivatives, i.e. when $A_1(\mathfrak{a}_i)=(i,j)$,

$$\begin{aligned}\mathfrak{Z}^{\{z,z_1\}}f(Z,z,z_1)&=\mathbb{1}_{i\neq 0}\int_\Lambda\mathrm{d}\tilde{z}\int_0^1[\mathrm{d}\rho_{\tilde{z}-z}(t)]^i\int_0^t[\mathrm{d}\rho_{\tilde{z}-z}(u)]^j(\tilde{\mathfrak{J}}_{\sigma,\ell}^{\otimes 2,\{z,\tilde{z}\}}f)(Z,z,\tilde{z})\delta(z_1-z-\rho_{\tilde{z}-z}(u)),\\ &-\varepsilon\sum_{m\in\{1,\ldots,d\}}\frac{\mathbb{1}_{i=m-,j=m+}+\mathbb{1}_{i=m+,j=m-}}{2}\int_\Lambda\mathrm{d}\tilde{z}\,(\tilde{z}-z)^m(\tilde{\mathfrak{J}}_{\sigma,\ell}^{\otimes 2,\{z,\tilde{z}\}}f)(Z,z,\tilde{z})\delta(z_1-z),\end{aligned} \tag{116}$$

for $z,z_1\in\Lambda$, where, with $(\tilde{\mathfrak{J}}_{\sigma,\ell}^{\otimes 2,\{z,\tilde{z}\}}f)(Z,z,\tilde{z})$ we mean that the smoothing operators are applied to the variables z and $\tilde{z}$. We observe that, with respect to the operator $\mathscr{A}$ introduced in Equations (110) the variables Z are unaffected by the operator $\mathfrak{R}$.

Remark 46. We recall that, on account of Equation (66), the operator $\boldsymbol{R}$ might also account for a first order time derivative contribution, in which case the map $\mathfrak{Z}^{\{z,z_1\}}$ is of the form

$$\mathfrak{Z}^{\{z,z_1\}}f(Z,z,z_1)=\int_\Lambda\mathrm{d}\tilde{z}\int_0^1\mathrm{d}t(\tilde{z}-z)_0(\tilde{\mathfrak{J}}_{\sigma,\ell}^{\otimes 2,\{z,\tilde{z}\}}f)(Z,z,\tilde{z})\delta(z_1-z-\rho_{\tilde{z}-z}(t)).$$

In the forthcoming analysis we shall only discuss the case of the second order spatial Taylor remainder since the first order in time scenario is easier and can handled analogously.

Similarly

$$\begin{aligned}(\mathfrak{R}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}(Z,z,z_1)&=(k'+1)\int_{\Lambda^2}\mathrm{d}z_2\mathrm{d}z_3\mathscr{J}\dot{G}_\sigma(z_2,z_3)\times\\ &\times\ [\mathfrak{Z}^{\{z,z_1\}}\kappa_{|I_1|+1}(F_\sigma^{\mathfrak{a}_1},\ldots,F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(1)}(z;z_2))\kappa_{|I_2|+1}(F_\sigma^{[\ell'],(1)}(z_3,z_1),\ldots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}})+\\ &+\ \mathfrak{Z}^{\{z,z_1\}}\kappa_{|I_1|+1}(F_\sigma^{\mathfrak{a}_1},\ldots,F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(2)}(z;z_1,z_2))\kappa_{|I_2|+1}(F_\sigma^{[\ell'],(0)}(z_3),\ldots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}})].\end{aligned} \tag{117}$$

