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RELATIONS AND FUNCTIONS

EXERCISE 2.1

1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of x and y.

2. If the set A has 3 elements and the set $B=\{3,4,5\}$, find the number of elements in $(A \times B)$.

3. If $G=\{7,8\}$ and $H=\{5,4,2\}$, find $G \times H$ and $H \times G$.

4. State whether each of the following statements are true or false. If false, rewrite the statement correctly.
   (i) If $P=\{m,n\}$ and $Q=\{n,m\}$, then $P \times Q=\{(m,n),(n,m)\}$.
   (ii) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x,y) such that $x \in A$ and $y \in B$.
   (iii) If $A=\{1,2\}$, $B=\{3,4\}$, then $A \times (B \cap \phi)=\phi$.

5. If $A=\{-1,1\}$, find $A \times A \times A$.

6. If $A \times B=\{(a,x),(a,y),(b,x),(b,y)\}$, find A and B.

7. Let $A=\{1,2\}$, $B=\{1,2,3,4\}$, $C=\{5,6\}$ and $D=\{5,6,7,8\}$. Verify that:
   (i) $A \times (B \cap C)=(A \times B) \cap (A \times C)$
   (ii) $A \times C$ is a subset of $B \times D$.

8. Let $A=\{1,2\}$ and $B=\{3,4\}$. Write $A \times B$. How many subsets does $A \times B$ have? List them.

9. Let A and B be sets such that $n(A)=3$ and $n(B)=2$. If $(x,1),(y,2),(z,1) \in A \times B$, find A and B.

10. The Cartesian product $A \times A$ has 9 elements including $(-1,0)$ and $(0,1)$. Find the set A and the remaining elements of $A \times A$.

2.3 Relations

Consider the two sets $P=\{a,b,c\}$ and $Q=\{\text{Ali, Bhanu, Binoy, Chandra, Divya}\}$. The Cartesian product of P and Q has 15 ordered pairs: $P \times Q=\{(a,\text{Ali}),(a,\text{Bhanu}),\ldots,(c,\text{Divya})\}$.

We can now obtain a subset of $P \times Q$ by introducing a relation R as: $R=\{(x,y): x$ is the first letter of the name $y, x \in P, y \in Q\}$.

Then $R=\{(a,\text{Ali}),(b,\text{Bhanu}),(b,\text{Binoy}),(c,\text{Chandra})\}$. An arrow diagram of this relation R is shown in Fig. 2.4.

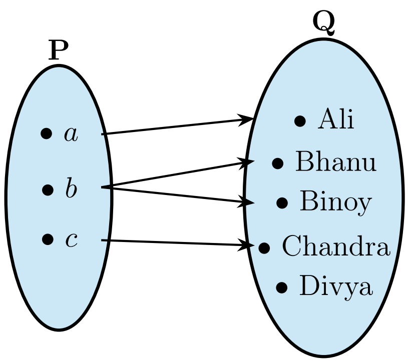


Fig 2.4

Definition 2 A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the *image* of the first element.

Definition 3 The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

Definition 4 The set of all second elements in a relation R from a set A to a set B is called the *range* of the relation R. The whole set B is called the *codomain* of the relation R. Note that range $\subset$ codomain.

Remarks

1. A relation may be represented algebraically either by the *Roster method* or by the *Set-builder method.*

2. An arrow diagram is a visual representation of a relation.

Example 7

Let $A = \{1, 2, 3, 4, 5, 6\}$**.** Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

Solution

(i) By the definition of the relation, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$. The corresponding arrow diagram is shown in Fig. 2.5.

(ii) We can see that the domain $= \{1, 2, 3, 4, 5\}$, the range $= \{2, 3, 4, 5, 6\}$, and the codomain $= \{1, 2, 3, 4, 5, 6\}$.

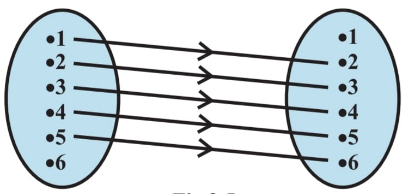


Fig 2.5

Example 8 The Fig 2.6 shows a relation between the sets P and Q. Write this relation

(i) in set-builder form,

(ii) in roster form.

