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بسم الله الرحمن الرحيم

MT132

- حِلف الطلاب العصاميين يقدم لكم النسخة الثالثة من تجميعات الامتحانات النهائية محلولةً بالكامل.
- جديد النسخة الثانية: إضافة نموذج الفصل الأول من العام الدراسي 24-23
- الشكر مُسدى لكل من صحَح وتعقَّب وراجع النسخ، ومَن جمع الملفات على مر السنين.
- نرجو ألا تكون هذه التجميعات محل اعتمادكم في المذاكرة، فشرائح المُقرر أولى وأكثر أهمية.
- يغلب على النماذج تكرار أفكار الأسئلة، لذا ننصح بالرجوع إليها والاهتمام بها.

وإنْ تَجِدْ عَيبًا فسُدَّ الخَلَلا   فجَلَّ مَن لا عَيبَ فيهِ وعَلا

حرر في:
24/06/1445 – 06/01/2024

النسخة الثالثة (3)

Q–1: [5×2 marks] Choose the correct answer:

1.1: If $W = Span\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \end{bmatrix}\right\}$, then $\dim(W) =$

a) 0
b) 1
c) 2
d) 3
e) None of the above

1.2: The vectors $\boldsymbol{v_1} = \begin{bmatrix} 1-a \\ 4 \end{bmatrix}, and\ \boldsymbol{v_2} = \begin{bmatrix} a \\ 4 \end{bmatrix}$ are linearly dependent if

a) $a = \frac{1}{2}$
b) $a = -2$
c) $a = \frac{-1}{2}$
d) $a = 2$
e) None of the above

1.3: The matrix $A = \begin{bmatrix} a^2 & 3 \\ 3 & 1 \end{bmatrix}$ is singular if

f) $a = 3$
g) $a = \pm 3$
h) $a \neq 3$
i) $a \neq 3$
j) None of the above

1.4: $|BAB^{-1}| =$

a) $|B|$
b) $|A|$
c) 1
d) 0
e) None of the above

1.5: The det $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

a) 0
b) 1
c) -2
d) 2
e) None of the above

Q–2: [10 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: If 0 (Zero) is an eigenvalue of a matrix A then A^2 is singular.
2.2: If $|A| = 0$, then $A = \mathbf{0}$.

Each question is worth 10 marks. Answer the following questions:

Q–3: [5+2+3 marks]

Let $W = \left\{ \begin{bmatrix} 3a+b \\ -2b \\ 4a-3b \end{bmatrix} ; a, b \in R \right\}$ be a subset of $\boldsymbol{R^3}$.

a) Show that W is a subspace of $\boldsymbol{R^3}$.
b) Find a set S span W.
c) Find the basis and the dimension of W?

Q–4: [4+3+3 marks]

Define the mapping $T: \boldsymbol{R^2} \to \boldsymbol{R^3}$ by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ x+y \\ 3y \end{bmatrix}$.

a) Show that T is a linear transformation.
b) Find a matrix A such that $T(X) = AX$, for each $X \in R^2$.
c) Find the null of T; $N(T)$.

Q–5: [10 marks]

Prove that the matrix $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$. Then find matrices P and D such that $A = PDP^{-1}$.

End of the questions

Answer Q2-1

λ is eigenvalue $\rightarrow |\lambda \ \mathrm{I} - \mathrm{A}| = 0$

$|(0) \ \mathrm{I} - \mathrm{A}| = 0$

$|(0) \ \ - \mathrm{A}| = 0$

$|-\mathrm{A}| = 0$

$(-1^n) \ |A| = 0$

$|A| = 0 \ \rightarrow A$ is singular so A^2 is also singular $\ \rightarrow$ True.

Answer Q2 – 2

If A = $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = (2)(6) - (4)(3) = 12 - 12 = 0 \ \rightarrow$ False.

Answer Q3-A+B

$$W = \left\{ \begin{bmatrix} 3a + b \\ -2b \\ 4a - 3b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 3a \\ 0 \\ 4a \end{bmatrix} + \begin{bmatrix} b \\ -2b \\ -3b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$W = \left\{ a \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \right\} = Span\ S \quad \longrightarrow \quad S = \left\{ \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \right\}$$

Answer Q3-C

$basis\ For\ \ W: \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \qquad Dim = 2.$

Answer Q4-A

$x, y \in R^2$

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$T(rx + y) = \begin{bmatrix} rx_1 + x_2 \\ ry_1 + y_2 \end{bmatrix} = \begin{bmatrix} 2rx_1 + 2x_2 \\ rx_1 + x_2 + ry_1 + y_2 \\ 3ry_1 + 3y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2rx_1 \\ rx_1 + ry_1 \\ 3ry_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 + y_2 \\ 3y_2 \end{bmatrix} \rightarrow rT(x) + T(Y)$$

Answer Q4-B

$$T(X) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x + y \\ 3y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$$

Answer Q4-C

$$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} 2X \\ X + Y \\ 3Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = 0, \;\; y = 0 \rightarrow N(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

Answer Q5

$|\lambda \ \mathrm{I} - \mathrm{A}| = 0$

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 2 & -3 \\ -2 & \lambda - 1 \end{vmatrix} = 0$$

$(\lambda - 2)\,(\lambda - 1) - 6 = 0$

$\lambda^2 - \lambda - 2\lambda + 2 - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$\lambda_1 = -1\,, \quad \lambda_2 = 4.$

$when\ \lambda = -1$

$$\begin{bmatrix} -1-2 & -3 & 0 \\ -2 & -1-1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 \\ -2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$

$x + \ t = 0 \quad \rightarrow \quad x = -t$

$$S.S = \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} t \in \mathbb{R} \right\} \quad \rightarrow \quad S.S = \left\{ t \begin{bmatrix} -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$when\ \lambda = 4$

$$\begin{bmatrix} 4-2 & -3 & 0 \\ -2 & 4-1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ -2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-3}{2} & 0 \\ -2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$

$$x + \frac{-3}{2} t = 0 \quad \rightarrow \quad x = \frac{3}{2} t$$

$$S.S = \left\{ \begin{bmatrix} \frac{3}{2} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \quad \rightarrow \quad S.S = \left\{ t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$$

Q-1: [5×2 marks] Choose the correct answer:

1.1 If A is an 3 x 3 matrix and $\det(A) = 2$, then $\det(2A^TA^{-1}) =$

a) 6
b) 0
c) 8
d) -6
e) None of the above

1.2: If $v = \begin{bmatrix} 1 \\ -2 \\ 9 \end{bmatrix}$ *in* $W = span\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}\right\}$, then the value of c_1, c_2 to make the combination $c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 9 \end{bmatrix}$ are:

a) $c_1 = 0, c_2 = 0$
b) $c_1 = 3, c_2 = -2$
c) $c_1 = -3, c_2 = 2$
d) $c_1 = -2, c_2 = 3$
e) None of the above

1.3: Which of the following is not a basis for R^3?

a) $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

b) $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right\}$

c) $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$

d) All are bases
e) None of the above

1.4: The dimension of the subspace $W = \left\{\begin{bmatrix} a-b \\ b-2a \\ 3a \end{bmatrix} : a, b \in \mathbb{R}\right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

1.5: If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation define by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$, then

$T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

c) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

e) None of the above

Q–2: [2×5 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: If A and B are 3×3 nonsingular matrices, then $|2A^4B^T| = 8|A|^4/|B|$.

2.2: The set $\left\{\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 10 \\ 30 \end{bmatrix}\right\}$ is a basis of R^2.

Each question is worth 10 marks. Answer the following questions:

Q–3: Let $W = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x - 2y + z = 0,\ x, y, z \in \mathbb{R}\right\}$ be a subset of $\mathbb{R}^3$.

a) **[4 marks]** Show that W is a subspace of $\mathbb{R}^3$.

b) **[3 marks]** Find a set S such that $W = \text{span } S$.

c) **[3 marks]** Is $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \in W$? Explain.

Q–4: Define a linear operator $T: R^2 \to R^3$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ 3x + y \\ 2y \end{bmatrix}$

a) **[3 marks]** Show that T is a linear transformation;

b) **[2 marks]** Give a description of the range of T;

c) **[3 marks]** Is $\begin{bmatrix}2\\-1\\-2\end{bmatrix}$ in $\boldsymbol{R}(\boldsymbol{T})$?

d) **[2 marks]** Find the null space $\boldsymbol{N}(\boldsymbol{T})$.

Q–5: Let $A=\begin{bmatrix}1 & -3\\-1 & 4\end{bmatrix}$.

a) **[3 marks]** Find A^{-1}

b) **[4 marks]** Find a matrix X such that $AX-\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2 & 1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\\1 & 1\end{bmatrix}$

c) **[3 marks]** Find the eigenvalues of A.

End of questions

Answer Q2-1

$|2A^4B^t|$

$2^3|A^4B^t|$

$8|A^4B^t|$

$8|A^4|\,|B^t|$

$8|A|^4\,|B|$ $\quad\rightarrow\quad$ False.

Answer Q2-2

$\begin{bmatrix}2 & 10\\6 & 30\end{bmatrix}=\begin{vmatrix}2 & 10\\6 & 30\end{vmatrix}$

$|A|=(2)(30)-(10)(6)=0$

Since the $|A|=0$ so, the vectors are linearly dependent therefore the statement is False.

Answer Q3-A and Q3-B

$$W = \left\{ \begin{bmatrix} x \\ y \\ -3x + 2y \end{bmatrix}, x, y \in R \right\}$$

$$W = \left\{ \begin{bmatrix} x \\ 0 \\ -3x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 2y \end{bmatrix}, x, y \in R \right\}$$

$$W = \left\{ x \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, x, y \in R \right\}$$

$$W = \ span\ S \quad \rightarrow \quad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Answer Q3-C

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -3 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$c_1 = 1, \qquad c_2 = 3 \qquad \rightarrow \qquad \text{True.}$$

Answer Q4-A

$$x, y \ \in R^2$$

$$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$T(rx + y) = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ \ + y_2 \end{bmatrix} = \begin{bmatrix} rx_1 \ + x_2 - 2ry_1 - 2y_2 \\ 3rx_1 \ + 3x_2 + ry_1 \ \ + y_2 \\ 2ry_1 \ \ + 2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} rx_1 \ - 2ry_1 \\ 3rx_1 \ + ry_1 \\ 2ry_1 \end{bmatrix} + \begin{bmatrix} x_2 - 2y_2 \\ 3x_2 + y_2 \\ 2y_2 \end{bmatrix}$$

$$= \ rT(x) \ + \ T(Y)$$

Answer Q4-B

$$R(T) = span\left\{T\begin{bmatrix}1\\0\end{bmatrix}, T\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$$R(T) = span\left\{\begin{bmatrix}1\\3\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\2\end{bmatrix}\right\}$$

Basis for $R(T) = \left\{\begin{bmatrix}1\\3\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\2\end{bmatrix}\right\}$

Answer Q4-C

$$\begin{bmatrix}1 & -2 & 2\\3 & 1 & -1\\0 & 2 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 2\\0 & 7 & -7\\0 & 2 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

$c_1 = 0, \quad c_2 = -1 \quad \rightarrow$ True.

Answer Q4-D

$$N(T) = \left\{\begin{bmatrix}X\\Y\end{bmatrix} : T\left(\begin{bmatrix}X\\Y\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$N(T) = \left\{\begin{bmatrix}X\\Y\end{bmatrix} : \left(\begin{bmatrix}X-2Y\\3X+Y\\2Y\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$\begin{bmatrix}1 & -2 & 0\\3 & 1 & 0\\0 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0\\0 & 7 & 0\\0 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

$x = 0, y = 0,$

$$N(T) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

Answer Q5-A

$$\begin{bmatrix}4 & 3\\1 & 1\end{bmatrix}$$

Answer Q5-B

$$AX - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

$$AX = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1}AX = A^{-1}\begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$$

$$X = A^{-1}\begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 19 & 21 \\ 5 & 6 \end{bmatrix}$$

Answer Q5-C

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = 0$$

$$\left|\begin{bmatrix} \lambda - 1 & 3 \\ 1 & \lambda - 4 \end{bmatrix}\right| = 0$$

$$(\lambda - 1)(\lambda - 4) - (1)(3) = 0$$

$$\lambda^2 - 4\lambda - \lambda + 4 - 3 = 0$$

$$\lambda^2 - 5\lambda + 1 = 0$$

$$\lambda = \frac{5 + \sqrt{21}}{2} \quad \text{or} \quad \lambda = \frac{5 - \sqrt{21}}{2}$$

Q-1 [5×2 marks] Choose the correct answer:

1.1: The linear system $\begin{cases} x + 2y = 4 \\ 4x + 8y = -16 \end{cases}$

a) has no solution.

b) has an infinity many solutions.

c) has a unique solution $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d) has a unique solution $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

e) None of the above

1.2: The matrix $A = \begin{bmatrix} a & 4 \\ 2 & a-2 \end{bmatrix}$is singular if

a) $a = 2$
b) $a = -4$
c) $a = 2, -4$
d) $a = -2$
e) None of the above

1.3: The vectors $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $v_2 = \begin{bmatrix} a \\ 1-a \end{bmatrix}$ are linearly dependent if

a) $a = \frac{5}{2}$
b) $a = \frac{2}{5}$
c) $a = \frac{-5}{2}$
d) $a = \frac{-2}{5}$
e) None of the above

1.4: A spanning set for $W = \left\{ \begin{bmatrix} 2a \\ 0 \end{bmatrix}; a \in \mathbb{R} \right\}$ is

a) $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
e) None of the above

1.5: If $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A^9 =$

a) $-A$
b) I_2
c) $-I_2$
d) A
e) None of the above

Q-2: [10 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: $\lambda = 0$ is an eigenvalue of a matrix A if and only if A is singular.

2.2: $W = \left\{ \begin{bmatrix} a \\ b \\ ab \end{bmatrix} ; a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

Each question is worth 10 marks. Answer the following questions:

Q-3: [5+3+2 marks]

Let $W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} ; b = 3a, \ \ and \ \ a, b \in \mathbb{R} \right\}$ be a subset of $\mathbb{R}^2$.

a) Show that W is a subspace of $\mathbb{R}^2$.
b) Find a basis for W. What is the dimension of W.
c) Is $\begin{bmatrix} 17 \\ 50 \end{bmatrix} \in W$? Explain.

Q-4: [4+3+3 marks]

Define the mapping $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ y - z \end{bmatrix}$.

a) Show that T is a linear transformation.
b) Find a matrix A such that $T(X) = AX$ for each $X \in \mathbb{R}^3$.
c) Find the null of T; $N(T)$.

Q-5: [4+6 marks]

Let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

a) Show that $\lambda_1 = -1$, and $\lambda_2 = 3$ are eigenvalues of A.
b) Find matrices P and D such that $P^{-1}AP = D$; D is a diagonal matrix.

Answer Q2-1

λ is eigenvalue → $|\lambda \ \mathrm{I}-\mathrm{A}|=0$

$|(0) \ \mathrm{I}-\mathrm{A}|=0$

$|(0) \ \ -\mathrm{A}|=0$

$|-\mathrm{A}|=0$

$(-1^{n}) \ |A|=0$

$|A|=0$ *means* A is singular → *True*.

Answer Q2-2

$Let \ a=2 \quad and \quad b=3$

$$W=\left\{\begin{bmatrix}2\\3\\6\end{bmatrix}: a, b \in \mathbb{R}\right\}$$

$$if \ Scaler=2 \ then \ 2\begin{bmatrix}2\\3\\6\end{bmatrix}=\begin{bmatrix}4\\6\\12\end{bmatrix}$$

$$\begin{bmatrix}4\\6\\12\end{bmatrix} \notin W \quad \rightarrow \quad False.$$

Answer Q3-A

$$W=\left\{\begin{bmatrix}\mathrm{a}\\3\mathrm{a}\end{bmatrix}: a \in \mathbb{R}\right\}$$

$$W=\left\{a\begin{bmatrix}1\\3\end{bmatrix}: a \in \mathbb{R}\right\}$$

$$W=span \ S \ \rightarrow \ S=\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$$

Answer Q3-B

$basis\ For\ \ W: \begin{bmatrix}1\\3\end{bmatrix}, \qquad Dim = 1.$

Answer Q3-C

$$\begin{bmatrix}1 & 17\\3 & 50\end{bmatrix} \sim \begin{bmatrix}1 & 17\\0 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 7\\5 & 0\end{bmatrix} \notin W$$

Answer Q4-A

$x, y \in \mathbb{R}^3$

$$X = \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}, \ \ Y = \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$$

$$T(rx + y) = \begin{bmatrix}rx_1 + x_2\\ry_1 + y_2\\rz_1 + z_2\end{bmatrix} = \begin{bmatrix}2rx_1 + 2x_2\\ry_1 + y_2 - rz_1 - z_2\end{bmatrix}$$

$$= \begin{bmatrix}2rx_1\\ry_1 - rz_1\end{bmatrix} + \begin{bmatrix}2x_2\\y_2 - z_2\end{bmatrix}$$

$$= rT(x) + T(Y)$$

Answer Q4-B

$$T(X) = T\left(\begin{bmatrix}x\\y\\Z\end{bmatrix}\right) = \begin{bmatrix}2x\\y - z\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\0 & 1 & -1\end{bmatrix} \times \begin{bmatrix}x\\y\\Z\end{bmatrix}, \ \ A = \begin{bmatrix}2 & 0 & 0\\0 & 1 & -1\end{bmatrix}$$

Answer Q4-C

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} 2X \\ Y - Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$let\ z = t$$

$$x = 0,\ \ y - t = 0 \quad \rightarrow \quad y = t,\ \ z = t$$

$$N(T) = \left\{ \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \ \rightarrow \ \left\{ t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

Answer Q5-A

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} = 0$$

$$\left| \begin{bmatrix} \lambda - 1 & -4 \\ -1 & \lambda - 1 \end{bmatrix} \right| = 0$$

$$(\lambda - 1)(\lambda - 1) - (-4)(-1) = 0$$

$$\lambda^2 - \lambda - \lambda + 1 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 3 \quad \text{or} \quad \lambda = -1$$

Answer Q5-B

$when\ \lambda = 3$

$$\begin{bmatrix} 3-1 & -4 & 0 \\ -1 & 3-1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$

$x - 2t = 0 \quad \rightarrow \quad x = 2t$

$S.S = \left\{ \begin{bmatrix} 2t \\ t \end{bmatrix} t \in \mathbb{R} \right\} \quad \rightarrow \quad S.S = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

$when\ \lambda = -1$

$$\begin{bmatrix} -1-1 & -4 & 0 \\ -1 & -1-1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 0 \\ -1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$

$x + 2t = 0 \quad \rightarrow \quad x = -2t$

$S.S = \left\{ \begin{bmatrix} -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$S.S = \left\{ t \begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \ P = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$

Q-1 [5×2 marks] Choose the correct answer:

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1.1: If the order of A is (4×3), the order of B is (4×5), and the order of C is (7×3), then the order of $(A^T B)^T C^T$ is:

A. 5×3

B. 4×5

C. 4×3

D. 5×7

E. None of the above

1.2: The value if m such that the vector $(m, 7, -4)$ is a linear combination of vectors $(-2,2,1)$ and $(2,1,-2)$ is:

A. -2

B. 2

C. 0

D. 3

E. None of the above

1.3: Which of the following is not subspace of $\mathbb{R}^2$:

A. $W = \left\{ \begin{bmatrix} a \\ 2a \end{bmatrix} : a \in \mathbb{R} \right\}$

B. $W = \left\{ \begin{bmatrix} a \\ -2a \end{bmatrix} : a \in \mathbb{R} \right\}$

C. $W = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} : a \in \mathbb{R} \right\}$

D. $W = \left\{ \begin{bmatrix} a \\ a^2 \end{bmatrix} : a \in \mathbb{R} \right\}$

E. None of the above

1.4: The dimension of the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} : a \in \mathbb{R} \right\}$ is:

A. 0

B. 1

C. 2

D. 3

E. None of the above

1.5: If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then $T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right)$:

A. $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

C. $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$

E. None of the above

Q-2: [10 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: If A and B are (4×4) nonsingular matrices, then $|-A^4 B| = -|A|^4|B|$.

2.2: The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} \right\}$ is linearly independent.

Each question is worth 10 marks. Answer the following questions:

Q-3: [5+3+2 marks]

Let $W = \left\{ \begin{bmatrix} x \\ 2x+y \\ -x+y \end{bmatrix} : x, y \in \mathbb{R} \right\}$ be a subset of $\mathbb{R}^3$

A. Show that W is a subspace of $\mathbb{R}^3$.
B. Find a set S such that $W = spanS$
C. Is $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in W$? Explain.

Q-4: [5+3+2 marks]

Define the mapping $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2x \end{bmatrix}$.

A. Show that T is a linear transformation.
B. Find a matrix A such that $T(X) = AX$ for each $X \in \mathbb{R}^2$
C. Describe the null space $N(T)$.

Q-5: [5+3+2 marks]

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

A. Find the eigenvalues of A.
B. If possible, find a matrix P such that $P^{-1}AP = D$, D is diagonal matrix.

Answer Q2-1

$|-A^4B| = -|A|^4|B|$

$|-A^4B| = (-1)^4|A^4||B|$

$|A|^4|B|$

Therefore, the statement is false.

Answer Q2-2

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix} = 1 \neq 0$$

$\longrightarrow$ The vectors are linearly independent.

Answer Q3-A+B

$$W = \left\{ \begin{bmatrix} x \\ 2x+y \\ -x+y \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ 2x \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$W = \left\{ x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} = Span\ S \quad \longrightarrow \quad S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Answer Q3-C

is $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$ is linear combinations of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

$$\xrightarrow{-2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ -1 & 1 & -5 \end{bmatrix} \xrightarrow{-1\,R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \therefore\ C_1 = 2,\ C_2 = -3$$

Answer Q4-A

$$T(rX+Y) = rT(X) + T(Y)$$

$$Let\ X = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \qquad Y = \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} \qquad r \in \mathbb{R}$$

$$T(rX+Y) = T\left(r\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} + \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} rX_1 + X_2 \\ rY_1 + Y_2 \end{bmatrix}\right) = \begin{bmatrix} rX_1 + X_2 + rY_1 + Y_2 \\ rX_1 + X_2 - rY_1 - Y_2 \\ 2\,rX_1 + 2X_2 \end{bmatrix}$$

$$= \begin{bmatrix} r(X_1+Y_1) + X_2 + Y_2 \\ r(X_1 - Y_1) + X_2 - Y_2 \\ r(2X_1) + 2X_2 \end{bmatrix} = \begin{bmatrix} r(X_1+Y_1) \\ r(X_1-Y_1) \\ r(2X_1) \end{bmatrix} + \begin{bmatrix} X_2+Y_2 \\ X_2-Y_2 \\ 2X_2 \end{bmatrix} = r\begin{bmatrix} X_1+Y_1 \\ X_1-Y_1 \\ 2X_1 \end{bmatrix} + \begin{bmatrix} X_2+Y_2 \\ X_2-Y_2 \\ 2X_2 \end{bmatrix}$$

$$rT\left(\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}\right) + \left(\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}\right) = rT(X) + T(Y)$$

Answer Q4-B

$$T(X) = T\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} X+Y \\ X-Y \\ 2X \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X+Y \\ X-Y \\ 2X \end{bmatrix}$$

Answer Q4-C

$$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : \begin{bmatrix} X+Y \\ X-Y \\ 2X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

$$\xrightarrow{-2\,R_1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{2\,R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ X = 0, \qquad Y = 0 \rightarrow N(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

Answer Q5-A

$$|\lambda\ \mathrm{I} - \mathrm{A}| = 0$$

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 1)\,(\lambda - 1) - 4 = 0$$

$$\lambda^2 - \lambda - \lambda + 1 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda_1 = 3\,, \qquad \lambda_2 = -1$$

Answer Q5-B

$$When\ \lambda = 3 \qquad \begin{bmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Let\ y = t \qquad x - y = 0 \quad \longrightarrow \quad x = y = t$$

$$S.S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \ \mathbb{R} \right\}$$

$$When\ \lambda = -1 \qquad \begin{bmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Let\ y = t \qquad x + y = 0 \quad \longrightarrow \quad x = -y = -t$$

$$S.S = \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} -1 \\ 1 \end{bmatrix} : t \in \ \mathbb{R} \right\}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad\qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Q-1 [20×5 marks] Choose the correct answer:

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1: The linear system corresponding to the augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & 12 \end{array}\right)$

a) Has infinitely many solutions
b) Has a unique solution
c) Has two solutions
d) Is inconsistent.
e) None of the above

2: If A is a 3×3 skew symmetric matrix, then $|A| =$

a) 0
b) 3
c) 9
d) −3
e) None of the above

3: The inverse of the nonsingular matrix $2A^{-1}$ is

a) $2A$
b) $2A^{-1}$
c) $0.5A$
d) 20.5
e) None of the above

4: If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $(A^2)^T =$

a) A
b) I_2
c) A^T
d) $-I_2$
e) None of the above

5: The matrix $\begin{bmatrix} x & 1 \\ 1 & x \end{bmatrix}$ is singular if

a) $x = 0$
b) $x = 2$
c) $x = -1$
d) $x = -2$
f) None of the above

6: If A is a 3×3 nonsingular matrix, then $|2A^T A^{-1}| =$

a) 1
b) 6
c) 9
d) 8
e) None of the above

7: The Solution set of the linear system $\begin{cases} x - 2y = 0 \\ 2x - 4y = 0 \end{cases}$ is

a) $\left\{ \begin{bmatrix} t \\ 2t \end{bmatrix} | t \in \mathbb{R} \right\}$
b) $\left\{ \begin{bmatrix} 2t \\ t \end{bmatrix} | t \in \mathbb{R} \right\}$
c) $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} | t \in \mathbb{R} \right\}$
d) $\left\{ \begin{bmatrix} 2t \\ 2t \end{bmatrix} | t \in \mathbb{R} \right\}$
e) None of the above

