



Language:
English

Day: **1**

Wednesday, July 7, 2010

Problem 1. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the equality

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z .)

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D . Let E be a point on the arc \overarc{BDC} and F a point on the side BC such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF . Prove that the lines DG and EI intersect on Γ .

Problem 3. Let \mathbb{N} be the set of positive integers. Determine all functions $g : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(g(m) + n)(m + g(n))$$

is a perfect square for all $m, n \in \mathbb{N}$.



Language:
English

Day: 2

Thursday, July 8, 2010

Problem 4. Let P be a point inside the triangle ABC . The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S . Suppose that $SC = SP$. Prove that $MK = ML$.

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed: **Type 1:** Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1} .

Type 2: Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2} .

Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

Problem 6. Let a_1, a_2, a_3, \dots be a sequence of positive real numbers. Suppose that for some positive integer s , we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\}$$

for all $n > s$. Prove that there exist positive integers ℓ and N , with $\ell \leq s$, such that for all $n \geq N$, $a_n = a_\ell + a_{n-\ell}$.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points



Language:
English

Day: **1**

Monday, July 18, 2011

Problem 1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A . Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A . Find all sets A of four distinct positive integers which achieve the largest possible value of n_A .

Problem 2. Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A windmill is a process that starts with a line l going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S . This point, Q , takes over as the new pivot, and the line now rotates clockwise about Q , until it next meets a point of S . This process continues indefinitely. Show that we can choose a point P in S and a line l going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

Problem 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y . Prove that $f(x) = 0$ for all $x \leq 0$.

Language: English

*Time: 4 hours and 30 minutes
Each problem is worth 7 points*



53rd International Mathematical Olympiad
MAR DEL PLATA - ARGENTINA

Language:
English

Day: **2**

Tuesday, July 19, 2011

Problem 4. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \dots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

Problem 5. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n , the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

Problem 6. Let ABC be an acute triangle with circumcircle Γ . Let l be a tangent line to Γ , and let l_a , l_b and l_c be the lines obtained by reflecting l in the lines BC , CA and AB , respectively. Show that the circumcircle of the triangle determined by the lines l_a , l_b and l_c is tangent to the circle Γ .

Language: English

*Time: 4 hours and 30 minutes
Each problem is worth 7 points*



International
Mathematical
Olympiad Am
sterdam 2011

Language: English

Day: 1

Tuesday, July 10, 2012

Problem 1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A . This excircle is tangent to the side BC at M , and to the lines AB and AC at K and L , respectively. The lines LM and BJ meet at F , and the lines KM and CJ meet at G . Let S be the point of intersection of the lines AF and BC , and let T be the point of intersection of the lines AG and BC . Prove that M is the midpoint of ST .

(The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC , to the ray AB beyond B , and to the ray AC beyond C .)

Problem 2. Let $n \geq 3$ be an integer, and let a_2, a_3, \dots, a_n be positive real numbers such that $a_2 a_3 \dots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

Problem 3. The liar's guessing game is a game played between two players A and B . The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B . Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S . Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X , then B wins; otherwise, he loses. Prove that:

1. If $n \geq 2^k$, then B can guarantee a win.
2. For all sufficiently large k , there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points



International
Mathematical
Olympiad Am
sterdam 2011

Language: English

Day: 1

Wednesday, July 11, 2012

Problem 4. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c = 0$ the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here \mathbb{Z} denotes the set of integers.)

Problem 5. Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C . Let X be a point in the interior of the segment CD . Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK . Prove that $MK = ML$.

Problem 6. Find all positive integers n for which there exist non-negative integers a_1, a_2, \dots, a_n such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

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