Special Topics in Logic and Security 1

Domain Interaction

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Memory Access

What happens with the *Pts* domain in the program below?

```
int A[4][8] = {...};
uint i, j;
uint sum = 0;

for (i = 0; i < 4; i++)
    for (j = 0; j < 8; j++)
        sum += A[i][j];

printf("sum = %d\n", sum);
```

Memory Access

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```
int A[4][8] = {...};
uint i, j;
uint sum = 0;

for (i = 0; i < 4; i++)
    for (j = 0; j < 8; j++)
        sum += A + 8*sizeof(*A)*i + sizeof(*A)*j;

printf("sum = %d\n", sum);
```

Memory Access

What happens with the *Pts* domain in the program below?

```
for (i = 0; i < 4; i++)
    for (j = 0; j < 8; j++)
        sum += A + 8*sizeof(*A)*i + sizeof(*A)*j;
```

On an architecture where `int` has 64-bits:

```
for (i = 0; i < 4; i++)
    for (j = 0; j < 8; j++)
        sum += A + 8*8*i + 8*j
```

If A starts at *byte* 0, it leads to the following memory access sequence:

$$\underbrace{0, 8, 16, 24, \ldots, 56}_{\texttt{A[0]}} \quad \underbrace{64, 72, \ldots, 120}_{\texttt{A[1]}} \quad \underbrace{128, 136, \ldots, 184}_{\texttt{A[2]}} \quad \underbrace{192, 200, \ldots, 248}_{\texttt{A[3]}}$$

Memory Access

$$\underbrace{0, 8, 16, 24, \ldots, 56}_{\texttt{A[0]}} \quad \underbrace{64, 72, \ldots, 120}_{\texttt{A[1]}} \quad \underbrace{128, 136, \ldots, 184}_{\texttt{A[2]}} \quad \underbrace{192, 200, \ldots, 248}_{\texttt{A[3]}}$$

We can rewrite

```
for (i = 0; i < 4; i++)
    for (j = 0; j < 8; j++)
        sum += A + 64*i + 8*j
```

in the *Lin* domain where ***a*** and ***x*** are:

$$\boldsymbol{a} = [64\ 8]^\top, \quad \boldsymbol{x} = [i\ j]^\top$$

which leads to two half-spaces in the *Ineq* domain:

$$0 \leq i \leq 3, \quad 0 \leq j \leq 7 \quad, [\![64i \leq 192]\!], \quad [\![8j \leq 56]\!]$$

that define a convex surface in the polyhedra domain *Poly*.

But what can we say about the *Pts* domain, what can we say about the values taken by pointers accessing array `A`?

Aligned versus Unaligned Access

The way we declare the array

```
int A[4][8];
A[i][j] == *(A + 8*sizeof(*A)*i + sizeof(*A)*j)
```

offers information regarding the way memory access is intended to happen: 32 integers laid out in 4 contiguous memory blocks.

What happens when we write:

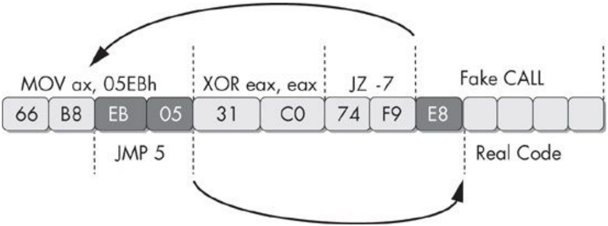
```
u = *((char *)A + 2);
v = *((char *)A + 111);
t = *((char *)A + 254);
```

what are the values of `u`, `v` and `t`?

Some architectures constrain memory access to be aligned in hardware.

Intel, AMD and ARM allow unaligned access. Thus, the entire range $[0, 255]$ has to be taken into consideration, not just the set $\{0, 8, 16, \ldots, 248\}$.

Example: Unaligned Access



`https://stackoverflow.com/questions/30192694/jump-to-the-middle-of-an-instruction`

Polyhedra Connection

Coming back to the remark: "Thus, the entire range $[0, 255]$ has to be taken into consideration, not just the set $\{0, 8, 16, \ldots, 248\}$."

In fact the Polyhedra domain is given us exactly this information: half-planes include all byte ranges, not just the type-induced multiples.

Example

In the *Pts* domain of `s = strings[i]` we showed that `s` can only point to $\{s_1, s_2, s_3\}$ because `i` can only take the values $\{0, 1, 2\}$ inside the loop.

But in fact the *Poly* half-plane is actually $[\![8i \leq 16]\!]$ and we have no analysis to show that other bytes inside `strings` can not be accessed.

We now need to equip *Poly* to check if pointers access intended memory addresses (or not) within the arrays.

Granger Domain

The *Granger* domain is used for variables inference and analysis that take the form $c + m\mathbb{Z}$

$$\{\ldots, -2m + c, -m + c, c, c + m, c + 2m, \ldots\}$$

the domain is abstract enough to permit constructions of the type