For what concerns the operator $\mathfrak{L}$, we observe that by definition of $\boldsymbol{L}$, *cf.* Equation (66), if $k(\mathfrak{a}_i)\neq 1$, it holds that

$$(\mathfrak{L}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}=\mathscr{A}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}),\qquad(\mathfrak{L}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}=\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}), \tag{118}$$

while if $k(\mathfrak{a}_i)=1$ and $0\leq|A_1(\mathfrak{a}_i)|\leq 1$, it holds that

$$
\begin{aligned}
\big(\mathfrak{L}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}})\big)^{\boldsymbol{a}}(Z,z) &= (k'+1)\int_{\Lambda^3}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\,\mathscr{J}\dot G_\sigma(z_2,z_3)\{(z_1-z)^{A(\mathfrak{a}_i)}\times\\
&\times\ \big[\kappa_{n(\boldsymbol{a})+1}\big(F_\sigma^{\mathfrak{a}_1},\dots,\big(\tilde{\mathfrak{J}}^{\{z\}}_{\sigma,\ell(\mathfrak{a}_i)-1-\ell'}F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(1)}\big)(z;z_2),\big(\tilde{\mathfrak{J}}^{\{z_1\}}_{\sigma,\ell'}F_\sigma^{[\ell'],(1)}\big)(z_3,z_1),\dots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}\big)\\
&+\ \kappa_{n(\boldsymbol{a})+1}\big(F_\sigma^{\mathfrak{a}_1},\dots,\big(\tilde{\mathfrak{J}}^{\otimes 2,\{z,z_1\}}_{\sigma,\ell(\mathfrak{a}_i)-1-\ell'}F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(2)}\big)(z;z_1,z_2),F_\sigma^{[\ell'],(0)}(z_3),\dots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}\big)\big]\big\},
\end{aligned}
\tag{119}
$$

and similarly

$$
\begin{aligned}
\big(\mathfrak{L}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}})\big)^{\boldsymbol{a}}(Z,z) &= (k'+1)\int_{\Lambda^3}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\,\mathscr{J}\dot G_\sigma(z_2,z_3)\{(z_1-z)^{A(\mathfrak{a}_i)}\times\\
&\times\ \big[\kappa_{|I_1|+1}\big(F_\sigma^{\mathfrak{a}_1},\dots,\tilde{\mathfrak{J}}^{\{z\}}_{\sigma,\ell(\mathfrak{a}_i)-1-\ell'}F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(1)}(z;z_2)\big)\kappa_{|I_2|+1}\big(\big(\tilde{\mathfrak{J}}^{\{z_1\}}_{\sigma,\ell'}F_\sigma^{[\ell'],(1)}\big)(z_3,z_1),\dots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}\big)\\
&+\ \kappa_{|I_1|+1}\big(F_\sigma^{\mathfrak{a}_1},\dots,\big(\tilde{\mathfrak{J}}^{\otimes 2,\{z,z_1\}}_{\sigma,\ell(\mathfrak{a}_i)-1-\ell'}F_\sigma^{[\ell(\mathfrak{a}_i)-1-\ell'],(2)}\big)(z;z_1,z_2)\big)\kappa_{|I_2|+1}\big(F_\sigma^{[\ell'],(0)}(z_3),\dots,F_\sigma^{\mathfrak{a}_{n(\boldsymbol{a})}}\big)\big]\big\}.
\end{aligned}
\tag{120}
$$

Lemma 47. *With the above notations, it holds that, for $\sigma\geq\mu$,*

$$
[\![\sigma]\!]^{-[\boldsymbol{a}]}\|\tilde{\mathscr{A}}_{\boldsymbol{b}}^{\boldsymbol{a},(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}})\|_{\mu,\sigma}\lesssim[\![\sigma]\!]^{-[\boldsymbol{b}]-1}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}.
$$

Proof. With reference to Equation (113), we start from the contribution involving $\mathfrak{R}$ and we recall that $(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}$ is non-vanishing only if $\mathfrak{a}_i$ is such that $k(\mathfrak{a}_i)=1$ and $1<|A(\mathfrak{a}_i)|\leq 2$. We set moreover $\boldsymbol{a}'\in\boldsymbol{A}$ to be equivalent to $\boldsymbol{a}$ except for the fact that $A(\mathfrak{a}_i')=0$.