What is its domain and range?

Solution It is obvious that the relation R is "x is the square of y".

(i) **In set-builder form,** $R = \{(x, y) : x$ is the square of $y,\ x \in P,\ y \in Q\}$

(ii) **In roster form,** $R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$

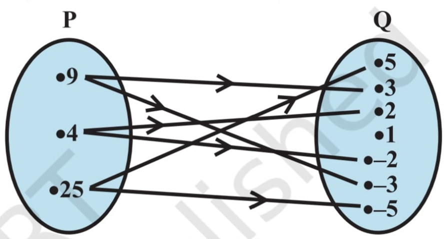


Fig 2.6

The domain of this relation is $\{4, 9, 25\}$.
The range of this relation is $\{-2, 2, -3, 3, -5, 5\}$.
Note that the element 1 is not related to any element in set P. The set Q is the codomain of this relation.

> **Note** The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.

Example 9 Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the number of relations from A to B.

Solution We have, $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. Since $n(A \times B) = 4$, the number of subsets of $A \times B$ is 2^4. Therefore, the number of relations from A into B will be 2^4.

Remarks A relation R from A to A is also stated as a relation on A.

EXERCISE 2.2

1. Let $A = \{1, 2, 3, \ldots, 14\}$. Define a relation R from A to A by $R = \{(x, y) : 3x - y = 0,$ where $x, y \in A\}$. Write down its domain, codomain and range.

2. Define a relation R on the set $\mathbb{N}$ of natural numbers by $R = \{(x, y) : y = x + 5, x$ is a natural number less than $4; x, y \in \mathbb{N}\}$. Depict this relationship using roster form. Write down the domain and the range.

3. $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. Define a relation R from A to B by $R = \{(x, y) :$ the difference between x and y is odd$; x \in A, y \in B\}$. Write R in roster form.

4. The Fig 2.7 shows a relationship between the sets P and Q. Write this relation
   (i) in set-builder form
   (ii) in roster form.

   What is its domain and range?

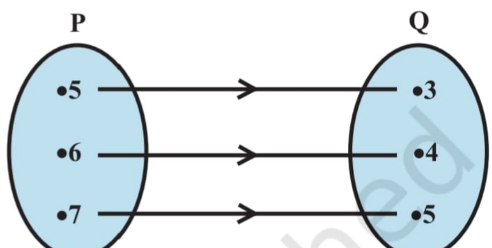


Fig 2.7

5. Let $A = \{1, 2, 3, 4, 6\}$. Let R be the relation on A defined by $\{(a, b) : a, b \in A, b$ is exactly divisible by $a\}$.
   (i) Write R in roster form
   (ii) Find the domain of R
   (iii) Find the range of R.

6. Determine the domain and range of the relation R defined by $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.

7. Write the relation $R = \{(x, x^3) : x$ is a prime number less than $10\}$ in roster form.

8. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A to B.

9. Let R be the relation on $\mathbb{Z}$ defined by $R = \{(a, b) : a, b \in \mathbb{Z}, a - b$ is an integer$\}$. Find the domain and range of R.

2.4 Functions In this chapter, we studied relations and functions. It is one of the most important concepts in mathematics. We can visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as 'map' or 'mapping' used to denote a function.

Definition 5 A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B. In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the **image** of a under f and a is called the **preimage** of b under f.

The function f from A to B is denoted by $f : A \to B$.

Looking at the previous examples, we can easily see that the relation in Example 7 is not a function because the element 6 has no image.

Again, the relation in Example 8 is not a function because the elements in the domain are connected to more than one images. Similarly, the relation in Example 9 is also not a function. (Why?) In the examples given below, we will see many more relations some of which are functions and others are not.

Example 9 Let $\mathbb{N}$ be the set of natural numbers and the relation R be defined on $\mathbb{N}$ such that $R = \{(x, y) : y = 2x, x, y \in \mathbb{N}\}$.
What is the domain, codomain and range of R? Is this relation a function?

