A VARIATIONAL METHOD FOR Φ_3^4

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Abstract. We introduce an explicit description of the Φ_3^4 measure on a bounded domain. Our starting point is the interpretation of its Laplace transform as the value function of a stochastic optimal control problem along the flow of a scale regularization parameter. Once small scale singularities have been renormalized by the standard counterterms, Γ-convergence allows to extend the variational characterization to the unregularized model.

1. Introduction

The Φ_d^4 Gibbs measure on the d-dimensional torus $\Lambda = \Lambda_L = \mathbb{T}_L^d = (\mathbb{R}/(2\pi L\mathbb{Z}))^d$ is the probability measure ν obtained as the weak limit for $T \to \infty$ of the family $(\nu_T)_{T>0}$ given by

$$\nu_T(\mathrm{d}\phi) = \frac{\exp[-V_T(\phi_T)]}{\mathscr{Z}_T} \vartheta(\mathrm{d}\phi), \tag{1}$$

where

$$V_T(\varphi) := \lambda \int_\Lambda (|\varphi(\xi)|^4 - a_T |\varphi(\xi)|^2 + b_T) \mathrm{d}\xi, \qquad \mathscr{Z}_T := \int e^{-V_T(\phi_T)} \vartheta(\mathrm{d}\phi).$$

Here $\lambda \geqslant 0$ is a fixed constant, Δ is the Laplacian on Λ, ϑ is the centered Gaussian measure with covariance $(1-\Delta)^{-1}$, $\mathscr{Z}_T$ is a normalization factor, a_T, b_T given constants and $\phi_T = \rho_T * \phi$ with ρ_T some appropriate smooth and compactly supported cutoff function such that $\rho_T \to \delta$ as $T \to \infty$. The measures ϑ and ν_T are realized as probability measures on $\mathscr{S}'(\Lambda)$, the space of tempered distributions on Λ. They are supported on the Hölder–Besov space $\mathscr{C}^{(2-d)/2-\kappa}(\Lambda)$ for all small $\kappa > 0$. The existence of the limit ν is conditioned on the choice of a suitable sequence of *renormalization constants* $(a_T, b_T)_{T>0}$. The constant b_T is not necessary, but is useful to decouple the behavior of the numerator from that of the denominator in eq. (1).

The aim of this paper is to give a proof of convergence using a variational formula for the partition function $\mathscr{Z}_T$ and for the generating function of the measure ν_T. As a byproduct we obtain also a variational description for the generating function of the limiting measure ν via Γ-convergence of the

Date: September 27th, 2019

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Key words and phrases. Constructive Euclidean quantum field theory, Boué–Dupuis formula, renormalization group, paracontrolled calculus, Γ-convergence.

variational problem. Let us remark that, to our knowledge, it is the first time that such explicit description of the unregulated Φ_3^4 measure is available.

Our work can be seen as an alternative realization of Wilson's [49] and Polchinski's [45] continuous renormalization group (RG) method. This method has been made rigorous by Brydges, Slade et al. [11, 14, 10] and as such witnesses a lot of progress and successes [15, 16, 4, 17, 18]. The key idea is the nonperturbative study of a certain infinite dimensional Hamilton–Jacobi–Bellman equation [13] describing the effective, scale dependent, action of the theory. Here we avoid the analysis involved in the direct study of the PDE by going to the equivalent stochastic control formulation, well established and understood in finite dimensions [23]. The time parameter of the evolution corresponds to an increasing amount of small scale fluctuations of the Euclidean field and our main tool is a variational representation formula, introduced by Boué and Dupuis [7], for the logarithm of the partition function interpreted as the value function of the control problem. See also the related papers of Üstünel [48] and Zhang [50] where extensions and further results on the variational formula are obtained. The variational formula has been used by Lehec [38] to prove some Gaussian functional inequalities, following the work of Borell [6]. In this representation we can avoid the analysis of an infinite dimensional second order operator and concentrate more on path-wise properties of the Euclidean interacting fields. We are able to leverage techniques developed for singular SPDEs, in particular the paracontrolled calculus developed in [31], to perform the renormalization of various non-linear quantities and show uniform bounds in the $T\to\infty$ limit.

Define the normalized free energy $\mathcal{W}_T$ for the cutoff Φ_3^4 measure, as the functional

$$\mathcal{W}_T(f) := -\frac{1}{|\Lambda|}\log \int_{\mathscr{S}'(\Lambda)} \exp[-|\Lambda| f(\phi) - V_T(\phi_T)]\vartheta(\mathrm{d}\phi), \tag{2}$$

for all $f\in C(\mathscr{S}'(\Lambda);\mathbb{R})$. The main result of the paper is the following

Theorem 1. *Let $d=3$ and take a small $\kappa>0$. There exist renormalization constants a_T, b_T (which depend polynomially on λ) such that the limit*

$$\mathcal{W}(f) := \lim_{T\to\infty} \mathcal{W}_T(f),$$

exists for every $f\in C(\mathscr{C}^{-1/2-\kappa};\mathbb{R})$ with linear growth. Moreover the functional $\mathcal{W}(f)$ has the variational form

$$\mathcal{W}(f) = \inf_{u\in\mathbb{H}_a^{-1/2-\kappa}} \mathbb{E}\Big[f(W_\infty + Z_\infty(u)) + \Psi_\infty(u) + \lambda\|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|l(u)\|^2_{L^2([0,\infty)\times\Lambda)}\Big]$$

where

- $\mathbb{E}$ *denotes expectations on the Wiener space of a cylindrical Brownian motion* $(X_t)_{t\geqslant 0}$ *on* $L^2(\Lambda)$ *with law* $\mathbb{P}$*;*
- $(W_t)_{t\geqslant 0}$ *is a Gaussian martingale process adapted to* $(X_t)_{t\geqslant 0}$ *and such that* $\mathrm{Law}_{\mathbb{P}}(W_t) = \mathrm{Law}_{\vartheta}(\phi_t)$*;*
- $\mathbb{H}_a^{-1/2-\kappa}$ *is the space of predictable processes (wrt. the Brownian filtration) in* $L^2(\mathbb{R}_+; H^{-1/2-\kappa})$*;*
- $(Z_t(u), l_t(u))_{t\geqslant 0}$ *are explicit (non-random) functions of* $u \in \mathbb{H}_a^{-1/2-\kappa}$ *and* W*;*
- $\Psi_\infty(u)$ *a nice polynomial (non-random) functional of* (W, u)*, independent of* f*.*

See Section 4 and in particular Lemma 22 and Theorem 23 for precise definitions of the various objects and a more detailed statement of this result. With respect to the notations in Lemma 22, observe that

$$f(W_\infty + Z_\infty(u)) + \Psi_\infty(u) = \Phi_\infty(\mathbb{W}, Z(u), K(u)).$$

Theorem 1 implies directly the convergence of $(\nu_T)_T$ to a limit measure ν on $\mathscr{S}'(\Lambda)$. Taking f in the linear dual of $\mathscr{C}^{-1/2-\kappa}$ it also gives the following formula for the Laplace transform of ν:

$$\int_{\mathscr{S}'(\Lambda)} \exp(-f(\phi))\nu(\mathrm{d}\phi) = \exp(-|\Lambda|(\mathcal{W}(f/|\Lambda|) - \mathcal{W}(0))). \tag{3}$$

To our knowledge this is the first such explicit description (i.e. without making reference of the limiting procedure). The difficulty is linked to the conjectured singularity of the Φ_3^4 measure with respect to the reference Gaussian measure. Another possible approach to an explicit description goes via integration by parts (IBP) formulas, see [2] for an early proof and a discussion of this approach. More recently [28] gives a self–contained proof of the IBP formula for any accumulation point of the Φ_3^4 in the full space. However is still not clear how to use these formulas directly to obtain uniqueness of the measure and/or other properties (either on the torus or on the more difficult situation of the full space). Therefore, while our approach here is limited to the finite volume situation, it could be used to prove additional results, like large deviations or weak universality very much like in the case of SPDEs, see e.g. [34, 25].

The parameter L, which determines the size of the spatial domain $\Lambda = \Lambda_L$, will be kept fixed all along the paper and we will not attempt here to obtain the infinite volume limit $L \to \infty$. For this reason we will avoid to explicitly show the dependence of $\mathcal{W}_T$ with Λ. However some care will be taken to obtain estimates uniform in the volume $|\Lambda|$.

An easy consequence of the estimates needed to establish the main theorem is the following corollary (well known in the literature, see e.g. [5]):

Corollary 2. *There exists functions $E_+(\lambda)$, $E_-(\lambda)$ not depending on $|\Lambda|$, such that*

$$\lim_{\lambda \rightarrow 0+} \frac{E_{\pm}(\lambda)}{\lambda^3} = 0,$$

and, for any $\lambda > 0$,

$$E_-(\lambda) \leqslant \mathcal{W}_T(0) \leqslant E_+(\lambda).$$

A similar statement for $d=2$ will be sketched below in order to introduce some of the ideas on which the $d=3$ proof is based.

The construction of the $\Phi^4_{2,3}$ measure in finite volume is a basic problem of constructive quantum field theory to which many works have been devoted, especially in the $d=2$ case. It is not our aim to provide here a comprehensive review of this literature. As far as the $d=3$ case is concerned, let us just mention some of the results that, to different extent, prove the existence of the limit as the ultraviolet (small scale) regularization is removed. After the early work by Glimm and Jaffe [26, 27], in part performed in the Hamiltonian formalism, all the subsequent research has been formulated in the Euclidean setting: i.e. as the problem of construction and study of the probability measures ν on a space of distributions. Feldman [21], Park [44], Feldman and Osterwalder [22], Magnén and Sénéor [39], Benfatto et al. [5], Brydges, Fröhlich and Sokal [12] and Brydges, Dimock and Hurd [14] obtained the main results we are aware of. Recent advances in the analysis of singular SPDEs put forward by the invention of regularity structures by M. Hairer [33] and related approaches [31, 19, 43], or even RG–inspired ones [37], have allowed to pursue the stochastic quantization program to a point where now it can be used to prove directly the existence of the finite volume Φ^4_3 measure in two different ways [41, 1]. Uniqueness by these methods requires additional efforts but seems at reach. Some results on the existence of the infinite volume measure [28] and dynamics [30] have been obtained recently. For an overview of the status of the constructive program wrt. the analysis of the $\Phi^4_{2,3}$ models the reader can consult the introduction to [1] and [28]

This paper is organized as follows. In Section 2 we set up our main tool, the Boué–Dupuis variational formula of Theorem 4. Then, as a warmup exercise, we use the formula to show bounds and existence of the Φ^4_2 measure in Section 3. We then pass to the more involved situation of three dimensions in Section 4 where we introduce the renormalized variational problem. In Section 5 we establish uniform bounds for this new problem and in Section 6 we prove Theorem 1. Section 7 and Section 8 are concerned with some details of the analytic and probabilistic estimates needed throughout the paper. Appendix A gather background material on functional spaces, paraproducts and related functional analytic background material.

Acknowledgments. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program SRQ: Scaling limits, Rough paths, Quantum field theory during which part of the work on this paper was undertaken. This work was supported by the German DFG via CRC 1060 and by EPSRC via Grant Number EP/R014604/1.

Convention. Let us fix some notations and objects.

- For $a \in \mathbb{R}^d$ let $\langle a \rangle := (1 + |a|^2)^{1/2}$.
- The various constants appearing in the estimates will be understood uniform in $|\Lambda|$, unless otherwise stated.
- The constant $\kappa > 0$ represents a small positive number which can be different from line to line.
- Denote with $\mathscr{S}(\Lambda)$ the space of Schwartz functions on Λ and with $\mathscr{S}'(\Lambda)$ the dual space of tempered distributions. The notation $\hat{f}$ or $\mathscr{F} f$ stands for the space Fourier transform of f and we will write $g(\mathrm{D})$ to denote the Fourier multiplier operator with symbol $g: \mathbb{R}^n \to \mathbb{R}$, i.e. $\mathscr{F}(g(\mathrm{D}) f) = g \mathscr{F} f$.
- In order to easily keep track of the volume dependence of various objects we normalize the Lebesgue measure on Λ to have unit mass. We denote the normalized integral and measure by

$$\fint f := \frac{1}{|\Lambda|} \int_\Lambda f, \quad \fint x = \frac{1}{|\Lambda|} \mathrm{d}x,$$

  where $|\Lambda|$ is the volume of Λ. Norms in all the related functional spaces (Lebesgue, Sobolev and Besov spaces) are understood similarly normalized unless stated otherwise. This normalization of the functional spaces is used not because it is the most convenient one but because it the one relevant to obtain uniform estimates in the volume of the variational functional. For example, another normalization of H^1 norm it would no longer be controlled by the L^2 norm of the drift appearing in Theorem 4 below uniformly in $|\Lambda|$. Note that that with our choice of normalization the Sobolev embedding no longer holds uniformly in $|\Lambda|$. This is the reason why we carefully avoid to use it in the estimates of Section 7.

The reader is referred to Appendix A for an overview of the functional spaces and the additional notations used in the paper.

2. A stochastic control problem

We begin by constructing a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ endowed with a process $(W_t)_{t \in [0, \infty]}$ belonging to $C([0, \infty], \mathscr{C}^{(2-d)/2-\kappa}(\Lambda))$ and such that $\mathrm{Law}_\vartheta(\phi_T) = \mathrm{Law}_{\mathbb{P}}(W_T)$ for all $T \geqslant 0$ and $\mathrm{Law}_{\mathbb{P}}(W_\infty) = \vartheta$, the Gaussian free field with covariance $(1 - \Delta)^{-1}$.

Fix $\alpha < -d/2$ and let $\Omega := C(\mathbb{R}_+; H^{-\alpha})$, $(X_t)_{t \geqslant 0}$ the canonical process on Ω and $\mathscr{B}$ the Borel σ–algebra of Ω. On $(\Omega, \mathscr{B})$ consider the probability measure $\mathbb{P}$ which makes the canonical process X a cylindrical Brownian motion in $L^2(\Lambda)$. In the following $\mathbb{E}$ without any qualifiers will denote expectations wrt. $\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}$ will denote expectations wrt. some other measure $\mathbb{Q}$. On the probability space $(\Omega, \mathscr{B}, \mathbb{P})$ there exists a collection $(B_t^n)_{n \in (L^{-1}\mathbb{Z})^d}$ of complex (2-dimensional) Brownian motions, such that $\overline{B_t^n} = B_t^{-n}$, B_t^n, B_t^m independent for $m \neq \pm n$ and $X_t = |\Lambda|^{-1/2} \sum_{n \in (L^{-1}\mathbb{Z})^d} e^{i \langle n, \cdot \rangle} B_t^n$. Note that X has a.s. trajectories in $C(\mathbb{R}_+, \mathscr{C}^{-d/2-\varepsilon}(\Lambda))$ for any $\varepsilon > 0$ by standard arguments.

Fix some $\rho \in C_c^{\infty}(\mathbb{R}_+, \mathbb{R}_+)$, decreasing and such that $\rho(s) = 1$ for any $s \leqslant 1/2$ and $\rho(s) = 0$ for any $s \geqslant 1$. For $x \in \mathbb{R}^d$ and $t > 0$, let $\rho_t(x) := \rho(\langle x \rangle / t)$ and

$$\sigma_t(x) := \left(\frac{\mathrm{d}}{\mathrm{d}t} (\rho_t^2(x)) \right)^{1/2} = (-2(\langle x \rangle / t) \rho(\langle x \rangle / t) \rho'(\langle x \rangle / t))^{1/2} / t^{1/2},$$

where $\dot{\rho}_t$ is the partial derivative of ρ_t with respect to t. Consider the process $(W_t)_{t \geqslant 0}$ defined by

$$W_t := \frac{1}{|\Lambda|^{1/2}} \sum_{n \in (L^{-1}\mathbb{Z})^d} \int_0^t \frac{\sigma_s(n)}{\langle n \rangle} e^{i \langle n, \cdot \rangle} \mathrm{d}B_s^n, \qquad t \geqslant 0. \tag{4}$$

It is a centered Gaussian process with covariance

$$\begin{aligned} \mathbb{E}[\langle W_t, \varphi \rangle \langle W_s, \psi \rangle] &= \frac{1}{|\Lambda|} \sum_{n, m \in (L^{-1}\mathbb{Z})^d} \mathbb{E}\left[\int_0^t \frac{\sigma_u(n)}{\langle n \rangle} \mathrm{d}B_u^n \, \hat{\varphi}(n) \overline{\int_0^s \frac{\sigma_u(m)}{\langle m \rangle} \mathrm{d}B_s^m \, \hat{\psi}(m)} \right] \\ &= \frac{1}{|\Lambda|} \sum_{n \in (L^{-1}\mathbb{Z})^d} \frac{\rho_{\min(s,t)}^2(n)}{\langle n \rangle^2} \hat{\varphi}(n) \overline{\hat{\psi}(n)}, \end{aligned}$$

for any $\varphi, \psi \in \mathscr{S}(\Lambda)$ and $t, s \geqslant 0$, by Fubini theorem and Itô isometry. By dominated convergence $\lim_{t \rightarrow \infty} \mathbb{E}[\langle W_t, \varphi \rangle \langle W_t, \psi \rangle] = |\Lambda|^{-1} \sum_{n \in (L^{-1}\mathbb{Z})^d} \langle n \rangle^{-2} \hat{\varphi}(n) \overline{\hat{\psi}(n)}$ for any $\varphi, \psi \in L^2(\Lambda)$.

Note that up to any finite time T the r.v. W_T has a bounded spectral support and the stopped process $W_t^T = W_{t \wedge T}$ for any fixed $T > 0$, is in $C(\mathbb{R}_+, W^{k,2}(\Lambda))$ for any $k \in \mathbb{N}$. Furthermore $(W_t^T)_t$ only depends on a finite subset of the Brownian motions $(B^n)_n$. Denote

$$W_t = \int_0^t J_s \mathrm{d}X_s, \qquad t \geqslant 0, \tag{5}$$

with $J_s := \langle \mathrm{D} \rangle^{-1} \sigma_s(\mathrm{D})$. Observe that W_t has a distribution given by the pushforward $(\rho_t(\mathrm{D}))_* \vartheta$ of ϑ through $\rho_t(\mathrm{D})$. We write the measure ν_T in (1) in terms of expectations over $\mathbb{P}$ as

$$\int g(\phi) \nu_T(\mathrm{d}\phi) = \frac{\mathbb{E}[g(W_T) \, e^{-V_T(W_T)}]}{\mathscr{Z}_T}, \tag{6}$$

for any bounded measurable $g: \mathscr{S}'(\Lambda) \to \mathbb{R}$.

For fixed T the polynomial appearing in the expression for $V_T(W_T)$ is bounded below (since $\lambda > 0$) and $\mathscr{Z}_T$ is well defined and also bounded away from zero (this follows easily from Jensen's inequality). However as $T \to \infty$ we tend to loose both these properties due to the fact that we will be obliged to take $a_T \to +\infty$ to renormalize the non–linear terms. To obtain uniform upper and lower bounds we need a more detailed analysis and we proceed as follows.

Denote by $\mathbb{H}_a$ the space of progressively measurable processes which are $\mathbb{P}$–almost surely in $\mathcal{H} := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element v of $\mathbb{H}_a$ is a *drift*. Below we will need also (generalized) drifts belonging to $\mathcal{H}^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$, we denote the corresponding space with $\mathbb{H}_a^\alpha$. Consider the measure $\mathbb{Q}_T$ on $(\Omega, \mathscr{B})$ whose Radon–Nykodim derivative wrt. $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \frac{e^{-V_T(W_T)}}{\mathscr{Z}_T}.$$

Since W_T depends on finitely many Brownian motions $(B^n)_n$, it is well known [46, 24] that any $\mathbb{P}$–absolutely continuous probability can be expressed via Girsanov transform. In particular, by the Brownian martingale representation theorem there exists a drift $u^T \in \mathbb{H}_a$ such that

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^\infty u_s^T \mathrm{d}X_s - \frac{|\Lambda|}{2} \int_0^\infty \|u_s^T\|_{L^2}^2 \mathrm{d}s \right),$$

(recall that we normalized the $L^2(\Lambda)$ norm) and the entropy of $\mathbb{Q}_T$ wrt. $\mathbb{P}$ is given by

$$H(\mathbb{Q}_T|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}_T}\left[\log \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} \right] = \frac{|\Lambda|}{2} \mathbb{E}_{\mathbb{Q}_T}\left[\int_0^\infty \|u_s^T\|_{L^2}^2 \mathrm{d}s \right].$$

Here equality holds also if one of the two quantities is $+\infty$. By Girsanov theorem, the canonical process X is a semimartingale under $\mathbb{Q}_T$ with decomposition

$$X_t = \tilde{X}_t + \int_0^t u_s^T \mathrm{d}s, \qquad t \geqslant 0,$$

where $(\tilde{X}_t)_t$ is a cylindrical $\mathbb{Q}_T$–Brownian motion in $L^2(\Lambda)$. Under $\mathbb{Q}_T$ the process $(W_t)_t$ has the semimartingale decomposition $W_t = \tilde{W}_t + U_t$ with

$$\tilde{W}_t := \int_0^t J_s \mathrm{d}\tilde{X}_s, \quad \text{and} \quad U_t = I_t(u^T),$$

where for any drift $v \in \mathbb{H}_a$ we define

$$I_t(v) := \int_0^t J_s v_s \mathrm{d}s.$$

The integral in the density can be restricted to $[0, T]$ since $u_t^T = 0$ if $t > T$. Now

$$-\log \mathscr{Z}_T = -\log \left[e^{-V_T(W_T)} \left(\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} \right)^{-1} \right] = V_T(W_T) + \int_0^\infty u_s^T \mathrm{d}X_s - \frac{|\Lambda|}{2} \int_0^\infty \|u_s^T\|^2 \mathrm{d}s, \tag{7}$$

and taking expectation of (7) wrt $\mathbb{Q}_T$ we get

$$-\log \mathscr{Z}_T = \mathbb{E}_{\mathbb{Q}_T} \left[V_T(\tilde{W}_T + I_T(u^T)) + \frac{|\Lambda|}{2} \int_0^\infty \|u_s^T\|^2 \mathrm{d}s \right]. \tag{8}$$

For any $v \in \mathbb{H}_a$ define the measure $\mathbb{Q}^v$ by

$$\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^\infty v_s \mathrm{d}X_s - \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 \mathrm{d}s \right).$$

Denote with $\mathbb{H}_c \subseteq \mathbb{H}_a$ the set of drifts $v \in \mathbb{H}_a$ for which $\mathbb{Q}^v(\Omega) = 1$, in particular $u^T \in \mathbb{H}_c$. By Jensen's inequality and Girsanov transformation we have

$$\begin{aligned} -\log \mathscr{Z}_T &= -\log \mathbb{E}_{\mathbb{P}}[e^{-V_T(W_T)}] = -\log \mathbb{E}^v \left[e^{-V_T(W_T) - \int_0^\infty v_s \mathrm{d}X_s + \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 \mathrm{d}s} \right] \\ &\leqslant \mathbb{E}^v \left[V_T(W_T) + \int_0^\infty v_s \mathrm{d}X_s - \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 \mathrm{d}s \right], \end{aligned}$$

for all $v \in \mathbb{H}_c$, where $\mathbb{E}^v := \mathbb{E}_{\mathbb{Q}^v}$. We conclude that

$$-\log \mathscr{Z}_T \leqslant \mathbb{E}^v \left[V_T(W_T^v + I_T(v)) + \frac{|\Lambda|}{2} \int_0^\infty \|v_s\|^2 \mathrm{d}s \right], \tag{9}$$

where $W = W_T^v + I_T(v)$ and $\mathrm{Law}_{\mathbb{Q}^v}(W^v) = \mathrm{Law}_{\mathbb{P}}(W)$. The bound is saturated when $v = u^T$. We record this result in the following lemma which is a precursor of our main tool to obtain bounds on the partition function and related objects.

Lemma 3. *For any $f \in C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth, the following variational formula for the free energy holds:*

$$\mathcal{W}_T(f) = -\frac{1}{|\Lambda|} \log \mathbb{E} \left[e^{-V_T^f(W_T)} \right] = \min_{v \in \mathbb{H}_c} \mathbb{E}^v \left[\frac{1}{|\Lambda|} V_T^f(W_T^v + I_T(v)) + \frac{1}{2} \int_0^\infty \|v_s\|_{L^2}^2 \mathrm{d}s \right].$$

where $V_T^f := |\Lambda| f + V_T$.

This formula is nice and easy to prove but somewhat inconvenient for certain manipulations since the space $\mathbb{H}_c$ is indirectly defined and the reference measure $\mathbb{E}^v$ and the process W^v depend on the drift v. A more straightforward formula has been found by Boué–Dupuis [7] which involves the fixed canonical measure $\mathbb{P}$ and a general adapted drift $u \in \mathbb{H}_a$. This formula will be our main tool in the following.

Theorem 4. *For any $f \in C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth the Boué–Dupuis (BD) variational formula for the free energy holds:*

$$\mathcal{W}_T(f) = -\frac{1}{|\Lambda|}\log \mathbb{E}\left[e^{-V_T^f(W_T)}\right] = \inf_{v \in \mathbb{H}_a} \mathbb{E}\left[\frac{1}{|\Lambda|}V_T^f(W_T + I_T(v)) + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2}^2 \mathrm{d}s\right].$$

where the expectation is taken wrt to the measure $\mathbb{P}$ on Ω.

Proof. The original proof can be found in Boué–Dupuis [7] for functionals bounded from above. In our setting the formula can be proved using the result of Üstünel [48] by observing that $V_T^f(Y_T)$ is a *tame* functional, according to his definitions. Namely, for some $p, q \geqslant 1$ such that $1/p + 1/q = 1$ we have

$$\mathbb{E}[|V_T^f(W_T)|^p] + \mathbb{E}[e^{-qV_T^f(W_T)}] < +\infty. \qquad \square$$

Remark 5. Some observations on these variational formulas.

a) They originates directly from the variational formula for the free energy of a statistical mechanical systems: V_T^f playing the role of the internal energy and the quadratic term playing the role of the entropy.

b) The infimum might not be attained in Theorem 4 (see e.g. Theorem 8 in [48]) while it is attained in Lemma 3.

c) The drift generated by absolutely continuous perturbations of the Wiener measure has been introduced and studied by Föllmer [24].

d) They are a non–Markovian and infinite dimensional extension of the well known stochastic control problem representation of the Hamilton–Jacobi–Bellman equation in finite dimensions [23].

e) The BD formula is easier to use than the formula in Lemma 3 since the probability do not depend on the drift v. Going from one formulation to the other requires proving that certain SDEs with functional drift admits strong solutions and that one is able to approximate unbounded functionals V_T by bounded ones. See Üstünel [48] and Lehec [38] for a streamlined proof of the BD formula and for applications to functional inequalities on Gaussian measures. For example, from this formula it is not difficult to prove integrability of functionals which are Lipschitz in the Cameron–Martin directions.

The next lemma provides a deterministic regularity result for $I(v)$ which will be useful below. In particular, it says that the drift v generates shifts of the Gaussian free field in directions which belong to H^1 uniformly in the scale parameter up to ∞. The space H^1 is the Cameron–Martin space of the free field [36].