Our goal is to estimate nicely the norm $\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}})\,e^{T_{\bar\nu}(\boldsymbol{a})}\|$ and as a premise we need some pre-processing. We recall that $\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}$ is a tensor product of smoothing operators and that any of these operators is characterised by a scale, which we denote with ν_i^j, where $i\in\{1,\dots,n(\boldsymbol{a})\}$ while $j\in\{0,1,\dots,k(\mathfrak{a}_i)\}$.

For any such scale there are two regimes with respect to σ, namely $2\nu_i[\![\nu_i]\!]^{-1}\leq\sigma[\![\sigma]\!]^{-1}$ or $2\nu_i[\![\nu_i]\!]^{-1}>\sigma[\![\sigma]\!]^{-1}$. In the first regime, by Equation (19) it holds that $\tilde{\mathfrak{J}}_{\nu_i^j,\ell(\mathfrak{a}_i)}\tilde{\mathfrak{J}}_{\sigma,\ell(\mathfrak{a}_i)}=\tilde{\mathfrak{J}}_{\nu_i^j,\ell(\mathfrak{a}_i)}$ and thus we can insert this new operator at the scale σ for free. In addition we observe that thanks to weighted Sobolev inequality, see Remark 5, since $\tilde{\mathfrak{J}}_{\nu_i^j,\ell(\mathfrak{a}_i)}$ is a contraction, we can bound it away. In the second regime, namely when $2\nu_i[\![\nu_i]\!]^{-1}>\sigma[\![\sigma]\!]^{-1}$, we have $[\![\nu_i]\!]<2[\![\sigma]\!]$. We introduce a new family ν' of multi-indices derived from the original ν by replacing with the scale σ all the scales ν_i^j which are in the first regime, and by leaving untouched the ones in the second regime. Overall, this implies that, on account of Remark 5,

$$
\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}})\,e^{T_{\bar\nu}(\boldsymbol{a})}\|\leq\|(\tilde{\mathfrak{J}}_{\nu'}^{\boldsymbol{a}}(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}})\,e^{T_{\bar\nu}(\boldsymbol{a})}\|.
$$

In addition we observe that, as $\bar\nu'\geq\bar\nu$, it holds that $e^{T_{\bar\nu}(\boldsymbol{a})}\leq e^{T_{\bar\nu'}(\boldsymbol{a})}$ yielding

$$
\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}}(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}})\,e^{T_{\bar\nu}(\boldsymbol{a})}\|\leq\|(\tilde{\mathfrak{J}}_{\nu'}^{\boldsymbol{a}}(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}})\,e^{T_{\bar\nu'}(\boldsymbol{a})}\|,
$$

and as a consequence we reduced ourselves to estimating the right hand side of the above inequality. For notational simplicity, in the following analysis we shall rename the multi-index ν' and denote it with ν as there is no possible confusion. As a consequence, we stress that in the forthcoming analysis the scales ν_i^j in the multi-index ν are either equal to σ or satisfying $[\![\nu_i^j]\!]<2[\![\sigma]\!]$. We observe that in both cases $[\![\nu_i^j]\!]\lesssim[\![\sigma]\!]$. By Equation (115), we can decompose

$$
\begin{aligned}
(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}(Z,z,z_1) \ =:\ & (\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))_1^{\boldsymbol{a}}(Z,z,z_1)\\
&+(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot G_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))_2^{\boldsymbol{a}}(Z,z,z_1),
\end{aligned}
\tag{121}
$$