8: The columns of the matrix $\begin{bmatrix} 1 & a \\ 2 & a-1 \end{bmatrix}$ are dependent if $a =$

a) 2
b) 1
c) 0
d) -1
e) None of the above

9: The vector _____ could be a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
b) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
c) $\begin{bmatrix} 1 \\ 0 \\ \frac{3}{2} \end{bmatrix}$
d) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$
e) None of the above

10: Which of the following is not a subspace of $\mathbb{R}^2$:

a) $W = \left\{ \begin{bmatrix} a \\ 2a \end{bmatrix} : a \in \mathbb{R} \right\}$
b) $W = \left\{ \begin{bmatrix} a \\ -2a \end{bmatrix} : a \in \mathbb{R} \right\}$
c) $W = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} : a \in \mathbb{R} \right\}$
d) $W = \left\{ \begin{bmatrix} a \\ a^2 \end{bmatrix} : a \in \mathbb{R} \right\}$
e) None of the above

11: A spanning set for $W = \left\{\begin{bmatrix}2a\\4a\end{bmatrix} : a \in \mathbb{R}\right\}$ is

a) $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$
b) $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$
c) $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$
d) $\left\{\begin{bmatrix}1\\4\end{bmatrix}\right\}$
e) None of the above

12: If $W = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$, then $W =$

a) $\left\{\begin{bmatrix}a\\a\end{bmatrix} | a \in \mathbb{R}\right\}$
b) $\left\{\begin{bmatrix}a+2b\\a\end{bmatrix} | a, b \in \mathbb{R}\right\}$
c) $\left\{\begin{bmatrix}a\\a+2b\end{bmatrix} | a \in \mathbb{R}\right\}$
d) $\left\{\begin{bmatrix}2a-b\\2b\end{bmatrix} | a \in \mathbb{R}\right\}$
e) None of the above

13: The dimension of the subspace $W = \left\{\begin{bmatrix}a\\2a\\3a\end{bmatrix} : a \in \mathbb{R}\right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

14: The dimension of $O = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

15: Two vectors $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\r\end{bmatrix}$ does not form a basis for $\mathbb{R}^2$ if

a) $r = 6$
b) $r = 5$
c) $r = 4$
d) $r = 3$
e) None of the above

16: If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ b-a \end{bmatrix}$, then $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
b) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
c) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$
d) $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$
e) None of the above

17: If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 6 \\ 9 \end{bmatrix}$
b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
c) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$
d) $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$
e) None of the above

18: If $S, T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are linear transformation defined by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}$, then $(S+T)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
d) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
e) None of the above

19: The eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ are

a) $0, 2$
b) $1, 2$
c) $-1, 1$
d) $-1, 2$
e) None of the above

20: An eigenvalue of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is

a) 0
b) 1
c) 2
d) 4
e) None of the above

Q-1: [5×2 marks] Choose the correct answer:

1.1: If A, B and C are nonsingular matrices, then $(ABC)^{-1} =$

a) $(AB)^{-1}C^{-1}$
b) $A^{-1}(BC)^{-1}$
c) $A^{-1}B^{-1}C^{-1}$
d) $C^{-1}B^{-1}A^{-1}$
e) None of the above

1.2: The columns of the matrix $\begin{bmatrix} 2 & 1 \\ 6 & a \end{bmatrix}$ are linearly dependent if $a =$

a) 2
b) 3
c) 6
d) 12
e) None of the above

1.3: The dimension of the subspace $W = \left\{ \begin{bmatrix} a-b \\ b-2a \end{bmatrix} | a, b \in \boldsymbol{R} \right\}$ is

a) 1
b) 2
c) 3
d) 4
e) None of the above

1.4: ——— is a basis for the vector space $\boldsymbol{R}^2$.

a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
e) None of the above

1.5: Let $T: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ a+b \end{bmatrix}$. The matrix A such that $T(X) = AX, X \in \boldsymbol{R}^2$ is

a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
c) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
e) None of the above

Q-2: [2×5 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: If v_1 and v_2 are distinct vectors in R^n and A is an $n \times n$ matrix such that $Av_1 = Av_2$, then A is invertible.

2.2: If $S, T: R^2 \longrightarrow R^2$ are linear transformation defined by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y-x \\ y \end{bmatrix}$, then $(S+T)\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Each question is worth 20 marks. Answer the following questions:

Q–3: Let $W = \left\{ \begin{bmatrix} x \\ x+y \\ x+2y \end{bmatrix} : x, y \in \boldsymbol{R} \right\}$ be a subset of $\boldsymbol{R}^3$.

a) **[8 marks]** Show that W is a subspace of $\boldsymbol{R}^3$.

b) **[4 marks]** Find a set S such that $W = \text{span}\, S$.

c) **[8 marks]** Is $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in W$? Explain.

Q–4: Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_4 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

a) **[8 marks]** Are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ linearly independent? Justify your answer.

b) **[4 marks]** Is $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$? Justify your answer.

c) **[4marks]** Is $\boldsymbol{v}_4$ a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$? Justify your answer.

d) **[4 marks]** Find a subset of $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ that forms a basis for span S.

Q–5: Suppose $T: \boldsymbol{R}^3 \longrightarrow \boldsymbol{R}^3$ is a linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ z \end{bmatrix}.$$

a) **[12 marks]** Show that T is a linear transformation.

b) **[8 marks]** Find the null space $N(T)$.

Q–6: Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.

a) **[10 marks]** Show that A has eigenvalues $\lambda = 0, -1$.

b) **[10 marks]** Find an eigenvector corresponding to $\lambda = 0$.

End of Questions

Answer Q2-A

$v_1 \neq v_2$

$Av_1 = Av_2$

Suppose that A is invertible:

$Av_1 = Av_2$

$A^{-1}Av_1 = A^{-1}Av_2$

$v_1 = v_2$

Therefore, the statement is False.

Answer Q2-B

$S\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$

$\begin{bmatrix}1\\(1+2)\end{bmatrix} + \begin{bmatrix}(2-1)\\2\end{bmatrix}$

$\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$

Therefore, the statement is True.

Answer Q3-A

$$W = \left\{\begin{bmatrix}x\\x+y\\x+2y\end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix}x\\x\\x\end{bmatrix} + \begin{bmatrix}0\\y\\2y\end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$= \left\{x\begin{bmatrix}1\\1\\1\end{bmatrix} + y\begin{bmatrix}0\\1\\2\end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$W = span\ S$

$S = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$, So, W is a subspace

Answer Q3-B

$$span\ S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Answer Q3-C

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 = 3, \quad C_2 = -2$

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$yes, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \in \mathbb{R}$$

Answer Q4-A

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then, v_1, v_2 and v_3 are linearly independent.

Answer Q4-B

no, it is not a linear combination since the matrix has no solution.

Answer Q4-C

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ -1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 5 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & \frac{9}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{-6}{5} \\ 0 & 0 & 1 & \frac{9}{5} \end{bmatrix}$$

$C_1 = \frac{2}{5}, \quad C_2 = \frac{-6}{5}, \quad C_3 = \frac{9}{5}$

$\frac{2}{5} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{-6}{5} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{9}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ So, v_4 is a linear combinatin of v_1, v_2 and v_3

Answer Q4-D

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Answer Q5-A

$x, y \ \in \mathbb{R}^3$

$$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad Y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$T(rx + y) = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ \ + y_2 \\ rz_1 + z_2 \end{bmatrix} = \begin{bmatrix} rx_1 \ + x_2 \\ rx_1 \ + x_2 + \ ry_1 \ \ + y_2 \\ rz_1 + z_2 \end{bmatrix}$$

$$= \begin{bmatrix} rx_1 \\ rx_1 \ + \ ry_1 \\ rz_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \ + \ y_2 \\ z_2 \end{bmatrix}$$

$$= \ rT(x) \ + \ T(Y)$$

Answer Q5-B

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ X + Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$x = 0, \quad y = 0, \quad z = 0$$

$$N(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer Q6-A

$$|\lambda I - A| = 0$$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 1 & \lambda \end{bmatrix}$$

$$\lambda + 1 \begin{vmatrix} \lambda+1 & 0 \\ 1 & \lambda \end{vmatrix}$$

$$\lambda + 1(\lambda^2 + \lambda) = 0$$

$$\lambda^3 + \lambda^2 + \ \lambda^2 + \lambda = 0$$

$$\lambda^3 + 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 + 2\lambda + 1) = 0$$

$$\lambda = 0 \quad \text{or} \quad \lambda = -1$$

Answer Q6-B

$$when\ \lambda = 0$$

$$\begin{bmatrix} 0+1 & 0 & 0 \\ 0 & 0+1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ z = t$$

$$S.S = \left\{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} ; t \in \mathbb{R} \right\}$$

$$S.S = \left\{ \mathrm{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; \mathrm{t} \in \mathbb{R} \right\}$$

Q-1: [5×2 marks] Choose the correct answer: Fall 19-20 الفصل الأول

1.1: If A is a 2×2 nonsingular matrix, then $|-3A^T| =$

a) $3|A|$
b) $-3|A|$
c) $9|A|$
d) $-9|A|$
e) None of the above

1.2: An eigenvalue of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is

a) 0
b) 1
c) 2
d) 6
e) None of the above

1.3: A spanning set for $W = \left\{ \begin{bmatrix} 2a \\ 4a \end{bmatrix} : a \in \boldsymbol{R} \right\}$ is

a) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$
e) None of the above

1.4: The dimension of the subspace $W = \left\{ \begin{bmatrix} x \\ -x \\ 2x \end{bmatrix} : x \in \boldsymbol{R} \right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

1.5: If $T: \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^2$ is a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ b-a \end{bmatrix}$, then $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
b) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
c) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$
d) $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$
e) None of the above

Q-2: [2×5 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: $W = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} : a, b \in \mathbf{R} \right\}$ is a subspace of $\boldsymbol{R}^3$.

2.2: The vectors $\begin{bmatrix} a^2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent for all values of a.

Each question is worth 20 marks. Answer the following questions:

Q–3: Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -4 \\ 4 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_4 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 3 \end{bmatrix}$ be vectors in $\boldsymbol{R}^3$ and A be the matrix whose columns are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{v}_4$.

a) **[10 marks]** Find the matrix B in reduced row echelon form that is row equivalent to A.

b) **[5 marks]** Which vectors in $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ form a basis for the subspace $W = Span\ S$.

c) **[5 marks]** Find $|A|$.

Q–4: Let $W = \left\{ \begin{bmatrix} x \\ 2x+y \\ -x+y \end{bmatrix} : x, y \in \boldsymbol{R} \right\}$ be a subset of $\boldsymbol{R}^3$.

a) **[8 marks]** Show that W is a subspace of $\boldsymbol{R}^3$.

b) **[4 marks]** Find a set S such that $W = \text{span } S$.

c) **[8 marks]** Is $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in W$? Explain.

Q–5: Suppose $T: \boldsymbol{R}^3 \longrightarrow \boldsymbol{R}^3$ is a mapping defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ z \\ x+z \end{bmatrix}.$$

a) **[12 marks]** Show that T is a linear transformation.

b) **[8 marks]** Find the null space $N(T)$.

Q–6: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$.

a) **[8 marks]** Show that $\lambda = 1$ and $\lambda = 4$ are eigenvalues of A.

b) **[12 marks]** Find a matrix P such that $P^{-1}AP = D$, D is a diagonal matrix.

End of Questions

Answer Q2-A

$$\begin{bmatrix} 0 \\ 0 \\ 0+0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{so, it is not a subspace}$$

Answer Q2-B

$$\begin{vmatrix} a^2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= -0 + 1 \begin{vmatrix} a^2 & 1 \\ 1 & 1 \end{vmatrix} - 0$$

$$= \begin{vmatrix} a^2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$a^2 - 1 = 0$$

$$a^2 = 1$$

$$a = \pm 1$$

False, when $a = 1$ it is dependent.

Answer Q3-A

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 1 & 2 \\ 2 & -4 & 0 & -2 \\ -2 & 4 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer Q3-B

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Answer Q3-C

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 1 & 2 \\ 2 & -4 & 0 & -2 \\ -2 & 4 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Det(A) = 1(0)(0)(0) = 0$$

Answer Q4-A

$$W = \left\{ \begin{bmatrix} x \\ 2x + y \\ -x + y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ 2\ x \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$W = span\ S$$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{So, W is a subspace}$$

Answer Q4-B

$$span\ S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Answer Q4-C

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ -1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_1 = 2, \quad C_2 = -3$$

$$2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$$

$$so, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in W$$

Answer Q5-A

$x, y \ \in \mathbb{R}^3$

$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$T(rx + y) = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ \ + y_2 \\ rz_1 + z_2 \end{bmatrix} = \begin{bmatrix} rx_1 \ + x_2 \\ rz_1 + z_2 \\ rx_1 \ + x_2 + \ rz_1 + z_2 \end{bmatrix}$

$= \begin{bmatrix} rx_1 \\ rz_1 \\ rx_1 \ + rz_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ z_2 \\ x_2 + z_2 \end{bmatrix}$

$= \ rT(x) \ + \ T(Y)$

Answer Q5-B

$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ Z \\ X + Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$x = 0, \quad y = t, \quad z = 0$

$N(T) = \left\{ \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$

$N(T) = \left\{ t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$

Answer Q6-A

$$|\lambda I - A| = 0$$

$$\left|\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}\right| = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = 0$$

$$\left|\begin{bmatrix} \lambda - 1 & 0 \\ -2 & \lambda - 4 \end{bmatrix}\right| = 0$$

$$(\lambda - 1)(\lambda - 4) - 0(-2) = \lambda^2 - 4\lambda - \lambda + 4$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = 4$$

Answer Q6-B

$$when\ \lambda = 1$$

$$\begin{bmatrix} 1 - 1 & 0 \\ -2 & 1 - 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$

$$x + \frac{3}{2}t = 0$$

$$x = -\frac{3}{2}t$$

$$S.S = \begin{bmatrix} -\frac{3}{2}t \\ t \end{bmatrix} : t \in \mathbb{R}$$

$$= \left\{ t \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$when\ \lambda = 4$

$$\begin{bmatrix} 4-1 & 0 \\ -2 & 4-4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y\ = t$

$x = 0$

$S.S = \begin{bmatrix} 0 \\ t \end{bmatrix} : t \in \mathbb{R} \ \rightarrow \ t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \in \mathbb{R}$

$D\ = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad P\ = \begin{bmatrix} -\frac{3}{2} & 0 \\ 1 & 1 \end{bmatrix}$

Q–1: [5×2 marks] Choose the correct answer:

1.1: The matrix $\begin{bmatrix} 2a & 8 \\ 1 & 2 \end{bmatrix}$ is singular if

a) $a = 0$
b) $a = 2$
c) $a = -2$
d) $a = 4$
e) None of the above

1.2: The rows of the matrix $\begin{bmatrix} a & 1 \\ 6 & 3 \end{bmatrix}$ are linearly dependent if

a) $a = 2$
b) $a = 3$
c) $a = 6$
d) $a = 12$
e) None of the above

1.3: The dimension of the subspace $W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}\right\}$ is

a) 3
b) 3
c) 1
d) 0
e) None of the above

1.4: Two vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ c \end{bmatrix}$ form a basis for $\mathbf{R}^2$ if

a) $c = 2$
b) $c = 3$
c) $c = 4$
d) $c = 6$
e) None of the above

1.5: If $T: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then

$T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 6 \\ 9 \end{bmatrix}$
b) $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$
c) $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$
d) $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$
e) None of the above

Q–2: [2×5 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: If $W = \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_k\}$, then $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_k\}$ is a basis for W.

2.2: If $S, T: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ are linear transformation defined by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}$, then $(S+T)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Each question is worth 20 marks. Answer the following questions:

Q–3: Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

a) **[10 marks]** Compute, if possible, $AB + AC$ and $|B + C|$.
b) **[5 marks]** Find the matrix X such that $XC = B$.
c) **[5 marks]** Find one non-zero vector Y such that $BY = O$.

Q–4: Let $W = \text{span}\, S$, $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

a) **[10 marks]** Determine whether or not S is linearly independent.
b) **[6 marks]** Is $\boldsymbol{v}_1$ a linear combination of $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$? Explain.
c) **[4 marks]** Find a subset of S that forms a basis for W.

Q–5: Suppose $T: \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ is a mapping defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}.$$

a) **[12 marks]** Show that T is a linear transformation.
b) **[8 marks]** Find the null space $N(T)$.

Q–6: Let $A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$.

a) **[10 marks]** Show that A has eigenvalues $\lambda = 0, -1, -2$.
b) **[10 marks]** Find an eigenvector corresponding to $\lambda = 0$.

End of Questions

Answer Q2-A

for example: $W_1 = \begin{bmatrix}0\\0\end{bmatrix}, W_2 = \begin{bmatrix}2\\1\end{bmatrix}, W_3 = \begin{bmatrix}3\\4\end{bmatrix}$

$$\begin{bmatrix}0 & 2 & 3\\0 & 1 & 4\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 4\\0 & 2 & 3\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 4\\0 & 0 & -5\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 4\\0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

W_2 and W_3 are linearly independent wheres W_1 is linearly dependent,

so span $\{ W_1, W_2\ W_3\} =$ basis for $\{ W_2, W_3\}$, then the statment is False.

Answer Q2-B

$$S\left(\begin{bmatrix}2(1)\\3(1)\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$\left(\begin{bmatrix}2\\3\end{bmatrix}\right) + \left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \left(\begin{bmatrix}3\\2\end{bmatrix}\right)$, True.

Answer Q3-A

$$AB = \begin{bmatrix}1 & -1\\0 & 3\\2 & -2\end{bmatrix} \times \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} = \begin{bmatrix}-1 & -1\\6 & 6\\-2 & -2\end{bmatrix}$$

$$AC = \begin{bmatrix}1 & -1\\0 & 3\\2 & -2\end{bmatrix} \times \begin{bmatrix}1 & 2\\2 & 5\end{bmatrix} = \begin{bmatrix}-1 & -3\\6 & 15\\-2 & -6\end{bmatrix}$$

$$AB + AC = \begin{bmatrix}-1 & -1\\6 & 6\\-2 & -2\end{bmatrix} + \begin{bmatrix}-1 & -3\\6 & 15\\-2 & -6\end{bmatrix} = \begin{bmatrix}-2 & -4\\12 & 21\\-4 & -8\end{bmatrix}$$

$$B + C = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} + \begin{bmatrix}1 & 2\\2 & 5\end{bmatrix} = \begin{bmatrix}2 & 3\\4 & 7\end{bmatrix}$$

$$|B + C| = \left|\begin{bmatrix}2 & 3\\4 & 7\end{bmatrix}\right|$$

$$(2)(7) - (4)(3) = 14 - 12$$

$$|B + C| = 2$$

Answer Q3-B

$$XC = B$$

$$XCC^{-1} = BC^{-1}$$

$$X = BC^{-1}$$

$$C^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$$

Answer Q3-C

$$BY = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$

$$x + y = 0$$

$$x = -t$$

$$S.S = \{(-t, t) : t \in \mathbb{R}\}$$

$$let\ t = 1$$

$(-1,1)$ is a non-zero solution.

Note: you can choose any value for t.

Answer Q4-A

$$\begin{bmatrix} -1 & 2 & 1 \\ 5 & -1 & -2 \\ -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 9 & 3 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -1 \\ 0 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

S is not linearly independent.

Answer Q4-B

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & -2 & 5 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -3 & 9 \\ 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 = 1, \quad C_2 = -3$

$$1\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}, \text{True.}$$

Answer Q4-C

$$basis\ for\ S = \left\{ \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Answer Q5-A

$x, y \ \in R^3$

$$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$T(rx + y) = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ \ + y_2 \\ rz_1 + z_2 \end{bmatrix} = \begin{bmatrix} rx_1 \ + x_2 \\ 0 \\ rz_1 + z_2 \end{bmatrix}$$

$$= \begin{bmatrix} rx_1 \\ 0 \\ rz_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix}$$

$= \ rT(x) \ + \ T(Y)$

Answer Q5-B

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ 0 \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rightarrow \quad x = 0, \quad y = t, \quad z = 0$$

$$N(T) = \left\{ \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} \quad \rightarrow \quad N(T) = \left\{ t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

Answer Q6-A

$|\lambda I - A| = 0$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+1 & 0 & 1 \\ 0 & \lambda+1 & -1 \\ 1 & 0 & \lambda+1 \end{bmatrix}$$

$$\lambda + 1 \begin{vmatrix} \lambda+1 & -1 \\ 0 & \lambda+1 \end{vmatrix} - 0 + 1 \begin{vmatrix} 0 & \lambda+1 \\ 1 & 0 \end{vmatrix}$$

$\lambda + 1(\lambda^2 + \lambda + \lambda + 1) + 1(0 - \lambda - 1) = 0$

$\lambda^3 + \lambda^2 + \lambda^2 + \lambda + \lambda^2 + \lambda + \lambda + 1 - \lambda - 1 = 0$

$\lambda^3 + 3\lambda^2 + 2\lambda = 0$

$\lambda(\lambda^2 + 3\lambda + 2) = 0$

$\lambda = 0 \quad \text{or} \quad \lambda = -1 \quad \text{or} \quad \lambda = -2$

Answer Q6-B

$when\ \lambda = 0$

$$\begin{bmatrix} 0+1 & 0 & 1 & 0 \\ 0 & 0+1 & -1 & 0 \\ 1 & 0 & 0+1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$let\ z = t$

$x + t = 0 \quad \rightarrow \quad x = -t$

$y - t = 0 \quad so, \quad y = t$

$$S.S = \left\{ \begin{bmatrix} -t \\ t \\ t \end{bmatrix}; t \in \mathbb{R} \right\} \quad \rightarrow \quad S.S = \left\{ \text{t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; \text{t} \in \mathbb{R} \right\}$$

Q–1: [5×2 marks] Choose the correct answer:

1.1: The inverse of the matrix $\begin{bmatrix}1 & -3\\1 & -2\end{bmatrix}$ is

a) $\begin{bmatrix}-1 & 1\\3 & 2\end{bmatrix}$
b) $\begin{bmatrix}-2 & 3\\-1 & 1\end{bmatrix}$
c) $\begin{bmatrix}1 & -3\\1 & -2\end{bmatrix}$
d) $\begin{bmatrix}1 & 1\\-3 & -2\end{bmatrix}$
e) None of the above

1.2: The rows of the matrix $\begin{bmatrix}a & 1\\6 & 3\end{bmatrix}$ are linearly dependent if $a =$

a) 2
b) 3
c) 6
d) 12
e) None of the above

1.3: ---- is a spanning set for the subspace $W = \left\{\begin{bmatrix}a\\2a\end{bmatrix} : a \in \mathbf{R}\right\}$.

a) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$
b) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$
c) $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$
d) $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$
e) None of the above

1.4: The dimension of $W = \left\{\begin{bmatrix}x\\2x\\3x\end{bmatrix} : x \in \mathbf{R}\right\}$ is

a) 3
b) 2
c) 1
d) 0
e) None of the above

1.5: Let $T:\mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$. The matrix A such that $T(X) = AX, X \in \boldsymbol{R}^2$ is

a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

e) None of the above

Q–2: [2×5 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: If A is an $n \times n$ nonsingular matrix, then AA^T is also nonsingular matrix.

2.2: The set $\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -10 \\ 30 \end{bmatrix}\right\}$ is a basis for $\boldsymbol{R}^2$.

Each question is worth 20 marks. Answer the following questions:

Q–3: Let $S = \left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ be a set of vectors in $\boldsymbol{R}^3$.

a) [8 marks] Determine whether the vectors in S are linearly independent.

b) [12 marks] Write, if possible, the vector $\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ as a linear combination of the vectors in S.

Q–4: Let $W = \left\{\begin{bmatrix} x \\ 2x+y \\ -x+y \end{bmatrix} : x, y \in \boldsymbol{R}\right\}$ be a subset of $\boldsymbol{R}^3$.

a) [8 marks] Show that W is a subspace of $\boldsymbol{R}^3$.

b) [4 marks] Find a set S such that $W = \text{span } S$.

c) [8 marks] Is $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in W$? Explain.

Q-5: Define the mapping $T: \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$.

a) **[4 marks]** Find $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

b) **[10 marks]** Show that T is a linear transformation.

c) **[6 marks]** Find the null space $N(T)$.

Q-6: Let $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

a) **[8 marks]** Show that $\lambda = 8$ and $\lambda = -2$ are eigenvalues of A.

b) **[12 marks]** If possible, find a matrix P such that $P^{-1}AP = D$, D is a diagonal matrix.

End of Questions

Answer Q2-A

for example $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$Det(A) = (1)(2) - (0)(0) = 2$

since the $Det(A) \neq 0$, so, it is nonsingular.