```
int i = 5;
struct {
    int k;
    char v[100];
    short f;
} a[10];
a[i].f = 0;
*((char *)a + 13) = 2;
```

Exercise: Show how we can model the two references.

Multiplicity Domain

Proposition

For the analysis of simple vectors and arrays, that use basic types (e.g. `int`*,* `char`*), it suffices to show that a variable is a multiple of* 2^n *in order to prove that the memory access is aligned.*

Exercise: Prove the proposition on the example that was using array `A[4][8]`.

Definition

Let $Mult = \mathcal{X} \to \{0, \ldots, 64\}$ be the functional space that records the number of least significant bits (LSB) that are always zero, null.

Remark

The linear transform $M \in Mult$ *assigns a value* $n = M(x)$ *to all the variables* $x \in \mathcal{X}$.

Example: Multiplicity Domain

Remark

The linear transform $M \in Mult$ assigns a value $n = M(x)$ to all the variables $x \in \mathcal{X}$.

Suppose that all the variables are represented on at most 64-bits.

Type	Var.	Max Mult.
`uint`	`x`	$M(x) = 63$
`uint32_t`	`y`	$M(y) = 31$
`ushort`	`f`	$M(f) = 15$
`uchar`	`c`	$M(c) = 7$

Thus, no matter the type, `x=0` can be represented as $M(x) = 64$.

The Lattice $(Mult, \subseteq_M, \vee_M, \wedge_M)$

Let $M, M', M_1, M_2 \in Mult$.

Update: $M \to M' = M[x \to n'] \implies M'(x) = n'$ and $M'(y) = M(y), \forall y \neq x$.

Join: $M' = M_1 \vee_M M_2$ s.t. $M'(x) = \min(M_1(x), M_2(x)), \forall x \in \mathcal{X}$.

Inclusion: $M_1 \subseteq_M M_2 \iff M_1(x) \geq M_2(x), \forall x \in \mathcal{X}$.

Exercise: Find the $\top$ element: the largest element from the lattice. Explain.

Let $Equ = Lin \times \mathbb{Z}$ be the set of linear equations of the type $e = c$, where $e \in Lin, c \in \mathbb{Z}$.

Meet: The intersection operator adds the information provided by a new equation: $M' = M \wedge_M (e = c)$.

Definition

$\wedge_M : Mult \times Equ \to (Mult \cup \{\perp_M\})$, where $\perp_M$ denotes an unsatisfiable (impossible) state.

The $\wedge_M$ Operation

Definition

Let $\delta : \mathbb{Z} \to \{0, \ldots, 64\}$ s.t. $\delta(c)$ represents the number of unused (zero) LSB from c.

Let $e \equiv \boldsymbol{a}^\top \boldsymbol{x} = a_1 x_1 + \cdots + a_n x_n$ s.t. $a_i \neq 0, \forall i = 1, \ldots, n$. We recompute the multilpicity of variable x_j by rewriting $e = c$

$$-a_j x_j = a_1 x_1 + \cdots + a_{j-1} x_{j-1} + a_{j+1} x_{j+1} + \cdots + a_n x_n - c$$

Remark

The multiplicity of $a_i x_i$ is $\delta(a_i) + M(x_i)$, and the multiplicity of c is simply $\delta(c)$.

Intuition: *Mult* is similar to the exponent operations: $2^m 2^n = 2^{m+n}$.

Proposition

The operation $\wedge_M$ adds information, thus the number of null LSB from x_j can not decrease. On the contrary, this number can increase due to the presence of c.