We start by considering the second contribution in Equation (121). Since, by definition, in the norms for cumulants we have the smoothing operators $\tilde{\mathfrak{J}}^{\boldsymbol{a}}_{\nu}$, we consider, again at the level of integral kernel,

$$\tilde{\mathfrak{J}}^{\boldsymbol{a}}_{\nu}(\Re\mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma,\mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}_2(Z,z,z_1)=\int_{\Lambda^2}\mathrm{d}\tilde{z}\mathrm{d}\tilde{z}_1\int_{\Lambda^2}\mathrm{d}z_2\mathrm{d}z_3\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-\tilde{z})\,\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1-\tilde{z}_1)\times$$
$$\times\mathscr{J}\dot{G}_\sigma(z_2,z_3)\,\mathfrak{Z}^{\{\tilde{z},\tilde{z}_1\}}K^{\boldsymbol{a}}_2(Z,\tilde{z},\tilde{z}_1,z_2,z_3),$$

where we introduced the short notation

$$K^{\boldsymbol{a}}_2(Z,\tilde{z},\tilde{z}_1,z_2,z_3):=\kappa_{n(\boldsymbol{a})+1}(\tilde{\mathfrak{J}}^{\mathfrak{a}_1}_{\nu_1}F^{\mathfrak{a}_1}_\sigma,\ldots,F^{[\ell(\mathfrak{a}_i)-1-\ell'],(2)}_\sigma(\tilde{z},\tilde{z}_1,z_2),F^{[\ell'],(0)}_\sigma(z_3),\ldots,\tilde{\mathfrak{J}}^{\mathfrak{a}_{n(\boldsymbol{a})}}_{\nu_{n(\boldsymbol{a})}}F^{\mathfrak{a}_{n(\boldsymbol{a})}}_\sigma).$$

We observe that we can insert for free smoothing operators over the variables z_2 and z_3 thanks to the propagator $\dot{G}_\sigma$, and thus we can write

$$K^{\boldsymbol{a}}_2(Z,\tilde{z},\tilde{z}_1,z_2,z_3)=\kappa_{n(\boldsymbol{a})+1}(\tilde{\mathfrak{J}}^{\mathfrak{a}_1}_{\nu_1}F^{\mathfrak{a}_1}_\sigma,\ldots,\tilde{\mathfrak{J}}^{\{z_2\}}_{\sigma.\ell(\mathfrak{a}_i)-1-\ell'}F^{[\ell(\mathfrak{a}_i)-1-\ell'],(2)}_\sigma(\tilde{z},\tilde{z}_1,z_2),\tilde{\mathfrak{J}}_{\sigma.\ell'}F^{[\ell'],(0)}_\sigma(z_3),\ldots,\tilde{\mathfrak{J}}^{\mathfrak{a}_{n(\boldsymbol{a})}}_{\nu_{n(\boldsymbol{a})}}F^{\mathfrak{a}_{n(\boldsymbol{a})}}_\sigma).$$

It follows that, resorting to Equation (116), we can estimate

$$\left|\tilde{\mathfrak{J}}^{\boldsymbol{a}}_{\nu}(\Re\mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma,\mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}_2(Z,z,z_1)\right|e^{T_{\check{\nu}}(\boldsymbol{a})(Z,z,z_1)}\leqslant(\mathbb{I})+(\mathbb{III})$$