$|AA^T| = |A||A^T| = |A||A| = (2)(2) = 4$, since the $Det(A) \neq 0$, so, it is also nonsingular.

$True$.

Answer Q2-B

the second vector is linearly dependent, False.

Answer Q3-A

$\begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 3 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, linearly independent.

Answer Q3-B

$$\begin{bmatrix}-1 & 1 & 1 & 4\\ 2 & 0 & 1 & 0\\ 1 & 1 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -1 & -4\\ 0 & 2 & 3 & 8\\ 0 & 2 & 2 & 6\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -1 & -4\\ 0 & 2 & 2 & 6\\ 0 & 2 & 3 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 1 & 3\\ 0 & 0 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$C_1 = -1, \quad C_2 = 1, \quad C_3 = 2$ True.

Answer Q4-A

$$W = \left\{ \begin{bmatrix} x \\ 2x + y \\ -x + y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ 2\ x \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \ y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$W = span\ S$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So, W is a subspace

Answer Q4-B

$$span\ S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Answer Q4-C

$$\begin{bmatrix}1 & 0 & 2\\ 2 & 1 & 1\\ -1 & 1 & -5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -3\\ 0 & 1 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -3\\ 0 & 0 & 0\end{bmatrix}$$

$C_1 = 2, \quad C_2 = -3$

$$2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$$

$$so, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in \ W$$

Answer Q5-A

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \left(\begin{bmatrix} 1-1 \\ 1+1 \end{bmatrix}\right) = \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

Answer Q5-B

$x, y \ \in \mathbb{R}^2$

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$T(rx + y) = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ \ + y_2 \end{bmatrix} = \begin{bmatrix} rx_1 \ + x_2 + ry_1 \ \ + y_2 \\ rx_1 \ + x_2 - ry_1 \ \ - y_2 \end{bmatrix}$

$= \begin{bmatrix} rx_1 \ + ry_1 \\ rx_1 \ - \ ry_1 \end{bmatrix} + \begin{bmatrix} x_2 \ + y_2 \\ x_2 \ - y_2 \end{bmatrix}$

$= \ rT(x) \ + \ T(Y)$

Answer Q5-C

$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X+Y \\ X-Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$x = 0, \quad y = 0$

$N(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Answer Q6-A

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} = 0$$

$$\left|\begin{bmatrix} \lambda - 7 & -3 \\ -3 & \lambda + 1 \end{bmatrix}\right| = 0$$

$$(\lambda - 7)(\lambda + 1) - (-3)(-3) = 0$$
$$\lambda^2 + \lambda - 7\lambda - 7 - 9 = 0$$
$$\lambda^2 - 6\lambda - 16 = 0$$
$$(\lambda - 8)(\lambda + 2) = 0$$
$$\lambda = 8 \quad \text{or} \quad \lambda = -2$$

Answer Q6-B

$$when\ \lambda = 8$$

$$\begin{bmatrix} 8-7 & -3 & 0 \\ -3 & 8+1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$
$$x - 3t = 0$$
$$x = 3t$$
$$S.S = \left\{ \begin{bmatrix} 3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$S.S = \left\{ t \begin{bmatrix} 3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$when\ \lambda = -2$$

$$\begin{bmatrix} -2-7 & -3 & 0 \\ -3 & -2+1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & -3 & 0 \\ -3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & {}^1\!/_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$
$$x + {}^1\!/_3\, t = 0$$
$$x = -{}^1\!/_3\, t$$
$$S.S = \left\{ \begin{bmatrix} -{}^1\!/_3\, t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$
$$S.S = \left\{ t \begin{bmatrix} -{}^1\!/_3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$D = \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & -{}^1\!/_3 \\ 1 & 1 \end{bmatrix}$$

Q-1: [5×2 marks] Choose the correct answer:

1.1: If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A^{100} =$

a) A
b) I_2
c) $-A$
d) $-I_2$
e) None of the above

1.2: The rows of the matrix $\begin{bmatrix} 1 & 2 \\ a & a-1 \end{bmatrix}$ are linearly dependent if $a =$

a) -1
b) -2
c) 1
d) 2
e) None of the above

1.3: A spanning set for $W = \left\{ \begin{bmatrix} a \\ 2b \end{bmatrix} : a, b \in \boldsymbol{R} \right\}$ is

a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$
e) None of the above

1.4: The dimension of the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \\ 3b \end{bmatrix} : a, b \in \boldsymbol{R} \right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

1.5: If $T: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ 2a+3b \end{bmatrix}$, then $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$

e) None of the above

Q-2: [2×5 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: For the nonsingular matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+c & -a \\ b+d & -b \end{vmatrix}$.

2.2: The set $\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \end{bmatrix}\right\}$ is a basis for $\mathbf{R}^2$.

Each question is worth 20 marks. Answer the following questions:

Q-3: Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -4 & 2 \end{bmatrix}$.

a) **[12 marks]** Find A^{-1}.

b) **[8 marks]** Find the matrix C such that $AC = B^T$.

Q-4: Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$, $W = Span\ S$, where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_4 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$.

a) **[8 marks]** Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}$ is linearly independent.

b) **[8 marks]** Write $\boldsymbol{v}_3$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_4$.

c) **[4 marks]** Find a subset of S that form a basis for W.

Q–5: Suppose that $T: R^2 \longrightarrow R^3$ is a linear operator defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x-y \end{bmatrix}.$$

a) **[10 marks]** Show that T is a linear transformation.
b) **[6 marks]** Show that the null space $N(T)$ is trivial.
c) **[4 marks]** What is the range $R(T)$?

Q–6: Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

a) **[8 marks]** Find the eigenvalues of A.
b) **[12 marks]** If possible, find a matrix P such that $P^{-1}AP = D$, D is a diagonal matrix.

End of Questions

Answer Q2-A

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

for example the matrix A: $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$

$Det(A) = (1)(2) - (0)(0) = 2$

$\begin{vmatrix} a+c & -a \\ b+d & -b \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$

$\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = (1)(0) - (-1)(2) = 2$

so, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+c & -a \\ b+d & -b \end{vmatrix}$, True

Answer Q2-B

the second vector is linearly dependent, False.

Answer Q3-A

$$\begin{bmatrix}1&1&0&1&0&0\\0&1&1&0&1&0\\2&1&0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&1&0&1&0&0\\0&1&1&0&1&0\\0&-1&0&-2&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&-1&1&-1&0\\0&1&1&0&1&0\\0&0&1&-2&1&1\end{bmatrix}\sim$$

$$\begin{bmatrix}1&0&0&-1&0&1\\0&1&1&0&1&0\\0&0&1&-2&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1&0&1\\0&1&0&2&0&-1\\0&0&1&-2&1&1\end{bmatrix},A^{-1}=\begin{bmatrix}-1&0&1\\2&0&-1\\-2&1&1\end{bmatrix}$$

Answer Q3-B

$AC = B^T$

$A^{-1}AC = A^{-1}B^T$

$C = A^{-1}B^T$

$$B^T = \begin{bmatrix}1&-1\\3&-4\\0&2\end{bmatrix}$$

$$C = \begin{bmatrix}-1&0&1\\2&0&-1\\-2&1&1\end{bmatrix}\times\begin{bmatrix}1&-1\\3&-4\\0&2\end{bmatrix}=\begin{bmatrix}-1&3\\2&-4\\1&0\end{bmatrix}$$

Answer Q4-A

$$\begin{bmatrix}1&2&1\\-2&-1&-2\\1&1&-2\end{bmatrix}\sim\begin{bmatrix}1&2&1\\0&3&0\\0&-1&-3\end{bmatrix}\sim\begin{bmatrix}1&0&1\\0&1&0\\0&0&-3\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Answer Q4-B

$$\begin{bmatrix}1&2&1&-2\\-2&-1&-2&-2\\1&1&-2&0\end{bmatrix}\sim\begin{bmatrix}1&2&1&-2\\0&3&0&-6\\0&-1&-3&2\end{bmatrix}\sim\begin{bmatrix}1&0&1&2\\0&1&0&-2\\0&0&-3&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&0&-2\\0&0&1&0\end{bmatrix}$$

$C_1 = 2, \quad C_2 = -2$

$X_3 = 2X_1 - 2X_2$

Answer Q4-C

basais for $W = \begin{bmatrix}1\\-2\\1\end{bmatrix},\begin{bmatrix}2\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-2\\-2\end{bmatrix}$

Answer Q5-A

$x, y \ \in \mathbb{R}^2$

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$T(rx + y) = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ + y_2 \end{bmatrix} = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ + y_2 \\ rx_1 \ + x_2 - ry_1 \ - y_2 \end{bmatrix}$

$= \begin{bmatrix} rx_1 \\ ry_1 \\ rx_1 \ - ry_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ x_2 - y_2 \end{bmatrix}$

$= \ rT(x) \ + \ T(Y)$

Answer Q5-B

$N(T) = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$N(T) = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} x \\ y \\ \mathrm{x - y} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x = 0, \quad y = 0$

$N(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad N(T) = \{[0]\}$

Answer Q5-C

$R(T) = span \left\{ T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$R(T) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

$basis\ for\ R(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

Answer Q6-A

$|\lambda I - A| = 0$

$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$

$\left| \begin{bmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{bmatrix} \right| = 0$

$(\lambda - 1)(\lambda - 4) - (-1)(2) = 0$
$\lambda^2 - 4\lambda - \lambda + 4 + 2 = 0$
$\lambda^2 - 5\lambda + 6 = 0$
$(\lambda - 3)(\lambda - 2) = 0$
$\lambda = 3 \quad \text{or} \quad \lambda = 2$

Answer Q6-B

$when\ \lambda = 3$

$\begin{bmatrix} 3-1 & -1 & 0 \\ 2 & 3-4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$let\ y = t$

$x + \frac{-1}{2} t = 0$

$x = \frac{1}{2} t$

$S.S = \left\{ \begin{bmatrix} \frac{1}{2} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$S.S = \left\{ t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

$when\ \lambda = 2$

$\begin{bmatrix} 2-1 & -1 & 0 \\ 2 & 2-4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$let\ y = t$

$x - t = 0$

$x = t$

$S.S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$S.S = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{bmatrix}$

Q-1: [5×2 Marks] Choose the correct answers:

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1.1: if A is a 2×2 nonsingular matrix, then $|-2A^T| =$

a) $2|A|$
b) $-2|A|$
c) $4|A|$
d) $-4|A|$
e) none of the above

1.2: An eigenvalue of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is:

a) 0
b) 1
c) 2
d) 6
e) none of the above

1.3: A spanning set for $W = \left\{ \begin{bmatrix} a \\ 2a \end{bmatrix} : a \in \mathbb{R} \right\}$ is:

a) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$
e) none of the above

1.4: The dimension of the subspace $w = \left\{ \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} : a \in \mathbb{R} \right\}$ is:

a) 0
b) 1
c) 2
d) 6
e) none of the above

1.5: If $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \left(\begin{bmatrix} a-b \\ b-a \end{bmatrix}\right)$, $then\ T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
b) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
c) $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$
d) $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$
e) none of the above

Q-2: [5x2 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: The matrix $A = \begin{bmatrix} -2 & 2 \\ 1 & 3 \end{bmatrix}$ is invertible with $A^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -1 & -2 \end{bmatrix}$.

2.2: the set $\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

Each question is worth 10 marks. Answer the following questions:

Q-3: Let $X_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ and $X_4 = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$ be vectors in $\mathbb{R}^3$.

a) **[12 marks]** If A is the matrix whose columns are the vectors X_1, X_2, X_3 and X_4, find a matrix B in reduced roe echelon form that is Row equivalent to A.

b) **[8 marks]** Are the vectors X_1, X_2 and X_3linearly independent?

Q-4: $let\ W = \left\{\begin{bmatrix} x \\ 2x + y \\ -x + y \end{bmatrix} : x, y \in \mathbb{R}\right\}$ be a subset of $\mathbb{R}^3$

a) **[8 marks]** show that w is a subspace of $\mathbb{R}^3$
b) **[4 marks]** Find a set S Such that W = Span S
c) **[8 marks]** is $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in W$?

Q-5: Define the mapping T: $\mathbb{R}^2 \to \mathbb{R}^2$ by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ 2y \end{bmatrix}$.

a) **[12 marks]** Show that is a Linear transformation.
b) **[8 marks]** Find the null space.

Q-6: let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

a) **[8 marks]** show that λ=2 and λ=3 are eigenvalues of A.
b) **[12 marks]** Find a matrix P such that $P^{-1}AP = D,\ D$ is a diagonal matrix.

Answer Q2-A

$$AA^{-1} = \begin{bmatrix}-2 & 2\\ 1 & 3\end{bmatrix} \times \frac{1}{8}\begin{bmatrix}3 & -2\\ -1 & -2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$AA^{-1} = -I$, so, it is False.

Answer Q2-B

$$\begin{bmatrix}2 & 3\\ 3 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 3/2\\ 0 & -5/2\end{bmatrix} \sim \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$, True

Answer Q3-A

$$\begin{bmatrix}1 & 2 & 1 & -2\\ -2 & -1 & -2 & -2\\ 1 & 1 & -2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1 & -2\\ 0 & 3 & 0 & -6\\ 0 & -1 & -3 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1 & 2\\ 0 & 1 & 0 & -2\\ 0 & 0 & -3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 0\end{bmatrix}$$

Answer Q3-B

$$\begin{bmatrix}1 & 2 & 1\\ -2 & -1 & -2\\ 1 & 1 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 0\\ 0 & -1 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, yes$$

Answer Q4-A

$$W = \left\{ \begin{bmatrix} x\\ 2x+y\\ -x+y\end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x\\ 2\ x\\ -x\end{bmatrix} + \begin{bmatrix} 0\\ y\\ y\end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ x\begin{bmatrix} 1\\ 2\\ -1\end{bmatrix} + y\begin{bmatrix} 0\\ 1\\ 1\end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$W = span\ S$

$S = \left\{ \begin{bmatrix} 1\\ 2\\ -1\end{bmatrix}, \begin{bmatrix} 0\\ 1\\ 1\end{bmatrix} \right\}$, So, W is a subspace.

Answer Q4-B

$$span\ S = \left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

Answer Q4-C

$$\begin{bmatrix}1 & 0 & 2\\2 & 1 & 1\\-1 & 1 & -5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & -3\\0 & 1 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & -3\\0 & 0 & 0\end{bmatrix}$$

$C_1 = 2, \quad C_2 = -3$

$$2\begin{bmatrix}1\\2\\-1\end{bmatrix} - 3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\-5\end{bmatrix} \quad so, \begin{bmatrix}2\\1\\-5\end{bmatrix} \in W$$

Answer Q5-A

$x, y \in \mathbb{R}^2$

$$X = \begin{bmatrix}x_1\\y_1\end{bmatrix}, Y = \begin{bmatrix}x_2\\y_2\end{bmatrix}$$

$$T(rx + y) = \begin{bmatrix}rx_1 + x_2\\ry_1 + y_2\end{bmatrix} = \begin{bmatrix}rx_1 + x_2 - ry_1 - y_2\\2ry_1 + 2y_2\end{bmatrix}$$

$$= \begin{bmatrix}rx_1 - ry_1\\2ry_1\end{bmatrix} + \begin{bmatrix}x_2 - y_2\\2y_2\end{bmatrix}$$

$= rT(x) + T(Y)$

Answer Q5-B

$$N(T) = \left\{\begin{bmatrix}X\\Y\end{bmatrix} : T\left(\begin{bmatrix}X\\Y\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}\right\}$$

$$N(T) = \left\{\begin{bmatrix}X\\Y\end{bmatrix} : T\left(\begin{bmatrix}x - y\\2y\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}\right\}$$

$$\begin{bmatrix}1 & -1 & 0\\0 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

$x = 0, \quad y = 0$

$$N(T) = \begin{bmatrix}0\\0\end{bmatrix}$$

Answer Q6-A

$|\lambda I - A| = 0$

$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$

$\left| \begin{bmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{bmatrix} \right| = 0$

$(\lambda - 1)(\lambda - 4) - (-1)(2) = 0$
$\lambda^2 - 4\lambda - \lambda + 4 + 2 = 0$
$\lambda^2 - 5\lambda + 6 = 0$
$(\lambda - 3)(\lambda - 2) = 0$
$\lambda = 3 \quad \text{or} \quad \lambda = 2$

Answer Q6-B

when $\lambda = 3$

$\begin{bmatrix} 3-1 & -1 & 0 \\ 2 & 3-4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$let\ y = t$

$x + \frac{-1}{2} t = 0$

$x = \frac{1}{2} t$

$S.S = \left\{ \begin{bmatrix} \frac{1}{2} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$S.S = \left\{ t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

when $\lambda = 2$

$\begin{bmatrix} 2-1 & -1 & 0 \\ 2 & 2-4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$let\ y = t$
$x - t = 0$
$x = t$
$S.S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$S.S = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, P = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{bmatrix}$

Part1: [5×2 marks] Choose the correct answer:

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1.1: If A and B are 2×2 matrices with $|A| = 2$ and $|B| = -2$, then $|2A^t B^{-1}| =$

a) 4
b) 2
c) -2
d) -4
e) None of the above

1.2: The vectors $v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $v_2 = \begin{bmatrix} a \\ 6 \end{bmatrix}$ are linearly dependent if a =

a) 2
b) 3
c) 6
d) 12
e) None of the above

1.3: is a spanning set for the subspace W $= \left\{ \begin{bmatrix} a-b \\ b-2a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ is:

a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}$
e) None of the above

1.4: is a basis for the vector space $\mathbb{R}^2$

a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
e) None of the above

1.5: If T: $\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} x-y \\ 2y \end{bmatrix}\right), then\ T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$
b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
c) $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$
d) $\begin{bmatrix} -2 \\ 6 \end{bmatrix}$
e) None of the above

Part2: [10 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: $\lambda = 2$ is an eigenvalue for $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$.

2.2: the null space for the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} x-y \\ y-x \end{bmatrix}\right)$ is $N(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$.

Part3: Each question is worth 20 marks. Answer the following questions:

Q-1: let $X_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} -2 \\ 2 \\ -4 \\ 4 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $X_4 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 3 \end{bmatrix}$ be vectors in $\mathbb{R}^3$ and A the matrix whose columns are the vectors X_1, X_2, X_3 and X_4.

a) **[10 marks]** Find the matrix B in reduced echelon form that is row equivalent to A.
b) **[5 marks]** Which vectors in in S $\{X_1, X_2, X_3, X_4\}$ form basis for the subspace $W = span\ S$.
c) **[5 marks]** Find |A|.

Q-2: Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and let $S = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$,

$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right), T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \left(\begin{bmatrix} 8 \\ 7 \end{bmatrix}\right)$

a) **[5 marks]** Show that S is a linearly independent set.
b) **[5 marks]** Write $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in S.
c) **[5 marks]** Find $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.
d) **[5 marks]** Find $R(T)$, the range of T.

Q-3:

a) **[10 marks]** Show that $W = \left\{\begin{bmatrix} x \\ y \\ x+y \end{bmatrix} : x, y \in \mathbb{R}\right\}$ is a subspace of the vector space $\mathbb{R}^3$.
b) **[10 marks]** let $T: \mathrm{R}^2 \to \mathrm{R}^3$ be a mapping defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} x \\ y \\ x+y \end{bmatrix}\right)$. Show that T is a linear transformation.

Q-4:

Let $A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$.

c) Show that $\lambda_1 = 1$, and $\lambda_2 = 2$ are eigenvalues of A.
d) Find two independent eigenvectors of A.

Part 2 Answer Q2-1

$|\lambda I - A| = 0$

$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = 0$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = 0$

$\left|\begin{bmatrix} \lambda - 1 & 0 \\ -3 & \lambda - 2 \end{bmatrix}\right| = 0$

$(\lambda - 1)(\lambda - 2) - (0)(-3) = 0$

$\lambda^2 - 2\lambda - \lambda + 2 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1 \quad \text{or} \quad \lambda = 2 \quad \rightarrow \quad True.$

Part 2 Answer Q2-2

$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} \mathrm{X} - \mathrm{Y} \\ \mathrm{Y} - \mathrm{X} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x - t = 0 \quad \rightarrow \quad x = t, \quad y = t$

$N(T) = \left\{\begin{bmatrix} t \\ t \end{bmatrix}\right\} \quad \rightarrow \quad N(T) = \left\{t \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, False.

Part 3 Answer Q1-A

$$\begin{bmatrix}1 & -2 & 0 & -1\\ -1 & 2 & 1 & 2\\ 2 & -4 & 0 & -2\\ -2 & 4 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Part 3 Answer Q1-B

$$\begin{bmatrix}1\\ -1\\ 2\\ -2\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 0\\ 1\end{bmatrix}$$

Part 3 Answer Q1-C

$$\begin{bmatrix}1 & -2 & 0 & -1\\ -1 & 2 & 1 & 2\\ 2 & -4 & 0 & -2\\ -2 & 4 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$Det(A) = 1(0)(0)(0) = 0$$

Part 3 Answer Q2-A

$$\begin{bmatrix}-1 & 1\\ 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1\\ 0 & 3\end{bmatrix} \sim \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Part 3 Answer Q2-B

$$\begin{bmatrix}-1 & 1 & 2\\ 1 & 2 & 3\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -2\\ 1 & 2 & 3\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -2\\ 0 & 3 & 5\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -2\\ 0 & 1 & \frac{5}{3}\end{bmatrix} \sim \begin{bmatrix}1 & 0 & \frac{-1}{3}\\ 0 & 1 & \frac{5}{3}\end{bmatrix} \to C_1 = \frac{-1}{3} \ and \ C_2 = \frac{5}{3}$$

Part 3 Answer Q2-C

$$T\left(\begin{bmatrix}2\\ 3\end{bmatrix}\right) = T\left(\frac{-1}{3}\begin{bmatrix}-1\\ 1\end{bmatrix} + \frac{5}{3}\begin{bmatrix}1\\ 2\end{bmatrix}\right)$$

$$= -\frac{1}{3}T\left(\begin{bmatrix}-1\\ 1\end{bmatrix}\right) + \frac{5}{3}T\left(\begin{bmatrix}1\\ 2\end{bmatrix}\right)$$

$$= -\frac{1}{3}\begin{bmatrix}1\\ -1\end{bmatrix} + \frac{5}{3}\begin{bmatrix}8\\ 7\end{bmatrix} = \begin{bmatrix}\frac{-1}{3} + \frac{40}{3}\\ \frac{1}{3} + \frac{35}{3}\end{bmatrix} = \begin{bmatrix}13\\ 12\end{bmatrix}$$

Part 3 Answer Q2-D

Since S is linearly independent $\longrightarrow$ it is basis for $\mathbb{R}^2$

$$R(T) = Span\ \{T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right), T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = Span\ \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}8\\7\end{bmatrix}\right\}$$

Part 3 Answer Q3-A

$$W = \left\{\begin{bmatrix}x\\y\\x+y\end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix}x\\0\\x\end{bmatrix} + \begin{bmatrix}0\\y\\y\end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$= \left\{x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}0\\1\\1\end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$W = span\ S$$

$$S = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} \text{ So, W is a subspace}$$

Part 3 Answer Q3-B

$x, y \in \mathbb{R}^2$

$$X = \begin{bmatrix}x_1\\y_1\end{bmatrix}, \quad Y = \begin{bmatrix}x_2\\y_2\end{bmatrix}$$

$$T(rx+y) = \begin{bmatrix}rx_1 + x_2\\ry_1 + y_2\end{bmatrix} = \begin{bmatrix}rx_1 + x_2\\ry_1 + y_2\\rx_1 + x_2 + ry_1 + y_2\end{bmatrix}$$

$$= \begin{bmatrix}rx_1\\ry_1\\rx_1 + ry_1\end{bmatrix} + \begin{bmatrix}x_2\\y_2\\x_2 + y_2\end{bmatrix}$$

$$= rT(x) + T(Y)$$

Answer Part3: Q4-A

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = 0$$

$$\left|\begin{bmatrix} \lambda + 1 & -2 \\ 3 & \lambda - 4 \end{bmatrix}\right| = 0$$

$$(\lambda + 1)(\lambda - 4) - (-2)(3) = 0$$
$$\lambda^2 - 4\lambda + \lambda - 4 + 6 = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda - 1)(\lambda - 2) = 0$$
$$\lambda = 1 \quad \text{or} \quad \lambda = 2\,.$$

Answer Part3: Q4-B

$$when\ \lambda = 1$$

$$\left|\begin{bmatrix} 1+1 & -2 & 0 \\ 3 & 1-4 & 0 \end{bmatrix}\right| = \left|\begin{bmatrix} 2 & -2 & 0 \\ 3 & -3 & 0 \end{bmatrix}\right| \sim \left|\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right|$$

$$x - t = 0 \quad \rightarrow \quad x = t,\ \ y = t$$

$$S.S = \left\{\begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R}\right\}$$

$$S.S = \left\{t\begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$$

$$when\ \lambda = 2$$

$$\left|\begin{bmatrix} 2+1 & -2 & 0 \\ 3 & 2-4 & 0 \end{bmatrix}\right| = \left|\begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix}\right| \sim \begin{bmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x + \frac{-2}{3}t = 0 \quad \rightarrow \quad x = \frac{2}{3}t,\ \ y = t$$

$$S.S = \left\{\begin{bmatrix} \frac{2}{3}t \\ t \end{bmatrix} : t \in \mathbb{R}\right\}$$

$$S.S = \left\{t\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$$

Part 1: All Questions Are Required. Each Question Is Worth 2 Marks. Choose the Correct Answer:

Q–1: If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $(A^T)^2 =$

a) A
b) I_2
c) A^T
d) $-I_2$
e) None of the above

Q–2: An eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is

a) 0
b) 1
c) 2
d) 4
e) None of the above

Q–3: The columns of the matrix $\begin{bmatrix} 1 & a \\ 2 & a-1 \end{bmatrix}$ are dependent if $a =$

a) 2
b) 1
c) 0
d) −1
e) None of the above

Q–4: The dimension of the subspace $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix} \right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

Q–5: If $T:\mathbf{R}^3 \to \mathbf{R}^2$ is define by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a-3c \\ 3b+2c \end{bmatrix}$, then $T\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

c) $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$

d) $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

e) None of the above

Part 2: All Questions Are Required. Each Question Is Worth 5 Marks. Answer the Following Questions as True or False (1 Mark), Justifying Your Answer (4 Marks):

Q–1: The set $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$ is a basis for $\mathbf{R}^2$.