Example 10 The domain of R is the set of natural numbers $\mathbb{N}$. The codomain is also $\mathbb{N}$. The range is the set of even natural numbers. Since every natural number n has one and only one image, this relation is a function.

Example 11 Consider the following relations. For each relation, determine whether it is a function or not, giving reasons.

$$\text{(i)} \quad R = \{(2,1), (3,1), (4,2)\}, \text{ (ii)} \quad R = \{(2,2), (2,4), (3,3), (4,4)\}$$

$$\text{(iii)} \quad R = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,7)\}$$

Solution (i) Since 2, 3, 4 are the elements of domain of R having their unique images, this relation R is a function.
(ii) Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function.
(iii) Since every element has one and only one image, this relation is a function.

Definition 6 A function which has either $\mathbb{R}$ or one of its subsets as its range is called a real valued function. Further, if its domain is also either $\mathbb{R}$ or a subset of $\mathbb{R}$, it is called a real function.

Example 12 Let $\mathbb{N}$ be the set of natural numbers. Define a real valued function $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = 2x + 1$. Using this definition, complete the table given below.

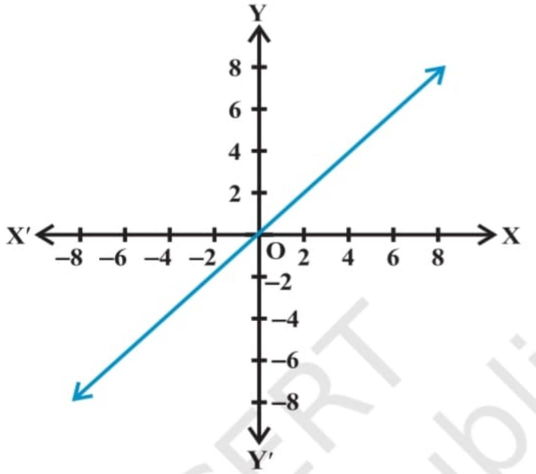
x	1	2	3	4	5	6	7
y	$f(1) =$	$f(2) =$	$f(3) =$	$f(4) =$	$f(5) =$	$f(6) =$	$f(7) =$

Solution The completed table is given by:

x	1	2	3	4	5	6	7
y	$f(1) = 3$	$f(2) = 5$	$f(3) = 7$	$f(4) = 9$	$f(5) = 11$	$f(6) = 13$	$f(7) = 15$

2.4.1 Some functions and their graphs

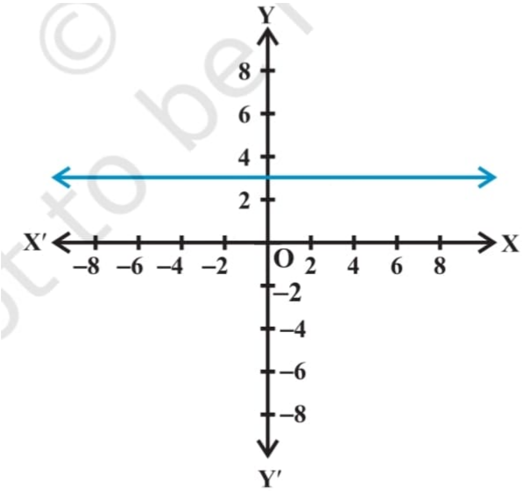
(i) Identity function Let $\mathbb{R}$ be the set of real numbers. Define the real valued function $f : \mathbb{R} \to \mathbb{R}$ by $y = f(x) = x$ for each $x \in \mathbb{R}$. Such a function is called the *identity function*. Here, the domain and range of f are $\mathbb{R}$. The graph is a straight line as shown in Fig 2.8. It passes through the origin.



$f(x) = x$

Fig 2.8

(ii) Constant function Define the function $f : \mathbb{R} \to \mathbb{R}$ by $y = f(x) = c$, $x \in \mathbb{R}$, where c is a constant. Here, the domain of f is $\mathbb{R}$ and its range is $\{c\}$.