where

$$(\mathbb{I}):=\int_{\Lambda^4}\mathrm{d}z'\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}\tilde{z}|\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-z')\mathscr{J}\dot{G}_\sigma(z_2,z_3)|\times$$
$$\times\int_0^1|\mathrm{d}\rho_{\tilde{z}-z'}(t)|\int_0^t|\mathrm{d}\rho_{\tilde{z}-z'}(u)|\,|(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell}K^{\boldsymbol{a}}_2)(Z,z',\tilde{z},z_2,z_3)|\,|\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1-z'-\rho_{\tilde{z}-z'}(u))|\,e^{T_{\check{\nu}}(\boldsymbol{a})(Z,z,z_1)}$$
$$\leqslant\int_{\Lambda^4}\mathrm{d}z'\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}\tilde{z}|\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-z')\mathscr{J}\dot{G}_\sigma(z_2,z_3)|\int_0^1|\mathrm{d}\rho_{\tilde{z}-z'}(t)|\int_0^t|\mathrm{d}\rho_{\tilde{z}-z'}(u)|\,|\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1-z'+\rho_{\tilde{z}-z'}(u))|\times$$
$$\times\left|(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell}K^{\boldsymbol{a}}_2)(Z,z',\tilde{z},z_2,z_3)\right|e^{T_{\check{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})}\,e^{T_{\check{\nu}}(z,z')}\,e^{T_{\check{\nu}}(\tilde{z},z_1)}=$$
$$=\int_{\Lambda^4}\mathrm{d}z'\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}\tilde{z}|\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-z')\mathscr{J}\dot{G}_\sigma(z_2,z_3)|\int_0^1|\mathrm{d}\rho_{\tilde{z}-z'}(t)|\int_0^t|\mathrm{d}\rho_{\tilde{z}-z'}(s)|\,|\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1)|\times$$
$$\times\left|(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell}K^{\boldsymbol{a}}_2)(Z,z',\tilde{z},z_2,z_3)\right|e^{T_{\check{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})}\,e^{T_{\check{\nu}}(z,z')}\,e^{T_{\check{\nu}}(\tilde{z}-z'-\rho_{\tilde{z}-z'}(s),z_1)}$$

where in the last step we performed the change of variable $z_1-z'-\rho_{\tilde{z}-z'}(s)\longmapsto z_1$ while in the first inequality we used $T_{\check{\nu}}(\boldsymbol{a})(Z,z,z_1)\leqslant T_{\check{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})+T_{\check{\nu}}(z,z')+T_{\check{\nu}}(\tilde{z},z_1)$. As a consequence,

$$(\mathbb{I})\lesssim\int_{\Lambda^6}\mathrm{d}z\mathrm{d}z_1\mathrm{d}z'\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}\tilde{z}|\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-z')|\,|\mathscr{J}\dot{G}_\sigma(z_2,z_3)|\times$$
$$\times\int_0^1|\mathrm{d}\rho_{\tilde{z}-z'}(t)|\int_0^t|\mathrm{d}\rho_{\tilde{z}-z'}(s)|\,|(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell}K^{\boldsymbol{a}}_2)(Z,z',\tilde{z},z_2,z_3)|\,|\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1)|\,e^{T_{\check{\nu}}(\boldsymbol{a})(Z,z')}\,e^{T_{\check{\nu}}(z,z')}\,e^{T_{\check{\nu}}(\tilde{z}-z'-\rho_{\tilde{z}-z'}(s),z_1)}$$
$$\leqslant\sup_{z'}\left[\left(\int_\Lambda\mathrm{d}z|\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-z')|\,e^{T_{\check{\nu}}(z,z')}\right)\left(\int\mathrm{d}z_1|\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1)|\,e^{\sup_s T_{\check{\nu}}(\tilde{z}-z'-\rho_{\tilde{z}-z'}(s),z_1)}\right)\right]\times$$
$$\times\int_{\Lambda^4}\mathrm{d}z'\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}\tilde{z}|\mathscr{J}\dot{G}_\sigma(z_2,z_3)|\int_0^1|\mathrm{d}\rho_{\tilde{z}-z'}(t)|\int_0^t|\mathrm{d}\rho_{\tilde{z}-z'}(s)|\,\left|(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell}K^{\boldsymbol{a}}_2)(Z,z',\tilde{z},z_2,z_3)\right|e^{T_{\check{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})}\lesssim$$
$$\lesssim\int_{\Lambda^4}\mathrm{d}z'\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}\tilde{z}|\mathscr{J}\dot{G}_\sigma(z_2,z_3)|\,|\tilde{z}-z'|^2\,\left|(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell}K^{\boldsymbol{a}}_2)(Z,z',\tilde{z},z_2,z_3)\right|e^{T_{\check{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})}.$$

where in the last inequality we used that

$$\sup_{z'}\left[\left(\int_\Lambda\mathrm{d}z|\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-z')|\,e^{T_{\check{\nu}}(z,z')}\right)\left(\int\mathrm{d}z_1|\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1)|\,e^{\sup_s T_{\check{\nu}}(\tilde{z}-z'-\rho_{\tilde{z}-z'}(s),z_1)}\right)\right]\lesssim 1.$$