Lemma 6. *Let $\alpha \in \mathbb{R}$. For any $v \in L^2([0,\infty), H^\alpha)$ we have*

$$\sup_{0 \leqslant t \leqslant T} \|I_t(v)\|_{H^{\alpha+1}}^2 + \sup_{0 \leqslant s < t \leqslant T} \frac{\|I_t(v) - I_s(v)\|_{H^{\alpha+1}}^2}{1 \wedge (t-s)} \lesssim \int_0^T \|v_r\|_{H^\alpha}^2 \mathrm{d}r.$$

Proof. Using the fact that $\sigma_s(\mathrm{D})$ is diagonal in Fourier space, and denoting with $(e_k)_{k \in (L^{-1}\mathbb{Z})^d}$ the basis of trigonometric polynomials, we have

$$\begin{aligned}
\left\| \int_r^t \sigma_s(\mathrm{D}) v_s \mathrm{d}s \right\|_{H^\alpha}^2 &= \frac{1}{|\Lambda|} \sum_{k \in (L^{-1}\mathbb{Z})^d} \langle k \rangle^{2\alpha} \left| \int_r^t \langle \sigma_s(\mathrm{D}) e_k, v_s \rangle \mathrm{d}s \right|^2 \\
&\leqslant \frac{1}{|\Lambda|} \sum_{k \in (L^{-1}\mathbb{Z})^d} \langle k \rangle^{2\alpha} \left(\int_r^t |\langle \sigma_s(\mathrm{D}) e_k, e_k \rangle|^2 \mathrm{d}s \right) \left(\int_r^t |\langle e_k, v_s \rangle|^2 \mathrm{d}s \right) \\
&\leqslant \int_r^t \|v_s\|_{H^\alpha}^2 \mathrm{d}s \sup_k \int_r^t \langle e_k, \sigma_s(\mathrm{D})^2 e_k \rangle \mathrm{d}s \\
&\leqslant \int_r^t \|v_s\|_{H^\alpha}^2 \mathrm{d}s \sup_k \langle e_k, \rho_t^2(\mathrm{D}) e_k \rangle \leqslant \int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.
\end{aligned}$$

On the other hand $\sigma_s(\mathrm{D})$ is a smooth Fourier multiplier and using Proposition 54 we have the estimate $\|\sigma_s(\mathrm{D}) f\|_{H^\alpha} \lesssim \|f\|_{H^\alpha} / \langle s \rangle^{1/2}$ uniformly in $s \geqslant 0$, therefore, for all $0 \leqslant r \leqslant t \leqslant T$, we have

$$\begin{aligned}
\left\| \int_r^t \sigma_s(\mathrm{D}) v_s \mathrm{d}s \right\|_{H^\alpha}^2 &\leqslant \left(\int_r^t \|\sigma_s(\mathrm{D}) v_s\|_{H^\alpha} \mathrm{d}s \right)^2 \leqslant (t-r) \int_r^t \|\sigma_s(\mathrm{D}) v_s\|_{H^\alpha}^2 \mathrm{d}s \\
&\lesssim (t-r) \int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.
\end{aligned}$$

We conclude that

$$\| I_t(v) - I_r(v)\|_{H^{\alpha+1}}^2 \lesssim \left\| \int_r^t \sigma_s(\mathrm{D}) v_s \mathrm{d}s \right\|_{H^\alpha}^2 \leqslant [1 \wedge (t-r)] \int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.$$

□

Notation 7. *In the estimates below the symbol $E(\lambda)$ will denote a generic positive deterministic quantity, not depending on $|\Lambda|$ and such that $E(\lambda) / \lambda^3 \to 0$ as $\lambda \to 0$. Moreover the symbol Q_T will denote a generic random variable measurable wrt. $\sigma((W_t)_{t \in [0,T]})$ and belonging to $L^p(\mathbb{P})$ uniformly in T and $|\Lambda|$ for any $1 \leqslant p < \infty$.*

3. Two dimensions

As a warm up consider here the case $d = 2$ setting $f = 0$ for simplicity. From Theorem 4 we see that the relevant quantity to bound is of the form

$$F_T(u) := \mathbb{E}\left[\frac{1}{|\Lambda|} V_T(W_T + I_T(u)) + \frac{1}{2} \|u\|_{\mathcal{H}}^2 \right], \tag{10}$$

for $u \in \mathbb{H}_a$. From now on we leave implicit the integration variable over the spatial domain Λ and let $Z_t = I_t(u)$ for brevity. Choosing

$$a_T = 6\mathbb{E}[W_T(0)^2], \qquad b_T = 3\mathbb{E}[W_T(0)^2]^2, \tag{11}$$

we have

$$\frac{1}{|\Lambda|}V_T(W_T+Z_T)=\lambda\fint [\![W_T^4]\!]+4\lambda\fint [\![W_T^3]\!]Z_T+6\lambda\fint [\![W_T^2]\!]Z_T^2+4\lambda\fint W_T Z_T^3+\lambda\fint Z_T^4,$$

where

$$\begin{aligned}
[\![W_T^4]\!] &:= W_T^4-6\mathbb{E}[W_T^2]\,W_T^2+3\mathbb{E}[W_T^2]^2,\\
[\![W_T^3]\!] &:= W_T^3-3\mathbb{E}[W_T^2]W_T,\\
[\![W_T^2]\!] &:= W_T^2-\mathbb{E}[W_T^2],
\end{aligned}$$

denote the Wick powers of the Gaussian r.v. W_T [36]. These polynomials, when seen as stochastic processes in T, are $\mathbb{P}$–martingales wrt. the filtration of $(W_t)_t$. In particular they have an expression as iterated stochastic integrals wrt. the Brownian motions $(B_t^n)_{t,n}$ introduced in eq. (4). Using Theorem 4 with $u=0$ we readily have an upper bound for the free energy:

$$-\frac{1}{|\Lambda|}\log \mathscr{Z}_T\leqslant \lambda\mathbb{E}\left[\fint [\![W_T^4]\!]\right]=0.$$

For a lower bound we need to estimate from below the average under $\mathbb{P}$ of the variational expression

$$\lambda\fint [\![W_T^4]\!]+4\lambda\fint [\![W_T^3]\!]Z_T+6\lambda\fint [\![W_T^2]\!]Z_T^2+4\lambda\fint W_T Z_T^3+\lambda\fint Z_T^4+\frac{1}{2}\|u\|_{\mathcal{H}}^2.$$

The strategy we adopt is to bound *path-wise*, and for a generic drift u, the contributions

$$\Phi_T(Z):=\underbrace{4\lambda\fint [\![W_T^3]\!]Z_T}_{\mathrm{I}}+\underbrace{6\lambda\fint [\![W_T^2]\!]Z_T^2}_{\mathrm{II}}+\underbrace{4\lambda\fint W_T Z_T^3}_{\mathrm{III}},$$

in term of quantities involving only the Wick powers of W which we can control in expectation and the last two *positive terms*

$$\frac{1}{2}\|u\|_{\mathcal{H}}^2+\lambda\fint Z_T^4.$$

Any residual positive contribution depending on u can be dropped in the lower bound making the dependence on the drift disappear. To control term I we see that by duality and Young's inequality, for any $\delta>0$,

$$\left|4\lambda\fint [\![W_T^3]\!]Z_T\right| \leqslant 4\lambda\|[\![W_T^3]\!]\|_{H^{-1}}\|Z_T\|_{H^1} \leqslant C(\delta,\ d)\lambda^2\|[\![W_T^3]\!]\|_{H^{-1}}^2+\delta\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s. \qquad (12)$$

For the term II the following fractional Leibniz rule is of help:

Proposition 8. *Let $1 < p < \infty$ and $p_1, p_2, p_1', p_2' > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p}$. Then for every $s, \alpha \geqslant 0$ there exists a constant C such that*

$$\|\langle \mathrm{D}\rangle^s (fg)\|_{L^p} \leqslant C \|\langle \mathrm{D}\rangle^{s+\alpha} f\|_{L^{p_2}} \|\langle \mathrm{D}\rangle^{-\alpha} g\|_{L^{p_1}} + C \|\langle \mathrm{D}\rangle^{s+\alpha} g\|_{L^{p_1'}} \|\langle \mathrm{D}\rangle^{-\alpha} f\|_{L^{p_2'}}.$$

Proof. See [32]. □

Using Proposition 8 we get, for any $\delta > 0$, $1 \geqslant \varepsilon > 0$,

$$\begin{aligned} \left| 6\lambda \fint [\![W_T^2]\!] Z_T^2 \right| &\lesssim \lambda \|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}} \|Z_T^2\|_{W^{\varepsilon,\frac{5}{4}}} \\ &\lesssim \lambda \|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}} \|Z_T\|_{W^{\varepsilon,2}} \|Z_T\|_{L^{\frac{10}{3}}} \\ &\lesssim \lambda \|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}} \|Z_T\|_{W^{1,2}} \|Z_T\|_{L^4} \\ &\leqslant \frac{C^2\lambda^3}{2\delta} \|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}^4 + \frac{\delta}{4} \|Z_T\|_{W^{1,2}}^2 + \frac{\delta\lambda}{4} \|Z_T\|_{L^4}^4. \end{aligned} \tag{13}$$

In order to bound the term III we observe the following:

Lemma 9. *For any $\varepsilon > 0$ there exists a $1 \leqslant p < \infty$, and $K < \infty$ such that for any $f \in W^{-1/2-\varepsilon,p}$ and $g \in W^{1,2} \cap L^4$*

$$\lambda \left| \fint f g^3 \right| \leqslant E(\lambda) \|f\|_{W^{-1/2-\varepsilon,p}}^K + \delta (\|g\|_{W^{1-\varepsilon,2}}^2 + \lambda \|g\|_{L^4}^4).$$

Proof. By duality $|\fint f g^3| \leqslant \|f\|_{W^{-1/2-\varepsilon,p}} \|g^3\|_{W^{1/2+\varepsilon,p'}}$. Applying again Proposition 8 and Proposition 55 of the appendix, we get

$$\begin{aligned} \|g^3\|_{W^{1/2+\varepsilon,14/13}} &\lesssim \|\langle \mathrm{D}\rangle^{1/2+\delta} g^3\|_{L^{14/13}} \lesssim \|\langle \mathrm{D}\rangle^{5/8} g\|_{L^{14/6}} \|g\|_{L^4}^2 \\ &\lesssim \|g\|_{H^{7/8}}^{5/7} \|g\|_{L^4}^{17/7}. \end{aligned}$$

So

$$\begin{aligned} \lambda \left| \fint f g^3 \right| &\lesssim \lambda \|f\|_{W^{-1/2-\varepsilon,14}} \|g\|_{H^{7/8}}^{5/7} \|g\|_{L^4}^{17/7} \\ &\lesssim \lambda^{11} \|f\|_{W^{-1/2-\varepsilon,14}}^{28} + \delta (\|g\|_{H^{7/8}}^2 + \lambda \|g\|_{L^4}^4). \end{aligned}$$

□

Using Lemma 9 we deduce

$$\left| 4\lambda \fint W_T Z_T^3 \right| \leqslant E(\lambda) \|W_T\|_{W^{-1/2-\varepsilon,p}}^K + \delta \, (\|Z_T\|_{W^{1-\varepsilon,2}}^2 + \lambda \|Z_T\|_{L^4}^4). \tag{14}$$

Remark 10. This estimate is not optimal for $d = 2$. Indeed in this case $(W_T)_T$ stays bounded in $W^{-\varepsilon,p}$ for any large p and it would have been enough to estimate Z_T^3 in $W^{\varepsilon,p'}$. The stronger estimate will be useful below for $d = 3$ since there we will only have $W_T \in W^{-1/2-\varepsilon,p}$.

Using eqs. (12), (13) and (14) we obtain, for δ small enough,

$$|\Phi_T(Z)| \leqslant Q_T + \delta \left[\frac{1}{2} \|u\|_{\mathcal{H}}^2 + \lambda \fint Z_T^4 \right], \tag{15}$$

where

$$Q_T = O(\lambda^2)[1 + \| [\![W_T^3]\!] \|_{H^{-1}}^2 + \| [\![W_T^2]\!] \|_{W^{-\varepsilon,5}}^4 + \| W_T \|_{W^{-1/2-\varepsilon,p}}^K].$$

Therefore

$$F_T(u) \geqslant -\mathbb{E}[Q_T] + (1-\delta)\left[\frac{1}{2}\|u\|_{\mathcal{H}}^2 + \lambda \fint Z_T^4\right] \geqslant -\mathbb{E}[Q_T].$$

This last average do not depends anymore on the drift and we are only left to show that

$$\sup_T \mathbb{E}[Q_T] < \infty.$$

However, it is well known that the Wick powers of the two dimensional Gaussian free field are distributions belonging to $L^a(\Omega, W^{-\varepsilon,b})$ for any $a \geqslant 1$ and $b \geqslant 1$ and hypercontractivity plus an easy argument gives the uniform boundedness of the above averages, see e.g. [42]. We have established:

Theorem 11. *For any $\lambda > 0$ we have*

$$\sup_T \frac{1}{|\Lambda|} |\log \mathscr{Z}_T| \lesssim O(\lambda^2),$$

where the constant in the r.h.s. is independent of Λ.

Remark 12. Observe that the argument above remains valid upon replacing λ with λp with $p \geqslant 1$. This implies that $e^{-V_T(Y_T)}$ is in all the L^p spaces wrt. the measure $\mathbb{P}$ uniformly in T and for any $p \geqslant 1$.

4. Three dimensions

In three dimensions the strategy we used above fails. Indeed here the Wick products are less regular: $[\![W_T^2]\!] \in \mathscr{C}^{-1-\kappa}$ uniformly in T for any small $\kappa > 0$ and $[\![W_T^3]\!]$ does not even converge to a well-defined random distribution. This implies that there is no straightforward approach to control the terms

$$\fint [\![W_T^3]\!] Z_T, \qquad \text{and} \qquad \fint [\![W_T^2]\!] Z_T^2, \tag{16}$$

like we did in Section 3. The only apriori estimate on the regularity of $Z_T = I_T(u)$ is in H^1, coming from Lemma 6 and the quadratic term in the variational functional $F_T(u)$. It is also well known that in three dimensions there are further divergences beyond the Wick ordering which have to be subtracted in order for the limiting measure to be non-trivial. For these reasons in the energy V_T we introduce further scale dependent renormalization constants γ_T, δ_T to have

$$\frac{1}{|\Lambda|} V_T^f(W_T + Z_T) = f(W_T + Z_T) + \fint (\lambda [\![(W_T + Z_T)^4]\!] - \lambda^2 \gamma_T [\![(W_T + Z_T)^2]\!] - \delta_T). \tag{17}$$

where we Wick products $⟦(W_T+Z_T)^4⟧, ⟦(W_T+Z_T)^2⟧$ are defined with respect to the Gaussian variable W_T.

Repeating the computation from Section 3 we arrive at

$$\begin{aligned} F_T(u) &= \mathbb{E}\left[f(W_T+Z_T)+\lambda\fint \mathbb{W}_T^3 Z_T+\frac{\lambda}{2}\fint \mathbb{W}_T^2 Z_T^2+4\lambda\fint W_T Z_T^3\right] \\ &\quad -\mathbb{E}\left[2\lambda^2\gamma_T\fint W_T Z_T+\lambda^2\gamma_T\fint Z_T^2+\delta_T\right]+\mathbb{E}\left[\lambda\fint Z_T^4+\frac{1}{2}\|u\|_{\mathcal{H}}^2\right]. \end{aligned} \tag{18}$$

where we introduced the convenient notations

$$\mathbb{W}_t^3 := 4⟦W_t^3⟧, \qquad \mathbb{W}_t^2 := 12⟦W_t^2⟧, \qquad t\geqslant 0,$$

and we recall that f is a fixed function belonging to $C(\mathscr{C}^{-1/2-\kappa};\mathbb{R})$ with linear growth.

As already observed, this form of the functional is not very useful in the limit $T\to\infty$ since some of the terms, taken individually, are not expected to behave well. We will perform a change of variables in the variational functional in order to obtain some explicit cancellations which will leave only quantities well behaved as $T\to\infty$. The main drawback is that the functional will have a less compact and canonical form.

Some care has to be taken in order for the resulting quantities to be still controlled by the coercive terms. To this end we need to introduce a regularization which keeps Fourier cutoffs compatible with suitable L^4 estimates. For all $t\geqslant 0$ let $\theta_t\colon\mathbb{R}^d\to\mathbb{R}_+$ be a smooth function such that

$$\begin{aligned} \theta_t(\xi)\sigma_s(\xi) &= 0 \text{ for } s\geqslant t, \\ \theta_t(\xi) &= 1 \text{ for } |\xi|\leqslant ct \text{ for some } c>0 \text{ provided that } t\geqslant T_0 \end{aligned} \tag{19}$$

for some $T_0>0$. For example one can fix smooth functions $\tilde\theta$, $\eta\colon\mathbb{R}^d\to\mathbb{R}_+$ such that $\tilde\theta(\xi)=1$ if $|\xi|\leqslant 1/4$ and $\tilde\theta(\xi)=0$ if $|\xi|\geqslant 1/3$, $\eta(\xi)=1$ if $|\xi|\leqslant 1$ and $\eta(\xi)=0$ if $|\xi|\geqslant 2$. Then let $\tilde\theta_t(\xi):=\tilde\theta(\xi/t)$ and define

$$\theta_t(\xi)=(1-\eta(\xi))\tilde\theta_t(\xi)+\zeta(t)\eta(\xi)\tilde\theta_t(\xi)$$

where $\zeta(t)\colon\mathbb{R}_+\to\mathbb{R}$ is a smooth function such that $\zeta(t)=0$ for $t\leqslant 10$ and $\zeta(t)=1$ for $t\geqslant 11$. Then eq (19) will hold with $T_0=11$.

By the Mihlin-Hörmander theorem we deduce that the operator $\theta_t=\theta_t(\mathrm{D})$ is bounded on L^p for any $1<p<\infty$, see Proposition 54. In the following, for any $f\in C([0,\infty],\mathscr{S}'(\Lambda))$ we define $f_t^\flat:=\theta_t f_t$ then

$$Z_t^\flat=\theta_t Z_t=\int_0^t \theta_t\langle\mathrm{D}\rangle^{-1}\sigma_s(\mathrm{D})u_s\,\mathrm{d}s=\int_0^T \theta_t\langle\mathrm{D}\rangle^{-1}\sigma_s(\mathrm{D})u_s\,\mathrm{d}s=\theta_t Z_T.$$

In this way we have $\|Z_t^\flat\|_{L^p}\lesssim\|Z_T\|_{L^p}$ for all $t\leqslant T$. In the sequel we will always assume $T\geqslant T_0$.

The renormalized functional will depend on some specific renormalized combinations of the martingales $(⟦\mathbb{W}_t^k⟧)_{t,k}$. Therefore it will be also convenient to introduce a collective notation for all the stochastic objects appearing in the functionals and specify the topologies in which they are expected to be well behaved. Let

$$\mathbb{W} := (\mathbb{W}^1, \mathbb{W}^2, \mathbb{W}^{\langle 3\rangle}, \mathbb{W}^{[3]\circ 1}, \mathbb{W}^{2\diamond[3]}, \mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}),$$

with $\mathbb{W}^1 := W$,

$$\mathbb{W}_t^{\langle 3\rangle} := J_t \mathbb{W}_t^3, \quad \mathbb{W}_t^{[3]} := \int_0^t J_s \mathbb{W}_s^{\langle 3\rangle} \mathrm{d}s, \quad \mathbb{W}_t^{[3]\circ 1} := \mathbb{W}_t^1 \circ \mathbb{W}_t^{[3]},$$

$$\mathbb{W}_t^{2\diamond[3]} := \mathbb{W}_t^2 \circ \mathbb{W}_t^{[3]} + 2\gamma_t \mathbb{W}_t^1, \quad \mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle} := (J_t \mathbb{W}_t^2) \circ (J_t \mathbb{W}_t^2) + 2\dot{\gamma}_t.$$

where $\circ$ denotes the resonant product (see Definition 56 in Appendix A). We do not need to include $\mathbb{W}^{[3]}$ in the data since it can be obtained as a function of $\mathbb{W}^{\langle 3\rangle}$ thanks to the bound

$$\|\mathbb{W}_t^{[3]} - \mathbb{W}_s^{[3]}\|_{\mathscr{C}^{1/2-2\kappa}} \leqslant \int_s^t \|J_r \mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{1/2-2\kappa}} \mathrm{d}r \leqslant \left[\int_0^T \|J_r \mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{1/2-2\kappa}}^2 \mathrm{d}r\right]^{1/2} |t-s|^{1/2}$$

$$\leqslant \left[\int_0^T \|\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{-1/2-\kappa}}^2 \frac{\mathrm{d}r}{\langle r\rangle^{1+2\kappa}}\right]^{1/2} |t-s|^{1/2} \lesssim \sup_{r\in[0,T]} \|\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{-1/2-\kappa}}^2 |t-s|^{1/2},$$

valid for all $0 \leqslant s \leqslant t \leqslant T$ which shows that the deterministic linear map $\mathbb{W}^{\langle 3\rangle} \mapsto \mathbb{W}^{[3]}$ is continuous from $C([0,\infty], \mathscr{C}^{-1/2-\kappa})$ to $C^{1/2}([0,\infty], \mathscr{C}^{1/2-2\kappa})$. The path-wise regularity of all the other stochastic objects follows from the next lemma, provided the function γ is chosen appropriately.

Lemma 13. *There exists a function $\gamma_t \in C^1(\mathbb{R}_+, \mathbb{R})$ such that*

$$|\gamma_t| + \langle t\rangle |\dot{\gamma}_t| \lesssim \log\langle t\rangle, \qquad t \geqslant 0. \tag{20}$$

and such that the vector $\mathbb{W}$ is almost surely in $\mathfrak{S}$ where $\mathfrak{S}$ is the Banach space

$$\mathfrak{S} = C([0,\infty], \mathfrak{W}) \cap \{\mathbb{W}^{\langle 3\rangle} \in L^2(\mathbb{R}_+, \mathscr{C}^{-1/2-\kappa}), \mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle} \in L^1(\mathbb{R}_+, \mathscr{C}^{-\kappa})\}$$

with

$$\mathfrak{W} = \mathfrak{W}_\kappa := \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-\kappa} \times \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-\kappa},$$

and equipped with the norm

$$\|\mathbb{W}\|_{\mathfrak{S}} := \|\mathbb{W}\|_{C([0,\infty],\mathfrak{W})} + \|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+, \mathscr{C}^{-1/2-\kappa})} + \|\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}\|_{L^1(\mathbb{R}_+, \mathscr{C}^{-\kappa})}.$$

The norm $\|\mathbb{W}\|_{\mathfrak{S}}$ belongs to all $L^p(\mathbb{P})$ spaces. Moreover the averages of the Besov norms $B_{q,r}^\alpha$ of the components of $\mathbb{W}$ of regularity α are uniformly bounded in the volume $|\Lambda|$ if $r < \infty$.

Proof. The proof is based on the observation that one can choose γ in such a way that every component $\mathbb{W}^{(i)}$ of the vector $\mathbb{W}$ is such that $(\Delta_q \mathbb{W}_t^{(i)}(x))_{t\geqslant 0}$ for $q \geqslant -1$ and $x \in \Lambda$ is a martingale wrt. the Brownian filtration (possibly modulo a deterministic term we can control). This can be seen by writing these terms as iterated stochastic integrals. For example, introducing the notation $\mathrm{d}w_s(k) = \langle k\rangle^{-1}\sigma_s(k)\mathrm{d}B_s^k$ we can write

$$\mathbb{W}_T^2(x) = 24\sum_{k_1,k_2} e^{i(k_1+k_2)\cdot x}\int_0^T\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)$$

so, recalling the definition of Littlewood-Paley kernels ϱ_i from Appendix A, we have

$$\Delta_i\mathbb{W}_T^2(x) = 24\sum_{k_1,k_2} e^{i(k_1+k_2)\cdot x}\varrho_i(k_1+k_2)\int_0^T\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2).$$

By Burkholder's inequality and Fubini's theorem

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T}\|\Delta_i\mathbb{W}_t^2\|_{L^p}^p\right] &\lesssim \left(\sum_{k_1,k_2}\varrho_i(k_1+k_2)\int_0^T\int_0^{s_2}\frac{\sigma_{s_1}^2(k_1)}{\langle k_1\rangle^2}\frac{\sigma_{s_2}^2(k_2)}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\right)^{p/2}\\ &\lesssim 2^{p(2+\kappa)i/2},\end{aligned}$$

uniformly in T and so

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T}\|\mathbb{W}_T^2\|_{B_{p,p}^{-1-\kappa}}^p\right] &\leqslant \mathbb{E}\left[\left(\sum_i 2^{p(-1-\kappa)i}\sup_{t\leqslant T}\|\Delta_i\mathbb{W}_t^2\|_{L^p}^p\right)\right]\\ &\lesssim \sum_i 2^{p(-1-\kappa)i}\,\mathbb{E}\left[\sup_{t\leqslant T}\|\Delta_i\mathbb{W}_t^2\|_{L^p}^p\right]\\ &\lesssim \sum_i 2^{p(-1-\kappa)i}2^{p(1+\kappa/2)i}\lesssim\sum_i 2^{-pi\kappa/2} < +\infty\end{aligned}$$

By Besov embedding this implies that $\mathbb{E}[\sup_{T<\infty}\|\mathbb{W}_T^2\|_{B_{p,q}^{-1-\kappa}}^p]$ is finite for any p, $q<\infty$ uniformly in the volume and $\mathbb{E}[\|\mathbb{W}_T^2\|_{C\mathscr{C}^{-1-}}^p]$ is finite. Since $\mathbb{W}_T^2$ is a continuous, L^2-bounded martingale, it converges and therefore it belongs to $C([0,\infty],\mathscr{C}^{-1-})$. The same reasoning can be carried out for the more complicated terms $\mathbb{W}^{\langle 3\rangle}$, $\mathbb{W}^{[3]\circ 1}$, $\mathbb{W}^{2\diamond[3]}$, $\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}$. The details can be found in Section 8. □

For convenience of the reader we summarize the probabilistic estimates in Table 1.

$\mathbb{W}^1$	$\mathbb{W}^2$	$\mathbb{W}^{\langle 3\rangle}$	$\mathbb{W}^{[3]}$	$\mathbb{W}^{[3]\circ 1}$	$\mathbb{W}^{2\diamond[3]}$	$\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}$
$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{-1-}$	$C\mathscr{C}^{-1/2-}\cap L^2\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{1/2-}$	$C\mathscr{C}^{0-}$	$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{0-}\cap L^1\mathscr{C}^{0-}$

Table 1. Regularities of the various stochastic objects, the domain of the time variable is understood to be $[0,\infty]$. Estimates in these norms hold a.s. and in $L^p(\mathbb{P})$ for all $p\geqslant 1$ (see Lemma 13).

Remark 14. The requirement that $\mathbb{W}^{\langle 3\rangle}\in L^2\mathscr{C}^{-1/2-}$ will be used in Section 6 to establish equicoercivity and to relax the variational problem to a suitable space of measures.

We are now ready to perform a change of variables which renormalizes the variational functional.

Lemma 15. *Define* $l = l^T(u) \in \mathbb{H}_a$, $Z = Z(u) \in C([0, \infty], H^{1/2-\kappa})$, $K = K(u) \in C([0,\infty], H^{1-\kappa})$ *such that*

$$
\begin{aligned}
&Z_t(u) := I_t(u), \\
&l_t^T(u) := u_t + \lambda \mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle} + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ Z_t^\flat(u)), \qquad t \geqslant 0. \\
&K_t(u) := I_t(w(u)), \ \text{with } w_t(u) := -\lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ Z_t^\flat(u)) + l_t^T(u),
\end{aligned}
\tag{21}
$$

Then the functional $F_T(u)$ *defined in eq. (18) takes the form*

$$
F_T(u) \;=\; \mathbb{E}\left[\Phi_T(\mathbb{W}, Z(u), K(u)) + \lambda \fint (Z_T(u))^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right],
$$

where

$$
\Phi_T(\mathbb{W}, Z, K) := f(W_T + Z_T) + \sum_{i=1}^{6} \Upsilon^{(i)},
$$

$$
\begin{aligned}
\Upsilon_T^{(1)} &:= -\frac{\lambda}{2}\mathfrak{K}_2(\mathbb{W}_T^2, K_T, K_T) + \frac{\lambda}{2}\fint (\mathbb{W}_T^2 \prec K_T)K_T - \lambda^2 \fint (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]})K_T, \\
\Upsilon_T^{(2)} &:= \lambda \fint (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat))K_T, \\
\Upsilon_T^{(3)} &:= \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat)K_t \mathrm{d}t, \\
\Upsilon_T^{(4)} &:= 4\lambda \fint W_T K_T^3 - 12\lambda^2 \fint W_T \mathbb{W}_T^{[3]} K_T^2 + 12\lambda^3 \fint W_T (\mathbb{W}_T^{[3]})^2 K_T, \\
\Upsilon_T^{(5)} &:= -2\lambda^2 \fint \gamma_T Z_T^\flat (Z_T - Z_T^\flat) - \lambda^2 \fint \gamma_T (Z_T - Z_T^\flat)^2 - 2\lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t, \\
\Upsilon_T^{(6)} &:= -\lambda^2 \fint \mathbb{W}_T^{2\diamond[3]} K_T - \frac{\lambda^2}{2}\int_0^T \fint \mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle}(Z_t^\flat)^2 \mathrm{d}t - \frac{\lambda^2}{2}\int_0^T \mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, \\
&\qquad Z_t^\flat, Z_t^\flat)\mathrm{d}t.
\end{aligned}
$$

Here $\mathfrak{K}_2$ *and* $\mathfrak{K}_{3,t}$ *are linear forms defined in Proposition 61 and 62 in Appendix A (and recalled in the proof below). Moreover we have chosen the renormalization constant* δ_T *appearing in equation (17) to be*

$$
\begin{aligned}
\delta_T \;:=\; & -\frac{\lambda^2}{2}\mathbb{E}\int_0^T \fint (\mathbb{W}_t^{\langle 3\rangle})^2 \mathrm{d}t + \frac{\lambda^3}{2}\mathbb{E}\fint \mathbb{W}_T^2 (\mathbb{W}_T^{[3]})^2 \\
& + 2\lambda^3 \gamma_T \mathbb{E}\fint W_T \mathbb{W}_T^{[3]} - 4\lambda^4 \mathbb{E}\fint W_T (\mathbb{W}_T^{[3]})^3.
\end{aligned}
\tag{22}
$$

Proof.

