K_0 for nonunital rings. Half- and split-exactness

(EXERCISES FOR LECTURES 10–11)

2.1. Let R and S be rings, and let $0\colon R \to S$ be the zero map. Show that the induced map $0_*\colon K_0(R) \to K_0(S)$ is zero.

2.2. Extend the isomorphisms $K_0(R \times S) \cong K_0(R) \times K_0(S)$ and $K_0(R) \cong K_0(R^{\mathrm{op}})$ (see Exercises 1.6 and 1.7) to nonunital rings.

2.3 (*matrix stability of K_0*)**. (a)** Given a ring R, define a (nonunital!) ring homomorphism $i_n\colon R \to M_n(R)$ by $i_n(a) = a \oplus 0_{n-1}$. Show that the induced map $(i_n)_*\colon K_0(R) \to K_0(M_n(R))$ is an isomorphism.
(b) Prove a similar result for the embedding $i_\infty\colon R \to M_\infty(R)$, $a \mapsto a \oplus 0$.
(c) Deduce that, if R is a field or if $R = \mathbb{Z}$ (or, more generally, if R is a principal ideal domain), then $K_0(M_\infty(R)) \cong \mathbb{Z}$.

2.4 (*a naive definition of K_0 for nonunital rings*)**.** Given a nonunital ring R, define $K_{00}(R)$ to be the Grothendieck group of the semigroup $\mathbb{V}(R)$ defined exactly as in the unital case (thus $\mathbb{V}(R)$ consists of equivalence classes of idempotents in $M_\infty(R)$ with operation $[e] + [f] = [e \oplus f]$ for idempotents $e, f \in M_\infty(R)$). Give an example showing that K_{00} is not half-exact.
(*Hint:* you may use the fact that $K^0(S^2) \cong \mathbb{Z}^2$.)

2.5. Let $p\colon R \to S$ be a unital ring homomorphism. Suppose that p is a retraction in the category of all (not necessarily unital) rings, i.e., that the ring extension $\operatorname{Ker} p \hookrightarrow R \to S$ splits. Is p necessarily a retraction in the category of unital rings?

2.6. Let X be a compact topological space, Y be a closed subset of X, and $U = X \setminus Y$.
(a) Show that the ideal $I_Y = \{f \in C(X) : f|_Y = 0\}$ is isometrically $*$-isomorphic to $C_0(U)$.
(b) Prove that the extension $C_0(U) \hookrightarrow C(X) \to C(Y)$ splits in the category of (not necessarily unital) $\mathbb{C}$-algebras iff Y is a retract of X.

2.7. Does the extension $C_0((0,1)) \hookrightarrow C_0([0,1)) \to \mathbb{C}$ split in the category of rings? (Here the first arrow extends $f \in C_0((0,1))$ to $[0,1)$ by $f(0) = 0$, and the second arrow is the evaluation at 0.)

2.8 (*an algebraic version of the index map*)**.** Let $I \xrightarrow{i} R \xrightarrow{p} S$ be a ring extension. Assume also that R, S, p are unital. Define a map $\operatorname{ind}\colon \mathrm{GL}_\infty(S) \to K_0(I)$ as follows. Given $a \in \mathrm{GL}_n(S)$, find (by Whitehead's lemma) $u \in \mathrm{GL}_{2n}(R)$ such that $p(u) = a \oplus a^{-1}$. Since $p(u)$ commutes with 1_n in $M_{2n}(S)$, it follows that $u1_nu^{-1} - 1_n \in M_{2n}(I)$ and that $u1_nu^{-1} \in M_{2n}(I_+)$. We let $\operatorname{ind}(a) = [u1_nu^{-1}] - [1_n] \in K_0(I)$ (where the brackets [..] denote classes in $K_0(I_+)$).
(a) Show that $\operatorname{ind}\colon \mathrm{GL}_\infty(S) \to K_0(I)$ is a well-defined group homomorphism.
(b) Show that the exact sequence $K_0(I) \to K_0(R) \to K_0(S)$ (see the lecture) fits into an exact sequence

$$\mathrm{GL}_\infty(R) \xrightarrow{p} \mathrm{GL}_\infty(S) \xrightarrow{\operatorname{ind}} K_0(I) \to K_0(R) \to K_0(S).$$

(c) Show that, if S is a field and $n \geqslant 3$, then the commutant of $\mathrm{GL}_n(S)$ is $\mathrm{SL}_n(S)$.
(d) Assuming that S is a field, show that the exact sequence from (b) induces an exact sequence

$$R^\times \xrightarrow{p} S^\times \xrightarrow{\operatorname{ind}} K_0(I) \to K_0(R) \to K_0(S).$$

(e) Using (d), calculate $K_0(p\mathbb{Z})$ where $p \in \mathbb{Z}$ is a prime. Is the map $K_0(p\mathbb{Z}) \to K_0(\mathbb{Z})$ injective?

2.9 (*an algebraic prototype of Fredholm operators)***.** Let V be a k-vector space of countable dimension. A linear operator $T: V \to V$ is *Fredholm* if $\operatorname{Ker} T$ and $\operatorname{Coker} T$ are finite-dimensional. The *Fredholm index* of T is $\operatorname{ind}(T) = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$.
(a) Show that T is Fredholm iff there exists a linear operator S on V such that the operators $1 - ST$ and $1 - TS$ are of finite rank.
(b) Show that, if T is Fredholm, then S in (a) can be chosen in such a way that $1 - ST$ and $1 - TS$ are idempotents with $\operatorname{Im}(1 - ST) = \operatorname{Ker} T$ and $\operatorname{Im}(1 - TS) \cong \operatorname{Coker} T$.
(c) Let $E = \operatorname{End}_k(V)$, and let $F \subset E$ be the ideal consisting of finite-rank operators (thus $F \cong M_\infty(k)$). Show that $T \in E$ is Fredholm iff $p(T)$ is invertible in E/F (where $p: E \to E/F$ is the quotient map).
(d) Given $a = p(T) \in (E/F)^\times$, show that the Fredholm index $\operatorname{ind}(T)$ defined in this exercise agrees with the K-theoretic index $\operatorname{ind}(a) \in K_0(F)$ defined in Exercise 2.8 modulo the identification $K_0(F) \cong \mathbb{Z}$ (see Exercise 2.3 (c)).

2.10 (*an algebraic version of the Calkin extension)***.** Let E and F be as in Exercise 2.9. Is the quotient map $E \to E/F$ a retraction **(a)** in the category of unital rings? **(b)** in the category of rings?

2.11 (*an algebraic version of the Toeplitz extension)***.** Let V be a k-vector space with a countable basis $\{e_0, e_1, \ldots\}$. Define linear operators u, v on V by $v(e_i) = e_{i+1}$ $(i \geqslant 0)$, $u(e_i) = e_{i-1}$ $(i \geqslant 1)$, $u(e_1) = 0$. Let T denote the subalgebra of $\operatorname{End}_k(V)$ generated by u and v (the *Toeplitz-Jacobson algebra*).
(a) Show that the ideal of T generated by all commutators $[a, b]$ $(a, b \in T)$ is precisely the algebra F of all finite-rank operators on V.
(b) Show that the quotient T/F is isomorphic to the Laurent polynomial algebra $k[t^{\pm 1}]$.
(c) Is $T \to T/F$ a retraction in the category of unital rings?
(d)* Is $T \to T/F$ a retraction in the category of rings?