$f(x) = 3$

Fig 2.9

The graph is a line parallel to the x-axis. For example, if $f(x) = 3$ for each $x \in \mathbb{R}$, then its graph will be a line as shown in Fig. 2.9.

(iii) Polynomial function A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a polynomial function if for each $x \in \mathbb{R}$, $y = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$.

The function defined by $f(x) = x^3 - x^2 + 2$ is a polynomial function, whereas the functions defined by $g(x) = x^4 + \dfrac{2}{x}$ and $h(x) = \dfrac{x^{2/3}}{2} + 2x$ are not polynomial functions.

Example 13 Define the function $f : \mathbb{R} \to \mathbb{R}$ by $y = f(x) = x^2,\ x \in \mathbb{R}$. Complete the Table given below by using this definition. What is the domain and range of this function? Draw the graph of f.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

Solution The completed Table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

Domain of $f = \{x : x \in \mathbb{R}\}$. Range of $f = \{x^2 : x \in \mathbb{R}\}$. The graph of f is given by Fig 2.10.

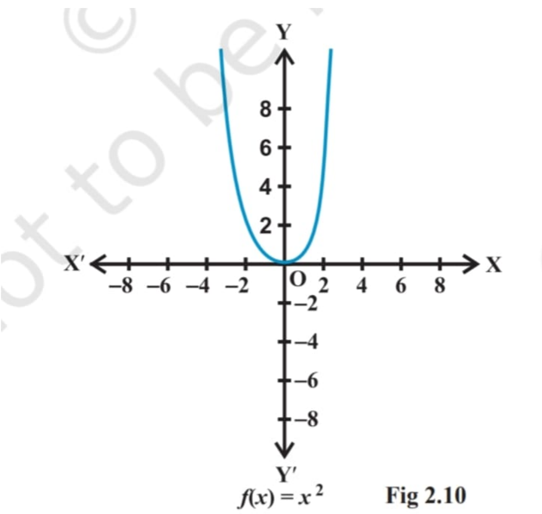
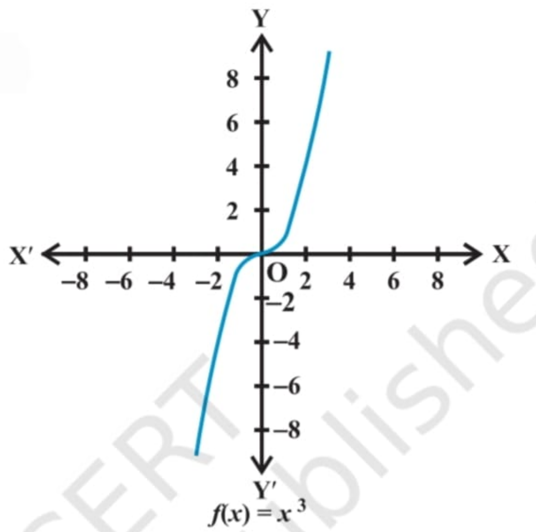


Fig 2.10

Example 14 Draw the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3,\ x \in \mathbb{R}$.

Solution We have $f(0) = 0,\ f(1) = 1,\ f(-1) = -1,\ f(2) = 8,\ f(-2) = -8,\ f(3) = 27,\ f(-3) = -27$, etc. Therefore, $f = \{(x, x^3) : x \in \mathbb{R}\}$. The graph of f is given in Fig. 2.11.



