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A Note on the Lyapunov Equation

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ABSTRACT

Some results on the sensitivity of the solution of the stable Lyapunov equation are shown to follow easily from well-known theorems of functional analysis. © Elsevier Science Inc., 1997

The Lyapunov equation, much studied because of its importance in differential equations and control theory [5], is the matrix equation

$$AX + XA^* = -W. \tag{1}$$

The matrix A is called *stable* if its spectrum is contained in the open left half plane. This condition is sufficient to ensure that the equation (1) has a unique solution X for every W. The solution can be expressed as

$$X = \int_0^\infty e^{At} W e^{A^* t}\, dt. \tag{2}$$

From this it is immediately clear that if W is positive (semidefinite) then so is X.

The sensitivity of the solution X to changes in W has been analyzed by several authors, including Hewer and Kenney [3]. Their result can be described as follows. Let $\mathbb{M}$ be the space of $n \times n$ complex matrices. For $A \in \mathbb{M}$ let $\|A\|$ be the norm of A as a linear operator on the euclidean space

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$\mathbb{C}^n$. If $\mathscr{A}$ is a linear operator on $\mathbb{M}$, let $\|\mathscr{A}\| = \sup\{\|\mathscr{A}(X)\| : X \in \mathbb{M}, \|X\| = 1\}$. Now, given $A \in \mathbb{M}$, define $\mathscr{L}_A(X) = AX + XA^*$. This is a linear operator on $\mathbb{M}$, and it is invertible if A is stable. The equation (2) can be written also as

$$\mathscr{L}_A^{-1}(W) = -\int_0^\infty e^{At} W e^{A^*t}\, dt.$$

The norm of the operator $\mathscr{L}_A^{-1}$ is a measure of the sensitivity of the solution. The theorem below gives a way of evaluating it.

THEOREM 1 (Hewer and Kenney [3]). *For every stable matrix A*

$$\|\mathscr{L}_A^{-1}\| = \|\mathscr{L}_A^{-1}(I)\|. \tag{3}$$

A quick and illuminating proof of this can be given using a standard result in C^*-algebras: Let $H = -\mathscr{L}_A^{-1}(I)$. As noted above, H is a positive matrix. Define a map Γ_A on $\mathbb{M}$ as

$$\Gamma_A(Y) = -H^{-1/2}\mathscr{L}_A^{-1}(Y)H^{-1/2}. \tag{4}$$

Then Γ_A is a *positive map* (it takes positive matrices to positive matrices) and is *unital* (it takes I to I). By a theorem of Russo and Dye [8] such a map on any C^*-algebra has norm 1. So $\|\Gamma_A\| = 1$ and hence $\|\mathscr{L}_A^{-1}\| \leqslant \|H\|$. But, by the definition of H, we must then have $\|\mathscr{L}_A^{-1}\| = \|H\|$. This proves (3). The same proof works equally well for operators in an infinite-dimensional Hilbert space.

More can be said on this. Let $\langle X, Y\rangle = \operatorname{tr} X^*Y$ be the Frobenius inner product on $\mathbb{M}$. If $\mathscr{A}$ is a linear map on $\mathbb{M}$, its adjoint $\mathscr{A}^*$, defined with respect to this inner product, is the map that satisfies

$$\langle \mathscr{A}(X), Y\rangle = \langle X, \mathscr{A}^*(Y)\rangle \qquad \text{for all } X, Y.$$

A linear map $\mathscr{A}$ on $\mathbb{M}$ is called *doubly stochastic* if it is (i) positive, (ii) unital, and (iii) trace preserving, i.e., $\operatorname{tr}\mathscr{A}(X) = \operatorname{tr} X$ for all X. The third condition is equivalent to the condition that $\mathscr{A}^*$ is unital. See [1]. It is natural to ask whether the map Γ_A defined in (4) is doubly stochastic.

A simple calculation with traces shows that

$$\mathscr{L}_A^* = \mathscr{L}_{A^*}, \tag{5}$$

$$\Gamma_A^*(Y) = -\mathscr{L}_{A^*}^{-1}(H^{-1/2}YH^{-1/2}). \tag{6}$$

Hence,

$$\begin{aligned}\Gamma_A^*(I) &= -\mathscr{L}_{A^*}^{-1}(H^{-1})\\ &= \int_0^\infty e^{A^*t}H^{-1}e^{At}\,dt\\ &= \int_0^\infty e^{A^*t}\left(\int_0^\infty e^{At}e^{A^*t}\,dt\right)^{-1} e^{At}\,dt. \end{aligned}\tag{7}$$

This shows that if A is normal then Γ_A^* is unital and hence Γ_A is doubly stochastic. The converse is also true:

THEOREM 2. *Let A be a stable matrix. Then the following three conditions are equivalent*:

(i) *the operator Γ_A is doubly stochastic,*
(ii) *A commutes with the matrix $H = \int_0^\infty e^{At}e^{A^*t}\,dt$,*
(iii) *A is normal.*

Proof. (i) $\Rightarrow$ (ii): Note that

$$AH + HA^* = -I. \tag{8}$$

From (7) we have

$$\Gamma_A^*(I) = \int_0^\infty e^{A^*t}H^{-1}e^{At}\,dt. \tag{9}$$

This is an integral like the one in (2), and hence $\Gamma_A^*(I)$ satisfies the Lyapunov equation:

$$A^*\Gamma_A^*(I) + \Gamma_A^*(I)A = -H^{-1}. \tag{10}$$

Hence, if Γ_A^* is unital, then from (10) we have $A^* + A = -H^{-1}$. Using (8), we get from this $HA = AH$.