Number of null LSB's from x_j cannot decrease

$$-a_j x_j = a_1 x_1 + \cdots + a_{j-1} x_{j-1} + a_{j+1} x_{j+1} + \cdots + a_n x_n - c$$

The equation's right-hand-side multiplicity has to be greater than or equal to the one of every individual term:

$$\min(\delta(c), \min_{i, i \neq j} \delta(a_i) + M(x_i))$$

Example (`A[i][j]`=*(A + 8*8*i + 8*j))

`64*i + 8*j` $\implies \min(\delta(64) + M(i), \delta(8) + M(j)) = \min(6 + M(i), 3 + M(j))$.

Let $i = 2, j = 4$, then $\min(6 + 1, 3 + 2) = 5$, and $64i + 8j = 160 = 1010\ 0000_2$.

If $a_j > 1 \vee a_j < -1$, then the number in the equation above has to be reduced by $\delta(a_j)$ in order to obtain the new $M' = M(x_j \to n')$

$$M' = M\left[x_j \to \max\left(M(x_j), \min(\delta(c), \min_{i, i \neq j} \delta(a_i) + M(x_i)) - \delta(a_j)\right)\right]$$

Example: Updating the Multiplicity

Let M be the initial multipliers state of the three variables x, y, z, where x is a multiple of 8, and y and z are multiples of 2.

We add the equation $x + y + 2z = 0 \in Equ$ that resulted from a pointer arithmetic problem in the domain of M.

$$M' = M \wedge_M \{x + y + 2z = 0\}$$

which we solve by iteratively updating the muliplicity of each variable.

	$M(x)$	$M(y)$	$\delta(2) + M(z)$	$\delta(0)$
M	3	1	$1+1$	64
$M'(x)$	3	1	$1+1$	64
$M'(y)$	3	2	$1+1$	64
$M'(z)$	3	2	$1+1$	64

Exercise: What happens if we add the equation $x + y + 2z = 1$?

Invalid State Exercise

Let M be the initial multipliers state of the three variables x, y, z, where x is a multiple of 8, and y and z are multiples of 2.

Exercise: What happens if we add the equation $x + y + 2z = 1$?

	$M(x)$	$M(y)$	$\delta(2) + M(z)$	$\delta(1)$
M	3	1	$1+1$	0
$M'(x)$	3	1	$1+1$	0
$M'(y)$	3	1	$1+1$	0
$M'(z)$	3	1	$1+1$	0

Remark: nothing changes due to $\delta(1) = 0$, but can we represent 1 with the left-hand side?

The equation $e \equiv \boldsymbol{a}^\top \boldsymbol{x} = c$ can be written:

$$\boldsymbol{a}^\top \boldsymbol{x} = a_1 2^3 + a_2 2^1 + a_3 2^2 = 2^0$$

where we substituted x, y, z with their multiplicities.

Conclusion: The closest we can get to 1 is with $\boldsymbol{a} = [0\ 1\ 0]^\top$, but $\boldsymbol{a}^\top \boldsymbol{x} \neq 1$.

Properties

Proposition

The $\wedge_M$ operation leads to an invalid state $\bot_M$ if

$$\min_{i=1,\dots,n} \delta(a_i) + M(x_i) > \delta(c)$$

Complexity: The update of a single variable has to take into account all the variables: quadratic cost. In practice we have to deal with at most 2–3 variables.

Remark

The special operations from Poly are solved similarly:

- $M' = M \triangleright x := e \implies M(x) = M(t) = 0$ *due to the projection operator*
- *updating terms other than* $M(x), M(t)$ *from* e *does not bring new information*
