A Forward Backward Approach to Stochastic Quantisation

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Motivation: Constructive QFT

- Osterwalder-Schrader reconstruction theorem ('75):

$$\text{Quantum Field Theory} \quad \underset{\text{rotation}}{\overset{\text{Wick}}{\longleftrightarrow}} \quad \text{Euclidean Quantum Field Theory}$$

- EQFT: Certain **Probability measures** on the space of **distributions** $\mathcal{S}'(\mathbb{R}^d)$.

$$\text{``}\ \mathbb{E}_\nu[\mathcal{O}(\Phi)] = \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)\,\nu(\mathrm{d}\varphi) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)\mathrm{e}^{-S(\varphi)}\mathrm{d}\varphi\ \text{''}$$

for

$$S(\varphi) = Q(\varphi,\varphi) + V(\varphi),$$

$$Q(\varphi,\varphi) = \int_{\mathbb{R}^d} (m^2|\varphi(x)|^2 + |\nabla\varphi(x)|^2)\mathrm{d}x \qquad V(\varphi) = \lambda \int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x$$

positive quadratic form $\qquad$ $U: \mathbb{R}^d \longrightarrow \mathbb{R}$ bounded from below

Simplest case: Gaussian Free Field

For $V(\varphi)=0$,

$$``\mu(\mathrm{d}\varphi)=\mathrm{e}^{-S_{\mathrm{free}}(\varphi)}\mathrm{d}\varphi", \quad S_{\mathrm{free}}(\varphi)=Q(\varphi,\varphi)=\int_{\mathbb{R}^d}(m^2|\varphi(x)|^2+|\nabla\varphi(x)|^2)\mathrm{d}x,$$

formally corresponds to a Gaussian measure on $\mathcal{S}'(\mathbb{R}^d)$ with

$$\mathrm{Cov}(\mu)=(m^2-\Delta)^{-1},$$

and $\mathrm{supp}(\mu)\subset H^{\alpha-}(\mathbb{R}^d)$ for $\alpha=(2-d)/2$

$\longrightarrow$only for $d=1$ supported on functions.

$\longrightarrow$ Starting point for more interesting EQFTs

Gibbsian pertubations of the GFF

$$``\nu(\mathrm{d}\varphi) = \frac{1}{\text{norm.}} \mathrm{e}^{-V(\varphi)} \mu(\mathrm{d}\varphi)" \quad \text{where} \quad V(\varphi) = \int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x$$

- Some possible starting points to obtain non-Gaussian models:
  - in $d = 2$:

$$U(x) = \lambda x^{2p} + \sum_{\ell}^{2p-1} a_\ell x^\ell \quad \text{for any } p > 0,$$
$$U(x) = \lambda \exp(\beta x),$$
$$U(x) = \lambda \cos(\beta x),$$

  - in $d = 2, 3$:

$$U(x) = \lambda x^4 - b x^2.$$

Goal: Make sense of

$$``\,\nu(\mathcal{O}) = \frac{1}{\text{norm.}}\int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)\mathrm{e}^{-S(\varphi)}\mathrm{d}\varphi$$

$$= \frac{1}{\text{norm.}}\int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi)\mathrm{e}^{-\int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x}\mu(\mathrm{d}\varphi)'',$$

with μ the Gaussian free field and,

$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}(|\nabla\varphi(x)|^2 + m^2|\varphi(x)|^2) + U(\varphi(x))\mathrm{d}x.$$

Problems:

Large Scales: No decay in space: $S(\varphi) = \infty$ at best (non-sense at worst)

Small Scales: ν not supported on **function** spaces but only on **distributions**

$\longrightarrow U(\varphi(x))$ ill-defined

Approximate Measures

With $V(\varphi) = \int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x$, define approximations

$$\mathrm{e}^{-V(\varphi)}\mu(\mathrm{d}\varphi) \quad \approx \quad \mathrm{e}^{-V_T^{\xi}(\varphi)}\mu^T(\mathrm{d}\varphi),$$