Step 1. We are going to absorb the mixed terms (16) via the quadratic cost function. To do so we develop them along the flow of the scale parameter via Itô formula. For the first we have

$$
\lambda \fint \mathbb{W}_T^3 Z_T = \lambda \int_0^T \fint \mathbb{W}_t^3 \dot{Z}_t \mathrm{d}t + \text{martingale},
$$

and we can cancel the first term on the r.h.s. by introducing

$$w_t := u_t + \lambda \mathbb{1}_{t \leqslant T} \mathbb{W}_t^{\langle 3 \rangle}, \qquad t \geqslant 0, \tag{23}$$

into the cost functional to get

$$\lambda \fint \mathbb{W}_T^3 Z_T + \frac{1}{2} \int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s = -\frac{\lambda^2}{2} \int_0^T \fint (\mathbb{W}_t^{\langle 3 \rangle})^2 \mathrm{d}t + \frac{1}{2} \int_0^\infty \|w_s\|_{L^2}^2 \mathrm{d}s + \text{martingale},$$

where we used that J_t is self-adjoint. Taking into account (here and below) that the martingale term will average to zero, we have replaced the divergent term $\fint \mathbb{W}_T^3 Z_T$ with a divergent but purely stochastic term $\int_0^T \fint (\mathbb{W}_t^{\langle 3 \rangle})^2 \mathrm{d}t$ which does not affect anymore the variational problem and can be explicitly removed by adding its average to δ_T. As a consequence, we are no more able to control $(Z_t)_t$ in H^1 and we should rely on the relation (21) and on a control over the H^1 norm of $(K_t)_t$ coming from the residual quadratic term $\|w\|_{\mathcal{H}}^2$.

Step 2. From (23) we have the relation

$$Z_T = -\lambda \mathbb{W}_T^{[3]} + K_T,$$

which can be used to expand the second mixed divergent term in (16) as

$$\frac{\lambda}{2} \fint \mathbb{W}_T^2 Z_T^2 = \frac{\lambda^3}{2} \fint \mathbb{W}_T^2 (\mathbb{W}_T^{[3]})^2 - \lambda^2 \fint \mathbb{W}_T^2 \mathbb{W}_T^{[3]} K_T + \frac{\lambda}{2} \fint \mathbb{W}_T^2 K_T^2. \tag{24}$$

Again, the first term on the r.h.s. a purely stochastic object and will give a contribution independent of the drift u and absorbed in δ_T. We are still not done since this operation has left two new divergent terms on the r.h.s. of eq. (24): the H^1 regularity of K_T is not enough to control the products with $\mathbb{W}^2$ which has regularity $\mathscr{C}^{-1-\kappa}$, a bit below -1. In order to proceed further we will isolate the divergent parts of these products via a paraproduct decomposition (see Appendix A for details) and expand

$$\begin{aligned} -\lambda^2 \fint \mathbb{W}_T^2 \mathbb{W}_T^{[3]} K_T + \frac{\lambda}{2} \fint \mathbb{W}_T^2 K_T^2 = {} & \lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T - \lambda^2 \fint (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}) K_T \\ & - \lambda^2 \fint (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]}) K_T + \frac{\lambda}{2} \fint (\mathbb{W}_T^2 \prec K_T) K_T \\ & + \frac{\lambda}{2} \left(\fint (\mathbb{W}_T^2 \circ K_T) K_T - \fint (\mathbb{W}_T^2 \succ K_T) K_T \right). \end{aligned}$$

The first two terms will require renormalizations which we put in place in Step 3 below. All the other terms will be well behaved and we collect them in $\Upsilon_T^{(1)}$. In particular we observe that the last one can be rewritten as

$$\frac{\lambda}{2} \left(\fint (\mathbb{W}_T^2 \circ K_T) K_T - \fint (\mathbb{W}_T^2 \succ K_T) K_T \right) = -\frac{\lambda}{2} \mathfrak{K}_2(\mathbb{W}_T^2, K_T, K_T)$$

introducing the trilinear form $\mathfrak{K}_2$ whose properties are detailed in Proposition 61 below.

Step 3. As we anticipated, the resonant term $\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}$ needs renormalization. In the expression of F_T in (18) we have the counterterm $-2\lambda^2\gamma_T \fint W_T Z_T$ available, which we put now in use writing

$$-\lambda^2 \fint (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}) K_T - 2\lambda^2\gamma_T \fint W_T Z_T = -\lambda^2 \fint \underbrace{(\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} + 2\gamma_T W_T)}_{\mathbb{W}_T^{2\diamond[3]}} K_T + 2\lambda^3\gamma_T \fint W_T \mathbb{W}_T^{[3]}.$$

The first contribution is collected in $\Upsilon_T^{(6)}$ and the expectation of the second will contribute to δ_T.

As far as the term $\lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T$ is concerned, we want to absorb it into $\int \|w_s\|^2 \mathrm{d}s$ like we did with the linear term in Step 2. Before we can do this we must be sure that, after applying Itô's formula, it will be still possible to use $\fint Z_T^4$ to control some of the growth of this term. Indeed the quadratic dependence in K_T (via Z_T) cannot be fully taken care of by the quadratic cost $\int \|w_s\|^2 \mathrm{d}s$.

We decompose

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T = \lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T + \lambda \fint (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat)) K_T$$

and using the fact that the functions $Z_T - Z_T^\flat$ and $K_T - K_T^\flat$ are spectrally supported outside of a ball or radius cT we will be able to show that the second term is nice enough as $T \to \infty$ to not require further analysis and we collect it in $\Upsilon_T^{(2)}$. For the first we apply Itô's formula to decompose it along the flow of scales as

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T = \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat) K_t \mathrm{d}t + \text{martingale}.$$

The second term will be fine and we collect it in $\Upsilon_T^{(3)}$.

Step 4. We are left with the singular term $\int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t$. Using eq. (21) and expanding w in the residual quadratic cost function obtained in Step 1, we compute

$$\begin{aligned}\lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \frac{1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t &= -\frac{\lambda^2}{2}\int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 \mathrm{d}t + \frac{1}{2}\int_0^\infty \|l_t\|_{L^2}^2 \mathrm{d}t \\ &= -\frac{\lambda^2}{2}\int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) \mathrm{d}t + \frac{1}{2}\|l\|_{\mathcal{H}}^2 \end{aligned} \quad (25)$$

To renormalize the first term on the r.h.s. we observe that the remaining counterterm can be rewritten as

$$-\lambda^2\gamma_T \fint Z_T^2 = -\lambda^2\gamma_T \fint (Z_T^\flat)^2 - 2\lambda^2\gamma_T \fint Z_T^\flat (Z_T - Z_T^\flat) - \lambda^2\gamma_T \fint (Z_T - Z_T^\flat)^2. \tag{26}$$

Differentiating in T the first term in the r.h.s. of eq. (26) we get

$$-\lambda^2\gamma_T \fint (Z_T^\flat)^2 = -\lambda^2 \int_0^T \fint \dot{\gamma}_t (Z_t^\flat)^2 \mathrm{d}t - 2\lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t. \tag{27}$$

The last term in eq. (27) and the last two contributions in (26) are collected in $\Upsilon_T^{(5)}$. The first contribution in eq. (27) has the right form to be used as a counterterm for the resonant product in (25). Using the commutator $\mathfrak{K}_{3,t}$ introduced in Proposition 62 we have

$$-\frac{\lambda^2}{2}\int_0^T \fint [(J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) + 2\dot{\gamma}_t (Z_t^\flat)^2]\mathrm{d}t$$

$$= -\frac{\lambda^2}{2}\int_0^T \fint \underbrace{[(J_t\mathbb{W}_t^2)\circ(J_t\mathbb{W}_t^2) + 2\dot{\gamma}_t]}_{\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}} (Z_t^\flat)^2 \mathrm{d}t - \frac{\lambda^2}{2}\int_0^T \mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)\mathrm{d}t$$

and we collect both terms in $\Upsilon_T^{(6)}$.

Step 5. Finally, we are left with the cubic term which we rewrite as

$$4\lambda \fint W_T Z_T^3 = -4\lambda^4 \fint W_T (\mathbb{W}_T^{[3]})^3 + 12\lambda^3 \fint W_T (\mathbb{W}_T^{[3]})^2 K_T - 12\lambda^2 \fint W_T \mathbb{W}_T^{[3]} K_T^2 + 4\lambda \fint W_T K_T^3.$$

The average of the first term is collected in δ_T while all the remaining terms in $\Upsilon_T^{(4)}$. At last we have established the claimed decomposition since the residual cost functional, from eq. (25) is indeed $\|l\|_{\mathcal{H}}^2/2$. □

5. Bounds

The aim of this section is to give upper and lower bounds on $\mathcal{W}_T(f)$ uniformly on T and $|\Lambda|$. In particular we will prove the bounds of Corollary 2 taking the explicit dependence on the coupling constant λ into account.

Lemma 16. *There exists a finite constant C, which does not depend on Λ, such that*

$$\sup_T |\mathcal{W}_T(f)| \leqslant C.$$

Proof. Observe that, from Lemma 15 and from the analysis in Section 7, we have that

$$|\Phi_T(\mathbb{W}, Z, K)| \leqslant Q_T + \varepsilon \left(\lambda \|Z_T\|_{L^4}^4 + \frac{1}{2} \int_0^\infty \|l_t^T(u)\|_{L^2}^2 \mathrm{d}t \right),$$

which immediately gives

$$-\mathbb{E}[Q_T] \leqslant -\mathbb{E}[Q_T] + (1-\varepsilon)\mathbb{E}\left(\lambda \|Z_T\|_{L^4}^4 + \frac{1}{2} \int_0^\infty \|l_t^T(u)\|_{L^2}^2 \mathrm{d}t \right) \leqslant \mathcal{W}_T(f). \tag{28}$$

On the other hand for any suitable drift $\check{u} \in \mathbb{H}_a$ we get the bound

$$\mathcal{W}_T(f) \leqslant \mathbb{E}[Q_T] + (1+\varepsilon)\mathbb{E}\left(\lambda \|I_T(\check{u})\|_{L^4}^4 + \frac{1}{2} \int_0^\infty \|l_t^T(\check{u})\|_{L^2}^2 \mathrm{d}t \right), \tag{29}$$

where

$$l_t^T(\check{u}) = \check{u}_t + \lambda \mathbb{1}_{t \leqslant T} J_t(\mathbb{W}_t^3 + \mathbb{W}_t^2 \succ (I_t(\check{u}))^\flat). \tag{30}$$

Therefore it remains to produce an appropriate drift $\check{u}$ for which the r.h.s. in eq. (29) is finite (and so uniformly in $|\Lambda|$ and of order $o(\lambda^3)$).

One possible strategy is to try and choose $\check{u}$ such that $l^T(\check{u}) = 0$, however this fails since estimates on this choice of drift via Gronwall's inequality would rely on the Besov-Hölder norm of $\mathbb{W}^2$ for which we do not have any uniform control in the volume. In order to overcome this problem we decompose $\mathbb{W}^2$ and use weighted estimates similarly as done in [30] in the SPDE context.

Consider the decomposition

$$\mathbb{W}_s^2 = \mathcal{U}_{\geqslant} \mathbb{W}_s^2 + \mathcal{U}_{\leqslant} \mathbb{W}_s^2,$$

where the random field $\mathcal{U}_{\geqslant} \mathbb{W}_s^2$ is constructed as follows. Let φ be smooth function, positive and supported on $[-2,2]^3$ and such that $\sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi^2(\bullet - m) = 1$. Denote $\varphi_m := \varphi(\bullet - m)$. Let $\tilde{\chi}$ be a smooth function supported in $B(0, 1)$, denote by $\mathcal{X}_{>N}$ the Fourier multiplier operator $\tilde{\chi}(\mathrm{D}/N)$ and similarly $\mathcal{X}_{\leqslant N} := (1 - \tilde{\chi}(\mathrm{D}/N))$. Set $L_m(s) := (1 + \|\varphi_m \mathbb{W}_s^2\|)_{\mathscr{C}^{-1-\delta}}^{\frac{1}{2\delta}}$, let

$$\mathcal{U}_{>} \mathbb{W}_s^2 := \sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi_m \mathcal{X}_{>L_m(s)}(\varphi_m \mathbb{W}_s^2)$$

and

$$\mathcal{U}_{\leqslant} \mathbb{W}_s^2 := \sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi_m \mathcal{X}_{\leqslant L_m(s)}(\varphi_m \mathbb{W}_s^2).$$

(with slight abuse of notation we drop the dependence on time of the operators $\mathcal{U}_{\leqslant}, \mathcal{U}_{>}$).

Observe that the laws of both $\mathcal{U}_{>}\mathbb{W}_s^2$ and $\mathcal{U}_{\leqslant}\mathbb{W}_s^2$ are translation invariant w.r.t to translations by $m \in \Lambda \cap \mathbb{Z}^d$. By [47], Theorem 2.4.7 and Bernstein inequality

$$\begin{aligned}\|\mathcal{U}_{>}\mathbb{W}_s^2\|_{\mathscr{C}^{-1-3\delta}} &\lesssim \sup_m \|\mathcal{X}_{>L_m(s)}(\varphi_m \mathbb{W}_s^2)\|_{\mathscr{C}^{-1-3\delta}}\\ &\lesssim \sup_m \frac{1}{1+\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}}\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}} \lesssim 1.\end{aligned}$$

Furthermore for a polynomial weight ρ (see Appendix A for precisions on the weights and the weighted spaces $L^p(\rho)$, $\mathscr{C}^\alpha(\rho)$ and $B_{p,q}^\alpha(\rho)$ used below):

$$\begin{aligned}\|\mathcal{U}_{\leqslant}\mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta}(\rho^2)} &\lesssim \sup_m \|\varphi_m \mathcal{U}_{\leqslant}\mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta}(\rho^2)}\\ &\lesssim \sup_m (1+\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}})\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho^2)}\\ &\lesssim \sup_m \rho(m)\,(1+\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}})\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)}\\ &\lesssim \sup_m (1+\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)})\|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)}\\ &\lesssim 1+\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)}^2,\end{aligned} \tag{31}$$

where we used the possibility to compare weighted and unweighted norms once localized via φ_m. We now let $\check{u}$ be the solution to the linear integral equation

$$\check{u}_t = -\lambda \mathbb{1}_{t\leqslant T}[\mathbb{W}_t^{\langle 3\rangle} + J_t \mathcal{U}_{>}\mathbb{W}_t^2 \succ \theta_t(I_t(\check{u}))], \qquad t \geqslant 0, \tag{32}$$

which can be solved globally. For $3\delta < 1/2$, $p \geqslant 1$ and $t \in [0,T]$, we have

$$\|I_t(\check{u})\|_{B_{p,p}^{1/2-3\delta}(\rho)} \lesssim \lambda \int_0^t \Big[\|J_s^2 \mathbb{W}_s^3\|_{B_{p,p}^{1/2-3\delta}(\rho)} + \lambda \|J_s^2 \mathcal{U}_{>}\mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{B_{p,p}^{1/2-\delta}(\rho)}\Big]\mathrm{d}s$$

$$\lesssim \lambda \int_0^t \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}}\|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)} + \lambda \int_0^t \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}}\|\mathcal{U}_{>}\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}\|I_s(\check{u})\|_{B_{p,p}^{1/2-3\delta}(\rho)}.$$

Gronwall's lemma implies that, for $t \in [0,T]$:

$$\begin{aligned}\|I_t(\check{u})\|_{B_{p,p}^{1/2-\delta}(\rho)} &\lesssim \left(\lambda \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}}\|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)}\right)\exp\left(\lambda \int_0^T \frac{\|\mathcal{U}_{>}\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}\mathrm{d}s}{\langle s\rangle^{1+\delta}}\right)\\ &\lesssim \left(\lambda \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}}\|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)}\right)\\ &\lesssim \lambda \|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1/2-\delta}(\rho))}.\end{aligned} \tag{33}$$

Note that eq. (33) is also valid replacing the weighted norm $B^{1/2-\delta}_{p,p}(\rho)$ with the standard (normalized) norm $B^{1/2-\delta}_{p,p}$, from which, using Besov embedding we deduce:

$$\sup_T \mathbb{E}\|I_T(\check{u})\|^4_{L^4} \lesssim \lambda^4 \mathbb{E}\left(\int_0^\infty \frac{\mathrm{d}s}{\langle s\rangle^{1/2+\delta}} \|J_s \mathbb{W}^3_s\|_{B^{-1/2-\delta}_{4,4}}\right)^4 \lesssim \lambda^4.$$

Computing $l^T(\check{u})$ from eq. (30) and (32), we obtain

$$l^T_t(\check{u}) = \lambda \mathbb{1}_{t \leqslant T} J_t \mathcal{U}_{\leqslant} \mathbb{W}^2_t \succ \theta_t(I_t(\check{u})), \qquad t \geqslant 0.$$

It remains to prove that $\mathbb{E}[\|l^T(\check{u})\|^2_{\mathcal{H}}] \lesssim O(\lambda^3)$ uniformly in $T > 0$. Note that, for $s \in [0, T]$,

$$\begin{aligned} \|J_s \mathcal{U}_{\leqslant} \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|_{L^2(\rho^3)} &\lesssim \frac{1}{\langle s\rangle^{1/2+\delta/2}} \|\mathcal{U}_{\leqslant} \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|_{B^{-1+\delta/2}_{2,2}(\rho^3)} \\ &\lesssim \frac{1}{\langle s\rangle^{1/2+\delta/2}} \|\mathcal{U}_{\leqslant} \mathbb{W}^2_s\|_{\mathscr{C}^{-1+\delta/2}(\rho^2)} \|I_s(\check{u})\|_{B^{1/2-3\delta}_{2,2}(\rho)}. \end{aligned} \tag{34}$$

We know that the distribution of $\check{u}$ is invariant under translation by $m \in \Lambda \cap \mathbb{Z}^d$. Recalling that $\sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi^2(\bullet - m) = 1$ and letting ρ be a polynomial weight with sufficient decay and such that $\rho^5 \geqslant \varphi^2$, we have

$$\begin{aligned} \mathbb{E}[\|l^T(\check{u})\|^2_{\mathcal{H}}] &= \lambda^2 \mathbb{E}[\|s \mapsto \mathbb{1}_{s \leqslant T} J_s \mathcal{U}_{\leqslant} \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{\mathcal{H}}] \\ &\leqslant \lambda^2 \sum_{m \in \Lambda \cap \mathbb{Z}^d} \mathbb{E}[\|s \mapsto \varphi(\bullet - m) J_s \mathcal{U}_{\leqslant} \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{\mathcal{H}}] \\ \text{(by trans. inv.)} &\lesssim \lambda^2 |\Lambda| \mathbb{E}[\|s \mapsto \mathbb{1}_{s \leqslant T} \varphi J_s \mathcal{U}_{\leqslant} \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{\mathcal{H}}] \\ \text{(using } \rho^5 \geqslant \varphi^2) &\lesssim \lambda^2 \int_0^T \mathrm{d}s \, \mathbb{E}[\|J_s \mathcal{U}_{\leqslant} \mathbb{W}^2_s \succ \theta_s(I_s(\check{u}))\|^2_{L^2(\rho^5)}] \\ \text{(by eq. (34))} &\lesssim \lambda^2 \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}} \mathbb{E}\Big[\|\mathcal{U}_{\leqslant} \mathbb{W}^2_s\|^2_{\mathscr{C}^{-1+\delta/2}(\rho^2)} \|I_s(\check{u})\|^2_{B^{1/2-3\delta}_{2,2}(\rho)}\Big] \\ &\lesssim \lambda^2 \int_0^T \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}} \mathbb{E}\Big[\lambda^2 \|\mathcal{U}_{\leqslant} \mathbb{W}^2_s\|^4_{\mathscr{C}^{-1+\delta/2}(\rho^2)} + \\ &\qquad \lambda^{-2} \|I_s(\check{u})\|^4_{B^{1/2-3\delta}_{2,2}(\rho)}\Big] \\ \text{(by eqs. (33),(31))} &\lesssim \lambda^4 \int_0^\infty \frac{\mathrm{d}s}{\langle s\rangle^{1+\delta}} \Big[1 + \mathbb{E}\|\mathbb{W}^2_s\|^8_{\mathscr{C}^{-1-\delta/2}(\rho)} + \\ &\qquad \lambda \mathbb{E}\|\mathbb{W}^{\langle 3\rangle}\|^4_{L^2(\mathbb{R}_+, B^{-1/2-\delta}_{p,p}(\rho))}\Big] \\ &\lesssim \lambda^4 \Big[1 + \sup_{s \geqslant 0} \mathbb{E}\|\mathbb{W}^2_s\|^8_{\mathscr{C}^{-1-\delta/2}(\rho)} + \\ &\qquad \lambda \mathbb{E}\|\mathbb{W}^{\langle 3\rangle}\|^4_{L^2(\mathbb{R}_+, B^{-1/2-\delta}_{p,p}(\rho))}\Big] \\ &\lesssim O(\lambda^4). \end{aligned}$$

The last inequality is the consequence of bounds on the two expectations on the r.h.s. obtained as follows. For p sufficiently large we have

$$\left[\mathbb{E}\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta/2}(\rho)}^8\right]^{p/8} \leqslant \mathbb{E}\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta/2}(\rho)}^p \leqslant \mathbb{E}\|\mathbb{W}_s^2\|_{B_{p,p}^{-1-\delta}(\rho^p)}^p$$

$$= \sum_{i\geqslant -1} 2^{i(-1-\delta/2)p} \int_\Lambda \mathrm{d}x |\rho(x)|^p \mathbb{E}|\Delta_i \mathbb{W}_s^2(x)|^p \lesssim \sum_{i\geqslant -1} 2^{i(-1-\delta/2)p} \mathbb{E}|\Delta_i \mathbb{W}_s^2(0)|^p \lesssim 1,$$

uniformly in $s\geqslant 0$. Similarly, we have

$$\left[\mathbb{E}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho))}^4\right]^{p/4} \leqslant \mathbb{E}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho)}^p \lesssim \mathbb{E}|\mathbb{W}_s^3(0)|^p.$$

By Lemma 47

$$\mathbb{E}|\mathbb{W}_s^3(0)|^p \lesssim (\mathbb{E}|\mathbb{W}_s^3(0)|^2)^{p/2} \lesssim \langle s\rangle^{3p/2},$$

and using the standard multiplier bounds for J_s we conclude

$$\begin{aligned}
\mathbb{E}\|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+,B_{p,p}^{-1/2-\delta}(\rho))}^4 &\lesssim \mathbb{E}\left(\int_0^\infty \|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)}^2 \mathrm{d}s\right)^2\\
&\lesssim \mathbb{E}\left(\int_0^\infty \left\|\frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D}\rangle}\mathbb{W}_s^3\right\|_{B_{p,p}^{-1/2-\delta}(\rho)}^2 \mathrm{d}s\right)^2\\
&\lesssim \mathbb{E}\left(\int_0^\infty \langle s\rangle^{-1-\delta}\left(\langle s\rangle^{-3/2}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho)}\right)^2 \mathrm{d}s\right)^2\\
&\lesssim \int_0^\infty \langle s\rangle^{-1-\delta}\mathbb{E}\left(\langle s\rangle^{-3/2}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho)}\right)^4 \mathrm{d}s\\
&\lesssim 1.
\end{aligned}$$

□

Remark 17. The decomposition of the noise is similar to the one given in [30] but differs in the fact that we choose the frequency cutoff dependent on the size of the noise instead of the point, to preserve translation invariance. The price to pay is that the decomposition is nonlinear in the noise, however this does not present any inconvenience in our context.

6. Gamma convergence

In this section we establish the Γ-convergence of the variational functional obtained in Lemma 15 as $T\to\infty$. Γ-convergence is a notion of convergence introduced by De Giorgi which is well suited for the study of variational problems. The book [8] is a nice introduction to Γ-convergence in the context of the calculus of variations. For the convenience of the reader we recall here the basic definitions and results.

Definition 18. *Let $\mathcal{T}$ be a topological space and let $F, F_n: \mathcal{T} \to (-\infty, \infty]$. We say that the sequence of functionals $(F_n)_n$ Γ-converges to F iff*

i. For every sequence $x_n \to x$ in $\mathcal{T}$

$$F(x) \leqslant \liminf_{n \to \infty} F_n(x_n);$$

ii. For every point x there exists a sequence $x_n \to x$ (called a recovery sequence) such that

$$F(x) \geqslant \limsup_{n \to \infty} F_n(x_n).$$

Definition 19. *A sequence of functionals $F_n: \mathcal{T} \to (-\infty, \infty]$ is called equicoercive if there exists a compact set $\mathcal{K} \subseteq \mathcal{T}$ such that for all $n \in \mathbb{N}$*

$$\inf_{x \in \mathcal{K}} F_n(x) = \inf_{x \in \mathcal{T}} F_n(x).$$

A fundamental consequence of Γ-convergence is the convergence of minima.

Theorem 20. *If $(F_n)_n$ Γ-converges to F and $(F_n)_n$ is equicoercive, then F admits a minimum and*

$$\min_{\mathcal{T}} F = \lim_{n \to \infty} \inf_{\mathcal{T}} F_n.$$

For a proof see [20].

In this section we allow all constants to depend on the volume $|\Lambda|$: this is not critical since, at this point, the aim is to obtain explicit formulas at fixed Λ.

We denote

$$\mathcal{H}^{\alpha,p} := L^2([0,\infty); W^{\alpha,p}), \qquad \alpha \in \mathbb{R}, 1 < p < \infty,$$

and by $\mathcal{H}_w^{\alpha,p}$ the reflexive Banach space $\mathcal{H}^{\alpha,p}$ endowed with the weak topology. With this definitions we have $\mathcal{H}^{\alpha} = \mathcal{H}^{\alpha,2}$ and $\mathcal{H} = \mathcal{H}^{0,2}$. Moreover for small enough $\kappa > 0$ (fixed once and for all) we let $\mathcal{L} := \mathcal{H}^{-1/2-\kappa,3}$. This space will be useful as it gives sufficient control over Z:

Lemma 21. *For κ small enough, $u \mapsto Z(u)$ is a compact map $\mathcal{L} \to C([0,\infty], L^4)$.*

Proof. By definition of Z we have for any $0<\varepsilon<1/8-\kappa/2$,

$$\begin{aligned}\|Z_{t_2}(u)-Z_{t_1}(u)\|_{W^{\varepsilon,4}} &= \left\|\int_{t_1}^{t_2} J_s u_s \mathrm{d}s\right\|_{W^{\varepsilon,4}} \leqslant \int_{t_1}^{t_2}\left\|\frac{\sigma_s(\mathrm{D})}{\langle\mathrm{D}\rangle}u_s\right\|_{W^{\varepsilon,4}}\mathrm{d}s\\ &\lesssim \int_{t_1}^{t_2}\|\langle\mathrm{D}\rangle^{-1+\varepsilon}u_s\|_{W^{\varepsilon,4}}\frac{\mathrm{d}s}{\langle s\rangle^{1/2+\varepsilon}}\\ &\lesssim \int_{t_1}^{t_2}\|\langle\mathrm{D}\rangle^{-1+\varepsilon}u_s\|_{W^{1/4+\varepsilon,3}}\frac{\mathrm{d}s}{\langle s\rangle^{1/2+\varepsilon}}\\ &\lesssim \left(\int_0^\infty\|u_s\|_{W^{-1/2-\kappa,3}}^2\mathrm{d}s\right)^{1/2}\left(\int_{t_1}^{t_2}\frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}\right)^{1/2}\\ &\lesssim \left(\int_{t_1}^{t_2}\frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}\right)^{1/2}\|u\|_{\mathcal{L}}.\end{aligned}$$

where we have used the Sobolev embedding $W^{1/4+\varepsilon,3}\hookrightarrow W^{\varepsilon,4}$. Since

$$\lim_{t_1\to t_2}\int_{t_1}^{t_2}\frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}=0, \qquad \int_0^\infty\frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}\mathrm{d}s<\infty,$$

for any $t_2\in[0,\infty]$, we can conclude by the Rellich–Kondrachov embedding theorem and the Ascoli–Arzelá theorem, that bounded sets in $\mathcal{L}$ are mapped to compact sets in $C([0,\infty],L^4)$, proving the claim. □

In the sequel, by an abuse of notation, we will denote both a generic element of $\mathfrak{S}$ and the canonical random variable on $\mathfrak{S}$ by

$$\mathbb{X}=(\mathbb{X}^1,\mathbb{X}^2,\mathbb{X}^{\langle 3\rangle},\mathbb{X}^{[3]\circ 1},\mathbb{X}^{\langle 2\rangle\diamond\langle 2\rangle},\mathbb{X}^{2\diamond[3]}).$$

We will need the following lemma, which establishes point-wise convergence for the functional Φ_T defined in Lemma 15.

