G = LCA group (2nd countable).

$$\hat{G} \xrightarrow[\gamma]{\sim} L^1(G)\widehat{\ } = \operatorname{Max} L^1(G)$$

$$\chi \longmapsto \tilde{\chi}$$

$\tilde{\chi} : M(G) \to \mathbb{C}$ $\quad \tilde{\chi}(\nu) = \hat{\nu}(\chi).$

$\tilde{\chi}(\delta_x) = \chi(x)$

$\hat{G}$ is loc. comp

$L^1(G)$ is hermitian

$$
\begin{tikzcd}
 & C_0(\hat{G}) \\
L^1(G) \arrow[ur, "\mathcal{F}"] \arrow[dr, "\Gamma"'] & \\
 & C_0(L^1(G)^\wedge) \arrow[uu, "\wr"', "\gamma^*"]
\end{tikzcd}
$$

$\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$

Prop G is 2nd countable $\Rightarrow$ so is $\hat{G}$.

Proof. G is 2nd countable $\Rightarrow$ $L^1(G)$ is separable (exer).

E = ~~sep~~. Ban. space $\Rightarrow (\mathbb{B}_{E^*}, wk^*)$ is comp. and metrizable

$\Longrightarrow$ separable and metr. $\Rightarrow$ 2nd countable.

$\Rightarrow \hat{G} \hookrightarrow (\mathbb{B}_{L^1(G)^*}, wk^*)$ is 2nd countable $\square$

Def. $A = *$-alg, $H =$ Hilb space.

A **$*$-representation** of A is a $*$-hom $\pi : A \to \mathcal{B}(H)$

π is **faithful** if $\operatorname{Ker}\pi = \{e\}$.

Lemma 1. $A =$ comm. Ban $*$-alg. Suppose A has a faithful $*$-rep on a Hilb. space. Then $\Gamma_A : A \to C_0(\operatorname{Max} A)$ is injective.

Proof $\pi : A \to \mathcal{B}(H)$ faithful $*$-rep; $B = \overline{\pi(A)} \subset \mathcal{B}(H)$

B is a comm. C^*-alg

$\Longrightarrow B \cong C_0(X) \Longrightarrow$ characters of B separate the points of B

$\Longrightarrow$ chars of A separate the points of A (because π is inj)

$\Longleftrightarrow \Gamma_A$ is inj. $\square$

<u>Lemma 2</u>. G = LCA group (2nd countable). Then

(1) $f \in L^1(G)$, $g \in L^2(G) \Longrightarrow f * g$ is defined a.e., $f * g \in L^2(G)$,

and $\|f * g\|_2 \leq \|f\|_1 \|g\|_2$.

(2) $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$, $\lambda(f)g = f * g$,

is a faithful $*$-rep.

Proof (1) exer.

(2) λ is a $*$-rep: exer.

Let (e_α) be an a.i. of $L^1(G)$ contained in $C_c(G)$.

Let $f \in \operatorname{Ker} \lambda$. Then:

$$0 = \lambda(f) e_\alpha = f * e_\alpha \longrightarrow f \implies f = 0 \quad \square$$

Thm. (uniqueness thm for $\mathcal{F}$)

$\mathcal{F} : L^1(G) \to C_0(\hat{G})$ is injective.

Proof: L1 & L2. $\square$

Cor. $\widehat{G}$ separates the points of G, that is, $\forall\, e \neq x \in G$ $\exists\, \chi \in \widehat{G}$ s.t. $\chi(x) \neq 1$.

Proof $\exists f \in C_c(G)$ s.t. $f(x^{-1}) \neq f(e) \Longrightarrow L_x f \neq f$

$$\left.\begin{array}{l} \Rightarrow \exists\, \chi \in \widehat{G} \text{ s.t. } \tilde{\chi}(L_x f) \neq \tilde{\chi}(f) \\ \qquad\qquad\qquad\quad \| \\ \qquad\qquad\quad \tilde{\chi}(\delta_x * f) = \chi(x)\tilde{\chi}(f) \end{array}\right\} \Longrightarrow \chi(x) \neq 1. \quad \square$$

Positive definite functions. Bochner's thm.

$A = *$-alg, $\omega : A \to \mathbb{C}$ linear.

<u>Def.</u> ω is <u>positive</u> $(\omega \geq 0)$ if $\omega(a^*a) \geq 0 \ \forall a \in A$.

<u>Notation</u> A = Banach $*$-alg.

$$A^*_{pos} = \{\omega \in A^* : \omega \geq 0\}.$$

This is a convex cone in A^*.