Large scale Problem	**Small Scale Problem**
$\int_{\mathbb{R}^d} U(\varphi(x))\mathrm{d}x = \infty$?	$\mathrm{supp}(\mu) \subset H^{(2-d)/2-}(\mathbb{R}^d)$
cut-off in space $\xi \in C_c^{\infty}(\mathbb{R}^{\mathrm{d}})$:	Regularise the measure:
$V^{\xi}(\varphi) = \int_{\mathbb{R}^d} \xi(x) U(\varphi(x))\mathrm{d}x$	$\mu^T \longrightarrow \mu$,
	μ^T supported on **functions**
	Additionally:
	Choose V_T depending on T

Question: Can we recover a EQFT?

$$\nu^{T,\xi}(\mathcal{O}) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) \mathrm{e}^{-V_T^{\xi}(\varphi)} \mu^T(\mathrm{d}\varphi)$$
$$\xrightarrow{???}$$
$$\text{``}\ \nu(\mathcal{O}) = \frac{1}{\text{norm.}} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) \mathrm{e}^{-V(\varphi)} \mu(\mathrm{d}\varphi)\ \text{''}.$$

Problem: In general, ν not absolutely continuous w.r.t. the Gaussian free field μ

$\longrightarrow$ Move to different characterisations for $\nu^{T,\xi}$ that do not rely on absolute continuity

Stochastic Quantisation: Basic Idea

Starting point:

Given a regularisation $\mu \mapsto \mu^T$ and a cut-off ξ we can construct $\nu^{T,\xi}$ (as the Gibbsian perturbation of the Free Field)

Think of a map

$$\text{``}\,\Phi^{\xi}\colon \mu^T \mapsto \nu^{T,\xi}\,\text{''}$$

Idea: Study the maps Φ^{ξ} to learn about the measures $\nu^{T,\xi}$ and (ideally) remove both regularisations T, ξ.

By now many, different approaches building on this perspective (e.g. via parabolic, elliptic SPDEs as introduced by Parisi/Wu-'81)

[G. Parisi, Y. Wu · Perturbation theory without gauge fixing · Sci. Sin. · 1981]

In this talk:

For a suitable potential V, and cut-offs $T < \infty, \xi \in C_c^\infty(\mathbb{R}^d)$:

If X solves the SDE

$$X_{t,T}^{\xi} = W_t - \int_0^t \dot{G}_s \mathbb{E}_s[\nabla V_T^{\xi}(X_{T,T})]\mathrm{d}s, \quad 0 \leqslant t \leqslant T.$$

and W_s is a Brownian motion with covariance G_s and $\mathrm{Law}(W_\infty) = \mu$.

Then, we can show,

$$\Phi^{\xi}(\mu^T) := \mathrm{Law}(X_{T,T}^{\xi}) = \nu^{\xi,T}.$$

Towards a limit

So far: Found the description $\nu^{\xi,T} = \mathrm{Law}\left(X^{\xi}_{T,T}\right)$, where

$$X^{\xi}_{t,T} = W_t - \int_0^t \dot{G}_s \mathbb{E}_s\left[\nabla V^{\xi}_T\left(X^{\xi}_{T,T}\right)\right] \mathrm{d}s.$$

Goal: Remove the regularisations ξ and T to recover $\nu = \mathrm{Law}(X^1_{\infty,\infty})$

- $\xi \longrightarrow 1$: Here: now mainly a technical problem[1]
- $T \longrightarrow \infty$: More delicate and more interesting (for this talk):

In dim. $d \geqslant 2$, covariance $G_T(0) := \int_0^T Q_s^2(0) \mathrm{d}s$ diverges as $T \longrightarrow \infty$, and so

$$\|\nabla V_T\|_\infty \longrightarrow \infty, \quad \text{as} \quad T \longrightarrow \infty.$$

1. so we drop it from now on.

Towards uniform bounds

$$X_{t,T} = W_t - \int_0^t \dot{G}_s \mathbb{E}_s[\nabla V_T(X_{T,T})]\mathrm{d}s \quad \text{where} \quad \lim_{T\to\infty} \|\nabla V_T\|_\infty \longrightarrow \infty.$$

Starting point: If X is a Markov process (as we would expect), for some $\wp$,

$$\mathbb{E}_s[\nabla V_T(X_{T,T})] = \wp_s^T(X_{s,T}).$$

Ansatz: Find a function F

$$\mathbb{E}_s[\nabla V_T(X_{T,T})] = F_{s,T}(X_{s,T}) + R_{s,T}, \quad R_{T,T} = 0,$$

to bring down the scales.