Lemma 22. *Define $l^\infty(u)=l^\infty(\mathbb{X},u)\in\mathbb{H}_a$ such that*

$$l_t^\infty(u):=u_t+\lambda\mathbb{X}_t^{\langle 3\rangle}+\lambda J_t(\mathbb{X}_t^2\succ Z_t^\flat(u)), \qquad t\geqslant 0. \tag{35}$$

For any sequence $(\mathbb{X}^T,u^T)_T$ such that $u^T\to u$ in $\mathcal{L}_w$, $l^T=l^T(\mathbb{X}^T,u^T)\to l=l^\infty(\mathbb{X},u)$ in $\mathcal{H}_w$ and

$$\begin{aligned}\mathbb{X}^T &= (\mathbb{X}^{T,1},\mathbb{X}^{T,2},\mathbb{X}^{T,\langle 3\rangle},\mathbb{X}^{T,[3]\circ 1},\mathbb{X}^{T,\langle 2\rangle\diamond\langle 2\rangle},\mathbb{X}^{T,2\diamond[3]})\\ \downarrow & \\ \mathbb{X} &= (\mathbb{X}^1,\mathbb{X}^2,\mathbb{X}^{\langle 3\rangle},\mathbb{X}^{[3]\circ 1},\mathbb{X}^{\langle 2\rangle\diamond\langle 2\rangle},\mathbb{X}^{2\diamond[3]})\end{aligned}$$

in $\mathfrak{S}$ we have

$$\lim_{T\to\infty}\Phi_T(\mathbb{X}^T,Z(u^T),K(u^T))=\Phi_\infty(\mathbb{X},Z(u),K(u)).$$

Here $Z_t(u)=I_t(u)$, we let $K_t(u):=Z_t(u)-\lambda\mathbb{X}_t^{[3]}$ and Φ_∞ is defined by

$$\Phi_\infty(\mathbb{X},Z(u),K(u)):=f(\mathbb{X}_\infty^1+Z_\infty(u))+\sum_{i=1}^6\Upsilon_\infty^{(i)}(\mathbb{X},Z(u),K(u)),$$

with $\Upsilon^{(i)}_\infty(\mathbb{X}, Z, K) = \Upsilon^{(i)}_\infty$ *given by*

$$
\begin{aligned}
\Upsilon^{(1)}_\infty &:= \frac{\lambda}{2}\mathfrak{K}_2(\mathbb{X}^2_\infty, K_\infty, K_\infty) + \frac{\lambda}{2}\fint(\mathbb{X}^2_\infty \prec K_\infty)K_\infty - \lambda^2\fint(\mathbb{X}^2_\infty \prec \mathbb{X}^{[3]}_\infty)K_\infty,\\
\Upsilon^{(2)}_\infty &= 0,\\
\Upsilon^{(3)}_\infty &:= \lambda\int_0^\infty\fint(\mathbb{X}^2_t \succ \dot{Z}^\flat_t)K_t \mathrm{d}t,\\
\Upsilon^{(4)}_\infty &:= 4\lambda\fint\mathbb{X}^1_\infty K^3_\infty - 12\lambda^2\fint(\mathbb{X}^1_\infty\mathbb{X}^{[3]}_\infty)K^2_\infty + 12\lambda^3\fint\mathbb{X}^1_\infty(\mathbb{X}^{[3]}_\infty)^2K_\infty,\\
\Upsilon^{(5)}_\infty &:= -2\lambda^2\int_0^\infty\fint\gamma_t Z^\flat_t\dot{Z}^\flat_t\mathrm{d}t,\\
\Upsilon^{(6)}_\infty &:= -\frac{\lambda^2}{2}\fint\mathbb{X}^{2\diamond[3]}_\infty K_\infty - \lambda^2\int_0^\infty\fint\mathbb{X}^{\langle 2\rangle\diamond\langle 2\rangle}_t(Z^\flat_t)^2\mathrm{d}t - \frac{\lambda^2}{2}\int_0^\infty\mathfrak{K}_{3,t}(\mathbb{X}^2_t, \mathbb{X}^2_t, Z^\flat_t,\\
&\qquad Z^\flat_t)\mathrm{d}t,
\end{aligned}
$$

where $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_{3,t}$ *are the multilinear forms defined in Proposition 60, Proposition 61 and Proposition 62 respectively and where, with abuse of notation, we let*

$$
\begin{aligned}
\mathbb{X}^1_\infty\mathbb{X}^{[3]}_\infty &:= \mathbb{X}^1_\infty \succ \mathbb{X}^{[3]}_\infty + \mathbb{X}^1_\infty \prec \mathbb{X}^{[3]}_\infty + \mathbb{X}^{[3]\circ 1}_\infty,\\
\mathbb{X}^1_\infty(\mathbb{X}^{[3]}_\infty)^2 &:= \mathbb{X}^1_\infty(\mathbb{X}^{[3]}_\infty \circ \mathbb{X}^{[3]}_\infty) + 2\mathbb{X}^{[3]\circ 1}_\infty\mathbb{X}^{[3]}_\infty + 2\mathfrak{K}_1(\mathbb{X}^{[3]}_\infty, \mathbb{X}^{[3]}_\infty, \mathbb{X}^1_\infty)\\
&\quad +2\mathbb{X}^1_\infty \succ (\mathbb{X}^{[3]}_\infty \succ \mathbb{X}^{[3]}_\infty) + 2\mathbb{X}^1_\infty \prec (\mathbb{X}^{[3]}_\infty \succ \mathbb{X}^{[3]}_\infty).
\end{aligned} \tag{36}
$$

Proof. Lemma 21 implies that for any $u^T \to u$ in $\mathcal{L}_w$ we have $Z(u^T) \to Z(u)$ in $C([0, \infty], L^4)$ and by the convergence of $l^T \to l$ in $\mathcal{H}_w$ we have also $K(u^T) \to K(u)$ in $C([0, \infty], H^{1-\kappa})$. The products $\mathbb{X}^{T,1}_T\mathbb{X}^{T,[3]}_T$ and $\mathbb{X}^{T,1}_T(\mathbb{X}^{T,[3]}_T)^2$ can be decomposed using paraproducts and, after identifying the resonant products with the corresponding stochastic objects in $\mathbb{X}^T$, we obtain the finite T analogs of the expressions in eq. (36). After this preprocessing, it is easy to see by continuity that we have $\mathbb{X}^{T,1}_T\mathbb{X}^{T,[3]}_T \to \mathbb{X}^1_\infty\mathbb{X}^{[3]}_\infty$ and $\mathbb{X}^{T,1}_T(\mathbb{X}^{T,[3]}_T)^2 \to \mathbb{X}^1_\infty(\mathbb{X}^{[3]}_\infty)^2$ in $\mathscr{C}^{1/2-\kappa}$. For $\Upsilon^{(1)}$ and $\Upsilon^{(4)}$ and the first term of $\Upsilon^{(6)}$ the statement follows from the fact that they are bounded multilinear forms on $\mathfrak{S} \times C([0, \infty], H^{1/2-\kappa}) \times C([0, \infty], H^{1-\kappa})$. For $\Upsilon^{(2)}$ and the first two terms of $\Upsilon^{(5)}$ convergence to 0 follows from the bounds established in Lemma 40 and the proof Lemma 43 (in particular eq. (55) and eq. (56)). For $\Upsilon^{(3)}$, the last term of $\Upsilon^{(5)}$ and the last two terms of $\Upsilon^{(6)}$ we can establish point-wise convergence under the time integrals since the integrands are again bounded (uniformly in time) multilinear forms, and conclude by dominated convergence. $\square$

Going back to our particular setting recall that from Lemma 15 we learned

$$\mathcal{W}_T(f) = \inf_{u \in \mathbb{H}_a} F_T(u),$$

with

$$F_T(u) = \mathbb{E}\left[\Phi_T(\mathbb{W}, Z(u), K(u)) + \lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right],$$

where $l^T(u), Z(u), K(u)$ are functions of u according to eq. (21). This form of the functional is appropriate to analyze the limit $T \to \infty$ and obtain the main result of the paper, stated precisely in the following theorem.

Theorem 23. *We have*

$$\lim_{T\to\infty} \mathcal{W}_T(f) = \mathcal{W}(f) := \inf_{u\in\mathbb{H}_a} F_\infty(u),$$

where

$$F_\infty(u) = \mathbb{E}\left[\Phi_\infty(\mathbb{W}, Z(u), K(u)) + \lambda \|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right],$$

with Φ_∞ and l^∞ introduced in Lemma 22.

Proof. The statement is a direct consequence of Theorem 27 below. □

In order to use Γ-convergence, we need to modify the variational setting to guarantee enough compactness and continuity uniformly as $T \to \infty$.

As long as T is finite, the original potential V_T is bounded below so in particular we have

$$-C_T + \mathbb{E}\left[\frac{1}{2}\|u\|_{\mathcal{H}}^2\right] \leqslant F_T(u). \tag{37}$$

which quantifies the coercivity of F_T. Unfortunately, this estimate does not survive the limit. However the analytic estimates contained in Section 7 below on the renormalized control problem allow to infer that there exists a small $\delta \in (0,1)$, and a finite constant $C > 0$ independent of T, such that

$$-C + (1-\delta)\mathbb{E}\left[\lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right] \leqslant F_T(u), \tag{38}$$

and

$$F_T(u) \leqslant C + (1+\delta)\mathbb{E}\left[\lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right]. \tag{39}$$

Moreover the cost functional $\lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2$ control the $\mathcal{L}$ norm of u uniformly in T, modulo constants depending only on $\|\mathbb{W}\|_{\mathfrak{S}}$ and which are bounded in average uniformly in T. More precisely we have (in a more general setting, useful below)

Lemma 24. *Let μ be a probability measure on $\mathfrak{S} \times \mathcal{L}$ with first marginal* $\mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ *and denote with $(\mathbb{X}, u)$ the canonical variable on $\mathfrak{S} \times \mathcal{L}$. Then there exists a constant C, depending only on λ, such that*

$$\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] \lesssim C + 2\lambda \mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4] + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2].$$

Proof. We use $\|l^T(u)\|_{\mathcal{L}} \lesssim \|l^T(u)\|_{\mathcal{H}}$ in the bound

$$\begin{aligned}
\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] &\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2] + \lambda\mathbb{E}_\mu[\|s \mapsto J_s(\mathbb{X}_s^2 \succ \theta_s Z_T(u))\|_{\mathcal{L}}^2] + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2] \\
&\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2] + \lambda\mathbb{E}_\mu\left[\int_0^\infty \frac{\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\kappa}}^2}{\langle s\rangle^{1+\kappa}} \|Z_T(u)\|_{L^4}^2 \mathrm{d}s\right] \\
&\quad + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2] \\
&\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2] + \frac{\lambda}{2}\mathbb{E}_\mu\left[\int_0^\infty \frac{\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\kappa}}^4}{\langle s\rangle^{1+\kappa}} \mathrm{d}s\right] + 2\lambda\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4] \\
&\quad + \mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2].
\end{aligned}$$

□

From this we conclude that we can relax the optimization problem and ask that $u \in \mathbb{L}_a$ where $\mathbb{L}_a$ is the space of predictable processes in $\mathcal{L}$:

$$\mathcal{W}_T(f) = \inf_{u \in \mathbb{L}_a} F_T(u).$$

For future reference note that eq. (38) implies also that for any sequence $(u^T)_T$ such that $F_T(u^T)$ remains bounded we must have that also

$$\sup_T \mathbb{E}[\|l^T(u^T)\|_{\mathcal{H}}^2] < \infty. \tag{40}$$

To prove Γ-convergence we need to set up the problem in a space with a topology which, on the one hand is strong enough to enable to prove the Γ-liminf inequality, and on the other hand allows to obtain enough compactness from F_T. Almost sure convergence on $\mathfrak{S} \times \mathcal{L}$ would allow for the former but is too strong for the latter. For this reason we need a setting based on convergence in law as made precise in the following definition.

Definition 25. *Denote by* $(\mathbb{X}, u)$ *be the canonical variables on* $\mathfrak{S} \times \mathcal{L}$ *and consider the space of probability measures*

$$\mathcal{Y} := \{\mu \in \mathcal{P}(\mathfrak{S} \times \mathcal{L}) \mid \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] < \infty\}$$

equipped with the following topology: $\mu_n \to \mu$ *iff*

a) μ_n *converges to* μ *weakly on* $\mathfrak{S} \times \mathcal{L}_w$,

b) $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$.

Let

$$\mathcal{X} := \{\mu \in \mathcal{Y} \mid \mu = \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, u) \text{ for some } u \in \mathbb{L}_a\}$$

and denote by $\bar{\mathcal{X}} \subseteq \mathcal{Y}$ *the closure of* $\mathcal{X}$ *in* $\mathcal{Y}$.

Remark 26. Condition (b) allows to exclude pathological points in $\bar{\mathcal{X}}$ and makes possible Lemma 34 below.

With these new notations we have

$$\mathcal{W}_T(f) = \inf_{\mu \in \mathcal{X}} \breve{F}_T(\mu), \tag{41}$$

where

$$\breve{F}_T(\mu) := \mathbb{E}_\mu\left[\Phi_T(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right]$$

and where $\mathbb{E}_\mu$ denotes the expectation on $\mathfrak{S} \times \mathcal{L}$ wrt. the probability measure μ. We also define the corresponding limiting functional as

$$\breve{F}_\infty(\mu) := \mathbb{E}_\mu\left[\Phi_\infty(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right]. \tag{42}$$

Finally we can state the key result of this section.

Theorem 27. *The family $(\breve{F}_T)_T$ Γ–converges to $\breve{F}_\infty$ on $\bar{\mathcal{X}}$. Moreover*

$$\lim_T \mathcal{W}_T(f) = \lim_T \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_T(\mu) = \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_\infty(\mu) = \mathcal{W}(f).$$

Proof.

Step 1. *(Relaxation)* We will prove below that:

a) the family $(\breve{F}_T)_T$ is indeed equicoercive on $\bar{\mathcal{X}}$ (Lemma 29);

b) the variational problems for $\breve{F}_T$ (with $T < \infty$ or $T = \infty$) on $\mathcal{X}$ and on $\bar{\mathcal{X}}$ are equivalent (Lemma 35 and Lemma 38).

Step 2. *(liminf inequality)* Consider a sequence $\mu^T \to \mu$ in $\bar{\mathcal{X}}$. We need to prove that

$$\liminf_{T \to \infty} \breve{F}_T(\mu^T) \geqslant \breve{F}_\infty(\mu).$$

It is enough to prove this statement for a subsequence, the full statement follows from the fact that every sequence has a subsequence satisfying the inequality. Take a subsequence (not relabeled) such that

$$\sup_T \breve{F}_T(\mu^T) < \infty. \tag{43}$$

If there is no such subsequence there is nothing to prove. Otherwise tightness for the subsequence follows like in the proof of equicoercivity in Lemma 29 below. Then invoking the Skorokhod representation theorem of [35] we can extract a subsequence (again, not relabeled) and find random variables $(\tilde{\mathbb{X}}^T, \tilde{u}^T)_T$ and $(\tilde{\mathbb{X}}, \tilde{u})$ on some probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ such that $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{\mathbb{X}}^T, \tilde{u}^T) = \mu^T$, $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{\mathbb{X}}, \tilde{u}) = \mu$ and almost surely $\tilde{\mathbb{X}}^T \to \tilde{\mathbb{X}}$ in $\mathfrak{S}$, $\tilde{u}^T \to \tilde{u}$ in $\mathcal{L}_w$. Note that $\tilde{l}^T := l^T(\tilde{\mathbb{X}}^T, \tilde{u}^T) \to l := l^\infty(\tilde{\mathbb{X}}, u)$ in $\mathcal{L}_w$ and using (43) we deduce that the almost sure convergence $l^T \to l$ in $\mathcal{H}_w$, maybe modulo taking another subsequence, again not relabeled. Note that, by the analytic estimates of Section 7 (which hold point-wise on the probability space) we have

$$\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda \|Z_T(\tilde{u}^T)\|_{L^4}^4 + \frac{1}{2}\|l^T(\tilde{u}^T)\|_{\mathcal{H}}^2 + Q(\tilde{\mathbb{X}}^T) \geqslant 0,$$

for some positive random variable $Q(\tilde{\mathbb{X}}^T)$ such that $\mathbb{E}_{\tilde{\mathbb{P}}}[Q(\tilde{\mathbb{X}}^T)] = \mathbb{E}[Q(\mathbb{W})] < \infty$ (for example we can take $Q(\tilde{\mathbb{X}}^T) = C(1 + \|\tilde{\mathbb{X}}^T\|^p_{\mathfrak{S}})$ for some large enough p). Fatou's lemma and Lemma 22 then give

$$\liminf_{T\to\infty} \breve{F}_T(\mu^T) = \liminf_{T\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}}\left[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2}\|l^T\|^2_{\mathcal{H}}\right]$$

$$= \liminf_{T\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}}\left[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2}\|l^T\|^2_{\mathcal{H}} + Q(\tilde{\mathbb{X}}^T)\right] - \mathbb{E}[Q(\mathbb{W})]$$

$$\geqslant \mathbb{E}_{\tilde{\mathbb{P}}} \liminf_{T\to\infty}\left[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2}\|l^T\|^2_{\mathcal{H}} + Q(\tilde{\mathbb{X}}^T)\right] - \mathbb{E}[Q(\mathbb{W})]$$

$$\geqslant \mathbb{E}_{\tilde{\mathbb{P}}}\left[\Phi_\infty(\tilde{\mathbb{X}}, Z(\tilde{u}), K(\tilde{u})) + \lambda\|Z_\infty(\tilde{u})\|^4_{L^4} + \frac{1}{2}\|l^\infty(\tilde{u})\|^2_{\mathcal{H}}\right] = \breve{F}_\infty(\mu),$$

which is the Γ-liminf inequality.

Step 3. *(limsup inequality)* Now all that remains is constructing a recovery sequence, for this we can again assume w.l.o.g that $\breve{F}_\infty(\mu) < \infty$. From Lemma 37 there is μ_L such that $|\breve{F}_\infty(\mu) - \breve{F}_\infty(\mu_L)| < \frac{1}{L}$ and (50) is satisfied. Then choosing $\mu_L^T = \mathrm{Law}_{\mu_L}(\mathbb{X}, \mathbb{1}_{\{t\leqslant T\}}u_t)$ we obtain that $l^T(\mathbb{1}_{\{\cdot\leqslant T\}}u) = \mathbb{1}_{\{\cdot\leqslant T\}}l^\infty(u)$, so $\|l^T(\mathbb{1}_{\{\cdot\leqslant T\}}u)\|_{\mathcal{H}} \leqslant \|l^\infty(u)\|_{\mathcal{H}}$, and $\|Z_T(\mathbb{1}_{\{\cdot\leqslant T\}}u)\|^4_{L^4} = \|Z_T(u)\|^4 \leqslant \|u\|^4_{\mathcal{L}}$, which is integrable by (50). By dominated convergence and Lemma 22 we obtain $\lim_{T\to\infty}\breve{F}_T(\mu_L^T) = \breve{F}_\infty(\mu_L)$. Extracting a suitable diagonal sequence gives the required recovery sequence. □

The rest of this section contains the auxiliary lemmas required to complete the proof of the previous theorem.

Lemma 28. *Let $G \subseteq \bar{\mathcal{X}}$ such that $\sup_{\mu\in G}\mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}] < \infty$. Then G is tight on $\mathfrak{S}\times\mathcal{L}_w$ and in particular compact in $\bar{\mathcal{X}}$.*

Proof. Observe that for all $\mu \in G$, $\mathrm{Law}_\mu(\mathbb{X}) = \mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ and that $\mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ on $\mathfrak{S}$ is tight since $\mathfrak{S}$ is a separable metric space, so for any $\varepsilon > 0$, we can find a compact set $\mathcal{K}^1_\varepsilon \subset \mathfrak{S}$ such that $\mu((\mathfrak{S}\setminus\mathcal{K}^1_\varepsilon)\times\mathcal{L}) < \varepsilon/2$. Now let $\mathcal{K}^2_\varepsilon := \mathcal{K}^1_\varepsilon \times B(0, C) \subset \mathfrak{S}\times\mathcal{L}$, for some large C to be chosen later. Then $\mathcal{K}^2_\varepsilon$ is a compact subset of $\mathfrak{S}\times\mathcal{L}_w$ and

$$\mathbb{P}_\mu[(\mathbb{X}, u) \notin \mathcal{K}^2_\varepsilon] \leqslant \frac{\varepsilon}{2} + \frac{1}{C}\mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}].$$

Choosing $C > \sup_{\mu\in G} 2\,\mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}]/\varepsilon$ gives tightness of the family G. □

Lemma 29. *The family $(\breve{F}_T)_T$ is equicoercive on $\bar{\mathcal{X}}$.*

Proof. Define for some $K>0$ large enough

$$\mathcal{K}:=\{\mu\in\bar{\mathcal{X}}:\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2]\leqslant K\}.$$

Note that $\mathcal{K}$ is compact from Lemma 28. From eq. (38) we have

$$\lambda\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4]+\frac{1}{2}\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2]\leqslant C+2\breve{F}_T(\mu).$$

Indeed, note that the analytic estimates of Section 7 are path-wise and holds also wrt. $(\mathbb{X},u)$ under the measure μ (the point is that here u is not necessarily adapted to $\mathbb{X}$), while for the probabilistic estimates on $Q_T(\mathbb{W})$ we have $\mathbb{E}[Q_T(\mathbb{W})]=\mathbb{E}_\mu[Q_T(\mathbb{X})]$ since $\mathrm{Law}_\mu(\mathbb{X})=\mathrm{Law}_{\mathbb{P}}(\mathbb{W})$. From this we deduce that for some $C,c>0$

$$\begin{aligned}\breve{F}_T(\mu) &\geqslant \frac{\lambda}{2}\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4]+\frac{1}{4}\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2]-C\\ &\geqslant c\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2]-C\end{aligned}$$

where in the last line we have used Lemma 24. Therefore $\inf_{\mu\in\mathcal{K}^c}\breve{F}_T(\mu)\geqslant cK-C$. On the other hand from eq. (39) it follows that $\sup_T\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_T(\mu)<\infty$. So for K large enough

$$\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_T(\mu)=\inf_{\mu\in\mathcal{K}}\breve{F}_T(\mu).$$

□

To be able to use this equicoercivity we will need to show that we can extend the infimum in (41) to $\bar{\mathcal{X}}$. For this we will first need some properties of the space $\bar{\mathcal{X}}$. In particular we will need to show that measures with sufficiently high moments are dense in $\bar{\mathcal{X}}$ in a way which behaves well with respect to $\breve{F}_T$. With this aim we introduce some useful approximations.

Definition 30. *Let $u\in\mathcal{L}$, $N\in\mathbb{N}$, and $(\eta_\varepsilon)_{\varepsilon>0}$ be a smooth Dirac sequence on Λ and $(\varphi_\varepsilon)_{\varepsilon>0}$ be another smooth Dirac sequence compactly supported on $\mathbb{R}_+\times\Lambda$. Denote by $*_\Lambda$ the convolution only wrt the space variable, and by $*$ the space-time convolution. Define the following approximations of the identity:*

$$\begin{aligned}(\mathrm{reg}_{x,\varepsilon}(u)) &:= u*_\Lambda\eta_\varepsilon,\\ (\mathrm{reg}_{t:x,\varepsilon}(u))(t) &:= e^{-\varepsilon t}u*\varphi_\varepsilon(t)=e^{-\varepsilon t}\int_0^t u(t-s)*_\Lambda\varphi_\varepsilon(s)\,\mathrm{d}s.\end{aligned}$$

Let

$$\tilde{T}^N(u):=\inf\left\{t\geqslant 0\middle|\int_0^t\|u(s)\|_{W^{-1/2-\kappa,3}}^2\mathrm{d}s\geqslant N\right\},$$

and

$$(\mathrm{cut}_N(u))(t):=u(t)\mathbb{1}_{\{t\leqslant\tilde{T}^N(u)\}}.$$

Observe the following properties of these maps:

- $\mathrm{reg}_{x,\varepsilon}$ *is a continuous map* $\mathcal{L}_w \to \mathcal{H}_w$ *and* $\mathcal{L} \to \mathcal{H}$*;*
- $\mathrm{reg}_{t:x,\varepsilon}$ *is a continuous map* $\mathcal{L}_w \to \mathcal{H}$*;*
- cut_N *is continuous as a map* $\mathcal{L} \to B(0,N) \subset \mathcal{L}$*;*
- *if u is a predictable process then* $\mathrm{reg}_{x,\varepsilon}(u)$*,* $\mathrm{reg}_{t:x,\varepsilon}(u)$*,* $\mathrm{cut}_N(u)$ *will also be predictable.*

Furthermore we have the bounds

$$\|\mathrm{reg}_{x,\varepsilon}(u)\|_{\mathcal{L}}, \|\mathrm{reg}_{t:x,\varepsilon}(u)\|_{\mathcal{L}}, \|\mathrm{cut}_N(u)\|_{\mathcal{L}} \leqslant \|u\|_{\mathcal{L}}.$$

uniformly in ε, N, and for every $u \in \mathcal{L}$,

$$\lim_{\varepsilon \to 0} \|\mathrm{reg}_{x,\varepsilon}(u) - u\|_{\mathcal{L}} = \lim_{\varepsilon \to 0} \|\mathrm{reg}_{t:x,\varepsilon}(u) - u\|_{\mathcal{L}} = \lim_{N \to \infty} \|\mathrm{cut}_N(u) - u\|_{\mathcal{L}} = 0.$$

With abuse of notation, for $\mu \in \mathcal{P}(\mathfrak{S} \times \mathcal{L})$ and $f: \mathcal{L} \to \mathcal{L}$, we let

$$f_* \mu = (\mathrm{Id}, f)_* \mu = \mathrm{Law}_\mu(\mathbb{X}, f(u)).$$

Remark 31. Let us briefly comment on the rationale for these approximation. $\mathrm{reg}_{t:x,\varepsilon}$ will be used when one wants to obtain a sequence of weakly convergent measures on $\mathfrak{S} \times \mathcal{H}$ or $\mathfrak{S} \times \mathcal{L}$ from a sequence of measures weakly convergent on $\mathfrak{S} \times \mathcal{L}_w$. $\mathrm{reg}_{x,\varepsilon}$ will be used when one wants to obtain a measure on $\mathfrak{S} \times \mathcal{H}$ from one on $\mathfrak{S} \times \mathcal{L}$, while preserving the estimates on the moments of $Z(u)$ since $Z(u *_\Lambda \eta_\varepsilon) = Z(u) *_\Lambda \eta_\varepsilon$.

Lemma 32. *Let $\mu \in \bar{\mathcal{X}}$. There exist $(\mu_n)_n$ in $\mathcal{X}$ such that $\mu_n \to \mu$ on $\mathfrak{S} \times \mathcal{L}$ (now with the norm topology) and* $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$.

Proof. By definition of $\bar{\mathcal{X}}$ of there exists $\tilde{\mu}_n \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}_w$. Then $(\mathrm{reg}_{t:x,\varepsilon})_* \tilde{\mu}_n \to (\mathrm{reg}_{t:x,\varepsilon})_* \mu$ on $\mathfrak{S} \times \mathcal{L}$ as $n \to \infty$, and since $(\mathrm{reg}_{t:x,\varepsilon})_* \mu \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}$ as $\varepsilon \to 0$, we obtain the statement by taking a diagonal sequence. □

Lemma 33. *Let $\mu_n \to \mu$ on $\mathfrak{S} \times \mathcal{L}$, such that* $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$*. Then*

1. *for every Lipschitz function f on $\mathcal{L}$, $\mathbb{E}_{\mu_n}[f(u)] \to \mathbb{E}_\mu[f(u)]$;*
2. *for every Lipschitz function f on $C([0,\infty], L^4)$ we have $\mathbb{E}_{\mu_n}[f(Z(u))] \to \mathbb{E}_\mu[f(Z(u))]$.*

Proof. Let f be a Lipschitz function on $\mathcal{L}$ with Lipschitz constant L. Let $\eta \in C(\mathbb{R}, \mathbb{R})$ be supported on $B(0, 2)$ with $\eta = 1$ on $B(0, 1)$, and $\eta_N(x) = \eta(x/N)$. Then $u \mapsto f(u)\, \eta_N(\|u\|_{\mathcal{L}})$ is bounded,

$$\lim_{n \to \infty} \mathbb{E}_{\mu_n}[f(u)\, \eta_N(\|u\|_{\mathcal{L}})] = \mathbb{E}_\mu[f(u)\, \eta_N(\|u\|_{\mathcal{L}})],$$

and

$$\begin{aligned}\mathbb{E}_{\mu_n}[f(u)\eta_N(\|u\|_{\mathcal{L}})]-\mathbb{E}_{\mu_n}[f(u)] &= \mathbb{E}_{\mu_n}[(f(u)\eta_N(\|u\|_{\mathcal{L}})-f(u))\mathbb{1}_{\{\|u\|_{\mathcal{L}}\geqslant N\}}\\ &\leqslant \mathbb{E}_{\mu_n}[2L\|u\|_{\mathcal{L}}\mathbb{1}_{\{\|u\|_{\mathcal{L}}\geqslant N\}}]\\ &\leqslant 2L\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2]^{1/2}\mu_n(\|u\|_{\mathcal{L}}\geqslant N)\\ &\leqslant \frac{2L}{N}\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2].\end{aligned}$$

Using that $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2]<\infty$ we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}_{\mu_n}[f(u)]-\mathbb{E}_{\mu}[f(u)] &\leqslant \left|\lim_{n\to\infty}\mathbb{E}_{\mu_n}[f(u)\ \eta_N(\|u\|_{\mathcal{L}}^2)]-\mathbb{E}_{\mu}[f(u)\ \eta_N(\|u\|_{\mathcal{L}}^2)]\right|\\ &\quad+\sup_n|\mathbb{E}_{\mu_n}[f(u)\ \eta_N(\|u\|_{\mathcal{L}}^2)]-\mathbb{E}_{\mu_n}[f(u)]|\\ &\quad+\sup_n|\mathbb{E}_{\mu}[f(u)\ \eta_N(\|u\|_{\mathcal{L}}^2)]-\mathbb{E}_{\mu}[f(u)]|\\ &\leqslant \frac{4L}{N}\sup_n\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2]\lesssim N^{-1},\end{aligned}$$

and sending $N\to\infty$ gives the statement. The second statement follows from the first and Lemma 21. □

The next lemma proves that we can approximate measures in $\bar{\mathcal{X}}$ by measures with bounded support in the second marginal which are still in $\bar{\mathcal{X}}$.