(ii) ⇒ (iii): If A commutes with H, the equation (8) gives $A + A^* = -H^{-1}$. From this we see that $A^{-1}(A + A^*) = (A + A^*)A^{-1}$. This shows that $A^*A = AA^*$.

We have already seen that (iii) ⇒ (i). ■

Doubly stochastic maps have several special properties. The one relevant to our discussion is that they are contractive with respect to every unitarily invariant norm on $\mathbb{M}$. (See [1].) Thus, if Γ_A is doubly stochastic we have

$$|||\Gamma_A(W)||| \leqslant |||W||| \tag{11}$$

for every unitarily invariant norm $|||\cdot|||$. All such norms satisfy the inequality $|||XYZ||| \leqslant \|X\|\,|||Y|||\,\|Z\|$ for any three matrices X, Y, Z. So, from (11) we get

$$|||\mathscr{L}_A^{-1}(W)||| = |||H^{1/2}\Gamma_A(W)H^{1/2}||| \leqslant \|H\|\,|||W||| = \|\mathscr{L}_A^{-1}(I)\|\,|||W|||. \tag{12}$$

This inequality can also be derived using other arguments. See, e.g., [2, Theorem 3.3].

We should take this opportunity to point out that many of the results in a recent paper on this topic [4] are simple consequences of elementary facts in functional analysis.

It is a basic fact [7, p. 93] that if T is an operator between normed spaces and T^* is its adjoint, then $\|T^*\| = \|T\|$. Lemma 3 of [4] is a very special case of this. Let

$$H_* = -(\mathscr{L}_A^{-1})^*(I).$$

Then from (5) above we have

$$H_* = -(\mathscr{L}_{A^*})^{-1}(I) = \int_0^\infty e^{A^*t}e^{At}\,dt.$$

Let X be the solution of (1) when A is stable and $W \geqslant 0$. Then

$$\begin{aligned} \operatorname{tr} X = \langle X, I\rangle &= \langle -\mathscr{L}_A^{-1}(W), I\rangle \\ &= \langle W, -(\mathscr{L}_A^{-1})^*(I)\rangle \\ &= \langle W, H_*\rangle = \operatorname{tr} WH_*. \end{aligned} \tag{13}$$

Choose an orthonormal basis in which H_* is diagonal, and calculate the traces. Since H_* is positive, this gives

$$\lambda_{\min}(H_*)\operatorname{tr}W \leqslant \operatorname{tr}X \leqslant \lambda_{\max}(H_*)\operatorname{tr}W. \tag{14}$$

This is Theorem 2 of [4]. The expression (13) not only gives an exact value of $\operatorname{tr}X$, it is also easier to compute than the quantities involved in the bounds (14).

The authors note towards the end of [4] (see the remark after Theorem 8) that this kind of argument could shorten the proofs of their Theorem 6 and Lemma 7 and also unify them. A more consistent use of this observation would achieve the same economy and simplicity for most of their results.

Finally we show how for each Schatten p-norm we could obtain a bound for $\|X\|_p$ in terms of $\|W\|_p$. Recall that, for $1 \leqslant p < \infty$, the norm $\|A\|_p$ is defined as

$$\|A\|_p = \left(\sum_{j=1}^{n}\left[s_j(A)\right]^p\right)^{1/p},$$

where $s_j(A)$, $1 \leqslant j \leqslant n$, are the singular value of A arranged in decreasing order. The operator norm $\|A\|$ that we have used above can also be thought of as

$$\|A\| = \|A\|_\infty = s_1(A).$$

If $\mathscr{A}$ is a linear operator on $\mathbb{M}$, let

$$\|\mathscr{A}\|_{p\to p} = \sup\left\{\|\mathscr{A}(X)\|_p : \|X\|_p = 1\right\}.$$

This is the norm of the operator $\mathscr{A}$ when the underlying space $\mathbb{M}$ is equipped with the norm $\|\cdot\|_p$.

A well-known theorem, sometimes called the Calderon-Lions interpolation theorem [6, p. 37], implies that for $1 \leqslant p \leqslant \infty$ we have

$$\|\mathscr{A}\|_{p\to p} \leqslant \|\mathscr{A}\|_{1\to 1}^{1/p}\|\mathscr{A}\|_{\infty\to\infty}^{1-1/p}. \tag{15}$$

Using this we can prove the following.

THEOREM 3. *Let A be a stable matrix. Let*

$$H = \int_0^\infty e^{At}e^{A^*t}\,dt, \qquad H_* = \int_0^\infty e^{A^*t}e^{At}\,dt.$$

Then, for $1 \leqslant p \leqslant \infty$,

$$\|\mathscr{L}_A^{-1}\|_{p \to p} \leqslant \|H_*\|^{1/p} \|H\|^{1-1/p}. \tag{16}$$

Proof. In this notation, Theorem 1 can be restated as

$$\|\mathscr{L}_A^{-1}\|_{\infty \to \infty} = \|H\|.$$

Since the norm $\|\cdot\|_1$ is dual to $\|\cdot\|_\infty$, we have

$$\|\mathscr{L}_A^{-1}\|_{1 \to 1} = \left\|\left(\mathscr{L}_A^{-1}\right)^*\right\|_{\infty \to \infty} = \|\mathscr{L}_{A^*}^{-1}\|_{\infty \to \infty} = \left\|\mathscr{L}_{A^*}^{-1}(I)\right\| = \|H_*\|.$$

This shows that for $p = 1, \infty$, we have equality in (16). For other values of p, we get this inequality using the interpolation result (15) cited above. ■

It would be interesting to know the exact values of $\|\mathscr{L}_A^{-1}\|_{p \to p}$ for $1 < p < \infty$.

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