Then, the remainder R satisfies a BSDE

$$R_{t,T} = \mathbb{E}_t[F_{T,T}(X_{T,T}) - F_{t,T}(X_{t,T})].$$

Derived the system

$$\begin{cases} X_{t,T} = W_t - \int_0^t \dot{G}_s(F_s(X_{s,T}) + R_{s,T})\mathrm{d}s, \\ R_{t,T} = \mathbb{E}_t[F_T(X_{T,T}) - F_t(X_{t,T})], \end{cases}$$

and from Itô's formula obtain an equation for R,

$$R_{t,T} = \mathbb{E}_t \int_t^T [H_s^F(X_{s,T}) - \mathrm{D}F_s(X_{s,T})\dot{G}_s R_{s,T}]\mathrm{d}s,$$

where

$$H_s^F(\varphi) = \left(\partial_s F_{s,T} + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_{s,T}) - \mathrm{D}F_{s,T}\dot{G}_s F_{s,T}\right)(\varphi).$$

A new problem: Approximate solutions to the flow equation

Goal: Find a "good enough" approximation F to the flow equation

$$H_s^F := \partial_s F_{s,T} + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_{s,T}) - \mathrm{D}F_{s,T}\dot{G}_s F_{s,T} \approx 0,$$

and solve

$$\begin{cases} X_{t,T} &= W_t - \int_0^t \dot{G}_s^2 (F_s(X_{T,T}) + R_{s,T}) \mathrm{d}s, \\ R_{t,T} &= \mathbb{E}_t [F_T(X_{T,T}) - F_t(X_{t,T})] \\ &= \mathbb{E}_t \int_t^T \mathrm{d}s H_s^F (X_{s,T}) - \mathbb{E}_t \int_t^T \mathrm{d}s \mathrm{D}F_s \dot{G}_s R_s. \end{cases}$$

with uniform bounds in T.

A concrete example: First order approximation for $V_t(x) = \lambda_t \int_{\mathbb{R}^2} \mathrm{d}x \cos(\beta\varphi(x))$

$$\begin{cases} X_{t,T} = W_t - \int_0^t \dot{G}_s (F_{s,T}(X_{s,T}) + R_{s,T}) \mathrm{d}s \\ R_{t,T} = \mathbb{E}_t \int_t^T [H_s^F(X_{s,T}) - \mathrm{D}F_s(X_{s,T}) \dot{G}_s R_{s,T}] \mathrm{d}s \end{cases} \quad \text{where} \quad H_s^F = \partial_s F_{s,T} + \frac{1}{2} \mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_{s,T}) - \mathrm{D}F_{s,T} \dot{G}_s F_{s,T}$$

Start by solving **only the linear equation**,

$$\partial_s F_s + \frac{1}{2} \mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) = 0, \quad F_T = \nabla V_T.$$

so that $H_s^F = \mathrm{D}F_s \dot{G}_s F_s$, and $F_t = \nabla V_t = -\lambda_t \beta \sin(\beta\varphi)$,

$$(\star) \begin{cases} X_{t,T} = W_t - \int_0^t \dot{G}_s (F_s(X_{s,T}) + R_{s,T}) \mathrm{d}s, \\ R_{t,T} = \mathbb{E}_t \int_t^T [\mathrm{D}F_{s,T} \dot{G}_s F_s - \mathrm{D}F_s(X_{s,T}) \dot{G}_s R_{s,T}] \mathrm{d}s. \end{cases}$$

Theorem

For any $T \in [0, \infty]$ *and* $\beta^2 < 4\pi$*, there is a solution* $(X_{\cdot,T}, R_{\cdot,T})$ *to the FBSDE* $(\star)$ *(unique for weak interactions) with* $\sup_{t,T} \|R_{t,T}\|_{L^\infty} \lesssim 1$*, Moreover, writing*

$$X_{t,\infty} = \mathcal{Z}_t + W_t \quad \text{where} \quad \mathcal{Z}_t = \int_0^t \dot{G}_s(F_{s,\infty}(X_{s,\infty}) + R_{s,\infty}) \mathrm{d}s.$$

we have convergence $\mathcal{Z}_t \longrightarrow \mathcal{Z}_\infty$ *in* $L^\infty(\mathrm{d}P; W^{1,\infty}(\mathbb{R}^d))$ *so that we obtain the sine-Gordon EQFT as a* ***random shift of the GFF***

$$\nu_{\mathrm{SG}} = \mathrm{Law}(X_{\infty,\infty}) = \mathrm{Law}\left(\mathcal{Z}_\infty + W_\infty\right).$$

Why this approach?