Lemma 34. *Let $\mu\in\bar{\mathcal{X}}$ such that $E_\mu[\|Z_T(u)\|_{L^4}^4]+E_\mu[\|u\|_{\mathcal{L}}^2]<\infty$. For any $L>0$ there exists $\mu_L\in\bar{\mathcal{X}}$ such that $\|u\|_{\mathcal{L}}\leqslant L$, μ_L-almost surely, $\mu_L\to\mu$ weakly on $\mathfrak{S}\times\mathcal{L}$ as $L\to\infty$,*

$$\mathbb{E}_{\mu_L}[\|Z_T(u)\|_{L^4}^4]\to\mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4],\quad \textit{and}\quad \mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^2]\to\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2].$$

Furthermore for any μ_L there exists $(\mu_{L,n})_n\subset\mathcal{X}$ such that $\|u\|_{\mathcal{L}}\leqslant L$, $\mu_{L,n}$-almost surely and $\mu_{L,n}\to\mu_L$ weakly on $\mathfrak{S}\times\mathcal{L}_w$.

Proof.

Step 1 First let us show how to approximate μ with $\tilde{\mu}_L$ which are defined such that $\|Z_T(u)\|_{L^4}\leqslant L$, $\tilde{\mu}_L$ almost surely. As $\mu\in\bar{\mathcal{X}}$, there exists $(\mu_n)_n\subset\mathcal{X}$ such that $\mu_n\to\mu$ on $\mathfrak{S}\times\mathcal{L}$ and $\sup_n\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2]<\infty$. Since $\mu_n\in\mathcal{X}$ there exist $(u^n)_n$ adapted such that $\mu_n=\mathrm{Law}(\mathbb{W},u^n)$. Define $\tilde{Z}_s^n:=\mathbb{E}\left[\int_0^T J_t u_t^n \mathrm{d}t\middle|\mathcal{F}_s\right]=\int_0^T J_t\mathbb{E}[u_t^n|\mathcal{F}_s]\mathrm{d}t$. Then $\tilde{Z}$ is a martingale with continuous paths in $L^4(\Lambda)$. Define the stopping time $T_{L,n}=\inf\{t\in[0,T]|\|\tilde{Z}_t^n\|_{L^4}\geqslant L\}$ where the infimum is equal to T if the set is empty. Observe that $\tilde{Z}_{T_{L,n}}=\int_0^T J_t\mathbb{E}[u_t^n|\mathcal{F}_{T_{L,n}}]\mathrm{d}t=Z_T(u^{L,n})$ with $u_t^{L,n}:=\mathbb{E}[u_t^n|\mathcal{F}_{T_{L,n}}]$ adapted, by optional sampling, and almost surely $\|\tilde{Z}_{T_L}\|_{L^4}\leqslant L$. Now set $\tilde{\mu}_{L,n}:=\mathrm{Law}_{\mathbb{P}}(\mathbb{W},u^{L,n})$.

Step 1.1 (Tightness) The next goal is to show that for fixed L, we can select a suitable convergent subsequence from $(\tilde{\mu}_{L,n})_n$. For this we first show that $(\tilde{\mu}_{L,n})_n$ is tight on $\mathfrak{S}\times\mathcal{L}_w$. From the definition of $\mathcal{X}$ we have that $\sup_n \mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}]<\infty$, and by construction

$$\sup_n \mathbb{E}_{\tilde{\mu}_{L,n}}[\|u\|^2_{\mathcal{L}}]\leqslant \sup_n \mathbb{E}_{\mathbb{P}}[\|\mathbb{E}[u^n_t|\mathcal{F}_{T_{L,n}}]\|^2_{\mathcal{L}}]\leqslant \sup_n \mathbb{E}_{\mathbb{P}}[\|u^n\|^2_{\mathcal{L}}]=\sup_n \mathbb{E}_{\mu_n}[\|u\|^2_{\mathcal{L}}]<\infty,$$

which gives tightness according to Lemma 28. We can then select a subsequence which converges on $\mathcal{L}_w$.

Step 1.2 (Bounds) Let $\tilde{\mu}_L$ be the limit of the sequence constructed in Step 1.1. In this step we prove bounds on the relevant moments of $\tilde{\mu}_L$. Let f_1^M, f_2^M be sequences of functions on $\mathbb{R}$ which are Lipschitz, convex and monotone for every M, while for every $x\in\mathbb{R}$

$$\begin{aligned}
&0\leqslant f_1^M(x)\leqslant x^2, \qquad \lim_{M\to\infty} f_1^M(x)=x^2,\\
&0\leqslant f_2^M(x)\leqslant x^4, \qquad \lim_{M\to\infty} f_2^M(x)=x^4.
\end{aligned}$$

Then $f_1^M(\|u\|_{\mathcal{L}})$ is a lower-semi continuous positive function on $\mathcal{L}_w$ so by the Portmanteau lemma we have

$$\mathbb{E}_{\tilde{\mu}_L}[f_1^M(\|u\|_{\mathcal{L}})]\leqslant \liminf_{n\to\infty}\mathbb{E}_{\tilde{\mu}_{L,n}}[f_1^N(\|u\|_{\mathcal{L}})],$$

and since it is also Lipschitz continuous and convex we have

$$\begin{aligned}
\liminf_{n\to\infty}\mathbb{E}_{\tilde{\mu}_{L,n}}[f_1^M(\|u\|_{\mathcal{L}})] &= \liminf_{n\to\infty}\mathbb{E}_{\mathbb{P}}[f_1^M(\|\mathbb{E}[u_n|\mathcal{F}_{T_{L,n}}]\|_{\mathcal{L}})]\\
&\leqslant \liminf_{n\to\infty}\mathbb{E}_{\mathbb{P}}[f_1^M(\|u_n\|_{\mathcal{L}})]=\mathbb{E}_{\mu}[f_1^M(\|u\|_{\mathcal{L}})].
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}_{\tilde{\mu}_L}[\|u\|^2_{\mathcal{L}}] &= \lim_{M\to\infty}\mathbb{E}_{\tilde{\mu}_L}[f_1^M(\|u\|_{\mathcal{L}})]\\
&\leqslant \lim_{M\to\infty}\mathbb{E}_{\mu}[f_1^M(\|u\|_{\mathcal{L}})]=\mathbb{E}_{\mu}[\|u\|^2_{\mathcal{L}}].
\end{aligned}$$

Proceeding similarly for Z, we see that $f_2^M(\|Z_T\|_{L^4})$ is a continuous function on L^4 bounded below, Lipschitz continuous and convex on L^4 so we again can estimate

$$\mathbb{E}_{\tilde{\mu}_L}[f_2^M(\|Z_T\|_{L^4})]=\lim_{n\to\infty}\mathbb{E}_{\tilde{\mu}_{L,n}}[f_2^M(\|Z_T\|_{L^4})],$$

$$\begin{aligned}
\mathbb{E}_{\tilde{\mu}_L}[f_2^N(\|Z_T\|_{L^4})] &= \lim_{n\to\infty}\mathbb{E}_{\tilde{\mu}_{L,n}}[f_2^M(\|Z_T\|_{L^4})]\\
&= \lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}[f_2^M(\|\mathbb{E}[Z_T(u_n)|\mathcal{F}_{T_{L,n}}]\|_{L^4})]\\
&\leqslant \lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}[f_2^M(\|Z_T(u_n)]\|_{L^4})]=\mathbb{E}_{\mu}[f_2^M(\|Z_T(u_n)]\|_{L^4})].
\end{aligned}$$

Taking $N\to\infty$, we obtain

$$\mathbb{E}_{\tilde{\mu}_L}[\|Z_T\|^4_{L^4}]\leqslant \mathbb{E}_{\mu}[\|Z_T\|^4_{L^4}].$$

Step 1.3 (Weak convergence) Now we prove weak convergence of $\tilde{\mu}_L$ to μ on $\mathfrak{S} \times \mathcal{L}$. Let $f: \mathfrak{S} \times \mathcal{L} \to \mathbb{R}$ be bounded and continuous. By dominated convergence and continuity of f, $\lim_\varepsilon \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))] = \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, u)]$. Using furthermore that $(\mathbb{X}, u) \mapsto f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))$ is continuous on $\mathfrak{S} \times \mathcal{L}_w$ and Lemma 21 in the 5th line below, we can estimate

$$
\begin{aligned}
& \lim_{L\to\infty} |\mathbb{E}_\mu[f(\mathbb{X}, u)] - \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, u)]| \\
= & \lim_{L\to\infty}\lim_{\varepsilon\to 0} \left| \lim_{n\to\infty} \mathbb{E}_{\mu_n}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u^n))] - \mathbb{E}_{\tilde{\mu}_{L,n}}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u^n))] \right| \\
= & \lim_{L\to\infty}\lim_{\varepsilon\to 0} \left| \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[f(\mathbb{W}, \mathrm{reg}_{t:x,\varepsilon}(u^n)) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}])] \right| \\
= & \lim_{L\to\infty}\lim_{\varepsilon\to 0} \left| \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[f(\mathbb{W}, \mathrm{reg}_{t:x,\varepsilon}(u^n)) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}]) \mathbb{1}_{\{T_L<\infty\}}] \right| \\
\leqslant & \lim_{L\to\infty}\lim_{\varepsilon\to 0} \left| \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[f(\mathbb{W}, \mathrm{reg}_{t:x,\varepsilon}(u^n)) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}]) \mathbb{1}_{\{\|u^n\|_{\mathcal{L}}>cL\}}] \right| \\
\leqslant & \frac{2}{c}\left(\sup_{\mathfrak{S}\times\mathcal{L}} |f| \right) \lim_{L\to\infty} \sup_n \frac{\mathbb{E}[\|u^n\|_{\mathcal{L}}^2]}{L^2} = 0.
\end{aligned}
$$

Step 2 In this step we improve the approximation to have bounded support. Let $\mu_n \to \mu$ be the subsequence selected in Step 1.1. Recall that $\mu_n = \mathrm{Law}(\mathbb{W}, u^n)$ with adapted u^n. Define $\tilde{Z}_t^{n,N} := \mathbb{E}[Z_T(\mathrm{cut}_N(u)) \mid \mathcal{F}_t]$, and similarly to Step 1, $T_{n,L,N} := \inf \{t \geqslant 0 | \|\tilde{Z}_t^{n,N}\|_{L^4} \geqslant L\}$. Set $u^{n,N,L} := \mathbb{E}[\mathrm{cut}_N(u) \mid \mathcal{F}_{T_{n,L,N}}]$, then $\|u^{n,N,L}\|_{\mathcal{L}} \leqslant N$ uniformly in n and $\mathbb{P}$-almost surely, so $\mu_{n,L,N} = \mathrm{Law}(\mathbb{W}, u^{n,N,L})$ is tight on $\mathfrak{S} \times \mathcal{L}_w$ and we can select a weakly convergent subsequence. Denote the limit by $\mu_{L,N}$. Now we follow the strategy from Step 1.

Step 2.1 (Bounds) We now prove bounds on $\mu_{L,N}$ uniformly in L, N similarly to step 1.2. Let f_1^M be defined like in Step 1.2. Then again we have

$$
\begin{aligned}
\liminf_{n\to\infty} \mathbb{E}_{\mu_{n,L,N}}[f_1^M(\|u\|_{\mathcal{L}})] &= \liminf_{n\to\infty} \mathbb{E}_{\mathbb{P}}[f_1^M \|\mathbb{E}[\mathrm{cut}_N(u^n)|\mathcal{F}_{T_{n,L,N}}]\|_{\mathcal{L}})] \\
&\leqslant \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[f_1^M(\|\mathrm{cut}_N(u^n)\|_{\mathcal{L}})] \\
&= \lim_{n\to\infty} \mathbb{E}_{\mu_n}[f_1^M(\|\mathrm{cut}_N(u)\|_{\mathcal{L}})] \leqslant \mathbb{E}_\mu[f_1^M(\|u\|_{\mathcal{L}})].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathbb{E}_{\mu_{L,N}}[\|u\|_{\mathcal{L}}^2] &= \lim_{M\to\infty} \mathbb{E}_{\tilde{\mu}_{L,N}}[f_1^M(\|u\|_{\mathcal{L}})] \\
&\leqslant \lim_{M\to\infty} \liminf_{n\to\infty} \mathbb{E}_{\mu_{n,L,N}}[f_1^M(\|u\|_{\mathcal{L}})] \\
&\leqslant \lim_{M\to\infty} \mathbb{E}_\mu[f_1^M(\|u\|_{\mathcal{L}})] = \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2].
\end{aligned}
$$

Step 2.1 (Weak convergence) Now we prove that $\mu_{L,N} \to \tilde{\mu}_L$ weakly on $\mathcal{L}$. Let $f: \mathfrak{S} \times \mathcal{L} \to \mathbb{R}$ be bounded and continuous. By dominated convergence and continuity of f, $\lim_{\varepsilon} \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))] = \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, u)]$, and furthermore since $f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))$ is continuous on $\mathfrak{S} \times \mathcal{L}_w$ we have (recall that $\tilde{T}^N(u^n)$ is introduced in Definition 30)

$$
\begin{aligned}
& \lim_{N\to\infty} |\mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, u)] - \mathbb{E}_{\mu_{L,N}}[f(\mathbb{X}, u)]| \\
= & \lim_{N\to\infty} \lim_{\varepsilon\to 0} \left| \lim_{n\to\infty} \mathbb{E}_{\mu_{n,L}}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))] - \mathbb{E}_{\tilde{\mu}_{n,L,N}}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))] \right| \\
= & \lim_{N\to\infty} \lim_{\varepsilon\to 0} \left| \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}]) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(\bar{u}^{n,N})|\mathcal{F}_{T_{n,L,N}}])] \right| \\
= & \lim_{N\to\infty} \lim_{\varepsilon\to 0} \left| \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[(f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}]) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(\bar{u}^{n,N})|\mathcal{F}_{T_{n,L,N}}]))\mathbb{1}_{\{\tilde{T}^N(u^n)<\infty\}}] \right| \\
\leqslant & \lim_{N\to\infty} \sup_{\varepsilon} \left| \sup_n \mathbb{E}_{\mathbb{P}}[(f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}]) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(\bar{u}^{n,N})|\mathcal{F}_{T_{n,L,N}}]))\mathbb{1}_{\{\|u^n\|_{\mathcal{L}}\}>N\}}] \right| \\
\leqslant & \left(\sup_{\mathfrak{S}\times\mathcal{L}} |f| \right) \lim_{N\to\infty} \sup_n \frac{\mathbb{E}[\|u^n\|_{\mathcal{L}}^2]}{N^2} \\
= & 0
\end{aligned}
$$

Step 3. We now put everything together. Since all $\mu_{L,N}$ are supported on the set $\{u: \|Z_T(u)\|_{L^4} \leqslant L\}$, weak convergence and Lemma 21 imply

$$
\lim_{N\to\infty} \mathbb{E}_{\mu_{N,L}}[\|Z_T(u)\|_{L^4}^4] = \mathbb{E}_{\tilde{\mu}_L}[\|Z_T(u)\|_{L^4}^4].
$$

By the Portmanteau lemma,

$$
\liminf_{N\to\infty} \mathbb{E}_{\mu_{N,L}}[\|u\|_{\mathcal{L}}^2] \geqslant \mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2], \tag{44}
$$

and

$$
\liminf_{L\to\infty} \mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2] \geqslant \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]
$$

which together with Step 1.2 imply $\lim_{L\to\infty} \mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2] = \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]$, and by the same argument $\lim_{L\to\infty} \mathbb{E}_{\tilde{\mu}_L}[\|Z_T(u)\|_{L^4}^4] = \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]$. For any $\delta > 0$ we can choose a $\tilde{\mu}_L$ such that

$$
|\mathbb{E}_{\tilde{\mu}_L}[\|Z_T(u)\|_{L^4}^4] - \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]| + |\mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2] - \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]| \leqslant \delta.
$$

By (44)

$$
\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2] \geqslant \liminf_{N\to\infty} \mathbb{E}_{\mu_{N,L}}[\|u\|_{\mathcal{L}}^2] \geqslant \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2] - \delta,
$$

and we can choose N large enough so that

$$|\mathbb{E}_{\mu_{N,L}}[\|Z_T(u)\|_{L^4}^4] - \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]| + |\mathbb{E}_{\mu_{N,L}}[\|u\|_{\mathcal{L}}^2] - \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]| \leqslant \delta,$$

which implies the statement of the theorem. □

Lemma 35. *If $T < \infty$ we have*

$$\inf_{\mu \in \mathcal{X}} \breve{F}_T(\mu) = \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_T(\mu).$$

Proof. To prove the claim it is enough to show that for any $\mu \in \bar{\mathcal{X}}$, for any $\alpha > 0$, there exists a sequence $\mu_n \in \mathcal{X}$ such that $\limsup_{n \to \infty} \breve{F}_T(\mu_n) \leqslant \breve{F}_T(\mu) + \alpha$. W.l.o.g we can assume that $\breve{F}_T(\mu) < \infty$. Observe that, as long as $T < \infty$ we can also express

$$\breve{F}_T(\mu) = \mathbb{E}_{\mu}\left[\frac{1}{|\Lambda|} V_T(\mathbb{X}_T^1 + Z_T(u)) + \frac{1}{2}\|u\|_{\mathcal{H}}^2\right],$$

and deduce that $\mathbb{E}_{\mu}\|u\|_{\mathcal{H}}^2 < \infty$ since V_T is bounded below at fixed T. By Lemma 34 there exists a sequence $(\mu_L)_L \subset \bar{\mathcal{X}}$, such that $\mu_L(\|u\|_{\mathcal{L}} \leqslant L) = 1$, $\mu_L \to \mu$ on $\mathfrak{S} \times \mathcal{L}$ and by weak convergence and domination,

$$\mathbb{E}_{\mu_L}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4], \qquad \mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^2] \to \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2].$$

First we have to improve the regularity of μ_L to get convergence on $\mathfrak{S} \times \mathcal{H}_w$ but without affecting our control on the moments of Z_T, so let $\mu_L^{\varepsilon} := (\mathrm{reg}_{x,\varepsilon})_* \mu_L$ and $\mu^{\varepsilon} := (\mathrm{reg}_{x,\varepsilon})_* \mu$. Then

$$\mathbb{E}_{\mu_L^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4] \to E_{\mu^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4], \qquad \mathbb{E}_{\mu_L^{\varepsilon}}[\|u\|_{\mathcal{H}}^2] \to E_{\mu^{\varepsilon}}[\|u\|_{\mathcal{H}}^2],$$

and $\mu_L^{\varepsilon} \to \mu^{\varepsilon}$ on $\mathfrak{S} \times \mathcal{H}$. By continuity of $\breve{F}_T$ and the bound (39), $\breve{F}_T(\mu_L^{\varepsilon}) \to \breve{F}_T(\mu^{\varepsilon})$ as $L \to \infty$ and $\breve{F}_T(\mu^{\varepsilon}) \to \breve{F}_T(\mu)$ as $\varepsilon \to 0$. In particular we can find L and ε such that $|\breve{F}_T(\mu_L^{\varepsilon}) - \breve{F}_T(\mu)| < \alpha / 2$. By Lemma 34 there exists a sequence $(\mu_{n,L})_{n,L}$ such that each measure $\mu_{n,L}$ is supported on $\mathfrak{S} \times B(0, L)$ and $\mu_{n,L} \to \mu_L$ weakly on $\mathfrak{S} \times \mathcal{H}_w$. Setting $\mu_{n,L}^{\varepsilon,\delta} := (\mathrm{reg}_{t;x,\delta})_* (\mathrm{reg}_{x,\varepsilon})_* \mu_{n,L}$ and $\mu_L^{\varepsilon,\delta} := (\mathrm{reg}_{t;x,\delta})_* (\mathrm{reg}_{x,\varepsilon})_* \mu_L$ we have $\mu_{n,L}^{\varepsilon,\delta} \to \mu_L^{\varepsilon,\delta}$ on $\mathfrak{S} \times \mathcal{H}$ with norm topology. It is not hard too see that $V_T(\mathbb{X}_T^1 + Z_T(u)) \lesssim_T \|\mathbb{X}\|_{\mathfrak{S}}^4 + \|u\|_{\mathcal{H}}^4$ and since on the support of $\mu_{n,L}^{\varepsilon,\delta}$, $\|u\|_{\mathcal{H}} \leqslant L$ and the first marginal of $\mu_{n,L}^{\varepsilon,\delta}$ is fixed we have again by domination and weak convergence

$$\lim_{n \to \infty} \mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\frac{1}{|\Lambda|} V_T(\mathbb{X}_T^1 + Z_T(u)) + \frac{1}{2}\|u\|_{\mathcal{H}}^2\right] = \mathbb{E}_{\mu_L^{\varepsilon,\delta}}\left[\frac{1}{|\Lambda|} V_T(\mathbb{X}_T^1 + Z_T(u)) + \frac{1}{2}\|u\|_{\mathcal{H}}^2\right]$$

and by dominated convergence (since $\mu_L^{\varepsilon,\delta}$ is supported on $\mathfrak{S} \times B(0, L)$) we can find a δ such that $|\breve{F}_T(\mu_L^{\varepsilon,\delta}) - \breve{F}_T(\mu_L^{\varepsilon})| < \alpha / 2$ which proves the statement. □

The proof of Lemma 35 does not apply when $T = \infty$. An additional difficulty derives from the fact that in approximating the drift u we might destroy the regularity of $l^{\infty}(u)$, since now $l^{\infty}(u)$ needs to be more regular than u, contrary to the finite T case. To resolve this problem we need to be able to smooth out the remainder without destroying the bound on $Z_T(u)$. To do so smoothing $l^{\infty}(u)$ directly, and constructing a corresponding new u will not work, since $l^{\infty}(u)$ by itself does not give enough control on u and $Z(u)$. However we are still able to prove the following lemma by regularizing an "augmented" version of $l^{\infty}(u)$.

Lemma 36. *There exists a family of continuous functions* $\mathrm{rem}_{\varepsilon}: \mathfrak{S} \times \mathcal{L} \to \mathcal{L}$, *which are also continuous* $\mathfrak{S} \times \mathcal{L}_w \to \mathcal{L}_w$, *such that for any* $T \in [0, \infty]$,

$$\begin{aligned}
\|\mathrm{rem}_{\varepsilon}(\mathbb{X}, u)\|_{\mathcal{L}} &\lesssim \|\mathbb{X}\|_{\mathfrak{S}} + \|u\|_{\mathcal{L}},\\
\|Z_T(\mathrm{rem}_{\varepsilon}(\mathbb{X}, u))\|_{L^4} &\lesssim \|\mathbb{X}\|_{\mathfrak{S}} + \|Z_T(u)\|_{L^4},\\
\|l^{\infty}(\mathrm{rem}_{\varepsilon}(\mathbb{X}, u))\|_{\mathcal{H}}^2 &\lesssim_{\varepsilon} (1 + \|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_{\infty}(u)\|_{L^4}^4 + \|u\|_{\mathcal{L}}^2,
\end{aligned}$$

and $\|l^{\infty}(\mathrm{rem}_{\varepsilon}(\mathbb{X}, u))\|_{\mathcal{H}}$ *depends continuously on* $(\mathbb{X}, u) \in \mathfrak{S} \times \mathcal{L}$. *Furthermore*

$$\mathrm{rem}_{\varepsilon}(\mathbb{X}, u) \to u \quad \text{in } \mathcal{L},$$

and if $l^{\infty}(u) \in \mathcal{H}$

$$l^{\infty}(\mathrm{rem}_{\varepsilon}(\mathbb{X}, u)) \to l^{\infty}(u) \quad \text{in } \mathcal{H} \text{ as } \varepsilon \to 0.$$

Proof. Let $\mathbb{X}^2 = \mathcal{U}_{\leqslant}\mathbb{X}^2 + \mathcal{U}_{>}\mathbb{X}^2$ be the decomposition introduced in Section 5, and observe that for any $c > 0$ we can easily modify it to ensure that $\|\mathcal{U}_{>}\mathbb{X}^2\|_{\mathscr{C}^{-1-\kappa}} < c$, almost surely for any $\mu \in \bar{\mathcal{X}}$ and for any $1 \leqslant p < \infty$, $\mathbb{E}_{\mu}[\|\mathcal{U}_{\leqslant}\mathbb{X}^2\|^p_{\mathscr{C}^{-1+\kappa}}] \leqslant C$ where C depends on $|\Lambda|$, κ, c, p. Now set $\tilde{l}_t(u) = -\lambda J_s(\mathcal{U}_{\leqslant}\mathbb{X}_t^2 \succ Z_t^{\flat}(u)) + l^{\infty}(u)$. Then u satisfies

$$u_s = -\lambda \mathbb{X}_s^{\langle 3 \rangle} - \lambda J_s(\mathcal{U}_{>}\mathbb{X}_s^2 \succ Z_s^{\flat}) + \tilde{l}_s(u).$$

From this equation we can see that, like in Section 5,

$$\|u\|_{\mathcal{L}} \lesssim \lambda \|\mathbb{X}^{\langle 3 \rangle}\|_{\mathcal{L}} + \lambda \int_0^{\infty} \frac{1}{\langle s \rangle^{1+\varepsilon}} \|\mathcal{U}_{>}\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\kappa}} \mathrm{d}s \|u\|_{\mathcal{L}} + \|\tilde{l}_s(u)\|_{\mathcal{L}},$$

and choosing c small enough we get

$$\|u\|_{\mathcal{L}} \lesssim \lambda \|\mathbb{X}^{\langle 3 \rangle}\|_{\mathcal{L}} + \|\tilde{l}_s(u)\|_{\mathcal{L}}. \tag{45}$$

Similarly we observe that

$$Z_T(u) = -\lambda \mathbb{X}_T^{[3]} - \lambda \int_0^T J_s^2(\mathcal{U}_{>}\mathbb{X}_s^2 \succ Z_s^{\flat}) \mathrm{d}s + Z_T(\tilde{l}(u)),$$

so again with c small enough and since $Z_s^\flat = \theta_s Z_T$ for $s \leqslant T$:

$$\|Z_T(u)\|_{L^4} \lesssim \lambda \left\| \mathbb{X}_T^{[3]} \right\|_{L^4} + \|Z_T(\tilde{l}(u))\|_{L^4}. \tag{46}$$

Conversely, it is not hard to see that we have the inequalities

$$\|Z_T(\tilde{l}(u))\|_{L^4} \lesssim \lambda \left\| \mathbb{X}_T^{[3]} \right\|_{L^4} + \|Z_T(u)\|_{L^4}, \tag{47}$$

and

$$\|\tilde{l}(u)\|_{\mathcal{L}} \lesssim \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}. \tag{48}$$

Clearly the map $(\mathbb{X}, u) \mapsto (\mathbb{X}, \tilde{l}(u))$ is continuous as a map $\mathfrak{S} \times \mathcal{L} \to \mathfrak{S} \times \mathcal{L}$ and using Lemma 21 also as a map $\mathfrak{S} \times \mathcal{L}_w \to \mathfrak{S} \times \mathcal{L}_w$, and the inverse is clearly continuous $\mathfrak{S} \times \mathcal{L} \to \mathfrak{S} \times \mathcal{L}$. We now show that it is also continuous as a map $\mathfrak{S} \times \mathcal{L}_w \to \mathfrak{S} \times \mathcal{L}_w$. Assume that $\tilde{l}(u^n) \to l(u)$ weakly, since then $\|l(u^n)\|_{\mathcal{L}}$ bounded, this implies by (45) that also $\|u^n\|_{\mathcal{L}}$ is bounded, and so we can select a weakly convergent subsequence, converging to $u^\star$. Then $u^\star$ solves the equation

$$u_s^\star = -\lambda \mathbb{X}_s^{\langle 3\rangle} - \lambda J_s(\mathcal{U}_> \mathbb{X}_s^2 \succ Z_s^\flat(u^\star)) + \tilde{l}_s(u),$$