Q–2: The mapping $T:\mathbf{R}^2 \to \mathbf{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 1 \end{bmatrix}$ is a linear transformation.

Part 3: All Questions Are Required. Each Question Is Worth 20 Marks. Answer the Following Questions:

Q–1: Let $X_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ and $X_4 = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^3$.

a) **[10 marks]** Let A be the matrix whose columns are the vectors X_1, X_2, X_3 and X_4. Find a matrix B in reduced row echelon form that is row equivalent to A.

b) **[5 marks]** Are the vectors X_1, X_2 and X_3 linearly independent? Explain.

c) **[5 marks]** Which vectors in $S = \{X_1, X_2, X_3, X_4\}$ form a basis for $W =$ span S.

Q–2: Let $W = \left\{ \begin{bmatrix} x \\ 2x+y \\ -x+y \end{bmatrix} \middle| \, x, y \in \mathbf{R} \right\}$ be a subset of $\mathbf{R}^3$.

a) **[10 marks]** Show that W is a subspace of $\mathbf{R}^3$.

b) **[5 marks]** Find a set S such that $W =$ span S.

c) **[5 marks]** Is $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in W$? Explain.

Q–3: Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2y \end{bmatrix}$.

a) **[10 marks]** Show that T is a linear transformation.

b) **[5 marks]** Find the range $R(T)$.

c) **[5 marks]** Find the null space $N(T)$.

Q–4: Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

a) **[10 marks]** Show that $\lambda = 2$ and $\lambda = 3$ are eigenvalues of A.

b) **[10 marks]** Find a matrix P such that $P^{-1}AP = D$, D is a diagonal matrix.

End of Questions

Answer Q2-A

$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, True

Answer Q2-B

$\begin{bmatrix} 0+0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, False, it is not a linear transformation.

Answer Q3-A

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ -2 & -1 & -2 & -2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 3 & 0 & -6 \\ 0 & -1 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer Q3-B

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, yes$$

Answer Q3-C

basis for W: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

Answer Q4-A

$$W = \left\{ \begin{bmatrix} x \\ 2x + y \\ -x + y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ 2\ x \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$W = span\ S$$

$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, So, W is a subspace.

Answer Q4-B

$$span\ S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Answer Q4-C

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ -1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_1 = 2, \quad C_2 = -3$

$$2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$$

$$so, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \in \ W$$

Answer Q5-A

$x, y \ \in R^2$

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$T(rx + y) = \begin{bmatrix} rx_1 \ + x_2 \\ ry_1 \ + y_2 \end{bmatrix} = \begin{bmatrix} rx_1 \ + x_2 - \ ry_1 - \ y_2 \\ 2ry_1 \ + 2y_2 \end{bmatrix}$

$= \begin{bmatrix} rx_1 \ - \ ry_1 \\ 2ry_1 \end{bmatrix} + \begin{bmatrix} x_2 - \ y_2 \\ 2y_2 \end{bmatrix}$

$= \ rT(x) \ + \ T(Y)$

Answer Q5-B

$R(T) = span \ \left\{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$R(T) = span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$

basis for $R(T) = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$

Answer Q5-C

$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$N(T) = \left\{\begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X - Y \\ 2Y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$x = 0, \quad y = 0$

$N(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Answer Q6-A

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$$

$$\left|\begin{bmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{bmatrix}\right| = 0$$

$$(\lambda - 1)(\lambda - 4) - (-1)(2) = 0$$
$$\lambda^2 - 4\lambda - \lambda + 4 + 2 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 3)(\lambda - 2) = 0$$
$$\lambda = 3 \quad \text{or} \quad \lambda = 2$$

Answer Q6-B

when $\lambda = 3$

$$\begin{bmatrix} 3-1 & -1 & 0 \\ 2 & 3-4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$
$$x + \frac{-1}{2} t = 0$$
$$x = \frac{1}{2} t$$
$$S.S = \left\{ \begin{bmatrix} \frac{1}{2} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \quad \rightarrow \quad S.S = \left\{ t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

when $\lambda = 2$

$$\begin{bmatrix} 2-1 & -1 & 0 \\ 2 & 2-4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$
$$x - t = 0$$
$$x = t$$
$$S.S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$S.S = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{bmatrix}$$

Part 1: All Questions Are Required. Each Question Is Worth 2 Marks. Choose the Correct Answer:

Q–1: If $A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, then $|A^{100}|=$

a) 0
b) 1
c) −1
d) −100
e) None of the above

Q–2: The columns of the matrix $\begin{bmatrix}2 & a\\ 4 & a-1\end{bmatrix}$ are linearly dependent if $a=$

a) 2
b) 1
c) 0
d) −1
e) None of the above

Q–3: If $W=\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\}$, then $\dim W=$

a) 0
b) 1
c) 2
d) 3
e) None of the above

Q–4: An eigenvalue of $A=\begin{bmatrix}1 & -2\\ -3 & 6\end{bmatrix}$ is

a) 0
b) 1
c) 6
d) 4
e) None of the above

Q–5: If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $T:\mathbf{R}^2\to\mathbf{R}^2$ is a linear transformation define by $T(X)=AX$, then $T\left(\begin{bmatrix}3\\ -1\end{bmatrix}\right)=$

a) $\begin{bmatrix}1\\ 5\end{bmatrix}$

b) $\begin{bmatrix}1\\ 6\end{bmatrix}$

c) $\begin{bmatrix}1\\ 7\end{bmatrix}$

d) $\begin{bmatrix}-1\\ 5\end{bmatrix}$

e) None of the above

Part 2: All Questions Are Required. Each Question Is Worth 5 Marks. Answer the Following Questions as True or False (1 Mark), Justifying Your Answer (4 Marks):

Q–1: The linear system $\begin{array}{rcrcr} x & - & 2y & = & 0\\ 2x & - & 4y & = & 0\end{array}$ has only the trivial solution.

Q–2: The mapping $T:\mathbf{R}^3\to\mathbf{R}^2$ defined by $T\left(\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right)=\begin{bmatrix}x-y\\ z+1\end{bmatrix}$ is a linear transformation.

Part 3: All Questions Are Required. Each Question Is Worth 20 Marks. Answer the Following Questions:

Q–1: Let $A=\begin{bmatrix}1 & 2 & 1 & -2\\ -2 & -1 & -2 & -2\\ 1 & 1 & -2 & 0\end{bmatrix}$.

a) [10 marks] Find the matrix B in reduced row echelon form that is row equivalent to A.

b) **[5 marks]** Show that the first three columns of A are linearly independent.

c) **[5 marks]** Find a basis (from the columns of A) for the subspace of $\mathbf{R}^3$ spanned by the columns of A.

2: Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x = 2y \right\}$ be a subset of $\mathbf{R}^2$.

a) **[10 marks]** Show that W is a subspace of $\mathbf{R}^2$.

b) **[10 marks]** Find a basis for W. What is the dimension of W?

3: Define the mapping $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 2x - 2y \end{bmatrix}$.

a) **[10 marks]** Show that T is a linear transformation.

b) **[5 marks]** Find the range $R(T)$.

c) **[5 marks]** Find the null space $N(T)$.

4: [20 marks] Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Find the eigenvalues and the corresponding eigenvectors of A.

End of Questions

Answer Q2-A

$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, False, it has infinitely many solutions.

Answer Q2-B

$\begin{bmatrix} 0 - 0 \\ 0 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, False, it is not a linear transformation.

Answer Q3-A

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ -2 & -1 & -2 & -2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 3 & 0 & -6 \\ 0 & -1 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer Q3-B

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer Q3-C

basis for W: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

Answer Q4-A

$W = \left\{ \begin{bmatrix} 2y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

$= \left\{ y \begin{bmatrix} 2 \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}$

$W = span\ S$

$S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} So,$ W is a subspace.

Answer Q4-B

basis for W: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the dimension: 1

Answer Q5-A

$x, y \ \in \mathbb{R}^2$

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$T(rx + y) = \begin{bmatrix} rx_1 & + x_2 \\ ry_1 & + y_2 \end{bmatrix} = \begin{bmatrix} rx_1 + x_2 - ry_1 - y_2 \\ 2rx_1 + 2x_2 - 2ry_1 - 2y_2 \end{bmatrix}$

$= \begin{bmatrix} rx_1 - ry_1 \\ 2rx_1 - 2ry_1 \end{bmatrix} + \begin{bmatrix} x_2 - y_2 \\ 2x_2 - 2y_2 \end{bmatrix}$

$= \ rT(x) \ + \ T(Y)$

Answer Q5-B

$\mathrm{R(T)} = \mathrm{span}\ \left\{ \mathrm{T} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathrm{T} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathrm{R(T)} = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}$

basis for $R(T) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Answer Q5-C

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} x - y \\ 2x - 2y \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$x = 0, \quad y = 0$$

$$N(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Answer Q6-A

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = 0$$

$$\left| \begin{bmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{bmatrix} \right| = 0$$

$$(\lambda - 1)(\lambda - 4) - (-1)(2) = 0$$

$$\lambda^2 - 4\lambda - \lambda + 4 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 3 \quad \text{or} \quad \lambda = 2$$

when $\lambda = 3$

$$\begin{bmatrix} 3-1 & -1 & 0 \\ 2 & 3-4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$

$$x + \frac{-1}{2}t = 0$$

$$x = \frac{1}{2}t$$

$$S.S = \left\{ \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} S.S = \left\{ t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} \quad \rightarrow \quad \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

when $\lambda = 2$

$$\begin{bmatrix} 2-1 & -1 & 0 \\ 2 & 2-4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$

$$x - t = 0$$

$$x = t$$

$$S.S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \quad \rightarrow \quad S.S = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} \quad \rightarrow \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Q-1: [5×2 Marks] Choose the correct answers:

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1.1: Let $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$, then $v = av_1 + bv_2$, where the pair (a, b) is:

a) $(1,1)$
b) (-3,0)
c) (0,3)
d) Impossible
e) None of the above

1.2: The vectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ a \end{bmatrix}$ are linearly dependent if a is equal to:

a) -2
b) 2
c) -4
d) 4
e) None of the above

1.3: The eigenvalues (λ_1, λ_2) of the matrix $\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$ are:

a) $(3, -2)$
b) $(-3,2)$
c) $(1,0)$
d) $(1, -3)$
e) None of the above

1.4: Given two vectors $u = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then the dot product $u.v$ is:

a) -4
b) 2
c) 4
d) 0
e) None of the above

1.5: the inverse of the matrix $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

a) $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$
b) $\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$
c) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$
d) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$
e) None of the above

Q-2: [5x2 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: The vectors $\mathrm{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ forms a linear combination with the vectors $\mathrm{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathrm{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2.2: The set $S = \left\{ \begin{bmatrix} x \\ 2x+1 \end{bmatrix} : x \in \mathbb{R} \right\}$ is a subspace.

Each question is worth 10 marks. Answer the following questions:

Q-3: [8+12] Consider the linear System $\begin{cases} x - y - 2z = -3 \\ -x + 2y + 3z = 2 \\ 2x - 2y - 3z = -1 \end{cases}$

a) Find the inverse of the matrix A
b) If $AX = b$ is the matrix form of the system, find the matrix X and its elements.

Q-4: [10+10 marks] Let $\mathrm{v}_1 = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \mathrm{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ and $\mathrm{v}_4 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

a) Show that $S = \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ is linearly independent.
b) Determine whether v_4 forms a linear combination of $\mathrm{v}_1, \mathrm{v}_2$ and v_3

Q-5: [10+10 marks] Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, V = \mathbb{R}^3$

a) Determine whether S is a basis for the vector space V.
b) Is the vector $v = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$ in Span(S)?

Q-6: [8+5+7 marks] Define T: $\mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x - 3y \end{bmatrix}$

a) Show that T is a linear transformation.
b) Find $T\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.
c) Find the null space.

Q-7: [6+8+6 marks] Let $A = \begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 5 \\ 3 & 0 & 4 \end{bmatrix}$

a) Find the eigenvalues of the matrix A.
b) Find the inverse of A
c) Find the matrix X and its elements such that $AX = b$

Answer Q2-A

$$\begin{bmatrix} 1 & 1 & 3 \\ -2 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}$$

$C_1 = -5, \quad C_2 = 8, \quad \text{True.}$

Answer Q2-B

$\begin{bmatrix} 0 \\ 2(0)+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, False, it is not a subspace

Answer Q3-A

$$\begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 2 & -2 & -3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

Answer Q3-B

$AX = b$

$A^{-1}AX = A^{-1}b$

$X = A^{-1}b$

$$x = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix}$$

Answer Q4-A

$$\begin{bmatrix} 2 & 2 & 3 \\ 8 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 8 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer Q4-B

$$\begin{bmatrix} 2 & 2 & 3 & -2 \\ 8 & -2 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 2 \\ 8 & -2 & 0 & 0 \\ 2 & 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & -8 \\ 0 & 2 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 3 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$True, \quad C_1 = 1, \quad C_2 = 4, \quad C_3 = -4$

Answer Q5-A

$$\begin{bmatrix}1&1&0\\0&1&0\\1&0&1\end{bmatrix}\sim\begin{bmatrix}1&1&0\\0&1&0\\0&-1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \text{True.}$$

Answer Q5-B

$$\begin{bmatrix}1&1&0&-2\\0&1&0&2\\1&0&1&-3\end{bmatrix}\sim\begin{bmatrix}1&1&0&-2\\0&1&0&2\\0&-1&1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&0&-4\\0&1&0&2\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&-4\\0&1&0&2\\0&0&1&1\end{bmatrix}$$

$\mathrm{C}_1=-4,\quad \mathrm{C}_2=2,\quad \mathrm{C}_3=1,\quad yes.$

Answer Q6-A

$x, y \in R^2$

$X=\begin{bmatrix}x_1\\y_1\end{bmatrix}, Y=\begin{bmatrix}x_2\\y_2\end{bmatrix}$

$T(rx+y)=\begin{bmatrix}rx_1+x_2\\ry_1+y_2\end{bmatrix}=\begin{bmatrix}2rx_1+2x_2+ry_1+y_2\\rx_1+x_2-3ry_1-3y_2\end{bmatrix}$

$=\begin{bmatrix}2rx_1+ry_1\\rx_1-3ry_1\end{bmatrix}+\begin{bmatrix}2x_2+y_2\\x_2-3y_2\end{bmatrix}$

$= rT(x)+T(Y)$

Answer Q6-B

$$\mathrm{T}\left(\begin{bmatrix}-2\\1\end{bmatrix}\right)=\begin{bmatrix}2(-2)+1\\-2-3(1)\end{bmatrix}=\begin{bmatrix}-3\\-5\end{bmatrix}$$

Answer Q6-C

$\mathrm{N(T)}=\left\{\begin{bmatrix}\mathrm{X}\\\mathrm{Y}\end{bmatrix}:\mathrm{T}\left(\begin{bmatrix}\mathrm{X}\\\mathrm{Y}\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}\right\}$

$\mathrm{N(T)}=\left\{\begin{bmatrix}\mathrm{X}\\\mathrm{Y}\end{bmatrix}:\mathrm{T}\left(\begin{bmatrix}2x+y\\x-3y\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}\right\}$

$$\begin{bmatrix}2&1&0\\1&-3&0\end{bmatrix}\sim\begin{bmatrix}1&-3&0\\2&1&0\end{bmatrix}\sim\begin{bmatrix}1&-3&0\\0&7&0\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

$\mathrm{x}=0,\quad \mathrm{y}=0$

$\mathrm{N(T)}=\begin{bmatrix}0\\0\end{bmatrix}$

Answer Q7-A

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix} = 0$$

$$\left|\begin{bmatrix} \lambda - 1 & -3 \\ -3 & \lambda + 2 \end{bmatrix}\right| = 0$$

$$(\lambda - 1)(\lambda + 2) - (-3)(-3) = 0$$

$$\lambda^2 + 2\lambda - \lambda - 2 - 9 = 0$$

$$\lambda^2 + \lambda - 11 = 0$$

$$\lambda = -\frac{1 - 3\sqrt{5}}{2}, \quad \text{or} \quad \lambda = -\frac{1 + 3\sqrt{5}}{2}$$

Answer Q7-B

$$A^{-1} = \begin{bmatrix} \frac{2}{11} & \frac{3}{11} \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix}$$

Answer Q7-C

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

$$X = \begin{bmatrix} \frac{2}{11} & \frac{3}{11} \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 5 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-4}{11} & 2 \\ 0 & \frac{-6}{11} & 1 \end{bmatrix}$$

PART 1: MULTIPLE CHOICE QUESTIONS

[5×2 marks] Choose the correct answer:

I. Let $B=\begin{bmatrix}5 & 3\\ 2 & 4\end{bmatrix}$; then $|B|$ is:

- **a)** 26
- **b)** -14
- **c)** -26
- **d)** 14
- **e)** None of the above.

II. Let $A=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$ and $b=\begin{bmatrix}2\\ 1\end{bmatrix}$; then $A.b$ is:

- **a)** $\begin{bmatrix}7\\ 0\end{bmatrix}$
- **b)** $\begin{bmatrix}0\\ 7\end{bmatrix}$
- **c)** $\begin{bmatrix}4 & 6\\ -1 & 2\end{bmatrix}$
- **d)** $\begin{bmatrix}8\\ 3\end{bmatrix}$
- **e)** None of the above.

III. If $A=\begin{bmatrix}4 & 7\\ 1 & 2\end{bmatrix}$, then $(A^{-1})^T$ is equal to:

- **a)** $\begin{bmatrix}-4 & -7\\ -1 & -2\end{bmatrix}$
- **b)** $\begin{bmatrix}7 & 4\\ 2 & 1\end{bmatrix}$
- **c)** $\begin{bmatrix}2 & -7\\ -1 & 4\end{bmatrix}$
- **d)** $\begin{bmatrix}2 & -1\\ -7 & 4\end{bmatrix}$
- **e)** None of the above.

IV. The pair of eigenvalues (λ_1,λ_2) for the matrix $B=\begin{bmatrix}3 & 0\\ 1 & -2\end{bmatrix}$ is

- **a)** $(0,1)$
- **b)** $(-2,3)$
- **c)** $(-3,2)$
- **d)** $(-1,0)$
- **e)** None of the above.

V. If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ x+y \end{bmatrix}$; then $T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right)$ is:

a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e) None of the above

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×5 marks] Answer the following as True or False and justify your answer (Award 1 point for the correct answer (T/F) and 4 points for the correct justification):

I. The transformation $T: R^2 \to R^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y-1 \end{bmatrix}$ is a linear transformation:

II. The set $W = \left\{ \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix} \middle| a \in R \right\}$ is a subspace of R^3.

PART 3: ESSAY QUESTIONS

[4×20 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [6+6+8 marks] Consider the linear system $\begin{cases} y - z = -1 \\ 3x + y + z = 1 \\ x + 2y - z = 0 \end{cases}$

a) Find the determinant of the coefficient matrix A.
b) Find the inverse of the matrix A.
c) The matrix form of the system is $Ax = b$; Find the solution of the system.

Q–2: [4+8+8 marks] Let $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$

a) Show that $S = \{v_1, v_2, v_3, v_4\}$ is linearly dependent.

b) Show that $T = \{v_1, v_2, v_3\}$ is linearly independent.

c) Determine whether v_4 is a linear combination of the vectors in T. If it is, write the corresponding linear combination expression.

Q–3: [6+8+6 marks] Define the linear transformation $T : R^4 \to R^3$ by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}, \text{ where } a, b, c, \text{ and } d \in R.$$

a) Find the range space $R(T)$.

b) Find the null space $N(T)$, its basis, and its dimension.

c) Find a basis for $R(T)$ and its dimension.

Q–4: [8+4+8 marks] Define $T : R^2 \to R^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-2y \\ 3x+y \end{bmatrix}$

a) Show that T is a linear transformation.

b) Find $T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$.

c) Determine if the set $S = \left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right\}$ is a linear combination of $v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Q–5: [8+6+8 marks] Let $A = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

a) Find the eigenvalues of the matrix A.

b) Find the inverse of B.

c) Find the matrix C and its elements such that $BC = A$.

End of Questions

Part 2 - Answer Q2-A

$\begin{bmatrix} -0 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \notin \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, False, it is not a linear transformation.

Part 2 - Answer Q2-B

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, False, it is not a subspace.

Part 3 - Answer Q1-A

$$\begin{bmatrix} 0 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \sim (-) \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

$$1 \begin{vmatrix} -5 & 4 \\ 1 & -1 \end{vmatrix} = (-5)(-1) - (4)(1) = -1$$

Part 3 - Answer Q1-B

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & -5 & 4 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -5 & 4 & 0 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 5 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 & 1 & -2 \\ 0 & 1 & 0 & -4 & -1 & 3 \\ 0 & 0 & 1 & -5 & -1 & 3 \end{bmatrix}, A^{-1} = \begin{bmatrix} 3 & 1 & -2 \\ -4 & -1 & 3 \\ -5 & -1 & 3 \end{bmatrix}$$

Part 3 - Answer Q1-C

$AX = B$

$A^{-1}AX = A^{-1}B$

$X = A^{-1}B$

$$X = \begin{bmatrix} 3 & 1 & -2 \\ -4 & -1 & 3 \\ -5 & -1 & 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

Part 3 - Answer Q2-A

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ -1 & -1 & -1 & -4 \\ 0 & -1 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & -3 & 2 & 1 \\ 0 & -1 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & -1 & -3 & -7 \\ 0 & -3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 11 & 22 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Part 3 - Answer Q2-B

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 2 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -3 \\ 0 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Part 3 - Answer Q2-C

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ -1 & -1 & -1 & -4 \\ 0 & -1 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & -3 & 2 & 1 \\ 0 & -1 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & -1 & -3 & -7 \\ 0 & -3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 11 & 22 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \rightarrow \quad C_1 = 1, \quad C_2 = 1, \quad C_3 = 2 \quad \rightarrow \quad X_4 = X_1 + X_2 + 2X_3.$$

Part 3 - Answer Q3-A

$$R(T) = span\left\{T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, T\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, T\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, T\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

$$R(T) = span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

$$basis\ for\ R(T) = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}\right\}$$

Part 3 - Answer Q3-B

$$N(T)=\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}:T\left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$N(T)=\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}:T\left(\begin{bmatrix}a+b\\b-c\\\mathrm{b}+\mathrm{d}\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$\begin{bmatrix}1&1&0&0&0\\0&1&-1&0&0\\0&1&0&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&1&0&0\\0&1&-1&0&0\\0&0&1&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1&0\\0&1&0&1&0\\0&0&1&1&0\end{bmatrix}$$

$let\ d=t$

$a-t=0 \quad \rightarrow \quad a=t$

$b+t=0 \quad \rightarrow \quad b=-t$

$c+t=0 \quad \rightarrow \quad c=-t$

$$N(T)=\left\{\begin{bmatrix}t\\-t\\-t\end{bmatrix}:t\in\mathbb{R}\right\} \quad \rightarrow \quad N(T)=\left\{t\begin{bmatrix}1\\-1\\-1\end{bmatrix}:t\in\mathbb{R}\right\}$$

Basis for $N(T)=\left\{\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\}$

Dim for $N(T)=1$

Part 3 - Answer Q3-C

basis for $R(T)=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\end{bmatrix}\right\}$

Dim for $R(T)=3$

Part 3 - Answer Q4-A

$x,y\ \in\mathbb{R}^2$

$X=\begin{bmatrix}x_1\\y_1\end{bmatrix},Y=\begin{bmatrix}x_2\\y_2\end{bmatrix}$

$$T(rx+y)=\begin{bmatrix}rx_1\ +x_2\\ry_1\ +y_2\end{bmatrix}=\begin{bmatrix}rx_1\ +x_2-2ry_1-2\,y_2\\3rx_1\ +3x_2+ry_1\ +y_2\end{bmatrix}$$

$$=\begin{bmatrix}rx_1\ -2ry_1\\3rx_1\ +ry_1\end{bmatrix}+\begin{bmatrix}x_2-2\,y_2\\3x_2\ +y_2\end{bmatrix}$$

$=\ rT(x)\ +\ T(Y)$

Part 3 - Answer Q4-B

$$T\begin{bmatrix}-1-2(0)\\3(-1)+0\end{bmatrix}=\begin{bmatrix}-1\\-3\end{bmatrix}$$

Part 3 - Answer Q4-C

$$\begin{bmatrix}-1&3&1\\2&0&4\end{bmatrix}\sim\begin{bmatrix}1&-3&-1\\0&6&6\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&1&1\end{bmatrix}, yes,\quad C_1=2,\quad C_2=1$$

Part 3 - Answer Q5-A

$$|\lambda I-A|=0$$

$$\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}-2&1\\-3&2\end{bmatrix}=0$$

$$\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}-\begin{bmatrix}-2&1\\-3&2\end{bmatrix}=0$$

$$\left|\begin{bmatrix}\lambda+2&-1\\3&\lambda-2\end{bmatrix}\right|=0$$

$$(\lambda+2)(\lambda-2)-(-1)(3)=0$$
$$\lambda^2-2\lambda+2\lambda-4+3=0\quad\rightarrow\quad\lambda^2-1=0$$
$$\lambda=1\quad\text{or}\quad\lambda=-1$$

Part 3 - Answer Q5-B

$$\begin{bmatrix}\frac{-5}{2}&6\\-\frac{1}{2}&1\end{bmatrix}$$

Part 3 - Answer Q5-C

$$BC=A$$

$$B^{-1}BC=B^{-1}A$$

$$C=B^{-1}A$$

$$C=\begin{bmatrix}\frac{-5}{2}&6\\-\frac{1}{2}&1\end{bmatrix}\times\begin{bmatrix}-2&1\\-3&2\end{bmatrix}=\begin{bmatrix}-13&\frac{19}{2}\\-2&\frac{3}{2}\end{bmatrix}$$

Mock Exams

Fall 21-22 الفصل الأول

MT132/M132 - Linear Algebra
Final - Mock
Duration is 150 Minutes

Part 1: All Questions Are Required. Each Question Is Worth 2 Marks. Choose the Correct Answer:

Q–1: The linear system corresponding to the augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 7 \end{array}\right]$

a) Has infinitely many solutions
b) Has a unique solution
c) Has two solutions
d) Is inconsistent
e) None of the above

Q–2: If A is a 3×3 matrices such that $|A| = 2$, then $|2A^T| =$
a) 4
b) 8
c) 16
d) 32
e) None of the above

Q–3: The inverse of the nonsingular matrix $2A$ is
a) $2A$
b) $2A^{-1}$
c) $0.5A$
d) $0.5A^{-1}$
e) None of the above

Q–4: If $W = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\right\}$, then $\dim W =$

a) 0
b) 1
c) 2
d) 3
e) None of the above

Q–5: An eigenvalue of the matrix $\begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}$ is

a) 0
b) 2
c) 4
d) 8
e) None of the above

Part 2: All Questions Are Required. Each Question Is Worth 5 Marks. Answer the Following Questions as True or False (1 Mark), Justifying Your Answer (4 Marks):

Q–1: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Q–2: The vectors $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}$ are linearly dependent if $a = 0$.