- $M \triangleright x := y \gg n \implies M(x)$ *is at least* $(M(y) - n)$, $M(y)$ *does not change*

Properties

Alignment: We can verify if the access is aligned through the operation $M \wedge_M \{x = 2^n\}$. If the result is $\bot_M$ then we have an illegal access error.

Example: Let x denote the offset of pointer s inside `strings` and let the pointer type be 8 bytes long. Then $M' \wedge_M \{s = 8\}$ ensures that $M'(s) \geq 3$ or $M' = \bot_M$. If $M \subseteq_M M'$, then we have aligned access. Otherwise emit warning.

Projection: Let the function $\exists_x : Mult \to Mult$ and let $M' = \exists_x(M)$. Then $M'(x) = 0$ and $M'(y) = M(y), \forall y \neq x$.

Theorem

$(Mult, \subseteq_M, \wedge_M, \vee_M)$ *forms a complete lattice.*

Exercise: Prove the theorem.

Proof: $(Mult, \subseteq_M, \wedge_M, \vee_M)$ forms a lattice

Let $M, M_1, M_2, M_3 \in Mult$

$(Mult, \subseteq_M)$ POSET:

- reflexive: $M \subseteq_M M \iff M(x) \geq M(x) \quad \forall M$
- anti-symmetric:
  $M_1 \subseteq_M M_2, M_2 \subseteq_M M_1 \implies M_1(x) \geq M_2(x), M_2(x) \geq M_1(x)$
  $\implies M_1 = M_2$
- transitivity: $M_1 \subseteq_M M_2, M_2 \subseteq_M M_3 \implies M_1 \subseteq_M M_3$;
  $M_1(x) \geq M_2(x)$ and $M_2(x) \geq M_3(x) \implies M_1(x) \geq M_3(x) \implies M_1 \subseteq_M M_3$

Lattice:

- associative:
  $(M_1 \vee_M M_2) \vee_M M_3 = M_1 \vee_M (M_2 \vee_M M_3)$
  $\impliedby \min(a, min(b, c)) = \min(\min(a, b), c)$;
  $(M_1 \wedge_M M_2) \wedge_M M_3 = M_1 \wedge_M (M_2 \wedge_M M_3) \impliedby$ update can not decrease LSB and order does not affect the update operation
- commute: $M_1 \vee_M M_2 = M_2 \vee_M M_1 \impliedby \min(a, b) = \min(b, a)$;
  $M_1 \wedge_M M_2 = M_2 \wedge_M M_1 \impliedby$ update can not decrease LSB
- absorb: $M_1 \wedge_M (M_1 \vee_M M_2) = M_1$; $M_1 \vee_M (M_1 \wedge_M M_2) = M_1$

Poly and *Mult* Interaction

Let $Num = (Poly \times Mult) \cup \{\bot_N\}$, where $\bot_N$ represents an unreachable state, that is impossible to attain, in the program definition. We define:

- $(P, M) \subseteq_N (P', M') \iff (P \subseteq_P P') \wedge (M \subseteq_M M')$
- $(P', M') = (P_1, M_1) \vee_N (P_2, M_2) \iff (P' = P_1 \vee_P P_2) \wedge (M' = M_1 \vee_M M_2)$
- $(P', M') = (P, M) \rhd x := e \iff (P' = P \rhd x := e) \wedge (M' = M \rhd x := e)$
- $(P', M') = (P, M) \rhd x := e \gg n \iff (P' = P \rhd x := e \gg n) \wedge (M' = M \rhd x := e \gg n)$
- $(P', M') = \exists_x(P, M) \iff (P' = \exists_x(P)) \wedge (M' = \exists_x(M))$
- $(P, M) \wedge_N \{e = c\} = \begin{cases} \bot_N & \text{if } P' = \emptyset \text{ or } M' = \bot_M \\ (P', M') & \text{otherwise} \end{cases}$, where

  $$P' = P \wedge_P [\![\{e = c\}]\!] \text{ and } M' = M \wedge_M \{e = c\}.$$