(which can be seen for example by testing with some $h \in \mathcal{L}^*$) which implies that $u^\star = u$ (e.g. by Gronwall). Now define $\mathrm{rem}_\varepsilon(u)$ to be the solution to the equation

$$\mathrm{rem}_\varepsilon(u) = -\lambda \mathbb{X}_s^{\langle 3\rangle} - \lambda J_s(\mathcal{U}_> \mathbb{X}_s^2 \succ Z_s^\flat(\mathrm{rem}_\varepsilon(u))) + \mathrm{reg}_{x,\varepsilon}(\tilde{l}_s(u)).$$

Then by the properties discussed above $(\mathbb{X}, u) \mapsto (\mathbb{X}, \mathrm{rem}_\varepsilon(u))$ is continuous in both the weak and the norm topology and we also have from (45) and (48) that

$$\|\mathrm{rem}_\varepsilon(u)\|_{\mathcal{L}} \lesssim \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}.$$

From (46) we have

$$\|Z_T(\mathrm{rem}_\varepsilon(u))\|_{L^4} \lesssim \lambda \left\| \mathbb{X}_T^{[3]} \right\|_{L^4} + \|Z_T(u)\|_{L^4},$$

and by definition of $\mathrm{rem}_\varepsilon(u)$

$$\begin{aligned} \|\tilde{l}(\mathrm{rem}_\varepsilon(u))\|_{\mathcal{H}} &= \|\mathrm{reg}_{x,\varepsilon}(\tilde{l}(u))\|_{\mathcal{H}} \\ &\lesssim_\varepsilon \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}. \end{aligned} \tag{49}$$

Now observe that

$$\begin{aligned} \|l^\infty(\mathrm{rem}_\varepsilon(u))\|_{\mathcal{H}}^2 &\lesssim \|s \mapsto \lambda J_s(\mathcal{U}_\leqslant \mathbb{X}_s^2 \succ Z_s^\flat(\mathrm{rem}_\varepsilon(u)))\|_{\mathcal{H}}^2 + \|\tilde{l}(\mathrm{rem}_\varepsilon(u))\|_{\mathcal{H}}^2 \\ &\lesssim_\varepsilon \lambda \int \frac{1}{\langle s\rangle^{1+\kappa}} \|\mathcal{U}_\leqslant \mathbb{X}_s^2\|_{\mathscr{C}^{-1+\kappa}}^2 \|Z_s^\flat(\mathrm{rem}_\varepsilon(u))\|_{L^4}^2 \mathrm{d}s + \\ &\quad \lambda \|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2 + \|u\|_{\mathcal{L}}^2 \\ &\lesssim \lambda(1 + \|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_\infty(\mathrm{rem}_\varepsilon(u))\|_{L^4}^4 + \|u\|_{\mathcal{L}}^2 \\ &\lesssim \lambda(1 + \|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_\infty(u)\|_{L^4}^4 + \|u\|_{\mathcal{L}}^2. \end{aligned}$$

Since also $\|\lambda J_s(\mathcal{U}_{\leqslant}\mathbb{X}_t^2 \succ Z_t^{\flat}(\mathrm{rem}_{\varepsilon}(u)))\|_{\mathcal{H}}$ depends continuously on $(\mathbb{X}, u)$ (both in the weak and strong topology on $\mathcal{L}$) we obtain the statement. $\square$

Lemma 37. *For any $\mu \in \bar{\mathcal{X}}$ such that $\breve{F}_{\infty}(\mu) < \infty$ there exists a sequence of measures $\mu_L \in \bar{\mathcal{X}}$ such that*

i. For any $p < \infty$,

$$\mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^p] + \mathbb{E}_{\mu_L}[\|l^{\infty}(u)\|_{\mathcal{H}}^p] < \infty, \tag{50}$$

ii. $\mu_L \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}$ and $\mathrm{Law}_{\mu_L}(l^{\infty}(u)) \to \mathrm{Law}_{\mu}(l^{\infty}(u))$ weakly on $\mathcal{H}$,

iii.

$$\lim_{L \to \infty} \breve{F}_{\infty}(\mu_L) = \breve{F}_{\infty}(\mu),$$

iv. For any μ_L there exists a sequence $\mu_{n,L} \in \mathcal{X}$ such that

$$\sup_n (\mathbb{E}_{\mu_{n,L}}[\|u\|_{\mathcal{L}}^p] + \mathbb{E}_{\mu_{n,L}}[\|l^{\infty}(u)\|_{\mathcal{H}}^p]) < \infty, \tag{51}$$

$\mu_{n,L} \to \mu_L$ weakly on $\mathfrak{S} \times \mathcal{L}_w$ and $\mathrm{Law}_{\mu_{n,L}}(l^{\infty}(u)) \to \mathrm{Law}_{\mu}(l^{\infty}(u))$ weakly on $\mathcal{H}_w$.

Proof. By Lemma 34 there exists a sequence $\mu_{\tilde{L}} \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}$ such that

$$\mathbb{E}_{\mu_{\tilde{L}}}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4], \qquad \mathbb{E}_{\mu_{\tilde{L}}}[\|u\|_{\mathcal{L}}^2] \to \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2],$$

and $\mu_{\tilde{L}}$ is supported on $\mathfrak{S} \times B(0, \tilde{L}) \subset \mathfrak{S} \times \mathcal{L}$. Now set $\mu_{\tilde{L}}^{\varepsilon} := (\mathrm{rem}_{\varepsilon})_* \mu_{\tilde{L}}$. Then $\mu_{\tilde{L}}^{\varepsilon} \to \mu^{\varepsilon} := (\mathrm{rem}_{\varepsilon})_* \mu$ on $\mathfrak{S} \times \mathcal{L}$ and by the bounds from Lemma 36 also $\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_{\mu^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4]$ and $\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{H}}^2] \to \mathbb{E}_{\mu^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{H}}^2]$. The bounds from Lemma 36 imply also $\mathbb{E}_{\mu^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]$, $\mathbb{E}_{\mu^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{H}}^2] \to \mathbb{E}_{\mu}[\|l^{\infty}(u)\|_{\mathcal{H}}^2]$, and furthermore

$$\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|u\|_{\mathcal{L}}^p] \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p + \|u\|_{\mathcal{L}}^p) \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p) + \tilde{L}^p,$$

and similarly

$$\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{L}}^p] \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p + \|u\|_{\mathcal{L}}^p) \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p) + \tilde{L}^p,$$

and by continuity of $\breve{F}_{\infty}$ and domination using (39) we are also able to deduce that we can find ε small enough and $\tilde{L}$ large enough depending on ε such that $|\breve{F}_{\infty}(\mu^{\varepsilon}) - \breve{F}_{\infty}(\mu)| < 1/2L$ and $|\breve{F}_{\infty}(\mu_{\tilde{L}}^{\varepsilon}) - \breve{F}_{\infty}(\mu^{\varepsilon})| < 1/2L$. Choosing $\mu_L = \mu_{\tilde{L}}^{\varepsilon}$ we obtain the first three points of the Lemma. For the fourth point recall that from Lemma 34 we have sequences $\mu_{n,\tilde{L}} \to \mu_{\tilde{L}}$ weakly on $\mathfrak{S} \times \mathcal{L}_w$, and $\mu_{n,\tilde{L}} \in \mathcal{X}$, which have support in $\mathfrak{S} \times B(0, \tilde{L})$ and since $\mathrm{rem}_{\varepsilon}$ is continuous on $\mathfrak{S} \times \mathcal{L}_w$ setting $\mu_{n,\tilde{L}}^{\varepsilon} := (\mathrm{reg}_{\varepsilon})_* \mu_{n,\tilde{L}}$ we obtain the desired sequence. $\square$

Lemma 38. *If $T = \infty$ we have*

$$\inf_{\mu \in \mathcal{X}} \breve{F}_{\infty}(\mu) = \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_{\infty}(\mu).$$

Proof. One can now proceed very similarly to the proof of Lemma 35. Let $\mu \in \bar{\mathcal{X}}$ such that $\breve{F}_{\infty}(\mu) < \infty$. By Lemma 37, for any $L, \mu \in \bar{\mathcal{X}}$, there exists a μ_L such that $|\breve{F}_{\infty}(\mu) - \breve{F}_{\infty}(\mu_L)| < 1/L$, and a sequence $(\mu_{n,L})_n$ such that $\mu_{n,L} \in \mathcal{X}$, $\mu_{n,L} \to \mu_L$ weakly on $\mathfrak{S} \times \mathcal{L}_w$, and such that (51) is satisfied. Define $\mu_{n,L}^{\varepsilon,\delta} := \mathrm{Law}(\mathbb{X}, \mathrm{rem}_{\varepsilon}(\mathrm{reg}_{t:x,\varepsilon}(u)))$, and observe that now $\mu_{n,L}^{\varepsilon,\delta} \to \mu_L^{\varepsilon,\delta}$ on $\mathfrak{S} \times \mathcal{L}$, $\mathrm{Law}_{\mu_{n,L}^{\varepsilon,\delta}}(\mathbb{X}, l^{\infty}(u)) \to \mathrm{Law}_{\mu_L^{\varepsilon,\delta}}(\mathbb{X}, l^{\infty}(u))$ on $\mathfrak{S} \times \mathcal{H}$, and that we have $\sup_n (\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|u\|_{\mathcal{L}}^p] + \mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|l^{\infty}(u)\|_{\mathcal{H}}^p]) < \infty$. Then for some $\chi \in C(\mathbb{R}, \mathbb{R})$, $\chi = 1$ on $B(0,1)$ supported on $B(0,2)$, for any $N \in \mathbb{N}$, the function

$$\begin{aligned}
& \chi\left(\frac{\|\mathbb{X}\|_{\mathfrak{S}} + \|u\|_{\mathcal{L}} + \|l^{\infty}(u)\|_{\mathcal{H}}}{N}\right)\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda\|Z_{\infty}(u)\|_{L^4}^4 + \right.\\
& \left.\frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right)\\
=: \;& \tilde{\chi}_N(\mathbb{X}, u)\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda\|Z_{\infty}(u)\|_{L^4}^4 + \frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right)
\end{aligned}$$

is bounded and continuous on $\mathfrak{S} \times \mathcal{L}$, and so by weak convergence

$$\begin{aligned}
& \lim_{n \to \infty} |\breve{F}_{\infty}(\mu_{n,L}^{\varepsilon,\delta}) - \breve{F}_{\infty}(\mu_L^{\varepsilon,\delta})|\\
\leqslant \;& \lim_{n \to \infty} \left|\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\tilde{\chi}_N(\mathbb{X}, u)\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda\|Z_{\infty}(u)\|_{L^4}^4 + \right.\right.\right.\\
& \left.\left.\frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right)\right] -\\
& \quad -\mathbb{E}_{\mu_L^{\varepsilon,\delta}}\left[\tilde{\chi}_N(\mathbb{X}, u)\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda\|Z_{\infty}(u)\|_{L^4}^4 + \right.\right.\\
& \left.\left.\left.\frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right)\right]\right|\\
& + \sup_n \mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\left|(1 - \tilde{\chi}_N(\mathbb{X}, u))\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda\|Z_{\infty}(u)\|_{L^4}^4 + \right.\right.\right.\\
& \left.\left.\left.\frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right)\right|\right]\\
& + \mathbb{E}_{\mu_L^{\varepsilon,\delta}}\left[\left|(1 - \tilde{\chi}_N(\mathbb{X}, u))\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda\|Z_{\infty}(u)\|_{L^4}^4 + \right.\right.\right.\\
& \left.\left.\left.\frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right)\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2\sup_n \mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\mathbb{1}_{\{\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^{\infty}(u)\|_{\mathcal{H}}>N\}}\left|\Phi_{\infty}(\mathbb{X}, \quad Z(u), \quad K(u)) \quad + \lambda\|Z_{\infty}(u)\|_{L^4}^4+\frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right|\right]\\
&\lesssim \sup_n \left(\mu_{n,L}^{\varepsilon,\delta}(\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^{\infty}(u)\|_{\mathcal{H}}>N)\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|\mathbb{X}\|_{\mathfrak{S}}^p+\|u\|_{\mathcal{L}}^8+\|l^{\infty}(u)\|_{\mathcal{H}}^4]\right)\\
&\lesssim \sup_n \left(\frac{1}{N}\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^{\infty}(u)\|_{\mathcal{H}}]\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|\mathbb{X}\|_{\mathfrak{S}}^p+\|u\|_{\mathcal{L}}^8+\|l^{\infty}(u)\|_{\mathcal{H}}^4]\right)\\
&\to 0 \quad \text{as } N\to\infty
\end{aligned}
$$

As we can find ε,δ such that $|\breve{F}_{\infty}(\mu_L^{\varepsilon,\delta})-\breve{F}_{\infty}(\mu_L)|<1/L$ we conclude. □

7. Analytic estimates

In this section we collect a series of analytic estimate which together allow to establish the pointwise bounds (38) and (39) and the continuity required for Lemma 22. First of all note that

$$
\begin{aligned}
\|K_t\|_{H^{1-\kappa}}^2 &\lesssim \lambda^2\int_0^t \frac{1}{\langle t\rangle^{1+\delta}}\|\mathbb{W}_s^2\|_{B_{4,\infty}^s}^2 \mathrm{d}s\, \|Z_T\|_{L^4}^2+\int_0^t\|l_s\|_{L^2}^2\mathrm{d}s\\
&\lesssim \lambda^3\left(\int_0^t \frac{1}{\langle t\rangle^{1+\delta}}\|\mathbb{W}_s^2\|_{B_{4,\infty}^s}^2\mathrm{d}s\right)^2+\lambda\|Z_T\|_{L^4}^4+\int_0^t\|l_s\|_{L^2}^2\mathrm{d}s,
\end{aligned}
\tag{52}
$$

which implies that quadratic functions of the norm $\|K_t\|_{H^{1-\kappa}}$ with small coefficient can always be controlled, uniformly in $[0,\infty]$, by the coercive term

$$
\lambda\fint Z_T^4+\frac{1}{2}\int_0^{\infty}\|l_s\|_{L^2}^2\mathrm{d}s.
$$

Lemma 39. *For any small $\varepsilon>0$ there exists $\delta>0$ such that*

$$
|\Upsilon_T^{(1)}|\leqslant C(\varepsilon,\delta)E(\lambda)Q_T+\varepsilon\|K_T\|_{H^{1-\delta}}^2+\varepsilon\lambda\|Z_T\|_{L^4}^4.
$$

Proof. By Proposition 61,

$$
\begin{aligned}
\lambda|\mathfrak{K}_2(\mathbb{W}_T^2,K_T,K_T)| &\lesssim \lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{B_{7/3,2}^{9/16}}^2\lesssim\lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{B_{7/3,7/3}^{5/8}}^2\\
&\lesssim \lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{H^{7/8}}^{10/7}\|K_T\|_{B_{4,4}^0}^{4/7}\\
&\lesssim \lambda^6\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}^7+\|K_T\|_{H^{7/8}}^2+\lambda\|K_T\|_{L^4}^4.
\end{aligned}
\tag{53}
$$

By Proposition 57,

$$
\left|\lambda\fint(\mathbb{W}_T^2\prec K_T)K_T\right| \lesssim \lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{B_{7/3,2}^{9/16}}^2
$$

which can be estimated in the same way, and finally

$$\left|\lambda^2 \fint (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]}) K_T\right| \lesssim \lambda^2 \|\mathbb{W}_T^2\|_{B_{4,4}^{-1-\delta/2}} \|\mathbb{W}_T^{[3]}\|_{B_{4,4}^{1/2-\delta/2}} \|K_T\|_{H^{1/2+\delta}}$$
$$\leqslant C(\delta) \quad \lambda^4 \left(\|\mathbb{W}_T^2\|_{B_{4,4}^{-1-\delta/2}} \quad \|\mathbb{W}_T^{[3]}\|_{B_{4,4}^{1/2-\delta/2}} \right)^2 \quad + \delta \|K_T\|_{H^{1/2+\delta}}^2.$$

□

Lemma 40. *For any small $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|\Upsilon_T^{(2)}\right| \leqslant T^{-\delta} (C(\varepsilon, \delta) E(\lambda) Q_T + \varepsilon \|K\|_{H^{1-\delta}} + \varepsilon \lambda \|Z_T\|_{L^4})$$

Proof. Using the spectral support properties of the various terms we observe that

$$\|\mathbb{W}_T^2\|_{B_{p,q}^{-1+\delta}} \lesssim \|\mathbb{W}_T^2\|_{B_{p,q}^{-1+\delta}} T^{2\delta},$$

and

$$T^{2\delta} \|Z_T - Z_T^\flat\|_{L^2} \lesssim \|Z_T - Z_T^\flat\|_{H^{2\delta}} \lesssim \|Z_T - Z_T^\flat\|_{H^{1/2-\delta}}^{\frac{2\delta}{1/2-\delta}} \|Z_T - Z_T^\flat\|_{L^2}^{\frac{1/2-3\delta}{1/2-\delta}}$$
$$\lesssim \|Z_T\|_{H^{1/2-\delta}}^{\frac{2\delta}{1/2-\delta}} \|Z_T\|_{L^2}^{\frac{1/2-3\delta}{1/2-\delta}},$$

where we used also interpolation and the L^2 bound $\|Z_T^\flat\|_{L^2} \lesssim \|Z_T\|_{L^2}$. We recall also that

$$Z_T = K_T + \lambda \mathbb{W}_T^{[3]}. \tag{54}$$

Therefore we estimate as follows

$$\lambda \fint (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat)) K_T = \lambda \fint (\mathbb{W}_T^2 \succ (K_T - K_T^\flat)) K_T + \lambda^2 \fint \left(\mathbb{W}_T^2 \succ \left(\mathbb{W}_T^{[3]} - \mathbb{W}_T^{[3],\flat}\right)\right) K_T$$

For the second term we can estimate

$$\lambda^2 \fint \left(\mathbb{W}_T^2 \succ \left(\mathbb{W}_T^{[3]} - \mathbb{W}_T^{[3],\flat}\right)\right) K_T \lesssim \lambda^2 \|W_T^2\|_{B_{4,\infty}^{-1+\delta}} \left\|\mathbb{W}_T^{[3]} - \mathbb{W}_T^{[3],\flat}\right\|_{B_{4,2}^0} \|K_T\|_{H^{1-\delta}}$$
$$\lesssim \lambda^2 T^{-\delta} \|W_T^2\|_{B_{4,\infty}^{-1-\delta}} \left\|\mathbb{W}_T^{[3]}\right\|_{B_{4,2}^{3\delta}} \|K_T\|_{H^{1-\delta}},$$

while for the first term we get

$$\lambda \fint (\mathbb{W}_T^2 \succ (K_T - K_T^\flat)) K_T \lesssim \lambda \|W_T^2\|_{B_{7,\infty}^{-1/2-\delta}} \|K_T\|_{B_{7/3,2}^0} \|K_T\|_{B_{7/3,2}^{1/2+\delta}}$$
$$\lesssim \lambda \|W_T^2\|_{B_{7,\infty}^{-1-\delta}} T^{1/2} T^{-1/2-\delta} \|K_T\|_{B_{7/3,2}^{1/2+\delta}}^2$$
$$\lesssim \lambda T^{-\delta} \|W_T^2\|_{B_{7,\infty}^{-1-\delta}} \|K_T\|_{B_{7/3,2}^{1/2+\delta}}^2,$$

which we can again estimate like in Lemma 39. $\square$

Lemma 41. *For any small $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|\Upsilon_T^{(3)}\right| \leqslant C(\varepsilon,\delta)E(\lambda)Q_T + \varepsilon \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}}^2 + \varepsilon\lambda \|Z_T\|_{L^4}^4.$$

Proof. First note that for $t \geqslant T_0$ (recall T_0 has been defined in (19)) we have $\dot{\theta}_t(\mathrm{D}) = (\langle \mathrm{D}\rangle / t^2)\dot{\tilde{\theta}}(\langle \mathrm{D}\rangle / t)$. In particular $\dot{Z}_t^\flat$ is spectrally supported in an annulus with inner radius $t/4$ and outer radius $t/3$. Then for any $\beta \in [0,1]$

$$\|\dot{Z}_t^\flat\|_{B_{p,q}^{s+\beta}} = \left\|\dot{\tilde{\theta}}\left(\frac{\langle \mathrm{D}\rangle}{t}\right)\frac{\langle \mathrm{D}\rangle}{t^2} Z_T\right\|_{B_{p,q}^{s+\beta}} \lesssim \left\|\dot{\tilde{\theta}}\left(\frac{\langle \mathrm{D}\rangle}{t}\right)\frac{\langle \mathrm{D}\rangle^{1+\beta}}{t^{2+\beta}} Z_T\right\|_{B_{p,q}^{s+\beta}} \lesssim \frac{\|Z_T\|_{B_{p,q}^s}}{\langle t\rangle^{1+\beta}}.$$

The same estimate holds trivially for $t \leqslant T_0$.

By Proposition 57, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
\left|\lambda \int_0^T \!\!\!\!\!\!- \int (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat) K_t \,\mathrm{d}t\right| &\lesssim \lambda \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}} \|\dot{Z}_t^\flat\|_{B_{3,2}^0} \|K_t\|_{H^{1-\delta}} \mathrm{d}t \\
&\lesssim \lambda \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}} \|Z_T\|_{B_{3,2}^{3\delta}} \|K_t\|_{H^{1-\delta}} \frac{\mathrm{d}t}{\langle t\rangle^{1+3\delta}} \\
&\lesssim \lambda \|Z_T\|_{B_{3,3}^{4\delta}} \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}} \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}} \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \\
&\lesssim \lambda \|Z_T\|_{H^{1/2-\delta}}^{1/2} \|Z_T\|_{B_{4,4}}^{1/2} \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}} \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}} \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \\
&\lesssim \lambda \|Z_T\|_{L^4}^{1/2} \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}}^{3/2} \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}} \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \\
&\quad + \lambda^{3/2} \|Z_T\|_{L^4}^{1/2} \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}} \|\mathbb{W}_T^{[3]}\|_{H^{4\delta}}^{1/2} \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}} \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}
\end{aligned}$$

and again

$$\begin{aligned}
&\lambda \|Z_T\|_{L^4}^{1/2} \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}}^{3/2} \int_0^T \|\mathbb{W}_t^2\|_{B_{7,\infty}^{-1+\delta}} \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \\
&\quad \leqslant C\lambda^7 \int_0^T \|\mathbb{W}_t^2\|_{B_{7,\infty}^{-1+\delta}}^8 \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} + \varepsilon \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}}^2 + \varepsilon\lambda \|Z_T\|_{L^4}^4.
\end{aligned}$$

While

$$\begin{aligned}
&\lambda^{3/2} \|Z_T\|_{L^4}^{1/2} \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}} \|\mathbb{W}_T^{[3]}\|_{H^{4\delta}}^{1/2} \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}} \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \\
\leqslant\ & C\lambda^{11/3} \int_0^T \|\mathbb{W}_t^2\|_{B_{7,\infty}^{-1+\delta}}^{8/3} \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \|\mathbb{W}_T^{[3]}\|_{H^{4\delta}}^{8/6} + \sup_{0\leqslant t\leqslant T} \|K_t\|_{H^{1-\delta}}^2 + \lambda \|Z_T\|_{L^4}.
\end{aligned}$$

$\square$

Lemma 42. *For any small $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|\Upsilon_T^{(4)}| \leqslant C(\varepsilon, \delta) E(\lambda) Q_T + \varepsilon \|K_T\|_{H^{1-\delta}}^2 + \varepsilon \lambda \|Z_T\|_{L^4}^4.$$

Proof. Using Lemma 9 we establish that

$$\left|\lambda \fint W_T K_T^3\right| \leqslant E(\lambda) \|W_T\|_{W^{-1/2-\varepsilon,p}}^K + \delta(\|K_T\|_{H^{1-\varepsilon}}^2 + \lambda \|K_T\|_{L^4}^4).$$

Next, we can write,

$$\lambda^3 \left|\fint W_T (\mathbb{W}_T^{[3]})^2 K_T\right| \lesssim \lambda^3 \left|\fint W_T (\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}) K_T\right| + \lambda^3 \|W_T\|_{B_{6,\infty}^{-1/2-\delta}} \|\mathbb{W}_T^{[3]}\|_{B_{6,4}^{-1/2-\delta}}^2 \|K_T\|_{H^{1-\varepsilon}}.$$

which can be easily estimated by Young's inequality. Decomposing

$$W_T(\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}) = W_T \succ (\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}) + W_T \prec (\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}) + W_T \circ (\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}).$$

We can estimate the first two terms by

$$\lambda^3 \left|\fint W_T \succ (\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}) K_T\right| \lesssim \lambda^3 \|W_T\|_{B_{6,\infty}^{-1/2-\delta}} \|\mathbb{W}_T^{[3]}\|_{B_{6,2}^0}^2 \|K_T\|_{H^{1-\varepsilon}},$$

and

$$\lambda^3 \left|\fint W_T \prec (\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}) K_T\right| \lesssim \lambda^3 \|W_T\|_{B_{6,2}^{-1/2-\delta}} \|\mathbb{W}_T^{[3]}\|_{B_{6,\infty}^0}^2 \|K_T\|_{H^{1-\varepsilon}}.$$

Young's inequality gives then the appropriate result. For the final term we use Proposition 60 to get

$$\begin{aligned}
&\lambda^3 \left|\fint W_T \circ (\mathbb{W}_T^{[3]} \succ \mathbb{W}_T^{[3]}) K_T\right| \\
&\quad \lesssim \lambda^3 \left|\fint \mathbb{W}_T^{[3]} \mathbb{W}_T^{1\circ[3]} K_T\right| + \lambda^3 \|W_T\|_{B_{4,\infty}^{-1/2-\delta}} \|\mathbb{W}_T^{[3]}\|_{B_{4,2}^{-1/2-\delta}}^2 \|K_T\|_{H^{1-\delta}} \\
&\quad \lesssim \lambda^3 \|\mathbb{W}_T^{[3]}\|_{B_{4,\infty}^{1/2-\delta}} \|\mathbb{W}_T^{1\circ[3]}\|_{B_{4,2}^{-\delta}} \|K_T\|_{H^{1-\delta}} + \lambda^3 \|W_T\|_{B_{4,\infty}^{-1/2-\delta}} \|\mathbb{W}_T^{[3]}\|_{B_{4,2}^{-1/2-\delta}}^2 \|K_T\|_{H^{1-\delta}} \\
&\quad \lesssim \lambda^6 C(\delta, \varepsilon) \left[\|\mathbb{W}_T^{[3]}\|_{B_{4,\infty}^{1/2-\delta}} \|\mathbb{W}_T^{1\circ[3]}\|_{B_{4,2}^{-\delta}} + \|W_T\|_{B_{4,\infty}^{-1/2-\delta}} \|\mathbb{W}_T^{[3]}\|_{B_{4,2}^{-1/2-\delta}}^2 \right]^2 + \varepsilon \|K_T\|_{H^{1-\delta}}^2.
\end{aligned}$$

For the last term we estimate

$$\left|\lambda^2 \fint (W_T \mathbb{W}_T^{[3]}) K_T^2\right| \lesssim \lambda^2 \|W_T \mathbb{W}_T^{[3]}\|_{B_{7,\infty}^{-1/2-\delta}} \|K_T\|_{B_{7/3,2}^{1/2+\delta}}^2,$$

which can be estimated like in Lemma 39 after we observe that

$$\|W_T \mathbb{W}_T^{[3]}\|_{B^{-1/2-\delta}_{7,\infty}} \leqslant \|W_T \succ \mathbb{W}_T^{[3]}\|_{B^{-1/2-\delta}_{7,\infty}} + \|W_T \circ \mathbb{W}_T^{[3]}\|_{B^{-1/2-\delta}_{7,\infty}} + \|W_T \prec \mathbb{W}_T^{[3]}\|_{B^{-1/2-\delta}_{7,\infty}}$$

$$\lesssim \|W_T\|_{B^{-1/2-\delta}_{14,\infty}} \|\mathbb{W}_T^{[3]}\|_{B^0_{14,\infty}} + \|\mathbb{W}_T^{1\circ[3]}\|_{B^{-\delta}_{7,\infty}}$$

and use Lemma 46 to bound $\mathbb{W}_T^{1\circ[3]}$. □

Lemma 43. *For any small $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|\Upsilon_T^{(5)}| \leqslant C_\varepsilon E(\lambda) \left[\frac{|\gamma_T|}{\langle T \rangle^{1/4}} + \int_0^T \frac{|\gamma_t| \,\mathrm{d}t}{\langle t \rangle^{5/4}} \right]^2 + \varepsilon \|Z_T\|^2_{H^{1/2-\delta}} + \varepsilon\lambda \|Z_T\|^4_{L^4}.$$

Proof. We can estimate

$$\left| \lambda^2 \gamma_T \fint Z_T^\flat (Z_T - Z_T^\flat) \right| \leqslant \lambda^2 |\gamma_T| \|Z_T^\flat\|_{L^2} \|Z_T - Z_T^\flat\|_{L^2} \lesssim \lambda^2 \frac{|\gamma_T|}{\langle T \rangle^{1/4}} \|Z_T^\flat\|_{L^2} \|Z_T - Z_T^\flat\|_{H^{1/4}}, \tag{55}$$

and

$$\left| \lambda^2 \gamma_T \fint (Z_T - Z_T^\flat)^2 \right| \leqslant \lambda^2 |\gamma_T| \|Z_T - Z_T^\flat\|^2_{L^2} \lesssim \lambda^2 \frac{|\gamma_T|}{\langle T \rangle^{1/4}} \|Z_T^\flat - Z_T\|_{L^2} \|Z_T - Z_T^\flat\|_{H^{1/4}}. \tag{56}$$

For the last term we can apply the estimate

$$\left| \lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t \right| \leqslant \lambda^2 \int_0^T |\gamma_t| \|Z_t^\flat\|_{L^2} \|\dot{Z}_t^\flat\|_{L^2} \mathrm{d}t \lesssim \lambda^2 \|Z_T\|_{L^2} \|Z_T\|_{H^{1/4}} \int_0^T \frac{|\gamma_t| \,\mathrm{d}t}{\langle t \rangle^{5/4}}.$$

Collecting these bounds we get

$$|\Upsilon_T^{(5)}| \lesssim C_\varepsilon \lambda^7 \left[\frac{|\gamma_T|}{\langle T \rangle^{1/4}} + \int_0^T \frac{|\gamma_t| \,\mathrm{d}t}{\langle t \rangle^{5/4}} \right]^2 + \lambda\varepsilon \|Z_T\|^4_{L^4} + \varepsilon \|Z_T\|^2_{H^{1/2-\delta}}.$$ □

Remark 44. Note that

$$\sup_T \left[\frac{|\gamma_T|}{\langle T \rangle^{1/4}} + \int_0^T \frac{|\gamma_t| \,\mathrm{d}t}{\langle t \rangle^{5/4}} \right] < \infty,$$

provided γ_T does not grow too fast in T which is indeed guaranteed by the choice of renormalization made in Lemma 46.

