Harmonic analysis and Banach algebras

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Harmonic analysis on locally compact abelian groups is a natural generalization of the classical Fourier analysis usually studied by undergraduate students in mathematics (that is, of the theory of trigonometric Fourier series and of the Fourier transform on the real line). The most elegant approach to harmonic analysis on abelian groups is based on the theory of commutative Banach algebras, which was initiated by Gelfand in the early 1940ies and was further developed by Raikov, Naimark, Shilov and many other brilliant mathematicians. This approach, in particular, yields a relatively simple analytic proof of the Pontryagin duality based on the Plancherel theorem. In this course, we discuss the basics of Banach algebra theory and apply it to constructing the harmonic analysis on a locally compact abelian group. If time permits, some nonabelian groups will also be considered.

Prerequisites. The Lebesgue integration theory and the basics of functional analysis.

Syllabus

1. A toy example: harmonic analysis on a finite abelian group. Classical examples: harmonic analysis on the integers, on the circle, and on the real line.
2. Topological groups. The Haar measure. The modular character.
3. Banach algebras and an elementary spectral theory. The Gelfand spectrum and the Gelfand transform of a commutative Banach algebra. C^*-algebras and the 1st Gelfand-Naimark theorem.
4. Banach algebras associated to locally compact groups: the L^1-algebra, the measure algebra, and the group C^*-algebra. Representations of locally compact groups and of their group algebras.
5. The dual of a locally compact abelian group. The Fourier transform as a special case of the Gelfand transform. Positive functionals on a Banach $*$-algebra. Positive definite functions and Bochner's theorem. The Fourier inversion formula. The Plancherel theorem. The Pontryagin duality.
6. Harmonic analysis on the Heisenberg group and/or on some other nonabelian groups (if time permits).