The Bott periodicity

(EXERCISES FOR LECTURE 19)

7.1. Let A be a C^*-algebra, $B \subset A$ a closed $*$-subalgebra, and $I \subset A$ a closed two-sided ideal. Show that the $*$-subalgebra $B + I$ is closed in A. (This fact was used in the proof of the Bott periodicity, see the lectures.)

7.2. Let $\varphi_i \colon A_i \to B_i$ be C^*-algebra homomorphisms ($i = 1, 2$). Show that, if both φ_1 and φ_2 are injective (resp. surjective), then so is $\varphi_1 \otimes_* \varphi_2 \colon A_1 \otimes_* A_2 \to B_1 \otimes_* B_2$. (This fact was used in the proof of the Bott periodicity, see the lectures.)

7.3. Let $\varphi, \psi \colon A \to B$ be C^*-algebra homomorphisms. Assume that $\varphi \perp \psi$ (this means that $\varphi(A)\psi(A) = 0$). Show that $\varphi+\psi$ is a $*$-homomorphism, and that $(\varphi+\psi)_* = \varphi_*+\psi_* \colon K_i(A) \to K_i(B)$ ($i = 0, 1$). (This fact was used in the proof of the Bott periodicity, see the lectures.)

7.4. Using the cyclic 6-term exact sequence, find a simple proof of the fact that $K_0(\mathcal{Q}(H)) = 0$. (Compare with the brutal force proof hinted at in Exercise 4.12.)

7.5. Construct an extension $C_0(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}\mathrm{P}^2) \to C(S^1)$. Using this extension, calculate $K^i(\mathbb{R}\mathrm{P}^2)$ ($i = 0, 1$).

7.6. Construct extensions $C_0(\mathbb{R}^3) \hookrightarrow C(\mathbb{R}\mathrm{P}^3) \to C(\mathbb{R}\mathrm{P}^2)$ and $C_0(\mathbb{R}^3 \sqcup \mathbb{R}^3) \hookrightarrow C(S^3) \to C(S^2)$. Using these extensions and the naturality of the index map, calculate $K^i(\mathbb{R}\mathrm{P}^3)$ ($i = 0, 1$).

7.7. Construct an extension $C_0(\mathbb{C}^n) \hookrightarrow C(\mathbb{C}\mathrm{P}^n) \to C(\mathbb{C}\mathrm{P}^{n-1})$. Using this extension, calculate $K^i(\mathbb{C}\mathrm{P}^n)$ ($i = 0, 1$).

7.8. Let v be the right shift operator acting on the Hilbert space $H = \ell^2(\mathbb{Z}_{\geqslant 0})$. Given $n \in \mathbb{N}$, let $\mathcal{T}_n$ denote the C^*-subalgebra of $\mathscr{B}(H)$ generated by $\mathcal{K}(H)$ and v^n. Calculate $K_i(\mathcal{T}_n)$ ($i = 0, 1$).

7.9 (*the Bott map*)**.** Let A be a C^*-algebra. In **(a)**–**(c)** below, we assume that A is unital.

(a) Construct a group isomorphism

$$\mathrm{U}_n(SA) \cong \left\{ f \in C(S^1, \mathrm{U}_n(A)) : f(1) = 1_n \right\}.$$

(b) Given a projection $p \in M_n(A)$, define $f_p \colon S^1 \to M_n(A)$ by $f_p(z) = \bar{z}p + 1_n - p$. Show that f_p maps S^1 to $\mathrm{U}_n(A)$ and hence (by **(a)**) determines an element $f_p \in \mathrm{U}_n(SA)$.

(c) Prove that there exists a unique group homomorphism $\beta_A \colon K_0(A) \to K_2(A) = K_1(SA)$ taking $[p]$ to $[f_p]$, for every projection $p \in M_\infty(A)$.

(d) If A is not necessarily unital, show that β_{A_+} restricts to a homomorphism $\beta'_A \colon K_0(A) \to K_2(A)$. Prove that $\beta'_A = \beta_A$ if A is already unital. (Because of this, we write β_A for β'_A below.)

(e) Let $\alpha_A \colon K_2(A) \to K_0(A)$ be the natural isomorphism constructed in Cuntz's proof of the Bott periodicity (see the lectures). Show that $\alpha_A \beta_A = 1$, and so $\beta_A = \alpha_A^{-1}$ is an isomorphism.

Hint. Show that the matrix

$$\begin{pmatrix} (v^* - 1) \otimes p + 1 \otimes 1 & 0 \\ e_{00} \otimes p & (v - 1) \otimes p \end{pmatrix}$$

is unitary in $M_2(\mathcal{T}_0 \otimes_* A)$ and lifts $f_p \otimes f_p^*$ under the homomorphism induced by the quotient map $\mathcal{T}_0 \otimes_* A \to SA$ in the reduced Toeplitz extension tensored by A.

7.10 (*external product and the Bott map*)**.** Let A, B be C^*-algebras.

(a) Assuming that A and B are unital, show that there exists a $\mathbb{Z}$-bilinear map $\mu_{A,B}\colon K_0(A) \times K_0(B) \to K_0(A \otimes_* B)$ uniquely determined by $\mu_{A,B}([p],[q]) = [p \otimes q]$ for projections $p \in M_m(A)$, $q \in M_n(B)$. (Here we identify $M_m(A) \otimes_* M_n(B)$ with $M_{mn}(A \otimes_* B)$.)

(b) By using unitizations, extend the definition of $\mu_{A,B}$ to nonunital C^*-algebras.

(c) By using suspensions, extend $\mu_{A,B}$ to a map $K_i(A) \times K_j(B) \to K_{i+j}(A \otimes_* B)$ $(i, j \in \mathbb{Z}_{\geqslant 0})$.

(d) (*the Bott element*). Let L denote the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$, and let $b = [L^*] - [1] \in K^0(S^2)$, where 1 stands for the 1-dimensional trivial bundle. Identifying S^2 with $(\mathbb{R}^2)_+$, observe that b actually belongs to $K^0(\mathbb{R}^2) \subset K^0(S^2)$. Since $K^0(\mathbb{R}^2) \cong K_0(C_0(\mathbb{R}^2)) = K_0(S^2\mathbb{C}) = K_2(\mathbb{C})$, we have $b \in K_2(\mathbb{C})$.

(e) Show that, for each C^*-algebra A and each $x \in K_0(A)$, we have $\beta_A(x) = \mu_{\mathbb{C},A}(b, x)$ (where β_A is the Bott map, see Exercise 7.9).