$f(x) = x^3$

Fig 2.11

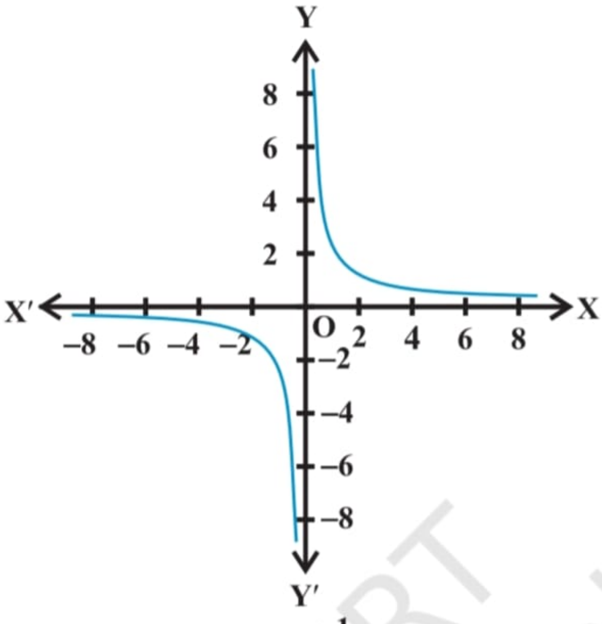
Example 15 Define the real valued function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, Complete the Table given below using this definition. What is the domain and range of this function?

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
$y = \frac{1}{x}$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Solution The completed Table is given by

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
$y = \frac{1}{x}$	-0.5	-0.67	-1	-2	4	2	1	0.67	0.5

Domain of $f = \mathbb{R} \setminus \{0\}$. Range of $f = \mathbb{R} \setminus \{0\}$. The domain of f is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig 2.12.



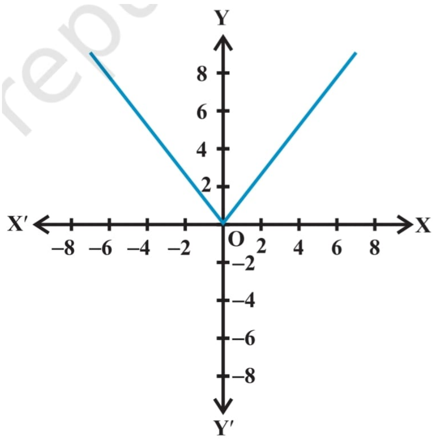
$f(x) = \frac{1}{x}$

Fig 2.12

(v) The Modulus function
The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ for each $x \in \mathbb{R}$ is called the *modulus function*. For each non-negative value of x, $f(x)$ is equal to x. But for negative values of x, the value of $f(x)$ is the negative of the value of x, i.e.,

$$f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

The graph of the modulus function is given in Fig. 2.13.



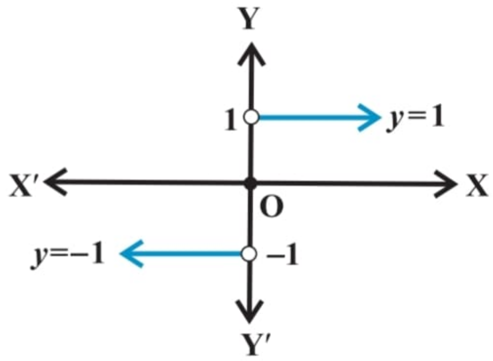
$f(x) = |x|$

Fig 2.13

(vi) Signum function The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \text{sgn}(x)$ is called the *signum function.*

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

The domain of the signum function is $\mathbb{R}$ and the range is the set $\{-1, 0, 1\}$. The graph of the signum function is given by Fig. 2.14.



$f(x) = \frac{|x|}{x}, x \neq 0$ and 0 for $x = 0$

Fig 2.14

(vii) Greatest integer function

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x],\ x \in \mathbb{R}$ assumes the value of the greatest integer, less than or equal to x. Such a function is called the *greatest integer function.*

$$
\begin{aligned}
[x] &= -1, \quad -1 \le x < 0,\\
[x] &= 0, \quad 0 \le x < 1,\\
[x] &= 1, \quad 1 \le x < 2,\\
[x] &= 2, \quad 2 \le x < 3 \text{ and so on.}
\end{aligned}
$$

The graph of the function is shown in Fig. 2.15.

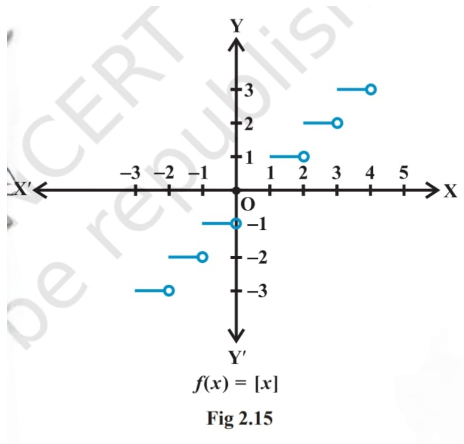


Fig 2.15

2.4.2 Algebra of real functions

In this Section, we shall learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real functions and divide one real function by another.