For (III) we can proceed similarly to get

$$
\begin{aligned}
(\mathrm{III}) := \varepsilon \int_{\Lambda^4} & \mathrm{d}z' \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}\tilde{z} |\check{j}_{\nu_0,\ell(\mathfrak{a}_i)}(z-z') \mathcal{J}\dot{G}_\sigma(z_2,z_3)| \times \\
& \times |(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell} K_2^{\boldsymbol{a}})(Z,z',\tilde{z},z_2,z_3)| \, |\tilde{z}-z'| \, |\check{j}_{\nu_1,\ell(\mathfrak{a}_i)}(z_1-z')| \, e^{T_{\bar{\nu}}(\boldsymbol{a})(Z,z,z_1)} \\
\lesssim \varepsilon \int_{\Lambda^4} & \mathrm{d}z' \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}\tilde{z} |\mathcal{J}\dot{G}_\sigma(z_2,z_3)| \, |\tilde{z}-z'| \, |(\tilde{\mathfrak{J}}^{\otimes 2,\{z,\tilde{z}\}}_{\sigma,\ell} K_2^{\boldsymbol{a}})(Z,z',\tilde{z},z_2,z_3)| \, e^{T_{\bar{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})}.
\end{aligned}
$$

When $[\![\bar{\nu}]\!] \approx [\![\sigma]\!] \gtrsim \varepsilon$ we conclude that

$$
\begin{aligned}
\|\tilde{\mathfrak{J}}^{\boldsymbol{a}}_{\nu} (\Re \mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}_2 \, e^{T_{\bar{\nu}}(\boldsymbol{a})}\| &\lesssim [\![\bar{\nu}]\!]^2 \|(\tilde{\mathfrak{J}}^{\boldsymbol{a}'}_{\nu} \mathscr{A}^{\boldsymbol{a}',(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma)) \, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})}\| \\
&\lesssim [\![\sigma]\!]^2 \|(\tilde{\mathfrak{J}}^{\boldsymbol{a}'}_{\nu} \mathscr{A}^{\boldsymbol{a}',(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma)) \, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})}\|,
\end{aligned}
$$

where in the first bound we absorbed the monomial $|\tilde{z}-z'|$ with the tree weight, namely

$$
|\tilde{z}-z'| \, e^{T_{\bar{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})} \lesssim [\![\bar{\nu}]\!] \, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})(Z,z',\tilde{z})}, \tag{122}
$$

while in the last step we used that, thanks to the pre-processing, $[\![\bar{\nu}]\!] \lesssim [\![\sigma]\!]$. The same analysis applies to the first term in Equation (121) as well as to the case of a time derivative, *cf.* Remark 46. Thus we overall have that

$$
\|\tilde{\mathfrak{J}}^{\boldsymbol{a}}_{\nu} (\Re \mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}} \, e^{T_{\bar{\nu}}(\boldsymbol{a})}\| \lesssim [\![\sigma]\!]^{|A(\mathfrak{a}_i)|} \|(\tilde{\mathfrak{J}}^{\boldsymbol{a}'}_{\nu} \mathscr{A}^{\boldsymbol{a}',(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma)) \, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})}\|. \tag{123}
$$