<u>Example 1.</u> $\chi : A \to \mathbb{C}$ $*$-char.

$$\chi(a^*a) = |\chi(a)|^2 \geq 0 \implies \chi \geq 0.$$

Example 2. X = loc. comp Hausd space. There is a bijection

$$C_0(X)^*_{pos} \cong \{\text{Fin. pos Radon measures on } X\} = M(X)_{pos}.$$

Indeed: $\mu \in M(X)$; $I_\mu \in C_0(X)^*$, $I_\mu(f) = \int f \, d\mu$.

$$I_\mu \geq 0 \iff \int_X |f|^2 d\mu \geq 0 \ \forall f \in C_0(X) \iff \mu \geq 0.$$

Example 3. H = Hilb. sp, $A \subset \mathcal{B}(H)$ $*$-subalg.

$v \in H$ $\quad \omega_v : A \to \mathbb{C}$ $\quad \omega_v(T) = \langle Tv | v \rangle$.

$$\omega_v(T^*T) = \|Tv\|^2 \geq 0 \implies \omega_v \geq 0.$$

<u>Example 4/exer.</u> $A = \mathbb{C}G$ or $A = \ell^1(G)$ (G = a group)

or $A = (C_c(G), \text{convol}) \subset L^1(G)$ (G = LC group)

$\omega : A \to \mathbb{C}$, $\omega(f) = f(e)$. Prove: $\omega \geq 0$.

<u>Notation</u> A = $*$-alg, $\omega : A \to \mathbb{C}$ pos lin func.

$\forall a, b \in A$ $\quad \langle a | b \rangle_\omega = \omega(b^* a)$

$\langle \cdot | \cdot \rangle_\omega$ is a sesquilin. form on A;

$\langle a | a \rangle_\omega \in \mathbb{R} \ \forall a \underset{\text{(exer)}}{\Longrightarrow} \langle \cdot | \cdot \rangle_\omega$ is hermitian, <u>that is</u>,

$\langle b | a \rangle_\omega = \overline{\langle a | b \rangle_\omega}$. $\forall a, b$

Hence $\langle \cdot | \cdot \rangle_\omega$ is a semi-inner product on A.

<u>Prop</u> (CBS ineq) $|\omega(b^*a)|^2 \leq \omega(a^*a)\,\omega(b^*b) \quad (a, b \in A)$.

G = a group ; $\varphi : G \to \mathbb{C}$.

$\forall n \in \mathbb{N} \quad \forall x = (x_1, \ldots, x_n) \in G^n$

define $\Phi_x : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $\Phi_x(u, v) = \overline{\sum_{i,j} \varphi(x_j^{-1} x_i) u_i \overline{v_j}}$.

This is a sesquilin. form on $\mathbb{C}^n$.

Def φ is <u>positive definite</u> (положительно определённая)
if $\forall n \in \mathbb{N}$ $\forall x \in G^n$ Φ_x is pos. def.
(that is, $\Phi_x(u,u) \geq 0$ $\forall u \in \mathbb{C}^n$). Suppose φ is pos def

<u>Observations</u> (1) $\varphi(e) \geq 0$ (let $n=1$)

(2) Φ_x is a semi-inner prod on $\mathbb{C}^n$. In part,

$$\Phi_x(v,u) = \overline{\Phi_x(u,v)} \quad \forall u,v.$$

(3) (CBS) $|\Phi_x(u,v)|^2 \leq \Phi_x(u,u)\,\Phi_x(v,v)$ $(u,v \in \mathbb{C}^n)$

(4) Let $n=2$, $x=(e,s)\in G^2$, $s\in G$. $u=(1,0)$, $v=(0,1)$

$$\begin{pmatrix} \varphi(e) & \varphi(s) \\ \varphi(s^{-1}) & \varphi(e) \end{pmatrix}$$

(2) $\Rightarrow \varphi(s^{-1})=\overline{\varphi(s)}$.

(3) $|\varphi(s)|^2 \leq \varphi(e)^2$, that is,

$|\varphi(s)|\leq \varphi(e)$.

In part, φ is bdd.

<u>Examples</u> (1) $\chi: G\to\mathbb{C}$ unitary char.

$$\sum_{i,j} \chi(x_j^{-1}x_i)\, u_i\overline{u_j} = \sum_{i,j} \overline{\chi(x_j)}\,\chi(x_i)\,u_i\overline{u_j} = \left|\sum \chi(x_i)u_i\right|^2 \geq 0$$

$\Rightarrow \chi$ is pos. def'

(2) H = Hilb. space; $\mathcal{U}(H) = \{$unitary opers on $H\}$

$\pi : G \to \mathcal{U}(H)$ unitary repres.