(i) Addition of two real functions Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be any two real functions, where $X \subset \mathbb{R}$. Then, we define $(f + g) : X \to \mathbb{R}$ by

$(f + g)(x) = f(x) + g(x), \quad$ for all $x \in X$.

(ii) Subtraction of a real function from another Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be any two real functions, where $X \subset \mathbb{R}$. Then, we define $(f - g) : X \to \mathbb{R}$ by

$$(f - g)(x) = f(x) - g(x), \quad \text{for all } x \in X.$$

(iii) Multiplication by a scalar Let $f : X \to \mathbb{R}$ be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product αf is a function from X to $\mathbb{R}$ defined by

$$(\alpha f)(x) = \alpha f(x), \quad x \in X.$$

(iv) Multiplication of two real functions The product (or multiplication) of two real functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ is a function $fg : X \to \mathbb{R}$ defined by

$$(fg)(x) = f(x)\,g(x), \quad \text{for all } x \in X.$$

This is also called *pointwise multiplication.*

(v) Quotient of two real functions Let f and g be two real functions defined from $X \to \mathbb{R}$, where $X \subset \mathbb{R}$. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{provided } g(x) \neq 0,\ x \in X.$$

Example 16 Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions. Find $(f+g)(x)$, $(f-g)(x)$,
$(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution We have,

$$(f+g)(x) = x^2 + 2x + 1,$$
$$(f-g)(x) = x^2 - 2x - 1,$$
$$(fg)(x) = x^2(2x+1) = 2x^3 + x^2,$$
$$\left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1}, \quad x \neq -\frac{1}{2}.$$

Example 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution We have,

$$(f+g)(x) = \sqrt{x} + x,$$
$$(f-g)(x) = \sqrt{x} - x,$$
$$(fg)(x) = \sqrt{x}\,x = x^{3/2},$$
$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-1/2}, \quad x \neq 0.$$

EXERCISE 2.3

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.
   (i) $\{(2,1),(5,1),(8,1),(11,1),(14,1),(17,1)\}$
   (ii) $\{(2,1),(4,2),(6,3),(8,4),(10,5),(12,6),(14,7)\}$
   (iii) $\{(1,3),(1,5),(2,5)\}$

2. Find the domain and range of the following real functions:
   (i) $f(x) = -|x|$ (ii) $f(x) = \sqrt{9-x^2}$

3. A function f is defined by $f(x) = 2x - 5$. Write down the values of
   (i) $f(0)$ (ii) $f(7)$ (iii) $f(-3)$

4. The function t which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by $t(C) = \frac{9C}{5} + 32$. Find (i) $t(0)$ (ii) $t(28)$ (iii) $t(-10)$ (iv) The value of C, when $t(C) = 212$

5. Find the range of each of the following functions:
   (i) $f(x) = 2 - 3x,\ x \in \mathbb{R},\ x > 0$
   (ii) $f(x) = x^2 + 2$, x is a real number
   (iii) $f(x) = x$, x is a real number

Miscellaneous Examples

Example 18

Let $\mathbb{R}$ be the set of real numbers. Define the real function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 10$ and sketch the graph of this function.

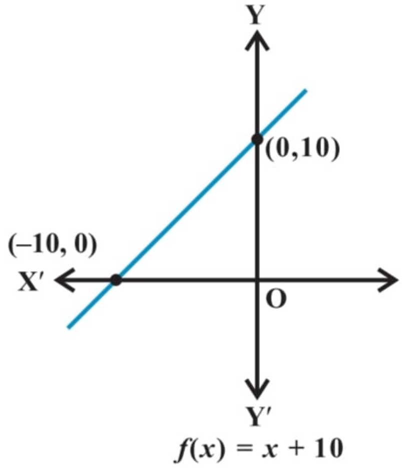


Fig 2.16

Solution Here $f(0) = 10$, $f(1) = 11$, $f(2) = 12$, $\ldots$, $f(10) = 20$, etc., and $f(-1) = 9$, $f(-2) = 8$, $\ldots$, $f(-10) = 0$ and so on.