Part 3: All Questions Are Required. Each Question Is Worth 20 Marks. Answer the Following Questions:

Q–1: Assume that $S = \left\{ \begin{bmatrix} a-2b \\ a \\ b \end{bmatrix} \middle| \; a, b \in \mathbf{R} \right\}$ can be written as a linear combination of X and Y in $\mathbf{R}^3$.

a) **[5 marks]** Find X and Y.
b) **[5 marks]** Are X and Y linearly independent? Explain.
c) **[10 marks]** Is the vector $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ a linear combination of X and Y? Explain.

Q–2:

a) **[10 marks]** Show that $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x+y+z=0 \right\}$ is a subspace of $\mathbf{R}^3$.

b) **[10 marks]** Let $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$. Find a basis for span S.

Q–3: Let $T: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$.

a) **[10 marks]** Show that T is a linear transformation.
b) **[10 marks]** Find the range $R(T)$ and the null space $N(T)$.

Q–4:

a) **[10 marks]** Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Solve the linear system $AX = B$.

b) **[10 marks]** Let $A = \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix}$. If possible, find a matrix P such that $P^{-1}AP = D$, D is a diagonal matrix.

Part 2 - Answer Q1

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = I, \quad \text{True.}$$

Part 2 - Answer Q2

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & a \\ 0 & 1 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & a-1 \\ 0 & 1 & 0 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & a-1 \\ 0 & 0 & -a+1 \end{vmatrix} \rightarrow (1)(1)(-a+1) = 0 \rightarrow a = 1, \quad \text{False.}$$

Part 3 - Answer Q1-A

$$\begin{bmatrix} a-2b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -2b \\ 0 \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ Y = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Part 3 - Answer Q1-B

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, yes$$

Part 3 - Answer Q1-C

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, yes, \quad C_1 = 4, \quad C_2 = 2$$

Part 3 - Answer Q2-A

$$W = \left\{\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ -\mathrm{x}-\mathrm{y} \end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$= \left\{\mathrm{x}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \mathrm{y}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : x, y \in \mathbb{R}\right\}$$

$$W = Span\, S \quad \rightarrow \quad S = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$$

Part 3 - Answer Q2-B

$$\begin{bmatrix} 2 & -2 & -2 & -4 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$basis\ for\ S = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Part 3 - Answer Q3-A

$x, y \in \mathbb{R}^3$

$$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$T(rx + y) = \begin{bmatrix} rx_1 + x_2 \\ ry_1 + y_2 \\ rz_1 + z_2 \end{bmatrix} = \begin{bmatrix} rx_1 + x_2 \\ 0 \\ rz_1 + z_2 \end{bmatrix}$$

$$= \begin{bmatrix} rx_1 \\ 0 \\ rz_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ z_2 \end{bmatrix}$$

$$= rT(x) + T(Y)$$

Part 3 - Answer Q3-B

$$R(T) = span \left\{ T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$R(T) = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$basis\ for\ R(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} : T\left(\begin{bmatrix} X \\ 0 \\ Z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$x = 0, \quad y = t, \quad z = 0$$

$$N(T) = \left\{ \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} \rightarrow \left\{ t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

Part 3 - Answer Q4-A

$AX = b$

$A^{-1}AX = A^{-1}b$

$X = A^{-1}b$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$$

Part 3 - Answer Q4-B

$|\lambda I - A| = 0$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix} = 0$$

$$\left| \begin{bmatrix} \lambda - 1 & 3 \\ 1 & \lambda + 1 \end{bmatrix} \right| = 0$$

$(\lambda - 1)(\lambda + 1) - (3)(1) = 0$
$\lambda^2 + \lambda - \lambda - 1 - 3 = 0$
$\lambda^2 - 4 = 0$
$\lambda = 2 \quad \text{or} \quad \lambda = -2$

$when\ \lambda = 2$

$$\begin{bmatrix} 2-1 & 3 & 0 \\ 1 & 2+1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$
$x + 3t = 0 \quad \rightarrow \quad x = -3t$

$$S.S = \left\{ \begin{bmatrix} -3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$S.S = \left\{ t \begin{bmatrix} -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$when\ \lambda = -2$

$$\begin{bmatrix} -2-1 & 3 & 0 \\ 1 & -2+1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$let\ y = t$

$x - t = 0 \quad \rightarrow \quad x = t$

$$S.S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$S.S = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$$

Q-1 [5x2 marks] Choose the correct answer:

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1.1: If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A^{10} =$

a) A
b) I_2
c) $-\mathrm{A}$
d) $-I_2$
e) None of the above

1.2: The columns of the matrix $\begin{bmatrix} 1 & a \\ 2 & a-1 \end{bmatrix}$ are dependent if $a\ =$

a) -1
b) -2
c) 1
d) 2
e) None of the above

1.3: a spanning set for $W = \left\{ \begin{bmatrix} 2a \\ 0 \end{bmatrix}, a \in \boldsymbol{R} \right\}$ is

a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
e) None of the above

1.4: The dimension of $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix},\ a, b \in \boldsymbol{R} \right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

1.5: If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ 2a+3b \end{bmatrix}$, then $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
b) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$
c) $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
d) $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$
e) None of the above

Q-2: [10 marks] Answer the following questions as True or False (1 mark), Justifying your answer (4 marks):

2.1: The subset $W = \left\{ \begin{bmatrix} \mathrm{x} \\ \mathrm{x}-1 \end{bmatrix}, x \in \mathbb{R} \right\}$ of $\mathbb{R}^2$ is a subspace.

2.2: The Range for the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \mathrm{x}-\mathrm{y} \\ \mathrm{y}-\mathrm{x} \end{bmatrix}$ is $\mathbb{R}^2$.

Each question is worth 20 marks. Answer the following questions:

Q-3: Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 0 \end{bmatrix}$.

a) **[12 Marks]** Find A^{-1}.
b) **[8 Marks]** Find the matrix X such that $AX = B^t$.

Q-4: Let S = span $\{v_1, v_2, v_3, v_4\}$ where $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, v_4 = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$.

a) **[8 marks]** Show that $\{v_1, v_2, v_3\}$ are linearly independent.
b) **[8 Marks]** write v_4 as a linear combination of v_1, v_2 and v_3.
c) **[4 Marks]** Find the subset of S that forms a basis for W.

Q-5: suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear operator defined by

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left\{ \begin{bmatrix} x+y \\ 0 \\ x-y \end{bmatrix} : x, y \in R \right\}$ be a subset of $\mathbb{R}^3$

a) **[8 marks]** show that T is a linear transformation.
a) **[8 Marks]** Show that the null space $N(T)$ is trivial.

Q-6: let $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

a) **[8 marks]** show that $\lambda = 8$ and $\lambda = -2$ are eigenvalues of A.
b) **[12 Marks]** Find two independent eigenvectors of A.

Answer Q2-1

$$\begin{bmatrix} 0 \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \notin \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathit{False, it\ is\ not\ a\ subspace}$$

Answer Q2-2

$$R(T) = span\left\{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$R(T) = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$$basis\ for\ R(T) = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \quad \rightarrow \quad False, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin \mathbb{R}^2$$

Answer Q3-A

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Answer Q3-B

$$AX = B^t$$

$$A^{-1}AX = A^{-1}B^t$$

$$X = A^{-1}B^t$$

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 6 \\ 5 & 6 \end{bmatrix}$$

Answer Q4-A

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer Q4-B

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ -2 & -1 & -2 & -2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 3 & 0 & -6 \\ 0 & -1 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_1 = 2, \quad C_2 = -2$

$X_3 = 2X_1 - 2X_2$

Answer Q4-C

basis for W: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

Answer Q5-A

$x, y \in \mathbb{R}^3$

$X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, Y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$T(rx + y) = \begin{bmatrix} rx_1 + x_2 \\ ry_1 + y_2 \end{bmatrix} = \begin{bmatrix} rx_1 + x_2 + ry_1 + y_2 \\ 0 \\ rx_1 + x_2 - ry_1 - y_2 \end{bmatrix}$$

$$= \begin{bmatrix} rx_1 + ry_1 \\ 0 \\ rx_1 - ry_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ 0 \\ x_2 - y_2 \end{bmatrix}$$

$= rT(x) + T(Y)$

Answer Q5-B

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} X \\ Y \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$N(T) = \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} : T\left(\begin{bmatrix} x + \mathrm{y} \\ 0 \\ \mathrm{x} - \mathrm{y} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = 0, \quad y = 0$

$N(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad N(T) = \{[0]\}$

Answer Q6-A

$$|\lambda I - A| = 0$$

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} = 0$$

$$\left| \begin{bmatrix} \lambda - 7 & -3 \\ -3 & \lambda + 1 \end{bmatrix} \right| = 0$$

$$(\lambda - 7)(\lambda + 1) - (-3)(-3) = 0$$
$$\lambda^2 + \lambda - 7\lambda - 7 - 9 = 0$$
$$\lambda^2 - 6\lambda - 16 = 0$$
$$(\lambda - 8)(\lambda + 2) = 0$$

$$\lambda = 8 \quad \text{or} \quad \lambda = -2$$

Answer Q6-B

$$when\ \lambda = 8$$

$$\begin{bmatrix} 8 - 7 & -3 & 0 \\ -3 & 8 + 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$
$$x - 3t = 0$$
$$x = 3t$$
$$S.S = \left\{ \begin{bmatrix} 3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$S.S = \left\{ t \begin{bmatrix} 3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$when\ \lambda = -2$$

$$\begin{bmatrix} -2 - 7 & -3 & 0 \\ -3 & -2 + 1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & -3 & 0 \\ -3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & {}^{1}/_{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$let\ y = t$$
$$x + {}^{1}/_{3}\, t = 0$$
$$x = -{}^{1}/_{3}\, t$$

$$S.S = \left\{ \begin{bmatrix} -{}^{1}/_{3}\, t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$S.S = \left\{ t \begin{bmatrix} -{}^{1}/_{3} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

PART 1: MULTIPLE CHOICE QUESTIONS

[5×2 marks] Choose the correct answer:

I. Let $B=\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$; then $|B|$ is:

a) 26
b) -14
c) -26
d) 14
e) None of the above.

II. Let $A=\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $b=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$; then $b.A$ is:

a) $\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 \\ 7 \end{bmatrix}$

c) $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$

d) $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$

e) None of the above.

III. If $A=\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$, then A^{-1} is equal to:

a) $\begin{bmatrix} -4 & -4 \\ -2 & -2 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -4 \\ -2 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}$

e) None of the above.

IV. An eigenvalue for the matrix $B=\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$ is

a) 0
b) 3
c) 1
d) -6
e) None of the above.

V. If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x \\ y^2 \end{bmatrix}$; then $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ is:

a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e) None of the above

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×5 marks] Answer the following as True or False and justify your answer (Award 1 point for the correct answer (T/F) and 4 points for the correct justification):

I. The matrices A and A^t have the same eigenvalues.

True since they have the same characteristic equation.

II. The set $W=\left\{\begin{bmatrix} s-2t \\ s+1 \\ s+t \end{bmatrix} \middle| s,t\in R\right\}$ is a subspace of R^3.

$(u+cv)\in W$; Let $u=\begin{bmatrix} s-2t \\ s+1 \\ s+t \end{bmatrix}; v=\begin{bmatrix} a-2b \\ a+1 \\ a+b \end{bmatrix}; u+v=\begin{bmatrix} s-2t+ca-2cb \\ s+1+ca+c \\ s+t+ca+cb \end{bmatrix}=$

$\begin{bmatrix} (s+ca)-2(t+cb) \\ (s+ca)+1+c \\ (s+ca)+(t+cb) \end{bmatrix}\notin W$. False.

PART 3: ESSAY QUESTIONS

[4×20 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [8+12 marks] Consider the linear system $\begin{cases} x+y-2z=-1 \\ -x+2y=1 \\ -y+z=0 \end{cases}$

a) Find the determinant of the coefficient matrix A;

b) Find the inverse of the matrix A.

a) Expand along the first column: $\begin{vmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1(2+0)+1(1-2)=1$

b) $\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow R1+R2 \to R2 \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow R2 \leftrightarrow R3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 3 & -2 & 1 & 1 & 0 \end{array}\right] \Rightarrow -R2 \to R2 \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 3 & -2 & 1 & 1 & 0 \end{array}\right] \Rightarrow -3R2+R3 \to R3$

$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \Rightarrow -R2+R1 \to R1 \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \Rightarrow \begin{matrix} R3+R1 \to R1 \\ R3+R2 \to R2 \end{matrix}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$

Q–2: [10+10 marks] Let $v_1 = \begin{bmatrix} -4 \\ 4 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$, and $v_4 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

a) Show that $T = \{v_1, v_3, v_4\}$ is linearly independent;

b) Find all vectors in R^3 that can be written as a linear combination of the vectors in T.

a) $\begin{vmatrix} -4 & -1 & -1 \\ 4 & 2 & 3 \\ -1 & -2 & 1 \end{vmatrix} = -4(2+6)+1(4+3)-1(-8+2) = -32+7+6 = -19 \neq 0;$

Therefore T is linearly independent.

b) $\left[\begin{array}{ccc|c} -4 & -1 & -1 & a \\ 4 & 2 & 3 & b \\ -1 & -2 & 1 & c \end{array}\right] \Rightarrow \begin{matrix} R1+R2 \to R2 \\ -R1+4R3 \to R3 \end{matrix} \left[\begin{array}{ccc|c} -4 & -1 & -1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -7 & 5 & -a+4c \end{array}\right] \Rightarrow 7R2+R3 \to R3$

$\left[\begin{array}{ccc|c} -4 & -1 & -1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 19 & 6a+7b+4c \end{array}\right]$. From the last row, the system has a solution. Therefore, every vector in R^3 is a linear combination of the vectors in T.

Q–3: [10+10 marks] Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, v_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, where $T = \{v_1, v_2, v_3\}$.

a) Is T linearly independent?

b) Is $\boldsymbol{v_4}$ in *Span(T)*?

a) $\begin{vmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \end{vmatrix} = 1(2-3)-1(-4-0) = 3 \neq 0$; Therefore, T is linearly independent.

b) $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 1 & 1 & 1 & 2 \\ 0 & 3 & 2 & 1 \end{array}\right] \Rightarrow -R1+R2 \to R2 \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{array}\right] \Rightarrow -R2+R3 \to R3 \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$;

$c_3 = 2; c_2 = -1$, and $c_1 = 1; v_1 - v_2 + 2v_3 = v_4$.

Q–4: [8+5+7 marks] Define $T: R^2 \to R^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ x+3y \end{bmatrix}$

a) Show that T is a linear transformation;

b) Find $T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$;

c) Find the null space *N(T)*.

a) $T(u+cv) = T(u)+cT(v)$; Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$;

$T(u+cv) = T\left(\begin{bmatrix} x+ca \\ y+cb \end{bmatrix}\right) = \begin{bmatrix} 2x+2ca-y-cb \\ x+ca+3y+3cb \end{bmatrix} = \begin{bmatrix} (2x-y)+c(2a-b) \\ (x+3y)+c(a+3b) \end{bmatrix} = \begin{bmatrix} 2x-y \\ x+3y \end{bmatrix} + c\begin{bmatrix} 2a-b \\ a+3b \end{bmatrix} =$

$T(u)+cT(v)$; Therefore T is a linear transformation.

b) $T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

c) $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 3 & 0 \end{array}\right] \Rightarrow R1-2R2 \to R2 \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & -7 & 0 \end{array}\right]$; Therefore, $N(T) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$.

Q–5: [6+8+6 marks] Let $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$.

a) Find the eigenvalues of the matrix *A*;

b) Find the inverse of *B*;

c) Find the matrix *C* and its elements such that *BC* = *A*.

a) $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(-5-\lambda)+12 = 0 \Rightarrow \lambda_1 = -1$ and $\lambda_2 = -2$.

b) $B^{-1} = \dfrac{1}{-4+3}\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$.

c) $BC = A \Rightarrow C = B^{-1}A = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} -3 & 19 \\ -4 & 26 \end{bmatrix}$.

End of Questions

PART 1: MULTIPLE CHOICE QUESTIONS

[5×2 marks] Choose the correct answer:

I. Let $A=\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$; then $|A|$ is:

a) 8
b) -8
c) 4
d) -4
e) None of the above.

II. Let $A=\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$ and $b=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; then $b.A$ is:

a) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} -6 \\ 7 \end{bmatrix}$

c) $\begin{bmatrix} 7 \\ -6 \end{bmatrix}$

d) $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$

e) None of the above.

III. If $A=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, then A^{-1} is equal to:

a) $\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

e) None of the above.

IV. An eigenvalue for the matrix A = $\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$ is

a) 0
b) 1
c) -2
d) -6
e) None of the above.

V. If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x+y \\ 2x \end{bmatrix}$; then $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ is:

a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e) None of the above

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×5 marks] Answer the following as True or False (Justify your answer):

I. The matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ have the same eigenvalues.

Both A and B have the same Characteristic Equation: $(a-\lambda)(d-\lambda)-bc=0$.
True.

II. The set $W = \left\{ \begin{bmatrix} s-t \\ s+t+1 \\ s+2t \end{bmatrix} \middle| s,t \in R \right\}$ is a subspace of R^3.

$(u+cv) \in W$; Let $u = \begin{bmatrix} s-t \\ s+t+1 \\ s+2t \end{bmatrix}$; $v = \begin{bmatrix} a-b \\ a+b+1 \\ a+2b \end{bmatrix}$; $u+cv = \begin{bmatrix} s-t+ca-cb \\ s+t+1+ca+cb+c \\ s+2t+ca+2cb \end{bmatrix} =$

$\begin{bmatrix} (s+ca)-(t+cb) \\ (s+ca)+(t+cb)+1+c \\ (s+ca)+2(t+cb) \end{bmatrix} \notin W$. False.

PART 3: ESSAY QUESTIONS

[4×20 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [8+12 marks] Consider the linear system $\begin{cases} x-2z=2 \\ x+y-2z=1 \\ 2x-2y-3z=-1 \end{cases}$

a) Find the determinant of the coefficient matrix A;
b) Find the inverse of the matrix A;

a) Expand along the 1st row: $|A| = \begin{vmatrix} 1 & 0 & -2 \\ 1 & 1 & -2 \\ 2 & -2 & -3 \end{vmatrix} = 1(-3-4) - 2(-2-2) = -7 + 8 = 1.$

b) $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \Rightarrow \begin{matrix} -R1 + R2 \to R2 \\ -2R1 + R3 \to R3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & -2 & 0 & 1 \end{array}\right] \Rightarrow 2R2 + R3 \to R3$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right] \Rightarrow 2R3 + R1 \to R1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 4 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -7 & 4 & 2 \\ -1 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}.$

c) $Ax = b \Rightarrow x = A^{-1}b = \begin{bmatrix} -7 & 4 & 2 \\ -1 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ -1 \\ -7 \end{bmatrix}.$

Q–2: [10+10 marks] Let $v_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and $v_4 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

a) Show that $S = \{v_1, v_2, v_3, v_4\}$ is linearly dependent;

b) Find all vectors in R^3 that can be written as a linear combination of the vectors in T.

a) S contains 4 vectors in R^3; Therefore, S is linearly dependent.

b) $\begin{vmatrix} 0 & -2 & -1 \\ -2 & -2 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 2(-4-0) - 1(2+2) = -8 - 4 = -12 \neq 0$; Therefore T is linearly independent.

c) $\left[\begin{array}{ccc|c} 0 & -2 & -1 & a \\ -2 & -2 & 0 & b \\ 1 & -1 & 2 & c \end{array}\right] \Rightarrow R1 \leftrightarrow R3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & c \\ -2 & -2 & 0 & b \\ 0 & -2 & -1 & a \end{array}\right] \Rightarrow 2R1 + R2 \to R2 \left[\begin{array}{ccc|c} 1 & -1 & 2 & c \\ 0 & -4 & 4 & b+2c \\ 0 & -2 & -1 & a \end{array}\right]$

$\Rightarrow -2R3 + R2 \to R3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & c \\ 0 & -4 & 4 & b+2c \\ 0 & 0 & 6 & -2a+b+2c \end{array}\right]$; From the last row, the system has a solution.

Therefore, every vector in R^3 is a linear combination of the vectors in T.

Q–3: [8+6+6 marks] Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, where $S = \{v_1, v_2, v_3\}$.

a) Find all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in *span(S)*;

b) Is S linearly independent?

c) Is $\mathbf{v_4}$ in *Span(S)*?

a) $\left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 2 & 0 & 1 & b \\ 1 & 3 & 1 & c \end{array}\right] \Rightarrow \begin{array}{c} -2R1+R2 \to R2 \\ -R1+R3 \to R3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 0 & 2 & 1 & b-2a \\ 0 & 4 & 1 & c-a \end{array}\right] \Rightarrow -2R2+R3 \to R3$

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 0 & 2 & 1 & b-2a \\ 0 & 0 & -1 & 3a-2b+c \end{array}\right]$; From the augmented matrix the original system is consistent.

Thus every vector in R^3 can be written as a linear combination of the vectors in S. Hence, $Span(S)$ is all of R^3.

b) $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 3 & 1 \end{vmatrix} = 1(0-3)+1(2-1) = -2 \neq 0$; Therefore, S is linearly independent.

c) $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{c} -2R1+R2 \to R2 \\ -R1+R3 \to R3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & -3 \\ 0 & 4 & 1 & -1 \end{array}\right] \Rightarrow -2R2+R3 \to R3$

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & 5 \end{array}\right] \Rightarrow c_1 = 3, c_2 = 1, c_3 = -5$ and $3v_1 + v_2 - 5v_3 = v_4$; Therefore, v_4 is in $Span(S)$.