(that is, a group homom.)

$\forall v \in H \quad \pi_v : G \to \mathbb{C}, \quad \pi_v(x) = \langle \pi(x) v \mid v \rangle.$

<u>Exer.</u> π_v is pos. def.

<u>Notation</u> $\mathcal{P}(G) = \{$pos def. functions on $G\}$.

$\mathbb{C}G$ = group alg of G.

$\mathbb{C}G \cong \left(\{ \text{finitely supported functions } G \to \mathbb{C} \}, \ * \right)$

$= \operatorname{span}\{ \delta_x : x \in G \}$; $\quad \delta_x * \delta_y = \delta_{xy}$;

$\delta_x^* = \delta_{x^{-1}}$.

$\mathbb{C}G$ is a $*$-alg.

<u>Observe</u>: $\exists$ a vec. space isom

$\alpha : \mathrm{Fun}(G) \xrightarrow{\sim} (\mathbb{C}G)^*$ (algebraic dual)

$\varphi \mapsto \alpha_\varphi$, $\quad \alpha_\varphi(\delta_x) = \varphi(x)$.

Prop $\alpha_\varphi \geq 0 \iff \varphi$ is pos. def.

Proof. $f \in \mathbb{C}G$ $\quad f = \sum_{i=1}^{n} c_i \delta_{x_i}$ $\quad f^* = \sum_j \overline{c_j} \delta_{x_j^{-1}}$

$$\alpha_\varphi(f^* * f) = \alpha_\varphi\Big(\sum_{i,j} c_i \overline{c_j} \delta_{x_j^{-1}x_i}\Big) = \sum_{i,j} \varphi(x_j^{-1}x_i) c_i \overline{c_j}. \quad \square$$

Rem. φ is pos. def $\iff \Phi_x$ is pos. def for every n-tuple $(x_1, .., x_n)$ of pairwise distinct elements of G.

Prop. G = a fin. abelian group. Then

$$\mathcal{P}(G) = \left\{ \sum_{\chi \in \hat{G}} c_\chi \chi : c_\chi \geq 0 \right\}.$$

Proof.

$$\mathbb{C}G \xrightarrow[\mathcal{F}]{\sim} \operatorname{Fun}(\hat{G}) \qquad * \text{-isomorphism}$$

$$\begin{array}{ccc} \operatorname{Fun}(\hat{G})^*_{pos} & \xrightarrow[\mathcal{F}^*]{\sim} & (\mathbb{C}G)^*_{pos} \\ \cong & & \cong \\ M(\hat{G})_{pos} & \xrightarrow{\sim} & \mathcal{P}(G) \\ \| & & \\ \left\{ \sum_{\chi \in \hat{G}} c_\chi \delta_\chi : c_\chi \geq 0 \right\} & & \end{array}$$

$$\mathcal{F}^*(\delta_\chi) = \chi \quad \text{(exer)}$$

$\square$

Exer. Give a proof which avoids using $\mathcal{F}$.

G = loc. comp. group (2nd countable); μ = Haar meas.

Recall: $\exists$ an isometric isom. of Ban. spaces

$$\alpha : L^\infty(G) \xrightarrow{\sim} L^1(G)^*, \quad \varphi \mapsto \alpha_\varphi, \quad \alpha_\varphi(f) = \int_G f\varphi \, d\mu.$$

Def. $\varphi \in L^\infty(G)$ is of positive type if $\alpha_\varphi \geq 0$.

Notation. $\mathcal{P}^\infty(G) = \{\varphi \in L^\infty(G) : \varphi \text{ is of pos. type}\}$

$\mathcal{P}(G) = \{\text{continuous pos. def. functions on } G\}$.

Thm. Let $\varphi \in C_b(G)$. Then φ is of pos. type $\Longleftrightarrow$ φ is pos. definite.