- Pathwise, scale-by-scale coupling of the GFF and the EQFT
- Ameable to stochastic analysis: e.g. coupling methods $\longrightarrow$ decay of correlations
- Approximate solutions to the infinite dimensional, non-linear PDE ("renormalisation flow equation"

$$\partial_s F_s + \frac{1}{2}\mathrm{Tr}(\dot{G}_s \mathrm{D}^2 F_s) - \mathrm{D}F_s \dot{G}_s F_s = 0, \quad F_T = \nabla V_T,$$

  are sufficient (if you can control the resulting FBSDE).
- closely linked to an optimisation problem $\longrightarrow$ large deviations
- Can verify OS axioms from studying the FBSDE (so we constructed a EQFT)
- Limit is non-Gaussia (i.e. the EQFT is non-trivial)

What's next?

For this specific model: Cover a wider parameter range for β^2?:

- For $\beta^2 \in (0, 8\pi)$: model is known to be renormalisable but with **infinitely many threshholds** requiring additional renormalisations (full control on the full space not yet achieved). [G. Benfatto, G. Gallavotti, F. Nicoló, et al.· On the massive sine-Gordon equation in {the first few/ higher/ all} regions of collapse · Comm. math. phys. {1982/ 1983/ 1986}]
- Beyond 4π: The linear approximation for the renormalisation flow is not enough $\longrightarrow$ requires better understanding of **approximations**

$$\partial_s F_s + \frac{1}{2}\mathrm{Tr}\dot{G}_s \mathrm{D}^2 F_s - \mathrm{D}F_s \dot{G}_s F_s \approx 0; \quad F_T = \lambda_T \sin(\beta\varphi).$$

- As critically is approached, this requires more and more 'non-linear' approximations F making the analysis of the forward equation more difficult.

What's next?

Better approximations of the renormalisation flow. Start from

$$F_s^{[0]}(\varphi) = 0,$$

and schematically expect better approximations by iterating for $\ell > 0$,

$$\partial_s F_s^{[\ell]} + \mathrm{Tr}\dot{G}_s \mathrm{D}^2 F_s^{[\ell]} = -\sum_{\ell_1+\ell_2=\ell} \mathrm{D}F_s^{[\ell_1]} \dot{G}_s F_s^{[\ell_2]}, \quad F_T^{[\ell]} = \begin{cases} \nabla V_T & , \ell = 1 \\ 0 & , \text{else} \end{cases}$$

(so even bounded initial conditions appear polynomial as ℓ increases!)

Then with $F_s = \sum_{q \leqslant \ell} F_s^{[q]}$, we need more and more terms as we approach criticality

$\longrightarrow$ FBSDEs appear nonlinear and the analysis becomes more involved.

Thanks!

Decompose the Gaussian free field as

$$\mathrm{Cov}(\mu) = (m^2 - \Delta)^{-1} = \int_0^\infty Q_s^2 \mathrm{d}s$$

for "nice"[2] operators Q_s, and a cylindrical Brownian motion B,

$$W_t := \int_0^t Q_s \mathrm{d}B_s \quad \text{is a Brownian motion with} \quad \mathrm{Cov}(W_t) = \int_0^t Q_s^2 \mathrm{d}s =: G_t,$$

e.g. $Q_t^2 = \frac{1}{t^2} \mathrm{e}^{-(m^2 - \Delta)/t^2}$.

Then, W_t is a **function** for any $t \in (0, \infty)$ with $W_\infty \sim \mu$ and we define

$$\mu^T := \mathrm{Law}(W_T)$$

2. self-adjoint, positive and Hilbert-Schmidt

Multiscale Decomposition

With

$$\mu^T = \mathrm{Law}(W_T) = \int_0^T Q_s \mathrm{d}B_s$$

we can write

$$\nu_{\mathrm{SG}}^{\xi,T}(\mathcal{O}) = \frac{1}{\mathrm{norm.}} \int_{\mathcal{S}'(\mathbb{R}^2)} \mathcal{O}(\varphi) \mathrm{e}^{-V_T^{\xi}(\varphi)} \mu^T(\mathrm{d}\varphi) = \frac{\mathbb{E}\left[\mathcal{O}(W_T) \mathrm{e}^{-V_T^{\xi}(W_T)}\right]}{\mathbb{E}\left[\mathrm{e}^{-V_T^{\xi}(W_T)}\right]},$$