We are now in position to exploit Lemma 44, which yields

$$
\|(\Re \mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{|A(\mathfrak{a}_i)|} \sup_A \|\partial^A \dot{G}_\sigma e^{\tau_\sigma(2)}\|_{\mathscr{L}(L^1,L^\infty)} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma},
$$

where the supremum is over A such that $|A(\boldsymbol{b})| = |A(\boldsymbol{a}')| + |A|$. It follows that

$$
[\![\sigma]\!]^{-[\boldsymbol{a}]} \|(\Re \mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{a}']} \sup_A \|\partial^A \dot{G}_\sigma e^{\tau_\sigma(2)}\|_{\mathscr{L}(L^1,L^\infty)} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma}
$$

where in the first inequality we used that, by definition of $\boldsymbol{a}'$, it holds that $[\boldsymbol{a}] - |A(\mathfrak{a}_i)| = [\boldsymbol{a}']$, while in the second one we used $\|\partial^A \dot{G}_\sigma e^{\tau_\sigma(2)}\|_{\mathscr{L}(L^1,L^\infty)} \lesssim [\![\sigma]\!]^{-|A|-1-d}$ as well as

$$
-[\boldsymbol{a}'] - 1 - d - |A| = -[\boldsymbol{b}] - 1 + \theta + \beta - \delta - d = -[\boldsymbol{b}] - 1,
$$

which follows from the compatibility condition $\theta + \beta - \delta - d = 0$.

When, on the other hand $[\![\bar{\nu}]\!] \approx [\![\sigma]\!] \lesssim \varepsilon$ we have only

$$
\begin{aligned}
\|\tilde{\mathfrak{J}}^{\boldsymbol{a}}_{\nu} (\Re \mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}_2 \, e^{T_{\bar{\nu}}(\boldsymbol{a})}\| &\lesssim (\varepsilon [\![\bar{\nu}]\!] + [\![\bar{\nu}]\!]^2) \|(\tilde{\mathfrak{J}}^{\boldsymbol{a}'}_{\nu} \mathscr{A}^{\boldsymbol{a}',(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma)) \, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})}\| \\
&\lesssim (\varepsilon [\![\sigma]\!]^{-1}) [\![\sigma]\!]^2 \|(\tilde{\mathfrak{J}}^{\boldsymbol{a}'}_{\nu} \mathscr{A}^{\boldsymbol{a}',(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma)) \, e^{{}^iT_{\bar{\nu}}(\boldsymbol{a})}\|.
\end{aligned}
$$

which would give us only the bound

$$
\|(\Re \mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}\|_{\mu,\sigma} \lesssim (\varepsilon [\![\sigma]\!]^{-1}) [\![\sigma]\!]^{|A(\mathfrak{a}_i)|} \sup_A \|\partial^A \dot{G}_\sigma e^{\tau_\sigma(2)}\|_{\mathscr{L}(L^1,L^\infty)} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma}.
$$

In this case, to conclude we need to exploit the more precise estimate on the operator $\dot{G}_\sigma$ given in Lemma 34 to infer that

$$
\|(\Re \mathscr{A}^{\cdot,(i)}_{\boldsymbol{b}}(\dot{G}_\sigma, \mathscr{F}^{\boldsymbol{b}}_\sigma))^{\boldsymbol{a}}\|_{\mu,\sigma} \lesssim (\varepsilon [\![\sigma]\!]^{-1}) (\varepsilon [\![\sigma]\!]^{-1} \vee 1)^{-d} [\![\sigma]\!]^{-[\boldsymbol{b}]-1} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma} \lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1} \|\mathscr{F}^{\boldsymbol{b}}_\sigma\|_{\mu,\sigma}
$$

which matches the other bound.