Lemma 45. *For any small $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left|\Upsilon_T^{(6)}\right| \leqslant C(\varepsilon, \delta) E(\lambda) Q_T + \varepsilon \|K_T\|_{H^{1-\delta}}^2 + \varepsilon \lambda \|Z_T\|_{L^4}^4.$$

Proof. We start by observing that

$$\lambda^2 \left| \fint (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} + 2\gamma_T W_T) K_T \right| \lesssim \lambda^2 \| \mathbb{W}_T^{2\diamond[3]} \|_{W^{-1/2-\varepsilon,2}} \|K_T\|_{W^{1/2+\varepsilon,2}}.$$

and using Lemma 46 and eq. (52) we have this term under control. Next split

$$\left| \frac{\lambda^2}{2} \mathbb{E} \int_0^T \fint [(J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 + 2\dot{\gamma}_t (Z_t^\flat)^2] \mathrm{d}t \right|$$

$$\lesssim \frac{\lambda^2}{2} \left| \int_0^T \fint ((J_t \mathbb{W}_t^2 \succ Z_t^\flat))^2 - (J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2)(Z_t^\flat)^2 \mathrm{d}t \right| + \lambda^2 \left| \int_0^T \fint \mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle} (Z_t^\flat)^2 \mathrm{d}t \right|.$$

Recall that $t^{1/2} J_t$ is a Fourier multiplier with symbol

$$\langle k\rangle^{-1} (-2\rho'(\langle k\rangle / t)\rho(\langle k\rangle / t)(\langle k\rangle / t))^{1/2} = \langle k\rangle^{-1} \eta(\langle k\rangle / t),$$

where η is a smooth function supported in an annulus of radius 1. From this we prove that $t^{1/2} J_t$ satisfies the assumptions of Proposition 59 with $m = -1$. Therefore

$$\|J_t(\mathbb{W}_t^2 \succ Z_t^\flat) - (J_t \mathbb{W}_t^2) \succ Z_t^\flat\|_{H^{1/4-2\delta}} \lesssim \langle t\rangle^{-1/2} \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}} \|Z_t^\flat\|_{B_{3,3}^{-1/4-\delta}},$$

and by Proposition 54,

$$\|J_t(\mathbb{W}_t^2 \succ Z_t^\flat)\|_{H^{-2\delta}} + \|J_t(\mathbb{W}_t^2 \succ Z_t^\flat)\|_{H^{-2\delta}} \lesssim \langle t\rangle^{-1/2-\delta} \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}} \|Z_t^\flat\|_{B_{3,3}^0}.$$

Therefore

$$\left| \frac{\lambda^2}{2} \int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 \mathrm{d}t - \frac{\lambda^2}{2} \int_0^T \fint ((J_t \mathbb{W}_t^2) \succ Z_t^\flat)^2 \mathrm{d}t \right|$$

$$\lesssim \lambda^2 \sup_{t \leqslant T} \left[\|Z_t^\flat\|_{B_{3,3}^0} \|Z_t^\flat\|_{B_{3,3}^{1/4-\delta}} \right] \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}$$

$$\lesssim \lambda^2 \sup_{t \leqslant T} [\|Z_t^\flat\|_{L^4} \|Z_t^\flat\|_{H^{1/2-\delta}}] \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}},$$

which can be easily estimated by Young's inequality. From Proposition 61 and Proposition 51

$$\left| \frac{\lambda^2}{2} \fint ((J_t \mathbb{W}_t^2 \succ Z_t^\flat))^2 - \frac{\lambda^2}{2} \fint (J_t \mathbb{W}_t^2 \succ Z_t^\flat) \circ J_t \mathbb{W}_t^2 \; Z_t^\flat \right| \lesssim \lambda^2 \|J_t \mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \|Z_t^\flat\|_{B_{3,\infty}^{-1/4-\delta}} \|Z_t^\flat\|_{B_{3,3}^0}$$

and by interpolation

$$\lesssim \lambda^2 \|J_t \mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \|Z_t^\flat\|_{L^4} \|Z_t^\flat\|_{H^{1/2-\delta}}.$$

The integrability of this term in time follows from the inequality

$$\|J_t \mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \lesssim \langle t\rangle^{-1-2\delta} \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2.$$

Using again Proposition 54 for $t^{1/2} J_t$ gives the estimate. Applying Proposition 60 and Proposition 51 we get

$$\lambda^2 \|(J_t \mathbb{W}_t^2 \succ Z_t^\flat) \circ J_t \mathbb{W}_t^2 - (J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2)(Z_t^\flat)\|_{B_{3/2,\infty}^0} \lesssim \lambda^2 \|J_t \mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \|Z_t^\flat\|_{B_{3,\infty}^{3\delta}}.$$

and after using duality and interpolation we obtain

$$\begin{aligned}
&\frac{\lambda^2}{2}\left| \int_0^T \fint ((J_t \mathbb{W}_t^2 \succ Z_t^\flat))^2 - (J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2)(Z_t^\flat)^2 \mathrm{d}t \right| \\
&\lesssim \lambda^2 \sup_{t \leqslant T} [\|Z_t^\flat\|_{L^4} \|Z_t^\flat\|_{H^{1/2-\delta}}] \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \\
&\lesssim \varepsilon \left(\frac{1}{2} \sup_{t \leqslant T} \|Z_t^\flat\|_{H^{1/2-\delta}}^2 + \lambda \|Z_T\|_{L^4}^4 \right) + C(\varepsilon, \delta) \lambda^7 \left(\int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^2 \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}} \right)^4 \\
&\lesssim \varepsilon \left(\frac{1}{2} \|Z_T\|_{H^{1/2-\delta}}^2 + \lambda \|Z_T\|_{L^4}^4 \right) + C(\varepsilon, \delta) \lambda^7 \int_0^T \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}}^8 \frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}.
\end{aligned}$$

Finally we have

$$\begin{aligned}
\lambda^2 \left| \int_0^T \fint \mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle} (Z_t^\flat)^2 \mathrm{d}t \right| &\lesssim \lambda^2 \left[\int_0^T \left\| \mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle} \right\|_{L^4} \mathrm{d}t \right] \|Z_T\|_{H^\varepsilon} \|Z_T\|_{L^4} \\
&\leqslant C(\varepsilon) \lambda^7 \left[\int_0^T \|\mathbb{W}_t^{\langle 2\rangle \diamond \langle 2\rangle}\|_{L^4} \mathrm{d}t \right]^4 + \lambda \varepsilon \|Z_T\|_{L^4}^4 + \varepsilon \|Z_T\|_{H^{1/2-\delta}}^2.
\end{aligned}$$

Using eq. (54) to control $\|Z_T\|_{H^{1/2-\delta}}$ in terms of K_T we obtain the claim. □

8. Stochastic estimates

In this section we close our argument proving the following lemmas which give uniform estimates as $T \to \infty$ of some of the stochastic terms appearing in our analytic estimates.

Lemma 46. *For any $\varepsilon > 0$ and any $p > 1, r < \infty, q \in [1, \infty]$, there exists a constant $C(\varepsilon, p, q)$ which does not depend on Λ such that*

$$\sup_T \mathbb{E}\left[\left\| W_T \circ W_T^{[3]} \right\|_{B_{r,q}^{-\varepsilon}}^p \right] \leqslant C(\varepsilon, p, q). \tag{57}$$

Moreover there exists a function $\gamma_t \in C^1(\mathbb{R}_+, \mathbb{R})$ *such that for any* $\varepsilon > 0$ *and any* $p > 1$*,*

$$\sup_T \mathbb{E}\Big[\|(\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} - 2\gamma_T W_T)\|^p_{B_{r,q}^{-1/2-\varepsilon}}\Big] \leqslant C(\varepsilon, p, q), \tag{58}$$

$$\mathbb{E}\Bigg[\bigg(\int_0^\infty \|J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2 - 2\dot{\gamma}_t\|_{B_{r,q}^{-\varepsilon}} \mathrm{d}t\bigg)^p\Bigg] \leqslant C(\varepsilon, p, q). \tag{59}$$

$$\sup_t \mathbb{E}\Big[\|J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2 - 2\dot{\gamma}_t\|_{B_{r,q}^{-\varepsilon}}\Big] \leqslant C(\varepsilon, p, q)$$

and

$$|\gamma_t| + \langle t \rangle |\dot{\gamma}_t| \lesssim 1 + \log\langle t \rangle, \qquad t \geqslant 0. \tag{60}$$

Furthermore γ *is independent of* Λ*. By Besov embedding, the Besov-Hölder norms of these objects are also uniformly bounded in* T *(but not uniformly in* Λ*).*

Proof. We will concentrate in proving the bounds on the renormalized terms in eqs. (58) and (59) and leave to the reader to fill the details for the easier term in eq. (57). Recall the representation of $(W_t)_t$ in terms of the family of Brownian motions $(B_t^n)_{t,n}$ in eq. (4). Wick's products of the Gaussian field W_T can be represented as iterated stochastic integrals wrt. $(B_t^n)_{t,n}$. In particular, if we let $\mathrm{d}w_s(k) = \langle k \rangle^{-1} \sigma_s(k) \mathrm{d}B_s^k$, we have

$$\mathbb{W}_T^2(x) = 12[\![W_T^2]\!](x) = 24 \sum_{k_1, k_2} e^{i(k_1 + k_2) \cdot x} \int_0^T \int_0^{s_2} \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2),$$

$$\mathbb{W}_T^{[3]}(x) = 24 \sum_{k_1, k_2, k_3} e^{ik_{(123)} \cdot x} \int_0^T \int_0^{s_3} \int_0^{s_2} \bigg(\int_{s_3}^T \frac{\sigma_u^2(k_{(123)})}{\langle k_{(123)} \rangle^2} \mathrm{d}u\bigg) \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2) \mathrm{d}w_{s_3}(k_3),$$

where we denote $k_{(1 \cdots n)} := k_1 + \cdots + k_n$ for any $n \geqslant 2$. Now products of iterated integrals can be decomposed in sums of iterated integrals and we get

$$\begin{aligned} \Delta_q(\mathbb{W}_T^{2 \diamond [3]})(x) &= \Delta_q\big(\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} - 2\gamma_T W_T\big)(x) \\ &= \sum_{k_1, \ldots, k_5} \int_{A_T^5} G_{0,q}^{2 \diamond [3]}((s,k)_{1 \cdots 5}) \mathrm{d}w_{s_1}(k_1) \cdots \mathrm{d}w_{s_5}(k_5) \\ &\quad + \sum_{k_1, \ldots, k_3} \int_{A_T^3} G_{1,q}^{2 \diamond [3]}((s,k)_{1 \cdots 3}) \mathrm{d}w_{s_1}(k_1) \cdots \mathrm{d}w_{s_3}(k_3) \\ &\quad + \sum_{k_1} \int_{A_T^1} G_{2,q}^{2 \diamond [3]}((s,k)_1) \mathrm{d}w_{s_1}(k_1), \end{aligned} \tag{61}$$

where $A_T^n := \{0 \leqslant s_1 < \cdots < s_n \leqslant T\} \subseteq [0, T]^n$ and where the deterministic kernels are given by

$$G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5}) := (24^2)\,\varrho_q(k_{(1\cdots 5)})e^{i(k_{(1\cdots 5)})\cdot x}\sum_{\sigma\in\mathrm{Sh}(2,3)}\sum_{i\sim j}\times$$
$$\times\,\varrho_i(k_{(\sigma_1\sigma_2)})\,\varrho_j(k_{(\sigma_3\sigma_4\sigma_5)})\left(\int_{s_{\sigma_5}}^T \frac{\sigma_u(k_{(\sigma_3\sigma_4\sigma_5)})^2}{\langle k_{(\sigma_3\sigma_4\sigma_5)}\rangle^2}\mathrm{d}u\right),$$

$$G_{1,q}^{2\diamond[3]}((s,k)_{1\cdots 3}) := (24^2)\,\varrho_q(k_{(1\cdots 3)})e^{i(k_{(1\cdots 3)})\cdot x}\sum_{\sigma\in\mathrm{Sh}(1,2)}\sum_{i\sim j}\sum_p\int_0^T \mathrm{d}r\frac{\sigma_r(p)^2}{\langle p\rangle^2}\times$$
$$\times\,\varrho_i(k_{\sigma_1}+q)\,\varrho_j(k_{(\sigma_2\sigma_3)}-q)\left(\int_{s_{\sigma_3}\vee r}^T \frac{\sigma_u(k_{(\sigma_2\sigma_3)}-p)^2}{\langle k_{(\sigma_2\sigma_3)}-p\rangle^2}\mathrm{d}u\right),$$

$$G_{2,q}^{2\circ[3]}((s,k)_1) := (24^2)\,\varrho_q(k_1)e^{ik_1\cdot x}\sum_{i\sim j}\sum_{p_1,p_2}\int_0^T\mathrm{d}r_1\int_0^T\mathrm{d}r_2\frac{\sigma_{r_1}(p_1)^2}{\langle p_1\rangle^2}\frac{\sigma_{r_2}(p_2)^2}{\langle p_2\rangle^2}\times$$
$$\times\,\varrho_i(p_1+p_2)\,\varrho_j(k_1-p_1-p_2)\left(\int_{r_1\vee r_2\vee s_1}^T\frac{\sigma_u(k_1-p_1-p_2)^2}{\langle k_1-p_1-p_2\rangle^2}\mathrm{d}u\right),$$

$$G_{2,q}^{2\diamond[3]}((s,k)_1) := G_{2,q}^{2\circ[3]}((s,k)_1) - 2\gamma_T\varrho_q(k_1)\,e^{ik_1\cdot x},$$

where $\mathrm{Sh}(k,l)$ is the set of permutations σ of $\{1,\ldots,k+l\}$ keeping the orders $\sigma(1)<\cdots<\sigma(k)$ and $\sigma(k+1)<\cdots<\sigma(k+l)$ and where, for any symbol z, we denote with expression of the form $z_{1\cdots n}$ the vector $(z_1,\ldots,z_n)$. Estimation of $\Delta_q(\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]})(x)$ reduces then to estimate each of the three iterated integrals using BDG inequalities to get, for any $p\geqslant 2$,

$$I_{0,q}=\left\{\mathbb{E}\left[\left|\sum_{k_1,\ldots,k_5}\int_{A_T^5}G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_5}(k_5)\right|^{2p}\right]\right\}^{1/p}$$
$$\lesssim\mathbb{E}\left[\left|\sum_{k_1,\ldots,k_5}\int_{A_T^5}G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_5}(k_5)\right|^2\right]$$
$$\lesssim\sum_{k_1,\ldots,k_5}\int_{A_T^5}|G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5.$$

The kernel $G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})$ being a symmetric function of its argument, we can simplify this expression into an integral over $[0,T]^5$:

$$I_{0,q}\lesssim\sum_{k_1,\ldots,k_5}\int_{[0,T]^5}|G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5.$$

Under the measure $\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_5$, we have

$$\left|\int_{s_{\sigma_5}}^T \frac{\sigma_u(k_{(\sigma_3\sigma_4\sigma_5)})^2}{\langle k_{(\sigma_3\sigma_4\sigma_5)}\rangle^2}\mathrm{d}u\right| \lesssim \frac{1}{\langle k_{\sigma_5}\rangle^2}.$$

Therefore with some standard estimates we can reduce us to consider

$$I_{0,q} \lesssim \sum_{k_1,\ldots,k_5}\int_{[0,T]^5}\frac{\varrho_q(k_{(1\cdots5)})^2}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5$$

$$\lesssim \sum_{k_1,\ldots,k_5}\int_{[0,T]^5}\frac{\varrho_q(k_{(1\cdots5)})^2}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5$$

$$\lesssim \sum_{k_1,\ldots,k_5}\frac{\varrho_q(k_{(1\cdots5)})^2}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_5\rangle^2}$$

$$\lesssim \sum_{p_1,p_2}\mathbb{1}_{p_1\sim p_2}\varrho_q(p_1+p_2)^2\sum_{k_1,\ldots,k_5}\frac{1}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}=p_1,k_{(345)}=p_2}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_5\rangle^2}$$

$$\lesssim \sum_{p_1,p_2}\mathbb{1}_{p_1\sim p_2}\varrho_q(p_1+p_2)^2\frac{1}{\langle p_1\rangle}\frac{1}{\langle p_2\rangle^4} \lesssim \sum_{p_1,r}\varrho_q(r)^2\frac{1}{\langle p_1\rangle}\frac{1}{\langle p_1+r\rangle^4} \lesssim \sum_r \varrho_q(r)^2\frac{1}{\langle r\rangle^2}\lesssim 2^q.$$

Now by similar reasoning we also have

$$|G^{2\diamond[3]}_{1,q}((s,k)_{1\cdots3})| \lesssim \sum_{\sigma\in\mathrm{Sh}(1,2)}|\varrho_q(k_{(1\cdots3)})|\sum_{i\sim j}\sum_p\int_0^T\mathrm{d}r\frac{\sigma_r(p)^2|\varrho_i(k_{\sigma_1}+p)\varrho_j(k_{(\sigma_2\sigma_3)}-p)|}{\langle p\rangle^2\langle k_{\sigma_1}+p\rangle^2}$$

$$\lesssim \sum_{\sigma\in\mathrm{Sh}(1,2)}\frac{|\varrho_q(k_{(1\cdots3)})|}{\langle k_{\sigma_1}\rangle}$$

so

$$I_{1,q} = \left\{\mathbb{E}\left[\left|\sum_{k_1,\ldots,k_3}\int_{A_T^3}G^{2\diamond[3]}_{1,q}((s,k)_{1\cdots5})\mathrm{d}y_{s_1}(k_1)\cdots\mathrm{d}y_{s_3}(k_3)\right|^{2p}\right]\right\}^{1/p}$$

$$\lesssim \sum_{k_1,\ldots,k_3}\int_{[0,T]^3}\left|\sum_{\sigma\in\mathrm{Sh}(1,2)}\frac{|\varrho_q(k_{(1\cdots3)})|}{\langle k_{\sigma_1}\rangle}\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_3$$

$$\lesssim \sum_{k_1,\ldots,k_3}\frac{|\varrho_q(k_{(1\cdots3)})|^2}{\langle k_1\rangle^4\langle k_2\rangle^2\langle k_3\rangle^2}\lesssim\sum_r\frac{\varrho_q(r)^2}{\langle r\rangle^2}\lesssim 2^q.$$

Finally, we note that the same strategy cannot be applied to the first chaos, since the kernel $G^{2\diamond[3]}_{2,q}$ cannot be uniformly bounded. We let

$$\begin{aligned} A_T(s_1,k_1) &:= (24^2)\sum_{i\sim j}\sum_{q_1,q_2}\int_0^T \mathrm{d}r_1\int_0^T \mathrm{d}r_2 \frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\times \\ &\quad \times \varrho_i(q_1+q_2)\,\varrho_j(k_1-q_1-q_2)\left(\int_{r_1\vee r_2\vee s_1}^T \frac{\sigma_u^2(k_1-q_1-q_2)}{\langle k_1-q_1-q_2\rangle^2}\mathrm{d}u\right), \end{aligned}$$

so

$$G^{2\diamond[3]}_{2,q}((s,k)_1) = \varrho_q(k_1)e^{ik_1\cdot x}[A_T(s_1,k_1)-2\gamma_T].$$

Observe that

$$\begin{aligned} A_T(0,0) &= (12^2\cdot 2)\sum_{q_1,q_2}\int_0^T \mathrm{d}r_1\int_0^T \mathrm{d}r_2 \frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\times \\ &\quad \times \int_{r_1\vee r_2}^T \frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\mathrm{d}u \sum_{i\sim j}\varrho_i(q_1+q_2)\,\varrho_j(-q_1-q_2). \end{aligned}$$

We choose γ_T as

$$\gamma_T = A_T(0,0) = (12^2\cdot 2)\sum_{q_1,q_2}\int_0^T \mathrm{d}u\int_0^u \mathrm{d}r_1\int_0^u \mathrm{d}r_2 \frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2} \tag{62}$$

where we used the fact that for all $q\in\mathbb{R}^d$ we have $\sum_{i\sim j}\varrho_i(q)\,\varrho_j(q)=1$, since $\int f\circ g=\int fg$. Note that, as claimed,

$$|\gamma_T| \lesssim \sum_{q_1,q_2}\frac{\mathbb{1}_{|q_1|,|q_2|,|q_1+q_2|\lesssim T}}{\langle q_1\rangle^2\langle q_2\rangle^2\langle q_1+q_2\rangle^2} \lesssim 1+\log\langle T\rangle.$$

Now

$$\begin{aligned} A_T(s_1,k_1)-2\gamma_T &= (24^2\cdot 6)\sum_{q_1,q_2}\int_0^T \mathrm{d}r_1\mathrm{d}r_2 \frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\sum_{i\sim j}\varrho_i(q_1+q_2)\times \\ &\quad \times\left(\varrho_j(k_1-q_1-q_2)\int_{s_1\vee r_1\vee r_2}^T \frac{\sigma_u^2(k_1-q_1-q_2)}{\langle k_1-q_1-q_2\rangle^2}\mathrm{d}u - \varrho_j(q_1+q_2)\int_{r_1\vee r_2}^T \frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\mathrm{d}u\right) \end{aligned}$$

so when $|q_1+q_2|\gg|k_1|$ the quantity in round brackets can be estimated by $|k_1|\langle q_1+q_2\rangle^{-4}$ while when $|q_1+q_2|\lesssim|k_1|$ it is estimated by $\langle q_1+q_2\rangle^{-2}$ so we have

$$|A_T(s_1, k_1) - \gamma_T| \lesssim \sum_{q_1, q_2} \frac{1}{\langle q_1 \rangle^2} \frac{1}{\langle q_2 \rangle^2} \frac{1}{\langle q_1 + q_2 \rangle^2} \left(\mathbb{1}_{|q_1 + q_2| \lesssim |k_1|} + \mathbb{1}_{|q_1 + q_2| \gtrsim |k_1|} \frac{|k_1|}{\langle q_1 + q_2 \rangle^2} \right)$$
$$\lesssim 1 + \log \langle k_1 \rangle.$$

And then with this choice of γ_T the kernel $\tilde{G}^{2\diamond[3]}_{2,q}$ stays uniformly bounded as $T \to \infty$ and satisfies

$$\left| G^{2\diamond[3]}_{2,q}((s,k)_1) \right| \lesssim \varrho_q(k_1) \log \langle k_1 \rangle.$$

From this we easily deduce that

$$I_{2,q} = \left\{ \mathbb{E} \left[\left| \sum_{k_1} \int_{A_T} G^{2\diamond[3]}_{2,q}((s,k)_1) \mathrm{d} y_{s_1}(k_1) \right|^{2p} \right] \right\}^{1/p} \lesssim q 2^q, \qquad q \geqslant -1.$$

All together these estimates imply that

$$\mathbb{E} \| \Delta_q \mathbb{W}^{2\diamond[3]}_T \|^{2p}_{L^{2p}} \lesssim (q 2^{q/2})^{2p}, \qquad q \geqslant -1.$$

Standard argument allows to deduce eq. (58). The analysis of the other renormalized product proceeds similarly. Let

$$V(t) := \mathbb{W}^{\langle 2 \rangle \diamond \langle 2 \rangle}_t = J_t \mathbb{W}^2_t \circ J_t \mathbb{W}^2_t - 2 \dot{\gamma}_t, \qquad t \geqslant 0.$$

First note that by definition of Besov spaces we have

$$\mathbb{E} \left[\left(\int_0^\infty \| V(t) \|_{B^{-\varepsilon - d/r}_{r,r}} \mathrm{d} t \right)^p \right] \lesssim \mathbb{E} \left[\left(\int_0^\infty \left(\sum_q 2^{-qr(\varepsilon + d/r)} \| \Delta_q V(t) \|_{L^r} \right)^{1/r} \mathrm{d} t \right)^p \right].$$

By Minkowski's integral inequality this is bounded by

$$\lesssim \left(\int_0^\infty \mathrm{d} t \left\{ \mathbb{E} \left[\left(\sum_q 2^{-qr(\varepsilon + d/r)} \| \Delta_q V(t) \|^r_{L^r} \right)^{p/r} \right] \right\}^{1/p} \right)^p.$$

When $r \geqslant p$ Jensen's inequality and Fubini's theorem give

$$\lesssim \left(\int_0^\infty \mathrm{d} t \left\{ \sum_q 2^{-qr(\varepsilon + d/r)} \int_\Lambda \frac{\mathrm{d} x}{|\Lambda|} \mathbb{E} [|\Delta_q V(t)(x)|^r] \right\}^{1/r} \right)^p.$$

Finally hypercontractivity and stationarity allow to reduce this to bound

$$\lesssim \left(\int_0^\infty \mathrm{d}t \left\{ \sum_q 2^{-qr(\varepsilon+d/r)} \left(\mathbb{E}[|\Delta_q V(t)(0)|^2] \right)^{r/2} \right\}^{1/r} \right)^p .$$

Letting $I_q(t) = \mathbb{E}[|\Delta_q V(t)(0)|^2]$ we have

$$\mathbb{E}\left[\left(\int_0^\infty \| \mathbb{W}_t^{\langle 2 \rangle \diamond \langle 2 \rangle} \|_{B^{-\varepsilon-d/r}_{r,r}} \mathrm{d}t \right)^p \right] \lesssim \left(\int_0^\infty \mathrm{d}t \left\{ \sum_q 2^{-qr(\varepsilon+d/r)} (I_q(t))^{r/2} \right\}^{1/r} \right)^p .$$