e.g. for the family of observables

$$\mathcal{O}(\varphi) = \mathrm{e}^{-g(\varphi)}.$$

$\longrightarrow$ study exponential functionals of Brownian motion

Variational Approach

Theorem. (Boué-Dupuis ('98)) *For a bounded functional F and a Q-Brownian motion W, the variational description*

$$-\log \mathbb{E}[e^{-F(W_\bullet)}] = \inf_{u \in \mathbb{H}^0} \mathbb{E}\left[F(X_\bullet(u)) + \frac{1}{2}\int_0^\infty \|u_s\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}s \right],$$

holds. Here, $\mathbb{H}^0$ is the space of adapted processes and

$$X_t(u) := W_t + \int_0^t Q_s u_s \mathrm{d}s.$$

[M. Boué, P. Dupuis · A variational representation for certain functionals of Brownian motion · Ann. Prob. 1998]

[N. Barashkov, M. Gubinelli · A variational method for φ^4_3 · Duke math. J. 2020]

[N. Barashkov, M. Gubinelli · On the variational method for EQFT in 2D · arXiv preprint · 2021]

Variational Description

Apply the BD-formula to the BM W and the functional,

$$V_T^{\xi}(\varphi) := \lambda_T \int_{\mathbb{R}^2} \xi(x)\cos(\varphi(x))\mathrm{d}x,$$

$$-\log \int \mathrm{e}^{-V_T^{\xi}(\varphi)} \mu^T(\mathrm{d}\varphi) = -\log \mathbb{E}\left[\mathrm{e}^{-V_T^{\xi}(W_T)}\right] = \inf_{u \in \mathbb{H}^0} \mathbb{E}\left[V_T^{\xi}(X_T(u)) + \int_0^{\infty} \|u_s\|_{L^2}^2 \mathrm{d}s \right],$$

where

$$X_T(u) = W_T + \int_0^T Q_s u_s \mathrm{d}s.$$

Now: Look for optimal control u, derive Euler-Lagrange equation.

Stochastic Control Problem

Theorem

The infimum is a minimum and the optimal control satisfies

$$u_s^{\xi,T} = -Q_s \mathbb{E}_s[\nabla V_T^{\xi}(X_T(u^{T,\xi}))],$$

and the optimal dynamics are

$$(*) \quad X_{t,T}^{\xi} = W_t - \int_0^t Q_s^2 \mathbb{E}_s[\nabla V_T^{\xi}(X_{T,T}^{\xi})]\mathrm{d}s.$$

Moreover, the solution to $(*)$ *satisfies*

$$\Phi^{\xi,T}(\mu) := \mathrm{Law}\left(X_{T,T}^{\xi}\right) = \nu_{\mathrm{SG}}^{\xi,T}.$$

Wick ordered cosine

For a centered Gaussian random variable W with covariance G define the Wick ordered exponentials

$$[\![\exp(i\beta W)]\!] := e^{\frac{\beta^2}{2}G} e^{i\beta W}.$$

Use this to define the Wick ordered cosine in the usual way (from $\cos(x) = \operatorname{Re}(e^{ix})$).

Theorem

For any $\delta > 0$, $p \geq 1$ and $\beta^2 < 4\pi$, the Wick ordered cosine satisfies

$$\sup_{t \geq 0} \mathbb{E}\left[\|[\![\cos(\beta W_t)]\!]\|^p_{B^{-\beta^2/4\pi-\delta}_{p,p}(\langle x\rangle^{-\ell})} \right] < \infty,$$

and converges in $L^p\big(\mathrm{d}P; B^{-\beta^2/4\pi-\delta}_{p,p}(\langle x\rangle^{-\ell})\big)$ and almost surely to a limit (denoted by $[\![\cos(\beta W_\infty)]\!]$

(i) Euclidean invariance (ii) Reflection positivity (iii) Exponential moment bounds