Lemma/exer 1. $\forall \varphi \in L^\infty(G)$, $f, g \in L^1(G)$. Then

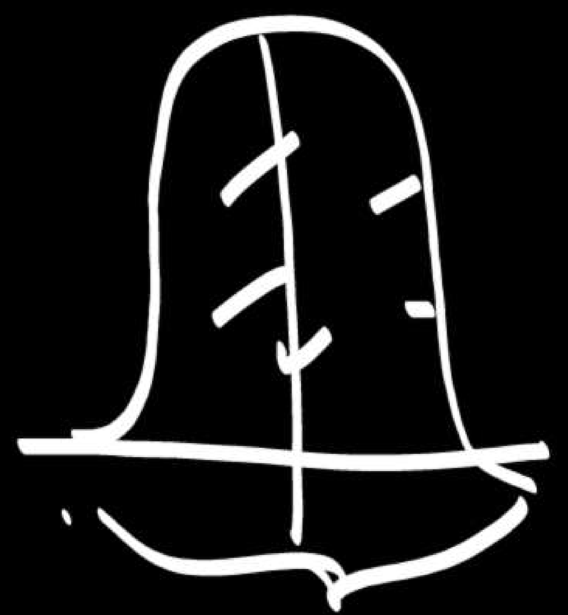
$$\alpha_\varphi(g^* * f) = \iint_{G\times G} \varphi(y^{-1}x) f(x) \overline{g(y)} \, d\mu(x) d\mu(y).$$

$g^*(x) = \overline{g(x^{-1})}\Delta(x^{-1})$

$g \mapsto g^*$ invol. on $L^1(G)$

Lemma/exer 2. β = a base of rel. comp, symm nbhds of $e \in G$.

$(u_V)_{V\in\beta}$ = a Dirac net in $L^1(G)$.



V

Then: (1) $\forall \varphi \in C_b(G)$ $\quad \int u_V \varphi \, d\mu \longrightarrow \varphi(e)$

In part, $u_V \xrightarrow{wk^*} \delta_e^G$ in $M(G)$

(2) $(u_V \otimes u_V)_{V\in\beta}$ is a Dirac net in $L^1(G\times G)$.

$\left((x,y) \longmapsto u_V(x)\, u_V(y) \right)$

Proof of Thm $(\Longrightarrow)$ Suppose φ is of pos. type

$x = (x_1, \dots, x_n) \in G^n \qquad t = (t_1, \dots, t_n) \in \mathbb{C}^n$.

Let $(u_\nu)_{\nu\in\beta}$ be a Dirac net in $L^1(G)$.

Let $f_\nu = \sum t_i L_{x_i} u_\nu$. $(\nu \in \beta)$ $\quad f_\nu^* = \sum_j \bar{t}_j (L_{x_j} u_\nu)^*$

$$0 \le \alpha_\varphi(f_\nu^* * f_\nu) \underset{L1}{=} \sum_{i,j} \bar{t}_j t_i \iint_{G\times G} \varphi(y^{-1}x)\, u_\nu(x_i^{-1}x)\, u_\nu(x_j^{-1}y)\, d\mu(x)\, d\mu(y)$$

$$= \sum_{i,j} \bar{t}_j t_i \iint_{G\times G} \varphi\big((x_j y)^{-1}(x_i x)\big)\, u_\nu(x)\, u_\nu(y)\, d\mu(x)\, d\mu(y) \longrightarrow$$

$$\longrightarrow \sum_{i,j} \varphi(x_j^{-1}x_i)\, t_i \bar{t}_j. \implies \varphi \text{ is pos def.}$$

($\Longleftarrow$) suppose φ is pos. def.

It suff to show that $\alpha_\varphi(f^* * f) \geq 0 \ \forall f \in C_c(G)$.

Take $f \in C_c(G)$; $K = \operatorname{supp} f$.

$F(x,y) = \varphi(y^{-1}x) f(x) \overline{f(y)}$. $F \in C_c(G \times G)$, $\operatorname{supp} F \subset K \times K$.

<u>Exer.</u> $\forall \varepsilon > 0 \ \exists$ disjoint Borel sets $E_1, \ldots, E_n$ and $x_i \in E_i$ $(i = 1, \ldots, n)$ s.t. $K = \bigsqcup_{i=1}^{n} E_i$ and

$$\Big\| F - \overbrace{\sum_{i,j} F(x_i, x_j) \chi_{E_i \times E_j}}^{G_\varepsilon} \Big\|_\infty < \varepsilon.$$

(Hint: unif continuity of F)

$$\left| \int F \, d(\mu\otimes\mu) - \int G_\varepsilon \, d(\mu\otimes\mu) \right| \leq \int_{K\times K} |F - G_\varepsilon| \, d(\mu\otimes\mu) < \varepsilon \mu(K)^2 .$$

$$\int_{K\times K} G_\varepsilon \, d(\mu\otimes\mu) = \sum_{i,j} F(x_i, x_j)\mu(E_i)\mu(E_j) =$$

$$= \sum_{i,j} \varphi(x_j^{-1}x_i) f(x_i)\mu(E_i)\overline{f(x_j)\mu(E_j)} \geq 0 .$$

$$\Longrightarrow \int F \, d(\mu\otimes\mu) \geq 0, \text{ that is, } \varphi \text{ is of pos. type } \square .$$