Q–4: [6+4+4+6 marks] Define $T: R^2 \to R^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-3y \\ x+y \end{bmatrix}$

a) Show that *T* is a linear transformation;

b) Find $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$;

c) Give a description of the range of *T*;

d) Find the null space *N(T)*.

a) $T\left(u+cv\right)=T(u)+cT(v)$; Let $u=\begin{bmatrix}x\\y\end{bmatrix}$ and $v=\begin{bmatrix}a\\b\end{bmatrix}\Rightarrow T\left(\begin{bmatrix}x+ca\\y+cb\end{bmatrix}\right)=\begin{bmatrix}2x+2ca-3y-3cb\\x+ca+y+cb\end{bmatrix}=$

$\begin{bmatrix}2x-3y+c\left(2a-3b\right)\\x+y+c\left(a+b\right)\end{bmatrix}=\begin{bmatrix}2x-3y\\x+y\end{bmatrix}+c\begin{bmatrix}2a-3b\\a+b\end{bmatrix}=T\left(u\right)+cT\left(v\right);$

Therefore, T is a linear transformation.

b) $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)=\begin{bmatrix}2\left(-1\right)-3\left(1\right)\\-1+1\end{bmatrix}=\begin{bmatrix}-2-3\\0\end{bmatrix}=\begin{bmatrix}-5\\0\end{bmatrix}.$

c) $\begin{bmatrix}2x-3y\\x+y\end{bmatrix}=x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}-3\\1\end{bmatrix}\Rightarrow R\left(T\right)=Span\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}-3\\1\end{bmatrix}\right\}.$

d) $\left[\begin{array}{cc|c}2&-3&0\\1&1&0\end{array}\right]\Rightarrow R1-2R2\to R2\left[\begin{array}{cc|c}2&-3&0\\0&-5&0\end{array}\right]\Rightarrow x=y=0\Rightarrow N(T)=\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}.$

Q–5: [6+8+6 marks] Let $A=\begin{bmatrix}-2&2\\3&-3\end{bmatrix}$ and $B=\begin{bmatrix}-2&1\\-3&1\end{bmatrix}$.

a) Find the eigenvalues of the matrix A;

b) Find the inverse of B;

c) Find the matrix X and its elements such that $BX = A$.

a) $\left|A-\lambda I\right|=0\Rightarrow\begin{vmatrix}-2-\lambda&2\\3&-3-\lambda\end{vmatrix}=0\Rightarrow\lambda^2+5\lambda=0\Rightarrow\lambda_1=0$ and $\lambda_2=-5.$

b) $B^{-1}=\begin{bmatrix}1&-1\\3&-2\end{bmatrix}.$

c) $BX=A\Rightarrow X=B^{-1}A=\begin{bmatrix}1&-1\\3&-2\end{bmatrix}\begin{bmatrix}-2&2\\3&-3\end{bmatrix}=\begin{bmatrix}-5&5\\-12&12\end{bmatrix}.$

End of Questions

PART 1: MULTIPLE CHOICE QUESTIONS

[5×1 marks] Choose the correct answer:

Q–1

If $T: R^2 \to R^2$ is a linear operator such that $T(v_1)=\begin{bmatrix}1\\2\end{bmatrix}$ and $T(v_2)=\begin{bmatrix}-1\\3\end{bmatrix}$, then

$T(v_1-2v_2)=$

a) $\begin{bmatrix}3\\-4\end{bmatrix}$

b) $\begin{bmatrix}-3\\-7\end{bmatrix}$

c) $\begin{bmatrix}-4\\3\end{bmatrix}$

d) $\begin{bmatrix}4\\-3\end{bmatrix}$

e) None of the above

Q-2 *Let* $A=\begin{bmatrix}1 & 0\\1 & -1\end{bmatrix}$, an eigenvalue of A is:

a) 2
b) 0
c) -2
d) 3
e) None of the above

Q-3 The vector $v=\begin{bmatrix}13\\-2\end{bmatrix}$ is a linear combination of the vectors $v_1=\begin{bmatrix}-1\\2\end{bmatrix}$ and $v_2=\begin{bmatrix}3\\0\end{bmatrix}$: $v=av_1+bv_2$ when *(a,b)* is:

a) (-1,-4)
b) (4,1)
c) (-1,4*)*
d) (1,4)
e) None of the above.

Q-4 The determinant of the matrix $A=\begin{bmatrix} 2 & -40 & 10 \\ 0 & 3 & 12 \\ 0 & 0 & 4 \end{bmatrix}$ is:

a) 0
b) 4
c) -4
d) 24
e) None of the above.

Q-5 The solution to the linear system $\begin{cases} x-y=0 \\ 2x+3y=-1 \end{cases}$ when the pair (x,y) is:

a) (1,1)
b) (-1,-1)
c) (-1,1)
d) (1,-1)
e) None of the above.

a,e,c,d,a

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×2.5 marks] Answer the following as True or False (Justify your answer):

Q-1 The set $W=\left\{\begin{bmatrix} s-t \\ s+t+1 \\ s+2t \end{bmatrix} \text{ s, t } \in \text{ R}\right\}$ is a subspace of R^3..

False. For any vectors u and v in W, $u \oplus (k \otimes v)$ is not in w. Therefore, w is not a subspace of R^3.

Q-2 If A, B, and C are n x n matrices such that C is invertible and $A=C^{-1}BC$, then $\det(A)=\det(B)$.

True. $\det(A)=\det(C^{-1}BC)=\det(C^{-1})\det(B)\det(C)=\dfrac{1}{\det(C)}\det(B)\det(C)=\det(B).$

PART 3: ESSAY QUESTIONS

[4×10 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1 [4+3+3 points]

$$Let\ A = \begin{bmatrix} 3 & 3 & -1 \\ -1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}.$$

a) Find $\det(A)$;

b) Are the column vectors of A independent? Explain.

c) Is $Ax = b$ consistent for all vectors $b \in R^3$? Explain.

a) $\det(A) = 24$.

b) Yes, since det(A) $\neq$ 0.

c) Yes, since det(A) $\neq 0$.

Q-2 [4+3+3 points]

Consider the matrix $B = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}$,

a) Find the solution to the system $Bx = 0$.

b) Are the column vectors of B linearly dependent or independent? Explain.

c) Is the set $S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ a basis for the vector space $V = R^3$? Explain.

a) Solving the augmented matrix: $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 0 \\ 3 & 6 & 3 & 9 & 0 \\ 2 & 4 & 2 & 9 & 0 \end{array}\right]$, we obtain the solution set

$$S = \left\{ \begin{bmatrix} -2s - t \\ s \\ t \\ 0 \end{bmatrix} \right\}.$$

b) The column vectors of the matrix B are linearly dependent since we have 4 vectors in R^3.

c) No, since the set S has only two vectors.

Q–3 [4+3+3 points]

$a)$ Let $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$, find B^{-1}.

b) $Let\ C = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 2 \\ 0 & 0 & 7 \end{bmatrix}$, find CB.

$c)$ Given $A^{-1} - I = B$, find A

$a)$ $B^{-1} = \begin{bmatrix} 3 & 1 & -2 \\ -4 & -1 & 3 \\ -5 & -1 & 3 \end{bmatrix}$

$b)$ $\text{CB} = \begin{bmatrix} 9 & 8 & -1 \\ 11 & 8 & 0 \\ 7 & 14 & -7 \end{bmatrix}$.

$c)$ We have $A^{-1} - I = B \Rightarrow A^{-1} = B + I \Rightarrow AA^{-1} = A(B+I) \Rightarrow I = A(B+I) \Rightarrow (B+I)^{-1} = A$

$\Rightarrow A = B^{-1} + I$. Therefore, $A = \begin{bmatrix} 4 & 1 & -2 \\ -4 & 0 & 3 \\ -5 & -1 & 4 \end{bmatrix}$.

Q–4 [4+3+3 points]

Suppose $T: R^3 \to R^3$ is a linear transformation and let

$$B = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\} \text{ be a basis for } \mathrm{R}^3, \textit{where } T(v_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, T(v_2) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, T(v_3) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

a) Find $T\left(\begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \right)$,

b) Find the Null space N(T);

c) Find a basis for R(T) and determine dim(R(T)).

a) Since B is a basis for R^3, there are uniques scalars $c_1, c_2,$ and c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}; \text{ In this case } c_1 = c_2 = c_3 = 1.$$

$$\text{Therefore, } T\left(\begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \right) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + T\left(\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right) + T\left(\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}.$$

$$b)\ aT(v_1) + bT(v_2) + cT(v_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Solving, we obtain:}$$

$a = b = -t; c = t$. Therefore, N(T) = Span $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

c) To find a basis for R(T), we row-reduce the matrix: $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ to obtain:

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore the basis for R(T) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. dim(R(T)) = 2.

PART 1: MULTIPLE CHOICE QUESTIONS

[5×1 marks] Choose the correct answer:

I. Let $T : R^2 \to R^2$ be a linear operator such that $T(v_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(v_2) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ then $T(2v_1 - v_2)$ is equal to:

a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

c) $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

e) None of the above.

II. If $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$, then $\det(A)$ is equal to:

a) 2
b) 0
c) -2
d) -1
e) None of the above.

III. If $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, then B^{-1} is equal to:

a) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

b) $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$

c) $\begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix}$

d) $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

e) None of the above.

IV. An eigenvalue for the matrix A = $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ is

a) 0
b) 2
c) 3
d) -1
e) None of the above.

V. If A and B are two ***2* x *2*** matrices where *det(A) = -2* and *det(B) = 3*, then $det(A^T A^{-1} B)$ is equal to:

a) 3
b) 12
c) -3
d) 0
e) None of the above

a,b,d,c,a

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×2.5 marks] Answer the following as True or False (Justify your answer):

I. If $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x-y \\ w-z & -3 \end{bmatrix}$, then $x=2,\ y=4,\ z=1,$ and $w=3.$

$x=-x+4 \Rightarrow 2x=4 \Rightarrow x=2;$

$y=6+x-y \Rightarrow 2y=6+2=8 \Rightarrow y=4;$

$w=2w-3 \Rightarrow w=3;$

$z=-1+w-z \Rightarrow 2z=w-1=3-1=2 \Rightarrow z=1;$ True.

II. Let S, T: $R^2 \to \mathrm{R}^2$ be two linear operators defined by: $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -x \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ x+3y \end{bmatrix}$. If (S + T)(v) = S(v) + T(v), then (S + T)(v) for $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is equal to $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

$(S+T)(v) = S(v) + T(v) = S\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2-1 \\ -2 \end{bmatrix} + \begin{bmatrix} 4+1 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$. True.

PART 3: ESSAY QUESTIONS

[4×10 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [3+3+4 marks] Consider the matrices $A=\begin{bmatrix} 0 & 4 & 3 \\ 1 & 2 & -2 \\ 0 & 0 & (t-4) \end{bmatrix}$ and $b=\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$

a) Evaluate the determinant of A in terms of $\boldsymbol{t}$;

b) For what values of $\boldsymbol{t}$ is the determinant of A zero?

c) For $\boldsymbol{t=2}$, solve the system $Ax = b$ by elimination.

a) $|A|=(t-4)\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix}=(t-4)(-4)=-4t+16.$

b) $-4t+16=0 \Rightarrow t=4.$

c) $\left[\begin{array}{ccc|c} 0 & 4 & 3 & -2 \\ 1 & 3 & -2 & 3 \\ 0 & 0 & -2 & 4 \end{array}\right]$; From Row 3: $-2z=4 \Rightarrow z=-2.$

From Row 1: $4y+3z=-2 \Rightarrow 4y=-3z-2=-3(-2)-2=4 \Rightarrow 4y=4 \Rightarrow y=1.$

From Row 2: $x+3y-2z=3 \Rightarrow x=-3y+2z+3=-3-4+3=-4.$

$S=\{-4,1,-2\}.$

Q–2: [4+4+2 marks] Given: $A=\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$; Find where possible:

a) $A.B$;

b) $B.A$.

c) What can you conclude?

a) $\begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}.$

b) $\begin{bmatrix} 15 & -21 \\ 10 & -3 \end{bmatrix}.$

c) $A.B \neq B.A$

Q–3: [2+4+4 marks] Let $v_1=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, v_2=\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_3=\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $v_4=\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

a) Show that $\boldsymbol{S=\{v_1,v_2,v_3,v_4\}}$ is linearly dependent;

b) Show that $\boldsymbol{T=\{v_1,v_2,v_3\}}$ is linearly independent;

c) Show that $\boldsymbol{v_4}$ can be written as a linear combination of $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, and $\boldsymbol{v_3}$.

a) The set S is made up of 4 vectors in R^3, therefore S is linearly dependent.

b) $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-2) - 2(-1-2) = -1+6 = 5 \neq 0 \Rightarrow T$ is linearly independent.

c) If v_4 can be written as a linear combination with v_1, v_2, and v_3 ; Find c_1, c_2, and c_3

where $c_1v_1 + c_2v_2 + c_3v_3 = v_4$: $\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ -1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{c} R1+R2 \to R2 \\ -R1+R3 \to R3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & 1 & 3 \end{array}\right] \Rightarrow$

$3R3+R2 \to R3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 5 & 9 \end{array}\right] \Rightarrow c_3 = \frac{9}{5}$; From Row 2: $3c_2 = -2c_3 \Rightarrow c_2 = -\frac{2}{3}\left(\frac{9}{5}\right) = -\frac{6}{5}$;

From Row 1: $c_1 + 2c_2 = -2 \Rightarrow c_1 = -2c_2 - 2 = -2\left(\frac{-6}{5}\right) - 2 = \frac{12}{5} - 2 = \frac{2}{5}$.

Q–4: [3+2+2+3 marks] Let ***H*** be the set of all vectors of the form $\begin{bmatrix} 2a+3b \\ -a \\ 2b \end{bmatrix}$, where ***a*** and ***b*** $\in R$.

a) Is ***H*** a subspace of $\boldsymbol{R^3}$?;

b) Find vectors ***u*** and ***v*** such that ***H = Span{u , v }***.

c) Determine whether ***u*** and ***v*** are linearly independent.

d) Is the vector $w = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ in ***Span(H)***?

a) If H is a subspace of R^3, then for any u and $v \in H$ and $c \in R$, we have $(u+cv) \in H$

Let $u = \begin{bmatrix} 2a+3b \\ -a \\ 2b \end{bmatrix}$ and $v = \begin{bmatrix} 2x+3y \\ -x \\ 2y \end{bmatrix} \Rightarrow (u+cv) = \begin{bmatrix} 2a+3b \\ -a \\ 2b \end{bmatrix} + c\begin{bmatrix} 2x+3y \\ -x \\ 2y \end{bmatrix} =$

$\begin{bmatrix} 2a+3b+2cx+3cy \\ -a-cx \\ 2b+2cy \end{bmatrix} = \begin{bmatrix} 2(a+cx)+3(b+cy) \\ -(a+cx) \\ 2(b+cy) \end{bmatrix} \in H \Rightarrow H$ is a subspace of R^3.

b) $\begin{bmatrix} 2a+3b \\ -a \\ 2b \end{bmatrix} = \begin{bmatrix} 2a \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} 3b \\ 0 \\ 2b \end{bmatrix} = a\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \Rightarrow H = Span\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right\}$.

c) $c_1v_1 + c_2v_2 = 0 \Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right] \Rightarrow c_1 = c_2 = 0 \Rightarrow u$ and v are linearly independent.

d) $c_1v_1 + c_2v_2 = w \Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & 0 & 3 \\ 0 & 2 & -2 \end{array}\right] \Rightarrow R1 \leftrightarrow R2 \Rightarrow \left[\begin{array}{cc|c} -1 & 0 & 3 \\ 2 & 3 & 1 \\ 0 & 2 & -2 \end{array}\right] \Rightarrow 2R1 + R2 \rightarrow R2 \Rightarrow$

$\left[\begin{array}{cc|c} -1 & 0 & 3 \\ 0 & 3 & 7 \\ 0 & 2 & -2 \end{array}\right] \Rightarrow w$ is not in $Span(H)$.

Q–5: [3+2+3+2 marks] Define a linear operator $T : R^3 \rightarrow R^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x+z \\ y \\ x-z \end{bmatrix}$

a) Show that ***T*** is a linear transformation;

b) Give a description of the range of ***T***;

c) Is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ in ***R(T)*** ?

d) Find the null space ***N(T)*** and its dimension.

a) $T(u+cv)=T(u)+cT(v)$; Let $u=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $v=\begin{bmatrix}r\\s\\t\end{bmatrix}$; Where u and $v\in R^3$ and $x,y,z,r,s,$ and $t\in R$:

$$T(u+cv)=T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}+c\begin{bmatrix}r\\s\\t\end{bmatrix}\right)=T\left(\begin{bmatrix}x+cr\\y+cs\\z+ct\end{bmatrix}\right)=\begin{bmatrix}-(x+cr)+(z+ct)\\(y+cs)\\(x+cr)-(z+ct)\end{bmatrix}=\begin{bmatrix}(-x+z)+c(-r+t)\\y+cs\\(x-z)+c(r-t)\end{bmatrix}=$$

$$\begin{bmatrix}(-x+z)\\y\\(x-z)\end{bmatrix}+c\begin{bmatrix}(-r+t)\\s\\(r-t)\end{bmatrix}=T(u)+cT(v)\Rightarrow T \text{ is a linear transformation.}$$

b) $R(T)=\begin{bmatrix}-x+z\\y\\x-z\end{bmatrix}=\begin{bmatrix}-x\\0\\x\end{bmatrix}+\begin{bmatrix}0\\y\\0\end{bmatrix}+\begin{bmatrix}z\\0\\-z\end{bmatrix}=x\begin{bmatrix}-1\\0\\1\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}1\\0\\-1\end{bmatrix}\Rightarrow$

$$R(T)=Span\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}.$$

c) $\left[\begin{array}{ccc|c}-1&0&1&2\\0&1&0&1\\1&0&-1&-2\end{array}\right]\Rightarrow R1+R3\rightarrow R3\Rightarrow\left[\begin{array}{ccc|c}-1&0&1&2\\0&1&0&1\\0&0&0&0\end{array}\right]$; $c_2=1$; $c_3=t\,|\,t\in R$;

and from Row 1: $-c_1+c_3=2\Rightarrow -c_1=c_3+2\Rightarrow c_1=c_3-2=t-2$; Therefore $\begin{bmatrix}2\\1\\-2\end{bmatrix}$ is in $R(T)$.

d) $\begin{cases}-x+z=0\\ y=0\\ x-z=0\end{cases}\Rightarrow\left[\begin{array}{ccc|c}-1&0&1&0\\0&1&0&0\\1&0&-1&0\end{array}\right]\Rightarrow R1+R3\rightarrow R3\Rightarrow\left[\begin{array}{ccc|c}-1&0&1&0\\0&1&0&0\\0&0&0&0\end{array}\right]\Rightarrow y=0;$

$z=t\,|\,t\in R$; From Row 1: $-x+z=0\Rightarrow x=z=t$;

$$N(T)=\left\{t\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}\Rightarrow N(T)=Span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\};\ \dim N(T)=1.$$

End of Questions

PART 1: MULTIPLE CHOICE QUESTIONS

[5×1 marks] Choose the correct answer:

I. Let $T:R^3 \to R^3$ be a linear operator such that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x+z \\ y \\ x-z \end{bmatrix}$, then a basis for $N(T)$ is:

a) $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$

b) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

e) None of the above.

II. If $T:R^2 \to R^2$ is a linear operator such that $T(v_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T(v_2) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $T(v_1 - 2v_2)$ is equal to:

a) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

c) $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

d) $\begin{bmatrix} -3 \\ -4 \end{bmatrix}$

e) None of the above.

III. Let $v=\begin{bmatrix}1\\1\end{bmatrix}, v_1=\begin{bmatrix}-2\\4\end{bmatrix}, v_2=\begin{bmatrix}3\\-7\end{bmatrix}$, then ***v* =*av*₁ + *bv*₂** if the pair ***(a , b)*** is:

a) (-5, -3)
b) (5, 3)
c) (-3, -5)
d) (3, 5)
e) None of the above.

IV. An eigenvalue for the matrix A = $\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}$ is

a) 0
b) 2
c) 1
d) -1
e) None of the above.

V. If *A*, *B*, and *C* are matrices of size (4 x 5), (4 x 5), and (5 x 4) respectively, then the size of $(BA^T - C^T)$ is:
a) (4 x 4)
b) (5 x 5)
c) (4 x 5)
d) (5 x 4)
e) None of the above

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×2.5 marks] Answer the following as True or False (Justify your answer):

I. Consider the system of equations: $\begin{cases}ax+by=0\\cx+dy=0\end{cases}$, if $x=x_0, y=y_0$, and $x=x_1$, $y=y_1$ are any two solutions, then $x=x_0+x_1$, $y=y_0+y_1$ is also a solution.

If (x_0+x_1) and (y_0+y_1) are solutions to the linear system, they must satisfy the equations: $\begin{cases}a(x_0+x_1)+b(y_0+y_1)=0\\c(x_0+x_1)+d(y_0+y_1)=0\end{cases} \Rightarrow$ Expanding, we have: $\begin{cases}ax_0+ax_1+by_0+by_1=0\\cx_0+cx_1+dy_0+dy_1=0\end{cases} \Rightarrow$ $\begin{cases}ax_0+by_0=0\\cx_0+dy_0=0\end{cases}$ and $\begin{cases}ax_1+by_1=0\\cx_1+dy_1=0\end{cases}$; True, since (x_0, y_0) and (x_1, y_1) are solutions to the system.

II. The set $W = \left\{ \begin{bmatrix} s-t \\ s+2t \end{bmatrix} \middle| s,t \in R \right\}$ is a subspace of R^2.

Let $u = \begin{bmatrix} x-y \\ x+2y \end{bmatrix}$ and $v = \begin{bmatrix} a-b \\ a+2b \end{bmatrix}$, where u and v $\in W$. If W is a subspace, then $(u+cv) \in W : (u+cv) = \begin{bmatrix} x-y+ca-cb \\ x+2y+ca+2cb \end{bmatrix} = \begin{bmatrix} x+ca-(y-cb) \\ x+cy+2(y+cb) \end{bmatrix} \in W$. True.

PART 3: ESSAY QUESTIONS

[4×10 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [3+4+3 marks] Consider the matrices $A = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 4 \end{bmatrix}$

a) Find A^{-2};
b) Find the matrix X such that $AX = B^T$;
c) Find BA.

a) $A^{-2} = \left(A^{-1}\right)^2$; $A^{-1} = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$; $\left(A^{-1}\right)^2 = \left(A^{-1}\right)\left(A^{-1}\right) = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 57 & 16 \\ 32 & 9 \end{bmatrix}$.

b) $AX = B^T \Rightarrow X = A^{-1}B^T = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 13 & -12 & 8 \\ 7 & -7 & 4 \end{bmatrix}$.

c) $BA = \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 11 & -19 \\ 6 & -11 \\ 16 & -28 \end{bmatrix}$.

Q–2: [4+3+3 marks] Consider the linear system: $\begin{cases} x+y+2z=9 \\ 2x+4y-3z=1 \\ 3x+6y-5z=0 \end{cases}$

a) Calculate the determinant of the coefficient matrix A;
b) Find $det(-2A^2A^TA^{-1})$.
c) Let $C = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$, find the eigenvalues of C.

a) Expand along the first row:

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{vmatrix} = (-1)^2(1)\begin{vmatrix} 4 & -3 \\ 6 & -5 \end{vmatrix} + (-1)^3(1)\begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix} + (-1)^4(2)\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} =$$

$$(-20+18)-(-10+9)+2(12-12) = -2+1+0 = -1$$

b) $\det\left(-2A^2A^TA^{-1}\right) = (-2)^3|A||A||A|\dfrac{1}{|A|} = -8(-1)(-1) = -8.$

c) $|C-\lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 5\lambda = 0 \Rightarrow \lambda_1 = 0$ and $\lambda_2 = -5.$

Q–3: [3+4+3 marks] Let $W = \left\{ \begin{bmatrix} s-t \\ s+t \\ s+2t \end{bmatrix} \middle| \, s,t \in R \right\}$ be a subset of $\mathbf{R}^3$.

a) Show that W is a subspace of $\mathbf{R}^3$;

b) Let $S = \{v_1, v_2, v_3\}$ where $v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$. Find all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $span(S)$;

c) Are the vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, and $\boldsymbol{v_3}$ linearly independent? Explain.

a) Let $u=\begin{bmatrix} a-b \\ a+b \\ a+2b \end{bmatrix}$ and $v=\begin{bmatrix} x-y \\ x+y \\ x+2y \end{bmatrix}$ where $a,b,x,$ and $y\in R$ and vectors u and $v\in W$.

If W is a subspace of R^3, then $(u+cv)\in W$.

$$(u+cv)=\begin{bmatrix} a-b+cx-cy \\ a+b+cx+cy \\ a+2b+cx+2cy \end{bmatrix}=\begin{bmatrix} a+cx-(b+cy) \\ a+cx+(b+cy) \\ a+cx+2(b+cy) \end{bmatrix}\in W.$$

b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ -1 & -3 & 3 & b \\ 2 & 1 & 4 & c \end{array}\right] \Rightarrow \begin{matrix} R1+R2\to R2 \\ -2R1+R3\to R3 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 2 & a+b \\ 0 & -3 & 6 & -2a+c \end{array}\right] \Rightarrow$

$$-3R2+R3\to R3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 2 & a+b \\ 0 & 0 & 0 & -5a-3b+c \end{array}\right] \Rightarrow -5a-3b+c=0$$

and $c=5a+3b \Rightarrow Span(S)=\left\{\left.\begin{bmatrix} a \\ b \\ c \end{bmatrix}\,\right|\, c=5a+3b\right\}$.

c) $\begin{vmatrix} 1 & 2 & -1 \\ -1 & -3 & 3 \\ 2 & 1 & 4 \end{vmatrix}=(-1)^2(1)\begin{vmatrix} -3 & 3 \\ 1 & 4 \end{vmatrix}+(-1)^3(2)\begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix}+(-1)^4(-1)\begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix}=$

$(-12-3)-2(-4-6)-(-1+6)=-15+20-5=0$. The vectors are linearly dependent since the determinant is equal to zero.