Moving to the contribution involving the operator $\mathfrak{L}$, we observe that if $k(\mathfrak{a}_i)\neq 1$, then on account of Equation (118), the result follows directly from Lemma 44. On the other hand, if $k(\mathfrak{a}_i)\neq 1$, recalling Equation (119), the proof follows the same lines of the case of the operator $\mathfrak{R}$, yielding

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\|(\mathfrak{L}\mathscr{A}_{\boldsymbol{b}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}\|_{\mu,\sigma}\lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-1}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}.$$

The thesis now follows by triangular inequality applied to Equation (113). □

Lemma 48. *With the above notations, it holds that, for $\sigma\geqslant\mu$,*

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\|\tilde{\mathscr{B}}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a},(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}})\|_{\mu,\sigma}\lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma}.$$

Proof. The proof of this lemma follows the same lines of the previous one and thus we shall omit some details, discussing just the main differences which are simply due to the slightly different form of the operator $\mathscr{B}$. For what concerns the term $(\mathfrak{R}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}$, the main difference with respect to $(\mathfrak{R}\mathscr{A}_{\boldsymbol{b}}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}}))^{\boldsymbol{a}}$ is that, for $\sigma\geqslant\mu$, after the initial manipulations which are the same, we have the analogous of Equation (123), namely

$$\|(\mathfrak{R}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}\|_{\mu,\sigma}\ \lesssim\ [\![\sigma]\!]^{|A(\mathfrak{a}_i)|}\sup_{\nu|\mu\leqslant\nu\leqslant\sigma}\|(\tilde{\mathfrak{J}}_\nu^{\boldsymbol{a}'}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\boldsymbol{a}',(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))\,e^{i\tilde{T}_\nu(\boldsymbol{a})}\|$$

where we bounded the monomial due to the term $(\tilde{z}-z')^{A(\mathfrak{a}_i)}$ with the exponential $e^{(\sqrt{2}-1)T_\nu(\mathfrak{a}_i)}$, which is responsible for the modified weight $e^{i\tilde{T}_\nu(\boldsymbol{a})}$. Exploiting Lemma 45, we get

$$\|(\mathfrak{R}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}\|_{\mu,\sigma}\ \lesssim\ [\![\sigma]\!]^{|A(\mathfrak{a}_i)|}\|\partial^A\dot{G}_\sigma e^{2\tau_\sigma(2)}\|_{\mathscr{L}(L^\infty,L^\infty)}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma}.$$

As in the previous lemma

$$\begin{aligned}[\![\sigma]\!]^{-[\boldsymbol{a}]}\|(\mathfrak{R}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}\|_{\mu,\sigma}\ &\lesssim\ [\![\sigma]\!]^{-[\boldsymbol{a}']}\sup_A\|\partial^A\dot{G}_\sigma e^{2\tau_\mu(2)}\|_{\mathscr{L}(L^\infty,L^\infty)}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma}\\ &\lesssim\ [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma},\end{aligned}$$

where in the first inequality the sup is taken over A such that $|A(\boldsymbol{b})|+|A(\boldsymbol{c})|=|A(\boldsymbol{a}')|+|A|$ and we used that

$$-[\boldsymbol{a}']+2s-1-|A|=-[\boldsymbol{b}]-[\boldsymbol{c}]-\rho+\theta+\beta-\delta+2s-1=-[\boldsymbol{b}]-[\boldsymbol{c}]-1,$$

exploiting the compatibility condition $-\rho+\theta+\beta-\delta+2s=0$. The case of the term $(\mathfrak{L}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}$ follows suit, giving

$$[\![\sigma]\!]^{-[\boldsymbol{a}]}\|(\mathfrak{L}\mathscr{B}_{\boldsymbol{b},\boldsymbol{c}}^{\cdot,(i)}(\dot{G}_\sigma,\mathscr{F}_\sigma^{\boldsymbol{b}},\mathscr{F}_\sigma^{\boldsymbol{c}}))^{\boldsymbol{a}}\|_{\mu,\sigma}\lesssim [\![\sigma]\!]^{-[\boldsymbol{b}]-[\boldsymbol{c}]-1}\|\mathscr{F}_\sigma^{\boldsymbol{b}}\|_{\mu,\sigma}\|\mathscr{F}_\sigma^{\boldsymbol{c}}\|_{\mu,\sigma}.$$

The thesis now follows by triangular inequality and Equation (114). □

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