Now we decompose the random field $\Delta_q(\mathbb{W}_t^{\langle 2 \rangle \diamond \langle 2 \rangle})(x)$ into homogeneous stochastic integral as above and obtain

$$\begin{aligned} \Delta_q(\mathbb{W}_t^{\langle 2 \rangle \diamond \langle 2 \rangle})(x) &= \sum_{k_1, \ldots, k_4} \int_{A_t^4} G_{0,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s,k)_{1\cdots 4}) \mathrm{d}w_{s_1}(k_1) \cdots \mathrm{d}w_{s_4}(k_4) \\ &\quad + \sum_{k_1, k_2} \int_{A_t^2} G_{1,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s,k)_{12}) \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2) \\ &\quad + G_{2,q}^{\langle 2 \rangle \diamond \langle 2 \rangle} \end{aligned} \tag{63}$$

with

$$\begin{aligned} G_{0,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s,k)_{1\cdots 4}) &= (24^2)\, \varrho_q(k_{(1\cdots 4)}) e^{i(k_{(1\cdots 4)}) \cdot x} \times \\ &\quad \times \sum_{\sigma \in \mathrm{Sh}(2,2)} \sum_{i \sim j} \varrho_i(k_{(\sigma_1 \sigma_2)})\, \varrho_j(k_{(\sigma_3 \sigma_4)}) \frac{\sigma_t(k_{(\sigma_1 \sigma_2)})}{\langle k_{(\sigma_1 \sigma_2)} \rangle} \frac{\sigma_t(k_{(\sigma_3 \sigma_4)})}{\langle k_{(\sigma_3 \sigma_4)} \rangle} \\ G_{1,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s,k)_{12}) &= (24^2)\, \varrho_q(k_{(12)}) e^{i(k_{(12)}) \cdot x} \sum_{\sigma \in \mathrm{Sh}(1,1)} \sum_{i \sim j} \sum_q \times \\ &\quad \times \int_0^t \mathrm{d}r \frac{\sigma_r^2(q)}{\langle q \rangle^2} \varrho_i(k_{\sigma_1} + q)\, \varrho_j(k_{\sigma_2} - q) \left(\frac{\sigma_t(k_{\sigma_1} + q)}{\langle k_{\sigma_1} + q \rangle} \frac{\sigma_t(k_{\sigma_2} - q)}{\langle k_{\sigma_2} - q \rangle} \right) \\ G_{2,q}^{\langle 2 \rangle \diamond \langle 2 \rangle} &= (24^2) \mathbb{1}_{q=-1} \sum_{i \sim j} \sum_{q_1, q_2} \int_0^t \mathrm{d}r_1 \int_0^t \mathrm{d}r_2 \times \\ &\quad \times \frac{\sigma_{r_1}(q_1)^2}{\langle q_1 \rangle^2} \frac{\sigma_{r_2}(q_2)^2}{\langle q_2 \rangle^2} \varrho_i(q_1 + q_2)\, \varrho_j(-q_1 - q_2) \frac{\sigma_t(q_1 + q_2)^2}{\langle q_1 + q_2 \rangle^2} \\ &\quad - 2 \dot{\gamma}_t \mathbb{1}_{q=-1} . \end{aligned}$$

Using our choice of γ_T in eq. (62) we have that

$$\dot{\gamma}_t = (12^2 \cdot 2) \sum_{q_1, q_2} \int_0^t \mathrm{d}r_1 \int_0^t \mathrm{d}r_2 \frac{\sigma_{r_1}(q_1)^2}{\langle q_1 \rangle^2} \frac{\sigma_{r_2}(q_2)^2}{\langle q_2 \rangle^2} \frac{\sigma_t^2(q_1+q_2)}{\langle q_1+q_2 \rangle^2},$$

which implies also that

$$G_{2,q}^{\langle 2 \rangle \diamond \langle 2 \rangle} = 0, \quad \text{and} \qquad |\dot{\gamma}_t| \lesssim \frac{1+\log \langle t \rangle}{\langle t \rangle}.$$

as claimed. We pass now to estimate the other two chaoses. The technique is the same we used above. Consider first

$$\begin{aligned}
I_{0,q}(t) :=& \mathbb{E}\left[\left| \sum_{k_1, \ldots, k_4} \int_{A_t^4} G_{0,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s,k)_{1\cdots 4}) \mathrm{d}w_{s_1}(k_1) \cdots \mathrm{d}w_{s_4}(k_4) \right|^2 \right] \\
\lesssim& \sum_{k_1, \ldots, k_4} \int_{A_t^4} |G_{0,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s,k)_{1\cdots 4})|^2 \frac{\sigma_{s_1}(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_{s_4}(k_4)^2}{\langle k_4 \rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_4 \\
\lesssim& \sum_{k_1, \ldots, k_4} \int_{[0,t]^4} |G_{0,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s,k)_{1\cdots 4})|^2 \frac{\sigma_{s_1}(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_{s_4}(k_4)^2}{\langle k_4 \rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_4 \\
\lesssim& \sum_{k_1, \ldots, k_4} \varrho_q(k_{(1\cdots 4)})^2 \int_{[0,t]^4} \frac{\sigma_t^2(k_{(12)})}{\langle k_{(12)} \rangle^2} \frac{\sigma_t^2(k_{(34)})}{\langle k_{(34)} \rangle^2} \frac{\sigma_{s_1}(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_{s_4}(k_4)^2}{\langle k_4 \rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_4 \\
\lesssim& \sum_{k_1, \ldots, k_4} \varrho_q(k_{(1\cdots 4)})^2 \frac{\sigma_t^2(k_{(12)})}{\langle k_{(12)} \rangle^2} \frac{\sigma_t^2(k_{(34)})}{\langle k_{(34)} \rangle^2} \frac{1}{\langle k_1 \rangle^2} \cdots \frac{1}{\langle k_4 \rangle^2} \\
\lesssim& \frac{\mathbb{1}_{2^q \lesssim t}}{\langle t \rangle^6} \sum_{k_1, \ldots, k_4} \frac{\varrho_q(k_{(1\cdots 4)})^2}{\langle k_1 \rangle^2 \langle k_2 \rangle^2 \langle k_3 \rangle^2 \langle k_4 \rangle^2} \lesssim \frac{\mathbb{1}_{2^q \lesssim t}}{\langle t \rangle^6} 2^{4q}
\end{aligned}$$

where we used that $|\sigma_t(x)| \lesssim t^{-1/2} \mathbb{1}_{x \sim t}$. Now taking $\varepsilon + d/r > 0$ we have

$$\begin{aligned}
\int_0^\infty \mathrm{d}t \left\{ \sum_q 2^{-qr(\varepsilon + d/r)} (I_{0,q}(t))^{r/2} \right\}^{1/r} \lesssim& \int_0^\infty \mathrm{d}t \left\{ \sum_{q: 2^q \lesssim t} \frac{2^{qr(2-\varepsilon-d/r)}}{\langle t \rangle^{3r}} \right\}^{1/r} \\
\lesssim& \int_0^\infty \frac{\mathrm{d}t}{\langle t \rangle^{1+\varepsilon+d/r}} \lesssim 1.
\end{aligned}$$

Taking into account that $|k_1|, |k_2| \lesssim t$ we can estimate

$$|G_{1,q}^{\langle 2 \rangle \diamond \langle 2 \rangle}((s, \ k)_{12})| \ \lesssim \ |\varrho_q(k_{(12)})| \sum_p \frac{\mathbb{1}_{|p| \lesssim t}}{\langle p \rangle^2} \left(\frac{\sigma_t(k_1+p)}{\langle k_1+p \rangle} \frac{\sigma_t(k_2-p)}{\langle k_2-p \rangle} \right) \ \lesssim$$
$$|\varrho_q(k_{(12)})| \langle t \rangle^{-2},$$

from which we deduce that

$$
\begin{aligned}
I_{1,q}(t) := {} & \mathbb{E}\left[\left|\sum_{k_1,k_2}\int_{A_t^2} G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\right|^2\right] \\
\lesssim {} & \sum_{k_1,k_2}\int_{A_t^2}|G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\frac{\sigma_{s_2}(k_2)^2}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2 \\
\lesssim {} & \langle t\rangle^{-4}\sum_{k_1,k_2}|\varrho_q(k_{(12)})|^2\int_{[0,t]^2}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\frac{\sigma_{s_2}(k_2)^2}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2 \\
\lesssim {} & \langle t\rangle^{-4}\sum_{k_1,k_2}|\varrho_q(k_{(12)})|^2\frac{\mathbb{1}_{k_1\lesssim t}}{\langle k_1\rangle^2}\frac{\mathbb{1}_{k_2\lesssim t}}{\langle k_2\rangle^2}\lesssim \langle t\rangle^{-4}2^{2q}\mathbb{1}_{2^q\lesssim t},
\end{aligned}
$$

and then, as for $I_{0,q}$, we have

$$
\int_0^\infty \mathrm{d}t\left(\sum_q 2^{-qr(\varepsilon+d/r)}(I_{1,q}(t))^{r/2}\right)^{1/r} \lesssim \int_0^\infty\frac{\mathrm{d}t}{\langle t\rangle^2}\left(\sum_q 2^{qr(1-\varepsilon-d/r)}\mathbb{1}_{2^q\lesssim t}\right)^{1/r}\lesssim 1,
$$

as claimed. From these estimates standard arguments give eq. (59). □

Lemma 47. *We have*

$$
\mathbb{E}[\|\mathbb{W}_T^3\|_{L^p}^p]^{1/p}\lesssim T^{3/2}.
$$

This implies that $\mathbb{W}^{\langle 3\rangle}\in C([0,\infty], B_{p,p}^{-1/2-\kappa})\cap L^2(\mathbb{R}_+, B_{p,p}^{-1/2-\kappa})$ *for any* $p<\infty$ *uniformly in the volume and* $\mathbb{W}^{\langle 3\rangle}\in C([0,\infty],\mathscr{C}^{-1/2-\kappa})\cap L^2(\mathbb{R}_+, \mathscr{C}^{-1/2-\kappa})$.

Proof. Observe that

$$
\mathbb{W}_T^3(x) = 12[\![W_T^3]\!](x) = 24\sum_{k_1,k_2,k_3}e^{i(k_1+k_2+k_3)\cdot x}\int_0^T\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3).
$$

By space homogeneity, we get for any p,

$$
\begin{aligned}
\mathbb{E}[\|\mathbb{W}_T^3(x)\|_{L^p}^p] &= \mathbb{E}[\|\mathbb{W}_T^3(0)\|^p] \\
&= \mathbb{E}\left[\left|\sum_{k_1,k_2,k_3}\int_0^T\int_0^{s_2}\int_0^{s_1}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\right|^p\right] \\
&\lesssim \left(\mathbb{E}\left[\left|\sum_{k_1,k_2,k_3}\int_0^T\int_0^{s_2}\int_0^{s_1}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\right|^2\right]\right)^{p/2}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{k_1,k_2,k_3} \int_0^T \int_0^{s_2} \int_0^{s_1} \frac{\sigma_{s_1}(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3 \rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3 \right)^{p/2} \\
&\lesssim (T^{3/2})^p.
\end{aligned}
$$

Since

$$
\begin{aligned}
& \sum_{k_1,k_2,k_3} \int_0^T \int_0^{s_2} \int_0^{s_1} \frac{\sigma_{s_1}(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3 \rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3 \\
\leqslant & \sum_{k_1,k_2,k_3} \int_0^T \int_0^{T} \int_0^{T} \frac{\sigma_{s_1}(k_1)^2}{\langle k_1 \rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3 \rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3 \\
= & \left(\sum_k \int_0^T \frac{\sigma_s(k)^2}{\langle k \rangle^2} \mathrm{d}s \right)^3 \lesssim T^3.
\end{aligned}
$$

Now the remaining properties follow by the fact that σ_t is supported in an annulus of radius t, so

$$
\| \mathbb{W}_t^{\langle 3 \rangle} \|_{B^{-1/2-\kappa}_{p,p}} = \left\| \frac{\sigma_t(\mathrm{D})}{\langle \mathrm{D} \rangle} \mathbb{W}_t^3 \right\|_{B^{-1/2-\kappa}_{p,p}} \lesssim \| \sigma_t(\mathrm{D}) \mathbb{W}_t^3 \|_{B^{-3/2-\kappa}_{p,p}} \lesssim \langle t \rangle^{-1/2-\kappa} \langle t \rangle^{-3/2} \| \mathbb{W}_t^3 \|_{L^p}
$$

and the Hölder estimates follow by Besov embedding (but with constants which depends on the volume). □

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Appendix A. Besov spaces and paraproducts

In this section we will recall some well known results about Besov spaces, embeddings, Fourier multipliers and paraproducts. The reader can find full details and proofs in [3, 31] and for weighted spaces in [30, 40]. First recall the definition of Littlewood–Paley blocks. Let χ, ϱ be smooth radial functions $\mathbb{R}^d \to \mathbb{R}$ such that

- $\operatorname{supp}\chi \subseteq B(0,R)$, $\operatorname{supp}\varrho \subseteq B(0,2R) \setminus B(0,R)$;
- $0 \leqslant \chi, \varrho \leqslant 1$, $\chi(\xi) + \sum_{j \geqslant 0} \varrho(2^{-j}\xi) = 1$ for any $\xi \in \mathbb{R}^d$;
- $\operatorname{supp} \varrho(2^{-j} \cdot) \cap \operatorname{supp} \varrho(2^{-i} \cdot) = \varnothing$ if $|i-j| > 1$.

Introduce the notations $\varrho_{-1} = \chi$, $\varrho_j = \varrho(2^{-j} \cdot)$ for $j \geqslant 0$. For any $f \in \mathscr{S}'(\Lambda)$ we define the operators $\Delta_j f := \varrho_j(\mathrm{D}) f$, $j \geqslant -1$.

Definition 48. *Let $s \in \mathbb{R}, p, q \in [1, \infty]$. For a Schwarz distribution $f \in \mathscr{S}'(\Lambda)$ define the norm*

$$\|f\|_{B^s_{p,q}} = \|(2^{js} \|\Delta_j f\|_{L^p})_{j \geqslant -1}\|_{\ell^q}$$

where $\|\|_{L^p}$ denotes the normalized $L^p(\Lambda)$ norm. The space $B^s_{p,q}$ is the set of functions $f \in \mathscr{S}'(\Lambda)$ such that $\|f\|_{B^s_{p,q}} < \infty$ moreover $H^s = B^s_{2,2}$ are the usual Sobolev spaces, and we denote by $\mathscr{C}^s$ the closure of smooth functions in the $B^s_{\infty,\infty}$ norm.

Definition 49. *A polynomial weight ρ is a function $\rho: \mathbb{R}^d \to \mathbb{R}_+$ of the form $\rho(x) = c\langle x\rangle^{-\sigma}$ for $\sigma, c \geqslant 0$. For a polynomial weight ρ let*

$$\|f\|_{L^p(\rho)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \rho(x)\, \mathrm{d}x \right)^{1/p}$$

and by $L^p(\rho)$ the space of functions for which this norm is finite. For function defined on a torus in $\mathbb{R}^d$ we consider their periodic extensions on $\mathbb{R}^d$.

Definition 50. *For a polynomial weight ρ let*

$$\|f\|_{L^p(\rho)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \rho(x)\, \mathrm{d}x \right)^{1/p}$$

and by $L^p(\rho)$ the space of functions for which this norm is finite. For functions defined on the torus Λ we consider their periodic extensions on $\mathbb{R}^d$. Similarly we define the weighted Besov spaces $B^s_{p,q}(\rho)$ as the set of elements of $\mathscr{S}'(\mathbb{R}^d)$ for which the norm

$$\|f\|_{B^s_{p,q}(\rho)} = \|(2^{js} \|\Delta_j f\|_{L^p(\rho)})_{j \geqslant -1}\|_{\ell^q}$$

is finite and by $\mathscr{C}^s(\rho)$ those such that the norm

$$\|f\|_{\mathscr{C}^s(\rho)} = \|(2^{js} \|\rho \Delta_j f\|_{L^\infty})_{j \geqslant -1}\|_{\ell^\infty}$$

is finite.

Proposition 51. *Let $\delta > 0$. We have for any $q_1, q_2 \in [1, \infty]$, $q_1 < q_2$*

$$\| f \|_{B^s_{p, q_2}} \leq \| f \|_{B^s_{p, q_1}} \leq \| f \|_{B^{s + \delta}_{p, \infty}} .$$

Furthermore, if we denote by $W^{s, p}$ the normalized fractional Sobolev spaces then for any $q \in [1, \infty]$

$$\| f \|_{B^s_{p, q}} \leq \| f \|_{W^{s + \delta, p}} \leq \| f \|_{B^{s + 2 \delta}_{p, \infty}} .$$

Proposition 52. *For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, any $p, q \in [1, \infty]$ the Besov space $B^{s_1}_{p,q}$ is compactly embedded into $B^{s_2}_{p,q}$.*

Definition 53. *A smooth function η is said to be an S^m multiplier if for every multi-index α there exists a constant C_α such that*

$$\left| \frac{\partial^\alpha}{\partial \xi^\alpha} \eta (\xi) \right| \lesssim_\alpha (1 + | \xi |)^{m - | \alpha |}, \qquad \xi \in \mathbb{R}^d. \tag{64}$$

We say that a family η_t is a uniformly S^m multiplier if (64) is satisfied for every t with C_α independent of t.

Proposition 54. *Let η be an S^m multiplier, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $f \in B^s_{p,q}$, then*

$$\| \eta (\mathrm{D}) f \|_{B^{s - m}_{p, q}} \lesssim \| f \|_{B^s_{p, q}} .$$

Furthermore the constant depends only on s, p, q, d and the constants C_α in eq. (64).

For a proof see [3] Lemma 2.78.

Proposition 55. *Let θ p, p_1, p_2 and s, s_1, s_2 be such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}$ and $s = \theta s_1 + (1 - \theta) s_2$ and assume that $f \in W^{s_1, p_1} \cap W^{s_2, p_2}$. Then*

$$\| f \|_{W^{s, p}} \leqslant \| f \|_{W^{s_1, p_1}}^\theta \| f \|_{W^{s_2, p_2}}^{1 - \theta} .$$

For a proof see [9].

Definition 56. *Let $f, g \in \mathscr{S} (\Lambda)$. We define the paraproducts and resonant product*

$$f \succ g = g \prec f := \sum_{j < i - 1} \Delta_i f \Delta_j g, \qquad \text{and} \qquad f \circ g := \sum_{| i - j | \leqslant 1} \Delta_i f \Delta_j g.$$

Then

$$f g = f \prec g + f \circ g + f \succ g.$$

Proposition 57. *For any polynomial weight ρ, $\beta \leqslant 0$, $\alpha \in \mathbb{R}$ and p_1, $p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ we have the estimate*

$$\| f \succ g \|_{B_{p, q}^{\alpha + \beta} (\rho)} \lesssim \| f \|_{B_{p_1, \infty}^{\alpha} (\rho)} \| g \|_{B_{p_2, q}^{\beta} (\rho)} .$$

For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$ the estimate

$$\| f \circ g \|_{B_{p, q}^{\alpha + \beta} (\rho)} \lesssim \| f \|_{B_{p_1, \infty}^{\alpha} (\rho)} \| g \|_{B_{p_2, q}^{\beta} (\rho)} .$$

For a proof see Theorem 3.17 and Remark 3.18 in [40].

Proposition 58. *For any polynomial weights ν, ρ and $\beta \leqslant 0, \alpha \in \mathbb{R}$ we have*

$$\| f \succ g \|_{B_{p, q}^{\alpha + \beta} (\rho^p \nu)} \lesssim \| f \|_{\mathscr{C}^{\alpha} (\rho)} \| g \|_{B_{p, q}^{\beta} (\nu)} .$$

The proof is an easy modification of the proof of Theorem 3.17 in [40].

Proposition 59. *Assume $m \leqslant 0$, $\alpha \in (0, 1), \beta \in \mathbb{R}$. Let η be an S^m multiplier and $q, p_1, p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $f \in B_{p_1, \infty}^{\beta}$, $g \in B_{p_1, \infty}^{\alpha}$. Then for any $\delta > 0$.*

$$\| \eta (\mathrm{D}) (f \succ g) - (\eta (\mathrm{D}) f \succ g) \|_{B_{p, q}^{\alpha + \beta - m - \delta}} \lesssim \| f \|_{B_{p_1, \infty}^{\beta}} \| g \|_{B_{p_1, \infty}^{\alpha}} .$$

The constant depends only on α, β, δ and the constants in (64).

For a proof see [3] Lemma 2.99.

Proposition 60. *Let $\alpha \in (0, 1)$ $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$, $\alpha + \beta + \gamma > 0$ and p_1, p_2, p_3, $p \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p}$. Then there exists a trilinear form $\mathfrak{K}_1 (f, g, h)$ such that ,*

$$\| \mathfrak{K}_1 (f, g, h) \|_{B_{p, \infty}^{\alpha + \beta + \gamma}} \lesssim \| g \|_{B_{p_1, \infty}^{\alpha}} \| f \|_{B_{p_2, \infty}^{\beta}} \| h \|_{B_{p_3, \infty}^{\gamma}},$$

and when $f, g, h \in \mathscr{S}$ it has the form

$$\mathfrak{K}_1 (f, g, h) = (f \succ g) \circ h - g (f \circ h) .$$

Proof. The proof is a slight modification of the one given in [31]. Lemma 2.97 from [3] and an interpolation imply that $\| \Delta_j f g - \Delta_j (f g) \|_{L^p} \leq 2^{- j \alpha} \| f \|_{W^{\alpha, p_1}} \| g \|_{L^{p_2}}$. This in turn gives after some algebraic computation (see [31] for details) that

$$\Delta_j (f \succ g) = (\Delta_j f) \succ g + R_j (f, g),$$

with $\| R_j (f, g) \|_{L^p} \lesssim 2^{- j (\alpha + \beta)} \| f \|_{B_{p_1, \infty}^{\alpha}} \| g \|_{B_{p_2, \infty}^{\beta}}$. Now to prove the statement of the proposition observe that for smooth f, g, h we have

$$\mathfrak{K}_1 (f, g, h) = \sum_{j, k \geqslant - 1} \sum_{| i - j | \leqslant 1} \Delta_j (f \succ \Delta_k g) \Delta_i h - \Delta_k g \Delta_j f \Delta_i h.$$

Now observe that the term $f \succ \Delta_k g$ has Fourier transform outside of $2^k B$ for some ball B independent of k, so choosing N large enough we can rewrite the sum as

$$\mathfrak{K}_1(f,g,h) = \sum_{j,k \geqslant -1} \sum_{|i-j| \leqslant 1} \mathbb{1}_{k \leqslant i+N}(\Delta_j f \Delta_k g \Delta_i h + R_j(f, \Delta_k g)) - \Delta_k g \Delta_j f \Delta_i h$$

$$\sum_{j,k \geqslant -1} \sum_{|i-j| \leqslant 1} \mathbb{1}_{k \leqslant i+N} R_j(f, \Delta_k g) \Delta_i h - \mathbb{1}_{k \geqslant i+N} \Delta_k g \Delta_j f \Delta_i h.$$

Now we estimate the norm of the two terms separately. First note that for fixed j

$$\sum_{k \geqslant -1} \sum_{|i-j| \leqslant 1} \mathbb{1}_{k \leqslant i+N} R_j(f, \Delta_k g)$$

has a Fourier transform supported in $2^j B$. By Lemma 2.69 from [3] it is enough to get an estimate on

$$\sup_k \left\| 2^{(\alpha+\beta+\gamma)j} \sum_{j \geqslant -1} \sum_{|i-j| \leqslant 1} \mathbb{1}_{k \leqslant i+N} R_j(f, \Delta_k g) \Delta_i h \right\|_{L^p}$$

to bound it in $B^{\alpha+\beta+\gamma}_{p,\infty}$, so by Hölder inequality,

$$\begin{aligned} \left\| \sum_{|i-j| \leqslant 1} R_j\left(f, \sum_{k \geqslant -1}^{i+N} \Delta_k g\right) \Delta_i h \right\|_{L^p} &\lesssim \sum_{|i-j| \leqslant 1} 2^{-j(\alpha+\beta)} 2^{-i\gamma} \|g\|_{B^\alpha_{p_1,\infty}} \|f\|_{B^\beta_{p_2,q_1}} \|h\|_{B^\gamma_{p_3,q_2}} \\ &\lesssim 2^{-j(\alpha+\beta+\gamma)} \|g\|_{B^\alpha_{p_1,\infty}} \|f\|_{B^\beta_{p_2,q_1}} \|h\|_{B^\gamma_{p_3,q_2}}. \end{aligned}$$

For the second term observe that for fixed k the Fourier transform of

$$\sum_{j \geqslant -1} \sum_{|i-j| \leqslant 1} \mathbb{1}_{k \geqslant i+N} \Delta_k g \Delta_j f \Delta_i h$$

is supported in $2^k B$. Now we can estimate again by Hölder inequality

$$\begin{aligned} &\lesssim \left\| \sum_{j \geqslant -1} \sum_{|i-j| \leqslant 1} \mathbb{1}_{k \geqslant i+N} \Delta_k g \Delta_j f \Delta_i h \right\|_{L^p} \\ &\lesssim 2^{-\alpha k} \sum_{j \geqslant -1}^{k+N} 2^{-(\beta+\gamma)k} \mathbb{1}_{k \geqslant i+N} \|g\|_{B^\alpha_{p_1,\infty}} \|f\|_{B^\beta_{p_2,\infty}} \|h\|_{B^\gamma_{p_1,\infty}} \\ &\lesssim 2^{-j(\alpha+\beta+\gamma)} \|g\|_{B^\alpha_{p_1,\infty}} \|f\|_{B^\beta_{p_2,q_1}} \|h\|_{B^\gamma_{p_3,q_2}}. \end{aligned}$$

□

Proposition 61. *Assume $\beta \in (0,1)$, $\alpha, \gamma \in \mathbb{R}$ such that $\alpha + \gamma < 0$, and $\alpha + \beta + \gamma = 0$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then there exists a trilinear form $\mathfrak{K}_2(f,g,h)$ for which*

$$|\mathfrak{K}_2(f,g,h)| \lesssim \|f\|_{B^\alpha_{p_1,\infty}} \|g\|_{B^\beta_{p_2,q_1}} \|h\|_{B^\gamma_{p_3,q_2}},$$

and on smooth functions

$$\mathfrak{K}_2(f,g,h) = \fint [(f \succ g)h - g(f \circ h)].$$

Proof. This is modification of the proof of Lemma A.6 in [29]. Repeating an algebraic computation given in the proof of that lemma, we get that for smooth f, g, h we have

$$\mathfrak{K}_2(f,g,h) = \left(\sum_{j \leqslant i-1, |i-k| \leqslant L} - \sum_{i \sim k, j < i+L} \right) \fint (\Delta_i f \Delta_j g \Delta_k h),$$

for some $L \geqslant 1$. Then we estimate

$$\begin{aligned} |\mathfrak{K}_2(f,g,h)| &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f \Delta_j g \Delta_k h\|_{L^1} \\ &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f\|_{L^{p_1}} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \sup_i (2^{\alpha i} \|\Delta_i f\|_{L^{p_1}}) \sum_{j \sim k} 2^{(\beta+\gamma)k} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \|f\|_{B^\alpha_{p_1,\infty}} \|g\|_{B^\beta_{p_2,q_1}} \|h\|_{B^\gamma_{p_3,q_2}}. \end{aligned}$$

□

Proposition 62. *There exists a family $(\mathfrak{K}_{3,t})_{t \geqslant 0}$ of bounded multilinear forms on $\mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-1-\kappa} \times H^{1/2-\kappa} \times H^{1/2-\kappa}$ such that for smooth φ, ψ, $g^{(1)}$, $g^{(2)}$ it holds*

$$\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)}) = \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t \varphi \circ J_t \psi) g^{(1)} g^{(2)}],$$

and

$$|\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)})| \lesssim \frac{1}{\langle t \rangle^{1+\delta}} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}.$$

Proof. Note that $\langle t \rangle^{1/2} J_t$ satisfies the assumptions of Proposition 59 and with $m = -1$, therefore using also Proposition 51

$$\|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \lesssim \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}}.$$

Therefore

$$\begin{aligned} & \left| \fint [J_t(\varphi \succ g^{(1)}) - (J_t \varphi \succ g^{(1)})] J_t(\psi \succ g^{(2)}) \right| \\ \lesssim\ & \|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \|J_t(\psi \succ g^{(2)})\|_{H^{-1/2+3\kappa}} \\ \lesssim\ & \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \langle t \rangle^{-1/2-\delta} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

and by symmetry also

$$\begin{aligned} & \left| \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t\varphi \succ g^{(1)})(J_t\psi \succ g^{(2)})] \right| \\ \lesssim \; & \langle t\rangle^{-1-\kappa} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

Furthermore from Proposition 61 and for sufficiently small $\kappa > 0$,

$$\begin{aligned} & \left| \fint (J_t\varphi \succ g^{(1)})(J_t\psi \succ g^{(2)}) - \fint ((J_t\varphi \succ g^{(1)}) \circ J_t\psi) g_t^{(2)} \right| \\ \lesssim \; & \|J_t\varphi\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t\psi\|_{\mathscr{C}^{-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \\ \lesssim \; & \langle t\rangle^{-1-\kappa} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}. \end{aligned}$$

Applying Proposition-60 we get

$$\begin{aligned} & \|(J_t\varphi^{(1)} \succ g^{(1)}) \circ J_t\psi_t - (J_t\varphi_t \circ J_t\psi_t)(g^{(1)})\|_{H^{-1/2+\kappa}} \\ \lesssim \; & \|J_t\varphi_t\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t\psi_t\|_{\mathscr{C}^{-\kappa}} \\ \lesssim \; & \langle t\rangle^{-1-\delta} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \end{aligned}$$

and putting things together gives the required estimate. □