Q–4: [4+3+3 marks] Define $T:\mathbf{R}^2\to\mathbf{R}^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} -x+y \\ y \end{bmatrix}$.

a) Show that T is a linear transformation;

b) Define a linear operator $T:R^3\to R^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$, is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ in $R(T)$?

c) For the transformation in part ***b***, give a description for the ***range*** of ***T***.

a) $T(u+cv)=T(u)+cT(v)$; Where $u=\begin{bmatrix}x\\y\end{bmatrix}$ and $v=\begin{bmatrix}r\\s\end{bmatrix}$;

$$T(u+cv)=T\begin{bmatrix}x+cr\\y+cs\end{bmatrix}=\begin{bmatrix}-x-cr+y+cs\\y+cs\end{bmatrix}=\begin{bmatrix}-x+y+c(-r+s)\\y+cs\end{bmatrix}=\begin{bmatrix}-x+y\\y\end{bmatrix}+c\begin{bmatrix}-r+s\\s\end{bmatrix}=T(u)+cT(v).$$

b) $\left[\begin{array}{ccc|c}1&1&0&2\\0&1&0&1\\1&0&-1&-2\end{array}\right]\Rightarrow c_2=1;\ c_1+c_2=2\Rightarrow c_1=-c_2+2=-1+2=1;\ c_1-c_3=-2\Rightarrow c_3=c_1+2=1+2=3.$

Therefore, $\begin{bmatrix}2\\1\\-2\end{bmatrix}\in R(T)$.

c) $R(T)=\begin{bmatrix}x+y\\y\\x-z\end{bmatrix}=x\begin{bmatrix}1\\0\\1\end{bmatrix}+y\begin{bmatrix}1\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\-1\end{bmatrix}\Rightarrow R(T)=span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\-1\end{bmatrix}\right\}.$

Q–5: [3+4+3 marks] Consider the linear system $\begin{cases}x+3y=-1\\x+2y+4z=2\\-y+3z=1\end{cases}$

a) Solve the system by the elimination method;

b) Write the coefficient matrix ***A*** and find its inverse;

c) If $b=\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ and ***A*** is the coefficient matrix from part ***b***, write the system in matrix form ***(Ax = b)*** and find its solution ***x***.

a) $\left[\begin{array}{ccc|c}1&3&0&-1\\1&2&4&2\\0&-1&3&1\end{array}\right]\Rightarrow -R1+R2\to R2\Rightarrow\left[\begin{array}{ccc|c}1&3&0&-1\\0&-1&4&3\\0&-1&3&1\end{array}\right]\Rightarrow -R2+R3\to R3\Rightarrow\left[\begin{array}{ccc|c}1&3&0&-1\\0&-1&4&3\\0&0&-1&-2\end{array}\right]\Rightarrow$

From Row 3: $z=2$; From Row 2: $-y+4z=3\Rightarrow y=-4z+3=5$; From Row 1: $x+3y=-1\Rightarrow x=-16$.

$S=\{-16,5,2\}$.

b) $\left[\begin{array}{ccc|ccc}1&3&0&1&0&0\\1&2&4&0&1&0\\0&-1&3&0&0&1\end{array}\right]\Rightarrow -R1+R2\to R2\Rightarrow\left[\begin{array}{ccc|ccc}1&3&0&1&0&0\\0&-1&4&-1&1&0\\0&-1&3&0&0&1\end{array}\right]\Rightarrow -R2+R3\to R3\Rightarrow\left[\begin{array}{ccc|ccc}1&3&0&1&0&0\\0&-1&4&-1&1&0\\0&0&-1&1&-1&1\end{array}\right]\Rightarrow$

$\begin{matrix}-R2\to R2\\-R3\to R3\end{matrix}\Rightarrow\left[\begin{array}{ccc|ccc}1&3&0&1&0&0\\0&1&-4&1&-1&0\\0&0&1&-1&1&-1\end{array}\right]\Rightarrow -3R2+R1\to R1\Rightarrow\left[\begin{array}{ccc|ccc}1&0&12&-2&3&0\\0&1&-4&1&-1&0\\0&0&1&-1&1&-1\end{array}\right]\Rightarrow\begin{matrix}-12R3+R1\to R1\\4R3+R2\to R2\end{matrix}\Rightarrow$

$\left[\begin{array}{ccc|ccc}1&0&0&10&-9&12\\0&1&0&-3&3&-4\\0&0&1&-1&1&-1\end{array}\right]\Rightarrow A^{-1}=\begin{bmatrix}10&-9&12\\-3&3&-4\\-1&1&-1\end{bmatrix}.$

c) $x=A^{-1}b=\begin{bmatrix}10&-9&12\\-3&3&-4\\-1&1&-1\end{bmatrix}\begin{bmatrix}-1\\-1\\1\end{bmatrix}=\begin{bmatrix}11\\-4\\-1\end{bmatrix}.$

End of Questions

PART 1: MULTIPLE CHOICE QUESTIONS

[5×1 marks] Choose the correct answer:

I. The inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is

a) $\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

b) $-1\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

c) $\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

e) None of the above

II. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & -3 & -4 \end{bmatrix}$, then $det(A) =$

a) -1
b) 1
c) 2
d) 0
e) None of the above

III. The vector $V = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ forms a linear combination with the vectors $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ where $V = c_1V_1 + c_2V_2$ when the pair $(c_1,c_2) =$

a) (2,3)
b) (2,1)
c) (1,2)
d) (2,-1)
e) None of the above

IV. An eigenvalue for the matrix A = $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

V. If $T:\mathbf{R}^2 \to \mathbf{R}^2$ is defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a-b \\ a+3b \end{bmatrix}$, then $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$

e) None of the above

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×2.5 marks] Answer the following as True or False (Justify your answer):

I. The determinant of the matrix $\begin{bmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{bmatrix}$ is zero.

$$\det = (1)\begin{vmatrix} x+z & x+y \\ y & z \end{vmatrix} - 1\begin{vmatrix} y+z & x+y \\ x & z \end{vmatrix} + \begin{vmatrix} y+z & x+z \\ x & y \end{vmatrix} =$$

$$xz + z^2 - xy - y^2 - yz - z^2 + x^2 + xy + y^2 + yz - x^2 - xz = 0$$

True.

II. The set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set of $\boldsymbol{R}^3$.

$$c_1v_1+c_2v_2=0\rightarrow\left[\begin{array}{cc|c}1&-1&0\\1&2&0\\0&0&0\end{array}\right]\rightarrow -R1+R2\rightarrow R2\rightarrow\left[\begin{array}{cc|c}1&-1&0\\0&3&0\\0&0&0\end{array}\right]$$

Therefore, $c_1=c_2=0$ and the vectors are linearly independent.
True.

PART 3: ESSAY QUESTIONS

[4×10 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [6+4 marks] Consider the linear system $\begin{cases} x-2z=2 \\ x+y-2z=1 \\ 2x-2y-2z=-1 \end{cases}$.

a) Write the coefficient matrix A and Find A^{-1}.

$$\left[\begin{array}{ccc|ccc}1&0&-2&1&0&0\\1&1&-2&0&1&0\\2&-2&-2&0&0&1\end{array}\right]\begin{array}{c}-R1+R2\rightarrow R2\\-2R1+R3\rightarrow R3\end{array}\left[\begin{array}{ccc|ccc}1&0&-2&1&0&0\\0&1&0&-1&1&0\\0&-2&2&-2&0&1\end{array}\right]2R2+R3\rightarrow R3$$

$$\left[\begin{array}{ccc|ccc}1&0&-2&1&0&0\\0&1&0&-1&1&0\\0&0&2&-4&2&1\end{array}\right]\frac{1}{2}R3\rightarrow R3\left[\begin{array}{ccc|ccc}1&0&-2&1&0&0\\0&1&0&-1&1&0\\0&0&1&-2&1&0.5\end{array}\right]2R3+R1\rightarrow R1$$

$$\left[\begin{array}{ccc|ccc}1&0&0&-3&2&1\\0&1&0&-1&1&0\\0&0&1&-2&1&0.5\end{array}\right]\rightarrow A^{-1}=\begin{bmatrix}-3&2&1\\-1&1&0\\-2&1&0.5\end{bmatrix}$$

b) Write the system in matrix form *(Ax = b)* and find the solution set.

$$x=A^{-1}b=\begin{bmatrix}-3&2&1\\-1&1&0\\-2&1&0.5\end{bmatrix}\begin{bmatrix}2\\1\\-1\end{bmatrix}=\begin{bmatrix}-5\\-1\\-3.5\end{bmatrix}\rightarrow S=\{-5,-1,-3.5\}$$

Q–2: [6+2+2 marks] Let $A=\begin{bmatrix}1&2&3&2\\2&3&3&1\\1&1&1&2\end{bmatrix}$.

a) Find a matrix B in reduced row echelon form that is row equivalent to A.

$$\begin{bmatrix}1&2&3&2\\2&3&3&1\\1&1&1&2\end{bmatrix}\begin{matrix}-2R1+R2\to R2\\-R1+R3\to R3\end{matrix}\begin{bmatrix}1&2&3&2\\0&-1&-3&-3\\0&-1&-2&0\end{bmatrix}-R2\to R2\begin{bmatrix}1&2&3&2\\0&1&3&3\\0&-1&-2&0\end{bmatrix}R2+R3\to R3$$

$$\begin{bmatrix}1&2&3&2\\0&1&3&3\\0&0&1&3\end{bmatrix}-2R2+R1\to R1\begin{bmatrix}1&0&-3&-4\\0&1&3&3\\0&0&1&3\end{bmatrix}\begin{matrix}3R3+R1\to R1\\-3R3+R2\to R2\end{matrix}\begin{bmatrix}1&0&0&5\\0&1&0&-6\\0&0&1&3\end{bmatrix}=B$$

b) Deduce a basis for the column vectors of matrix A.

From part a), a basis is given by $\{C_1, C_2, C_3\}$.

c) Are the column vectors C_1, C_2 and C_3 linearly independent? Explain.

Yes, from part a), C_1, C_2, and C_3 are linearly independent since they were row reduced to the identity matrix.

Q–3: [6+2+2 marks] Let $W = \left\{ \begin{bmatrix} a+b \\ b \\ a-3b \end{bmatrix} \middle| \, a, b \in R \right\}$ be a subset of $\mathbf{R}^3$.

a) Show that W is a subspace of $\mathbf{R}^3$.

$(u+cv) \in W$? Let $u = \begin{bmatrix} a+b \\ b \\ a-3b \end{bmatrix}, v = \begin{bmatrix} x+y \\ y \\ x-3y \end{bmatrix} \Rightarrow u+cv = \begin{bmatrix} a+b \\ b \\ a-3b \end{bmatrix} + c\begin{bmatrix} x+y \\ y \\ x-3y \end{bmatrix} =$

$\begin{bmatrix} a+b+cx+cy \\ b+cy \\ a-3b+cx-3cy \end{bmatrix} = \begin{bmatrix} a+cx+(b+cy) \\ b+cy \\ a+cx-3(b+cy) \end{bmatrix}$ is in W. Then W is a subspace.

b) Find a set S such that $W =$ span S.

$W = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ -3b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$. Therefore, $W = span\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}\right\}$.

c) Is the vector $V = \begin{bmatrix} 1 \\ -2 \\ 9 \end{bmatrix}$ in W? Explain.

Yes, $3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 9 \end{bmatrix}$

Q–4: [6+4 marks] Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ y+z \end{bmatrix}$.

a) Show that T is a linear transformation.

$$T(u+cv)=T(u)+cT(v);\ \text{Let } u=\begin{bmatrix} x\\ y\\ z\end{bmatrix}, v=\begin{bmatrix} d\\ e\\ f\end{bmatrix}.$$

$$T\left(\begin{bmatrix} x+cd\\ y+ce\\ z+cf\end{bmatrix}\right)=\begin{bmatrix} 2(x+cd)-(y+ce)\\ (y+ce)+(z+cf)\end{bmatrix}=\begin{bmatrix} 2x-y+c(2d-e)\\ (y+z)+c(e+f)\end{bmatrix}=T(u)+cT(v).$$

b) Find the null space $N(T)$.

$$\begin{bmatrix} 2x-y\\ y+z\end{bmatrix}=\begin{bmatrix} 0\\ 0\end{bmatrix}\to z=t, y=-t, x=-\frac{t}{2}\,|\,t\in R.$$

$$N(T)=\left\{\begin{bmatrix} -\frac{t}{2}\\ -t\\ t\end{bmatrix}\,|\,t\in R\right\}.$$

Q–5: [3+3+4 marks] Let $A=\begin{bmatrix} 1 & 4\\ 1 & 5\end{bmatrix}$, $B=\begin{bmatrix} 3 & -1\\ 2 & -2\\ 3 & 0\end{bmatrix}$

a) Find A^{-1}

$$A^{-1}=\begin{bmatrix} 5 & -4\\ -1 & 1\end{bmatrix}$$

b) Find a matrix X such that $XA = B$.

$$X=BA^{-1}=\begin{bmatrix} 3 & -1\\ 2 & -2\\ 3 & 0\end{bmatrix}\begin{bmatrix} 5 & -4\\ -1 & 1\end{bmatrix}=\begin{bmatrix} 16 & -13\\ 12 & -10\\ 15 & -12\end{bmatrix}$$

c) Find the eigenvalues of $C=\begin{bmatrix} 1 & 5\\ 1 & 5\end{bmatrix}$.

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix} 1-\lambda & 5\\ 1 & 5-\lambda\end{vmatrix}=0\Rightarrow\lambda(\lambda-6)=0\Rightarrow\lambda_1=0,\lambda_2=6.$$

PART 1: MULTIPLE CHOICE QUESTIONS

[5×1 marks] Choose the correct answer:

I. Given the vectors $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, then the vector $\vec{w} = 2\vec{u} - 3\vec{v}$ is equal to:

a) $\begin{bmatrix} -4 \\ -11 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 4 \\ 11 \\ -4 \end{bmatrix}$

c) $\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

e) None of the above

II. Suppose A is a (3×3) matrix such that $det(2A^{-1}) = 3$, *then* $det(A)$ is equal to:

a) -1
b) 1
c) 2
d) 0
e) None of the above

III. The vector $V = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$ forms a linear combination with the vectors $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ where $V = c_1V_1 + c_2V_2$ when the pair $(c_1,c_2) =$

a) (2,3)
b) (2,1)
c) (1,2)
d) (2,-1)
e) None of the above

IV. The eigenvalues for the matrix A = $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ are:

a) 0
b) 1
c) 2
d) 3
e) None of the above

V. If T is a linear operator; $T : \mathbf{R}^2 \to \mathbf{R}^2$ where $T(v_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T(v_2) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $T(v_1 - 2v_2)$ is equal to:

a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$

e) None of the above

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×2.5 marks] Answer the following as True or False (Justify your answer):

I. Let A and B be n x n matrices such that $B = I - A$ and $A^2 = A$; Then $AB = BA = 0$.

$AB = BA$

$A(I - A) = (I - A)A$

$AI - A^2 = IA - A^2$

$A - A^2 = A - A^2$; Where $A^2 = A$

Therefore, $A - A = A - A = 0 \Rightarrow$ True.

II. Let A and C be invertible matrices of the same size such that $det(A) = 4$; Then $det(C) = \pm 2$ if $A^2 = C^2A^T$.

$|A^2| = |C^2||A| \Rightarrow |C^2| = |A| = 4 \Rightarrow |C| = \pm 2$ True.

PART 3: ESSAY QUESTIONS

[4×10 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [6+4 marks] Consider the matrix $A=\begin{bmatrix}-3&0&5\\-1&1&2\\-1&2&2\end{bmatrix}$.

a) Compute the inverse of the matrix A.

$$\left[\begin{array}{ccc|ccc}-3&0&5&1&0&0\\-1&1&2&0&1&0\\-1&2&2&0&0&1\end{array}\right]\Rightarrow R1\leftrightarrow R3\Rightarrow\left[\begin{array}{ccc|ccc}-1&2&2&0&0&1\\-1&1&2&0&1&0\\-3&0&5&1&0&0\end{array}\right]\Rightarrow -1R1\to R1\Rightarrow$$

$$\left[\begin{array}{ccc|ccc}1&-2&-2&0&0&-1\\-1&1&2&0&1&0\\-3&0&5&1&0&0\end{array}\right]\Rightarrow\begin{array}{c}R1+R2\to R3\\3R1+R3\to R3\end{array}\Rightarrow\left[\begin{array}{ccc|ccc}1&-2&-2&0&0&-1\\0&-1&0&0&1&-1\\0&-6&-1&1&0&-3\end{array}\right]\Rightarrow$$

$$-1R2\to R2\Rightarrow\left[\begin{array}{ccc|ccc}1&-2&-2&0&0&-1\\0&1&0&0&-1&1\\0&-6&-1&1&0&-3\end{array}\right]\Rightarrow 6R2+R3\to R3\Rightarrow\left[\begin{array}{ccc|ccc}1&-2&-2&0&0&-1\\0&1&0&0&-1&1\\0&0&-1&1&-6&3\end{array}\right]\Rightarrow$$

$$-1R3\to R3\Rightarrow\left[\begin{array}{ccc|ccc}1&-2&-2&0&0&-1\\0&1&0&0&-1&1\\0&0&1&-1&6&-3\end{array}\right]\Rightarrow 2R2+R1\to R1\Rightarrow\left[\begin{array}{ccc|ccc}1&0&-2&0&-2&1\\0&1&0&0&-1&1\\0&0&1&-1&6&-3\end{array}\right]\Rightarrow$$

$$2R3+R1\to R1\Rightarrow\left[\begin{array}{ccc|ccc}1&0&-2&-2&10&-5\\0&1&0&0&-1&1\\0&0&1&-1&6&-3\end{array}\right]\Rightarrow A^{-1}=\begin{bmatrix}-2&10&-5\\0&-1&1\\-1&6&-3\end{bmatrix}.$$

b) Find all solutions to the system of linear equations: $\begin{cases}-3x+5z=-2\\-x+y+2z=3\\-x+2y+2z=-1\end{cases}$

The system can be written in matrix form as:

$$Ax=b\Rightarrow x=A^{-1}b\Rightarrow x=\begin{bmatrix}-2&10&-5\\0&-1&1\\-1&6&-3\end{bmatrix}\begin{bmatrix}-2\\3\\-1\end{bmatrix}=\begin{bmatrix}39\\-4\\23\end{bmatrix}$$

Therefore, $x=39, y=-4,$ and $z=23$.

Q–2: [6+2+2 marks] Let $A=\begin{bmatrix}1&1&2&2\\2&2&5&5\\0&0&3&3\end{bmatrix}$.

a) Find a matrix B in reduced row echelon form that is row equivalent to A.

$$A=\begin{bmatrix}1&1&2&2\\2&2&5&5\\0&0&3&3\end{bmatrix}\Rightarrow -2R1+R2\to R2\Rightarrow\begin{bmatrix}1&1&2&2\\0&0&1&1\\0&0&3&3\end{bmatrix}\Rightarrow\begin{matrix}-2R2+R1\to R1\\-3R2+R3\to R3\end{matrix}$$

$$\Rightarrow\begin{bmatrix}1&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}\Rightarrow -1R3+R2\to R2\Rightarrow\begin{bmatrix}1&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}=B.$$

b) Deduce a basis for the column vectors of matrix A.

A basis for the column vectors of A are C_1 and C_3, or C_2 and C_4.

c) Are the column vectors C_1 and C_3 linearly independent? Explain.

Yes.

$$aC_1+bC_3=0\Rightarrow\left[\begin{array}{cc|c}1&2&0\\2&5&0\\0&3&0\end{array}\right]\Rightarrow a=b=0;$$

Therefore, the column vectors C_1 and C_3 are linearly independent.

Q–3: [4+2+4 marks] Let $W=\left\{\begin{bmatrix}a-2b\\a\\a+b\end{bmatrix}\;\middle|\; a,b\in R\right\}$ be a subset of $\mathbf{R}^3$.

a) Show that W is a subspace of $\mathbf{R}^3$.

Let u and v be two vectors that belong to W. If $(u+cv)\in W\Rightarrow W$ is a subspace of W.

Let $u=\begin{bmatrix}a-2b\\a\\a+b\end{bmatrix}$ and $v=\begin{bmatrix}x-2y\\x\\x+y\end{bmatrix}\Rightarrow(u+cv)=\begin{bmatrix}a-2b\\a\\a+b\end{bmatrix}+\begin{bmatrix}x-2y\\x\\x+y\end{bmatrix}=$

$\begin{bmatrix}a+cx-2(b+cy)\\a+x\\(a+cx)+(b+cy)\end{bmatrix}\in W$. Therefore, W is a subspace.

b) Find a set S such that $W=\text{span } S$.

$W=\begin{bmatrix}a\\a\\a\end{bmatrix}+\begin{bmatrix}-2b\\0\\b\end{bmatrix}=a\begin{bmatrix}1\\1\\1\end{bmatrix}+b\begin{bmatrix}-2\\0\\1\end{bmatrix}$. Therefore, $W=span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}$.

c) Are the two vectors found in part (b) linearly independent? Explain.

If the vectors are linearly independent $\Rightarrow aV_1+bV_2=0$ where $a=b=0$.

$\left[\begin{array}{cc|c}1&-2&0\\1&0&0\\1&1&0\end{array}\right]\Rightarrow a=b=0$. Therefore, the vectors are linearly independent.

Q–4: [6+4 marks] Define $T:\mathbf{R}^3 \to \mathbf{R}^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ y-x \\ y \end{bmatrix}$.

a) Show that T is a linear transformation.

For any vector u and v in R^3, if $T(u+cv) = T(u)+cT(v) \Rightarrow T$ is a linear transformation.

Let $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$

$$T(u+cv) = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + c\begin{bmatrix} a \\ b \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} x+ac \\ y+bc \\ z+cd \end{bmatrix}\right) = \begin{bmatrix} (x=y+z)+c(a+b+d) \\ (y-x)+c(b-a) \\ y+bc \end{bmatrix} =$$

$$\begin{bmatrix} x+y+z \\ y-x \\ y \end{bmatrix} + c\begin{bmatrix} a+b+d \\ b-a \\ b \end{bmatrix} = T(u)+cT(v).$$

b) Find the null space $N(T)$ and its dimension.

$$\begin{bmatrix} x+y+z \\ y-z \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x=y=z=0 \Rightarrow N(T) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Its dimension is 1.

Q–5: [2+4+3 marks] Let $A = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$.

a) Find A^{-1}

$$A^{-1} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

b) Find a matrix X such that $AX - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$AX - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow AX = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}\begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 19 & 21 \\ 5 & 6 \end{bmatrix}.$$

c) Find the eigenvalues of A.

$$|A-\lambda I|=0\Rightarrow\left|\begin{bmatrix}1&-3\\-1&4\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right|=\begin{vmatrix}1-\lambda&-3\\-1&4-\lambda\end{vmatrix}=0$$

$$\Rightarrow(1-\lambda)(4-\lambda)-3=0\Rightarrow\lambda^2-5\lambda+1=0$$

$$\Rightarrow\lambda_1=\frac{5+\sqrt{21}}{2}\text{ and }\lambda_2=\frac{5-\sqrt{21}}{2}.$$

End of Questions

PART 1: MULTIPLE CHOICE QUESTIONS

[5×1 marks] Choose the correct answer:

I. The real value of x for which the matrix $\begin{bmatrix} 1-x & 2 \\ 1 & 2 \end{bmatrix}$ is singular is

a) -2
b) 0
c) 2
d) 4
e) None of the above

II. If $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \end{bmatrix}$, then $A^2 - 2A + 13I =$

a) -1
b) 1
c) 2
d) 0
e) None of the above

III. The vector $V = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$ forms a linear combination with the vectors

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ where $V = c_1V_1 + c_2V_2$ when the pair $(c_1,c_2) =$

a) (2,3)
b) (3,2)
c) (4,0)
d) (6,1)
e) None of the above

IV. An eigenvalue for the matrix A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

V. If $T:\mathbf{R}^2 \to \mathbf{R}^2$ is defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-b+2 \end{bmatrix}$, then $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

e) None of the above

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×2.5 marks] Answer the following as True or False (Justify your answer):

I. If A is a symmetric n x n real matrix and B is n x m, then B^TAB is a symmetric m x m matrix.

True. Size of B^TAB is $(m \times n)(n \times n)(n \times m) = m \times m$.

II. The subset $S = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| y \in R \right\}$ is a subspace of R^2.

$(u+cv) \in S$? Let $u = \begin{bmatrix} 0 \\ x \end{bmatrix}, v = \begin{bmatrix} 0 \\ y \end{bmatrix} \Rightarrow u + cv = \begin{bmatrix} 0 \\ x+cy \end{bmatrix} \in S.$

Therefore, S is a subspace of R^2. True.

PART 3: ESSAY QUESTIONS

[4×10 marks] Answer ALL FIVE questions. Your grade in this part is the sum of the best FOUR questions.