- Looking for Gaussian measures satisfying (i) and (ii) leaves us with only combinations of the GFF
- Given a RP measure μ (like the GFF) the perturbation

$$e^{-\int_\Lambda U(\varphi(x))\mathrm{d}x}\mu(\mathrm{d}\varphi)$$

  is again reflection postive for any $\Lambda \subset \mathbb{R}^d$
- Euclidean invariance means that we need $\Lambda = \mathbb{R}^d$

i.e. the cut-off ξ destroys (i), and the mollification T destroys (ii)

But: both properties can be recovered in the limit

Optimality for the BD variational problem

On the finite volume: We can show that the solution X^{ξ}_{∞} to the FBSDE $(\star)$ is optimal for

$$\mathcal{V}^{\xi} = \inf_u \underbrace{\mathbb{E}\left[\lambda_0 \int_{\mathbb{R}^2} \xi(x) [\![\cos(\beta X_t(u))]\!](x) \mathrm{d}x + \frac{1}{2}\int_0^{\infty} \|u_s\|^2_{L^2} \mathrm{d}s\right]}_{=:\mathcal{J}^{0,\xi}(u)}, \qquad X_t = W_t + \int_0^t Q_s u_s \mathrm{d}s.$$

Makes no sense for $\xi = 1$.

However: The solution is Lipschitz in small perturbations of the interaction term V, so we can hope that the variational problem for the Laplace transform

$$\mathcal{W}^{\xi,T}(\mathrm{g}) := v^{\xi,T}(\mathrm{e}^{-\mathrm{g}}) = \inf_u \left(\mathcal{J}^{\mathrm{g},\xi}_T(u) - \mathcal{J}^{0,\xi}_T(u)\right)$$

converges as the cut-off ξ is removed.

Variational Problem on $\mathbb{R}^2$

Theorem

For n sufficiently large, $\lambda > 0$ small enough, the limit of the Laplace transforms exists and satisfies the variational problem

$$\mathcal{W}(g) = \lim_{\substack{\xi \to 1 \\ T \to \infty}} \mathcal{W}^{\xi,T}(g) = \inf_{v \in \mathcal{A}(g)} \mathbb{E}\left[g(X_\infty(\bar{u}+v)) + \int_{\mathbb{R}^2} (U_\infty(X_\infty(\bar{u}+v)) - U_\infty(X_\infty(\bar{u}))) + \mathcal{E}(\bar{u}, v) \right].$$

Here $X_\infty(u) = I_\infty(u) + W_\infty$ is the shifted GFF and

- *$\bar{u}$ is an adapted process which does not depend on g and v*
- *I_∞ is a linear functional increasing regularity by 1*
- *$\mathcal{E}$ is a quadratic form*
- *$\mathcal{A}(g)$ is the set of adapted controls v s.t. $\mathbb{E}\int_0^\infty \|v_s\|^2_{L^2(\langle x \rangle^n)} ds \leq C_{\nabla g, n}$.*

Non-Gaussianity of the limit

For a Gaussian measure supported on $H^{-1}(\langle x\rangle^{-\ell})$ with Cameron-Martin space $H_{\mathrm{CM}}(\nu)\subset H^{-1}(\langle x\rangle^{-\ell})$,

$$\log\int\exp(-\langle\varphi,\psi\rangle)\,\nu(\mathrm{d}\varphi)=\frac{1}{2}\|\psi\|^2_{H_{\mathrm{CM}}(\nu)}+\langle m,\psi\rangle_{H^{-1}(\langle x\rangle^{-\ell})}$$

So it is sufficient to show that the lhs is not quadratic for ν_{SG}.

Applying the BD formula with $V^{\psi}=V+\langle\cdot,\psi\rangle$ we can write the lhs as the limit of the approximate measures $\nu_{\mathrm{SG}}^{\xi,T}$ and (after a Cameron Martin shift) obtain

$$=\lim_{\substack{T\to\infty\\ \xi\to 1}}\langle G_T\psi,G_T\psi\rangle_{(m^2-\Delta)^{-1}}+\mathcal{V}_T^{\xi}((m^2-\Delta)^{-1}\psi)-\mathcal{V}_T^{\xi}(0)$$

but $\nabla\mathcal{V}_T^{\xi}=\nabla V_0^{\xi}(X_{0,T}^{\xi})+R_{t,T}^{\xi}\sim T^{c(\beta)}\sin(\beta\,\cdot\,)+O(1)$ is not linear.