Therefore, the shape of the graph of the given function assumes the form as shown in Fig. 2.16.

Remark The function f defined by $f(x) = mx + c$, $x \in \mathbb{R}$, is called a *linear function*, where m and c are constants. Above function is an example of a linear function.

Example 19 Let R be a relation from $\mathbb{Q}$ to $\mathbb{Q}$ defined by $R = \{(a, b) : a, b \in \mathbb{Q} \text{ and } a - b \in \mathbb{Z}\}$. Show that

1. $(a, a) \in R$ for all $a \in \mathbb{Q}$
2. $(a, b) \in R$ implies that $(b, a) \in R$
3. $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

Solution

1. Since, $a - a = 0 \in \mathbb{Z}$, it follows that $(a, a) \in R$.
2. $(a, b) \in R$ implies that $a - b \in \mathbb{Z}$. So, $b - a \in \mathbb{Z}$. Therefore, $(b, a) \in R$.
3. (a, b) and $(b, c) \in R$ implies that $a - b \in \mathbb{Z}$. $b - c \in \mathbb{Z}$. So,

$$(a - c) = (a - b) + (b - c) \in \mathbb{Z}.$$

Therefore, $(a, c) \in R$.

Example 20 Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a linear function from $\mathbb{Z}$ into $\mathbb{Z}$. Find $f(x)$.

Solution Since f is a linear function, $f(x) = mx + c$. Also, since $(1, 1), (0, -1) \in R$, $f(1) = m + c = 1$ and $f(0) = c = -1$. This gives $m = 2$ and $f(x) = 2x - 1$.

Example 21 Find the domain of the function $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4}$.

Solution Since $x^2 - 5x + 4 = (x - 4)(x - 1)$, the function f is defined for all real numbers except at $x = 4$ and $x = 1$. Hence the domain of f is $\mathbb{R} \setminus \{1, 4\}$.

Example 22 The function f is defined by $f(x) = \begin{cases} 1-x, & x < 0, \\ 1, & x = 0, \\ x+1, & x > 0. \end{cases}$

Draw the graph of $f(x)$.

Solution

Here, $f(x) = 1 - x$, $x < 0$, this gives

$$f(-4) = 1 - (-4) = 5;$$
$$f(-3) = 1 - (-3) = 4;$$
$$f(-2) = 1 - (-2) = 3;$$
$$f(-1) = 1 - (-1) = 2;\ \text{etc.,}$$

and $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$and so on for$f(x) = x + 1$, $x > 0$.

Thus, the graph of f is as shown in Fig. 2.17.

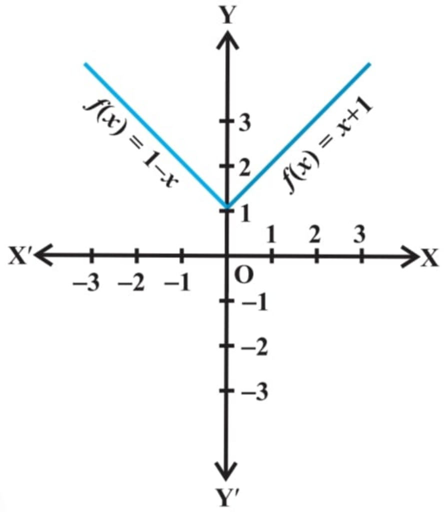


Fig 2.17

Miscellaneous Exercise on Chapter 2

1. The relation f is defined by $f(x) = \begin{cases} x^2, & 0 \leq x \leq 3, \\ 3x, & 3 < x \leq 10. \end{cases}$ The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \leq x \leq 2, \\ 3x, & 2 \leq x \leq 10. \end{cases}$ Show that f is a function and g is not a function.

2. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{1.1 - 1}$.

3. Find the domain of the function $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

4. Find the domain and the range of the real function f defined by $f(x) = \sqrt{x - 1}$.

5. Find the domain and the range of the real function f defined by $f(x) = |x - 1|$.

6. Let $f = \left\{ \left(x, \dfrac{x^2}{1 + x^2} \right) : x \in \mathbb{R} \right\}$ be a function from $\mathbb{R}$ into $\mathbb{R}$. Determine the range of f.

7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined, respectively, by $f(x) = x + 1$, $\quad g(x) = 2x - 3$. Find $f + g$, $f - g$ and $\dfrac{f}{g}$.

8. Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = ax+b$, for some integers a, b. Determine a and b.

9. Let R be a relation from $\mathbb{N}$ to $\mathbb{N}$ defined by $R = \{(a, b) : a, b \in \mathbb{N} \text{ and } a = b^2\}$. Are the following true?

   (a) $(a, a) \in R$, for all $a \in \mathbb{N}$

   (b) $(a, b) \in R$ implies $(b, a) \in R$

   (c) $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$

   Justify your answer in each case.

10. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$. Are the following true?

    (a) f is a relation from A to B

    (b) f is a function from A to B

    Justify your answer in each case.

11. Let f be the subset of $\mathbb{Z} \times \mathbb{Z}$ defined by $f = \{(ab, a + b) : a, b \in \mathbb{Z}\}$. Is f a function from $\mathbb{Z}$ to $\mathbb{Z}$? Justify your answer.

12. Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \to \mathbb{N}$ be defined by $f(n) =$ the highest prime factor of n. Find the range of f.

Summary

In this Chapter, we studied about relations and functions. The main features of this Chapter are as follows:

- **Ordered pair** A pair of elements grouped together in a particular order.
- **Cartesian product** $A \times B$ of two sets A and B is given by $A \times B = \{(a, b) : a \in A, b \in B\}$.

  In particular $R \times R = \{(x, y) : x, y \in R\}$

  and $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$.
- If $(a, b) = (x, y)$, then $a = x$ and $b = y$.
- If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.
- $A \times \phi = \phi$.
- In general, $A \times B \neq B \times A$.
- **Relation** A relation R from a set A to a set B is a subset of the cartesian product $A \times B$ obtained by describing a relationship between the first element x and the second element y of the ordered pairs in $A \times B$.
- The *image* of an element x under a relation R is given by y, where $(x, y) \in R$.
- The *domain* of R is the set of all first elements of the ordered pairs in a relation R.
- The *range* of the relation R is the set of all second elements of the ordered pairs in a relation R.
- **Function** A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B. We write $f : A \to B$, where $f(x) = y$.
- A is the domain and B is the codomain of f.
- The range of the function is the set of images.
- A real function has the set of real numbers or one of its subsets both as its domain and as its range.
- ***Algebra of functions*** For functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, we have

$$
\begin{aligned}
(f+g)(x) &= f(x) + g(x), && x \in X,\\
(f-g)(x) &= f(x) - g(x), && x \in X,\\
(f \cdot g)(x) &= f(x) \cdot g(x), && x \in X,\\
(kf)(x) &= k(f(x)), && x \in X,\ \text{where } k \text{ is a real number},\\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, && x \in X,\ g(x) \neq 0.
\end{aligned}
$$

Historical Note

The word **FUNCTION** first appears in a Latin manuscript *"Methodus tangentium* *inversa, seu de functionibus"* written by Gottfried Wilhelm Leibnitz (1646–1716) in 1673; Leibnitz used the word in the non-analytical sense. He considered a function in terms of "mathematical job" — the "employee" being just a curve.

On July 5, 1698, Johan Bernoulli, in a letter to Leibnitz, for the first time deliberately assigned a specialised use of the term *function* in the analytical sense. At the end of that month, Leibnitz replied showing his approval.

Function is found in English in 1779 in Chambers' Cyclopaedia: "The term function is used in algebra, for an analytical expression any way compounded of a variable quantity, and of numbers, or constant quantities".