Q–1: [6+4 marks] Consider the linear system $\begin{cases} -x+z=0 \\ x+y=3 \\ 3x+y-z=3 \end{cases}$.

a) Write the coefficient matrix A and Find A^{-1}.

$$A=\begin{bmatrix}-1&0&1\\1&1&0\\3&1&-1\end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc}-1&0&1&1&0&0\\1&1&0&0&1&0\\3&1&-1&0&0&1\end{array}\right]\xrightarrow{-R1\to R1}\left[\begin{array}{ccc|ccc}1&0&-1&-1&0&0\\1&1&0&0&1&0\\3&1&-1&0&0&1\end{array}\right]\xrightarrow[-3R1+R3\to R3]{-R1+R2\to R2}\left[\begin{array}{ccc|ccc}1&0&-1&-1&0&0\\0&1&1&1&1&0\\0&1&2&3&0&1\end{array}\right]$$

$$\xrightarrow{-R2+R3\to R3}\left[\begin{array}{ccc|ccc}1&0&-1&-1&0&0\\0&1&1&1&1&0\\0&0&1&2&-1&1\end{array}\right]\xrightarrow[-R3+R2\to R2]{R3+R1\to R1}\left[\begin{array}{ccc|ccc}1&0&0&1&-1&1\\0&1&0&-1&2&-1\\0&0&1&2&-1&1\end{array}\right].$$

$$A^{-1}=\begin{bmatrix}1&-1&1\\-1&2&-1\\2&-1&1\end{bmatrix}.$$

b) Write the system in matrix form *($Ax = b$)* and find the solution set.

$x=A^{-1}b=\begin{bmatrix}1&-1&1\\-1&2&-1\\2&-1&1\end{bmatrix}\begin{bmatrix}0\\3\\3\end{bmatrix}=\begin{bmatrix}0\\3\\0\end{bmatrix}$. The solution set is $S=\{0,3,0\}$.

Q–2: [6+2+2 marks] Let $A=\begin{bmatrix}1&1&5&1\\2&-1&1&2\\3&0&6&0\end{bmatrix}$.

a) Find a matrix B in reduced row echelon form that is row equivalent to A.

$$A=\begin{bmatrix}1&1&5&1\\2&-1&1&2\\3&0&6&0\end{bmatrix}\xrightarrow[-3R1+R3\to R3]{-2R1+R2\to R2}\begin{bmatrix}1&1&5&1\\0&-3&-9&0\\0&-3&-9&-3\end{bmatrix}\xrightarrow{-\frac{1}{3}R2\to R2}\begin{bmatrix}1&1&5&1\\0&1&3&0\\0&-3&-9&-3\end{bmatrix}$$

$$\xrightarrow{3R2+R3\to R3}\begin{bmatrix}1&1&5&1\\0&1&3&0\\0&0&0&-3\end{bmatrix}\xrightarrow{-\frac{1}{5}R3\to R3}\begin{bmatrix}1&1&1&5\\0&1&3&0\\0&0&0&1\end{bmatrix}\xrightarrow{-R2+R1\to R1}$$

$$\begin{bmatrix}1&0&2&1\\0&1&3&0\\0&0&0&1\end{bmatrix}\xrightarrow{-R3+R1\to R1}\begin{bmatrix}1&0&2&0\\0&1&3&0\\0&0&0&1\end{bmatrix}.$$

b) Deduce a basis for the column vectors of matrix A.

From part a), the basis are the column vectors C_1, C_2, and C_4.

c) Are the column vectors C_1, C_2 and C_3 linearly independent? Explain.

No, since they were not row reduced to the identity matrix.

Q–3: [6+2+2 marks] Let $W = \left\{ \begin{bmatrix} x-y \\ x \\ x+y \end{bmatrix} \middle| \; x, y \in R \right\}$ be a subset of $\mathbf{R}^3$.

a) Show that W is a subspace of $\mathbf{R}^3$.

$(u+cv) \in W$? Let $u = \begin{bmatrix} x-y \\ x \\ x+y \end{bmatrix}, v = \begin{bmatrix} a-b \\ a \\ a+b \end{bmatrix}. (u+cv) = \begin{bmatrix} x+a-(y+b) \\ x+a \\ (x+a)+(y+b) \end{bmatrix} \in W.$

Therefore, W is a subspace of R^3.

b) Find a set S such that $W =$ span S.

$W = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Therefore, $W = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

c) Is the vector $V = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ in W? Explain.

Yes, $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$.

Q–4: [6+4 marks] Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ y+z \end{bmatrix}$.

a) Show that T is a linear transformation.

$\mathrm{T(u+cv) = T(u) + cT(v)}$. Let $\mathrm{u} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}, \mathrm{v} = \begin{bmatrix} \mathrm{d} \\ \mathrm{e} \\ \mathrm{f} \end{bmatrix}$

$$\mathrm{T}\left(\begin{bmatrix} \mathrm{x+cd} \\ \mathrm{y+ce} \\ \mathrm{z+cf} \end{bmatrix} \right) = \begin{bmatrix} \mathrm{-(x+cd)} \\ \mathrm{y+ce+z+cf} \end{bmatrix} = \begin{bmatrix} \mathrm{-x-cd} \\ \mathrm{(y+z)+c(e+f)} \end{bmatrix} = \begin{bmatrix} \mathrm{-x} \\ \mathrm{y+z} \end{bmatrix} + \mathrm{c}\begin{bmatrix} \mathrm{-d} \\ \mathrm{e+f} \end{bmatrix}$$

$= \mathrm{T(u) + cT(v)}$.

b) Find the null space $N(T)$.

$\begin{bmatrix} -x \\ y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x = 0, y+z = 0 \Rightarrow z = t \,|\, t \in R$ and $y = -t$.

$N(T) = \left\{ \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \middle| \; t \in R \right\}$.

Q–5: [3+3+4 marks] Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}$

a) Find A^{-1}

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

b) Find a matrix X such that $AX = B$.

$$X = A^{-1}B = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 5 & 8 \\ -1 & -4 & -7 \end{bmatrix}.$$

c) Find the eigenvalues of A.

$$|A - \lambda I| = 0 \to \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0; \to \lambda_1 = 1, \lambda_2 = 3.$$

PART 1: MULTIPLE CHOICE QUESTIONS

[5×1 marks] Choose the correct answer:

I. The value of x for which $\begin{vmatrix} 2x & 8 \\ 1 & 2 \end{vmatrix} = 0$ is

a) -2
b) 0
c) 2
d) 4
e) None of the above

II. If A and B are nonsingular matrices, then the inverse of AB is
a) $A^{-1}B^{-1}$
b) $B^{-1}A^{-1}$
c) $A^{-1}B$
d) $B^{-1}A$
e) None of the above

III. Two vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ c \end{bmatrix}$ are linearly dependent if c =

a) 2
b) 3
c) 4
d) 6
e) None of the above

IV. The dimension of the subspace $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} \right\}$ is

a) 0
b) 1
c) 2
d) 3
e) None of the above

V. If $T : \mathbf{R}^3 \to \mathbf{R}^2$ is define by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a+b \\ b-c \end{bmatrix}$, then $T\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right) =$

a) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

e) None of the above

c,b,d,b,a

PART 2: TRUE AND FALSE WITH JUSTIFICATION

[2×2.5 marks] Answer the following as True or False (Justify your answer):

I. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, then there exists a 2×2 matrix B such that $AB = I_2$.

FALSE: Since A is not row equivalent to I_2, then A has no inverse.

II. The set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^2$.

True:

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \text{is linearly independent and hence a basis.}$$

PART 3: ESSAY QUESTIONS

[4×10 marks] Answer only FOUR out of the following five questions:

Q–1: [6+4 marks] Let $A=\begin{bmatrix}1 & -2 & 0\\ 0 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}$.

a) Find A^{-1}.

$$[A\,|\,I_3]=\left[\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 1 & -2 & 1 & 0 & 0 & 1\end{array}\right]\xrightarrow{R_3-R_1}\left[\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_1+2R_2}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 2 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right]=[I_3\,|\,A^{-1}]$$

$$A^{-1}=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

b) If $CA=\begin{bmatrix}1 & 2 & -1\\ 0 & 6 & 3\end{bmatrix}$, find C^T.

$$C=\begin{bmatrix}1 & 2 & -1\\ 0 & 6 & 3\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 2 & -1\\ 0 & 6 & 3\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 4 & -1\\ -3 & 6 & 3\end{bmatrix}$$

$$C^T=\begin{bmatrix}2 & -3\\ 4 & 6\\ -1 & 3\end{bmatrix}$$

Q–2: [6+4 marks] Let $A=\begin{bmatrix}1 & 2 & -1 & -2\\ -2 & -4 & 0 & 6\\ 2 & 4 & 1 & -7\end{bmatrix}$.

a) Find a matrix B in reduced row echelon form that is row equivalent to A.

$$A=\begin{bmatrix}1&2&-1&-2\\-2&-4&0&6\\2&4&1&-7\end{bmatrix}\xrightarrow[R_3-2R_1]{R_2+2R_1}\begin{bmatrix}1&2&-1&-2\\0&0&-2&2\\0&0&3&-3\end{bmatrix}\xrightarrow{-\frac{1}{2}R_2}\begin{bmatrix}1&2&-1&-2\\0&0&1&-1\\0&0&3&-3\end{bmatrix}$$

$$\xrightarrow[R_3-3R_2]{R_1+R_2}\begin{bmatrix}1&2&0&-3\\0&0&1&-1\\0&0&0&0\end{bmatrix}=B$$

b) Are the column vectors C_1, C_2 and C_3 linearly independent? Explain.
No, from the matrix B, C_1, C_2 and C_3 are linearly dependent.

Q–3: [6+2+2 marks] Let $W=\left\{\begin{bmatrix}a\\b+2a\\b-a\end{bmatrix}\middle|\ a,b\in R\right\}$ be a subset of $\mathbf{R}^3$.

a) Show that W is a subspace of $\mathbf{R}^3$.

$$W=\left\{\begin{bmatrix}a\\b+2a\\b-a\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\\-1\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}=\text{span}\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}\Rightarrow W \text{ is a subspace.}$$

b) Find a set S such that $W = \text{span}\ S$.

$$S=\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}.$$

c) Is the vector $V=\begin{bmatrix}2\\3\\-3\end{bmatrix}$ in W? Explain.

Yes, as $V=\begin{bmatrix}2\\3\\-3\end{bmatrix}=2\begin{bmatrix}1\\2\\-1\end{bmatrix}-1\begin{bmatrix}0\\1\\1\end{bmatrix}$

Q–4: [6+4 marks] Define $T:\mathbf{R}^3\to\mathbf{R}^2$ by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x-y\\2z\end{bmatrix}$.

a) Show that T is a linear transformation.

$$\bullet\, T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - y_1 - y_2 \\ 2z_1 + 2z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - y_1 \\ 2z_1 \end{bmatrix} + \begin{bmatrix} x_2 - y_2 \\ 2z_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right),$$

$$\bullet\, T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cy \\ 2cz \end{bmatrix} = c\begin{bmatrix} x - y \\ 2z \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

b) Find the null space $N(T)$.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x - y \\ 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = y = t,\ z = 0$$

$$\Rightarrow N(T) = \left\{\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \middle|\ t \in R\right\}$$

Q–5: [6+4 marks] Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation defined by $T(X) = AX$, for every vector X in $\mathbf{R}^2$.

i. Find $T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

ii. Find the range $R(T)$.

$$R(T) = \text{span}\left\{T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\} = \mathbf{R}^2$$

b) Find the eigenvalues of A.

$$|\lambda I_2 - A| = 0 \Rightarrow \begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 4) + 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2,\ 3$$

Part 1

[4×1 marks] Choose the correct answer:

I. The matrix $\begin{bmatrix} 2x & 8 \\ 1 & 2 \end{bmatrix}$ is singular if

a) $x = 0$
b) $x = 2$
c) $x = -2$
d) $x = 4$
e) None

II. If A and B are 2×2 matrices with $|A| = 2$ and $|B| = -2$, then $|2A^T B^{-1}| =$

a) 2
b) −2
c) 4
d) −4
e) None

III. A basis for the vector space $\mathbf{R}^2$ is

a) $\left\{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\}$

b) $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$

c) $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

d) $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

e) None

IV. If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation define by
$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$, then $T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

c) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

e) None

Part 2

[2×3 marks] Answer the following as True or False (Justify your answer):

I. For $a = -1$ the linear system corresponding to the augmented matrix $\left[\begin{array}{cc|c} 1 & a & 1 \\ 2 & a-1 & 1 \end{array}\right]$ is consistent.

False: $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right] \Rightarrow$ System is inconsistent.

II. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then $\lambda = 5$ is an eigenvalue of A.

True: $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda = 0 \Rightarrow \lambda = 0,5$.

Part 3

[40 marks] Answer only FOUR questions:

Q–1: [6+4 marks] Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

a) Find A^{-1}.

$$[A\,|\,I_3]=\left[\begin{array}{ccc|ccc}2&1&0&1&0&0\\0&1&-1&0&1&0\\1&1&0&0&0&1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1&1&0&0&0&1\\0&1&-1&0&1&0\\2&1&0&1&0&0\end{array}\right]\xrightarrow{R_3-2R_1}$$

$$\left[\begin{array}{ccc|ccc}1&1&0&0&0&1\\0&1&-1&0&1&0\\0&-1&0&1&0&-2\end{array}\right]\xrightarrow[R_3+R_1]{R_1-R_2}\left[\begin{array}{ccc|ccc}1&0&1&0&-1&1\\0&1&-1&0&1&0\\0&0&-1&1&1&-2\end{array}\right]\xrightarrow{-R_3}$$

$$\left[\begin{array}{ccc|ccc}1&0&1&0&-1&1\\0&1&-1&0&1&0\\0&0&1&-1&-1&2\end{array}\right]\xrightarrow[R_2+R_3]{R_1-R_3}\left[\begin{array}{ccc|ccc}1&0&0&1&0&-1\\0&1&0&-1&0&2\\0&0&1&-1&-1&2\end{array}\right]=[I_3\,|\,A^{-1}]$$

b) Find the matrix X such that $(XA-2I_3)^{-1}=B$.

$$X=(B^{-1}+2I_3)A^{-1}=\begin{bmatrix}4&0&1\\1&4&1\\-2&1&3\end{bmatrix}\begin{bmatrix}1&0&-1\\-1&0&2\\-1&-1&2\end{bmatrix}=\begin{bmatrix}3&-1&-2\\-4&-1&9\\-6&-3&10\end{bmatrix}$$

Q–2: [6+2+2 marks] Let $W=\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}\;\middle|\; a+b-c=0,\ a,b,c\in R\right\}.$

a) Show that W is a subspace of $\mathbf{R}^3$.

$$W=\left\{\begin{bmatrix}a\\b\\a+b\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\0\\1\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}=\text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}\Rightarrow W \text{ is a subspace.}$$

b) Find a set S such that $W=\text{span}\,S$.

$$S=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}.$$

c) What is $\dim W$?

$$S=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}\text{ is a linearly independent set}\Rightarrow \dim W=2.$$

Q–3: [6+2+2 marks] Let S = {X_1, X_2, X_3}, where $X_1=\begin{bmatrix}1\\-2\\1\end{bmatrix}$, $X_2=\begin{bmatrix}2\\-1\\1\end{bmatrix}$, and

$X_3=\begin{bmatrix}-1\\-4\\1\end{bmatrix}$.

a) Determine whether or not S is linearly independent.

$$c_1X_1+c_2X_2+c_3X_3=O\Rightarrow\begin{vmatrix}1&2&-1\\-2&-1&-4\\1&1&1\end{vmatrix}=0\Rightarrow S\text{ is linearly dependent set.}$$

b) Is X_3 a linear combination of X_1 and X_2? Explain.

Yes as S is linearly dependent set.

c) Find a basis for W = span S.

$$\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix},\begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$$

Q–4: [6+2+2 marks] Define $T:\mathbf{R}^3\to\mathbf{R}^3$ by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+y\\y-z\\2z\end{bmatrix}$.

a) Show that T is a linear transformation.

$$\bullet\, T\left(\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\end{bmatrix}\right)=\begin{bmatrix}x_1+x_2+y_1+y_2\\y_1+y_2-z_1-z_2\\2z_1+2z_2\end{bmatrix}$$

$$=\begin{bmatrix}x_1+y_1\\y_1-z_1\\2z_1\end{bmatrix}+\begin{bmatrix}x_2+y_2\\y_2-z_2\\2z_2\end{bmatrix}=T\left(\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}\right)+T\left(\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}\right),$$

$$\bullet\, T\left(c\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=T\left(\begin{bmatrix}cx\\cy\\cz\end{bmatrix}\right)=\begin{bmatrix}cx+cy\\cy-cz\\2cz\end{bmatrix}=c\begin{bmatrix}x+y\\y-z\\2z\end{bmatrix}=cT\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right).$$

b) Find the range R(T).

$$R(T)=span\left\{T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right),T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right),T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right\}=span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\2\end{bmatrix}\right\}=\mathbf{R}^3.$$

c) Find the null space N(T).

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}x+y\\y-z\\2z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow x=y=z=0\Rightarrow N(T)=\{O\}.$$

Q–5: [6+2+2 marks] Let $A^{-1}=\begin{bmatrix}1&2&-2&3\\1&2&-1&0\\2&5&-6&4\\-2&-4&4&-1\end{bmatrix}$.

a) Find $|A^T|$.

$$|A^{-1}|=\begin{vmatrix}1&2&-2&3\\1&2&-1&0\\2&5&-6&4\\-2&-4&4&-1\end{vmatrix}\overset{\substack{R_2-R_1\\R_3-2R_1\\R_4+2R_1}}{=}\begin{vmatrix}1&2&-2&1\\0&0&1&-3\\0&1&-2&-2\\0&0&0&5\end{vmatrix}\overset{R_2\leftrightarrow R_3}{=}-\begin{vmatrix}1&2&-2&1\\0&1&-2&-2\\0&0&1&-3\\0&0&0&5\end{vmatrix}=-5$$

$$\Rightarrow|A|=-\frac{1}{5}=|A^T|.$$

b) What is the matrix in reduced row echelon form that is row equivalent to A?

Since A is nonsingular, then A is row equivalent to I_4.

c) Find the solution set of the linear system $AX=O$.

Since A is nonsingular, then the solution set is $S=\{O\}$.

Part 1

[4×1 marks] Choose the correct answer:

I. The matrix $\begin{bmatrix} 2x & 8 \\ 1 & 2 \end{bmatrix}$ is singular if

a) $x = 0$
b) $x = 2$
c) $x = -2$
d) $x = 4$
e) None

II. If A and B are 2×2 matrices with $|A| = 2$ and $|B| = -2$, then $|2A^T B^{-1}| =$

a) 2
b) −2
c) 4
d) −4
e) None

III. A basis for the vector space $\mathbf{R}^2$ is

a) $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$
b) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$
c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
d) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
e) None

IV. If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation define by

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$, then $T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) =$

a) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

c) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

d) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

e) None

Part 2

[2×3 marks] Answer the following as True or False (Justify your answer):

I. For $a = -1$ the linear system corresponding to the augmented matrix $\left[\begin{array}{cc|c} 1 & a & 1 \\ 2 & a-1 & 1 \end{array}\right]$ is consistent.

False: $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right] \Rightarrow$ System is inconsistent.

II. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then $\lambda = 5$ is an eigenvalue of A.

True: $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda = 0 \Rightarrow \lambda = 0{,}5$.

Part 3

[40 marks] Answer only FOUR questions:

Q–1: [6+4 marks] Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

a) Find A^{-1}.

$$[A\,|\,I_3]=\left[\begin{array}{ccc|ccc}2 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 0 & 1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 0 & 0 & 1\\ 0 & 1 & -1 & 0 & 1 & 0\\ 2 & 1 & 0 & 1 & 0 & 0\end{array}\right]\xrightarrow{R_3-2R_1}$$

$$\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 0 & 0 & 1\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1 & 0 & -2\end{array}\right]\xrightarrow[R_3+R_1]{R_1-R_2}\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & -1 & 1\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 0 & -1 & 1 & 1 & -2\end{array}\right]\xrightarrow{-R_3}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & -1 & 1\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & -1 & 2\end{array}\right]\xrightarrow[R_2+R_3]{R_1-R_3}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & -1\\ 0 & 1 & 0 & -1 & 0 & 2\\ 0 & 0 & 1 & -1 & -1 & 2\end{array}\right]=[I_3\,|\,A^{-1}]$$

b) Find the matrix X such that $(XA-2I_3)^{-1}=B$.

$$X=(B^{-1}+2I_3)A^{-1}=\begin{bmatrix}4 & 0 & 1\\ 1 & 4 & 1\\ -2 & 1 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\ -1 & 0 & 2\\ -1 & -1 & 2\end{bmatrix}=\begin{bmatrix}3 & -1 & -2\\ -4 & -1 & 9\\ -6 & -3 & 10\end{bmatrix}$$

Q–2: [6+2+2 marks] Let $W=\left\{\begin{bmatrix}a\\ b\\ c\end{bmatrix}\;\middle|\; a+b-c=0,\ a,b,c\in R\right\}$.

a) Show that W is a subspace of $\mathbf{R}^3$.

$$W=\left\{\begin{bmatrix}a\\ b\\ a+b\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}+b\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}\right\}=\operatorname{span}\left\{\begin{bmatrix}1\\ 0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}\right\}\Rightarrow W \text{ is a subspace.}$$

b) Find a set S such that $W=\operatorname{span} S$.

$$S=\left\{\begin{bmatrix}1\\ 0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}\right\}.$$

c) What is $\dim W$?

$$S=\left\{\begin{bmatrix}1\\ 0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}\right\} \text{ is a linearly independent set} \Rightarrow \dim W=2.$$

Q–3: [6+2+2 marks] Let S = $\{X_1, X_2, X_3\}$, where $X_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, and $X_3 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$.

a) Determine whether or not S is linearly independent.

$$c_1X_1 + c_2X_2 + c_3X_3 = O \Rightarrow \begin{vmatrix} 1 & 2 & -1 \\ -2 & -1 & -4 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow S \text{ is linearly dependent set.}$$

b) Is X_3 a linear combination of X_1 and X_2? Explain.

Yes as S is linearly dependent set.

c) Find a basis for W = span S.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Q–4: [6+2+2 marks] Define $T : \mathbf{R}^3 \to \mathbf{R}^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \\ 2z \end{bmatrix}$.

a) Show that T is a linear transformation.

$$\bullet\ T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ y_1+y_2-z_1-z_2 \\ 2z_1+2z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+y_1 \\ y_1-z_1 \\ 2z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ y_2-z_2 \\ 2z_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right),$$

$$\bullet\ T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx+cy \\ cy-cz \\ 2cz \end{bmatrix} = c\begin{bmatrix} x+y \\ y-z \\ 2z \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right).$$

b) Find the range R(T).

$$R(T) = span\left\{T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right\} = span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\2\end{bmatrix}\right\} = \mathbf{R}^3.$$

c) Find the null space N(T).

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}x+y\\y-z\\2z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow x = y = z = 0 \Rightarrow N(T) = \{O\}.$$

Q–5: [6+2+2 marks] Let $A^{-1} = \begin{bmatrix}1 & 2 & -2 & 3\\ 1 & 2 & -1 & 0\\ 2 & 5 & -6 & 4\\ -2 & -4 & 4 & -1\end{bmatrix}$.

a) Find $|A^T|$.

$$|A^{-1}| = \begin{vmatrix}1 & 2 & -2 & 3\\ 1 & 2 & -1 & 0\\ 2 & 5 & -6 & 4\\ -2 & -4 & 4 & -1\end{vmatrix} \overset{\substack{R_2-R_1\\R_3-2R_1\\R_4+2R_1}}{=} \begin{vmatrix}1 & 2 & -2 & 1\\ 0 & 0 & 1 & -3\\ 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 5\end{vmatrix} \overset{R_2\leftrightarrow R_3}{=} -\begin{vmatrix}1 & 2 & -2 & 1\\ 0 & 1 & -2 & -2\\ 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 5\end{vmatrix} = -5$$

$$\Rightarrow |A| = -\frac{1}{5} = |A^T|.$$

b) What is the matrix in reduced row echelon form that is row equivalent to A?

Since A is nonsingular, then A is row equivalent to I_4.

c) Find the solution set of the linear system $AX = O$.

Since A is nonsingular, then the solution set is $S = \{O\}$.

Part 1

[10 points] Answer only FOUR of the following as True or False (Justify your answer):

a) [2.5 points] The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is the inverse of the matrix

$$B = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{TRUE: As } AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

b) [2.5 points] Let A and B be 3×3 matrices such that $|A| = 3$ and $|B| = -4$, then $|2A^T B^{-1}| = -6$.

$$\text{TRUE: } \left|2A^T B^{-1}\right| = 8\left|A^T B^{-1}\right| = \frac{8|A|}{|B|} = -6.$$

c) [2.5 points] The set $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$ is a basis for $\mathbf{R}^2$.

$$\text{TRUE: } c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \neq 0 \Rightarrow \text{Two vectors are independent, so basis.}$$

d) [2.5 points] The set $V = \mathbf{R}^2$ with addition defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix}$$

and scalar multiplication the standard operation on $\mathbf{R}^2$ is a vector space.

$$\text{FALSE: As} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

e) [2.5 points] The mapping $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1-x \\ y \end{bmatrix}$ is a linear transformation.

$$\text{FALSE: As T}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Part 2

[40 points] Answer only FOUR questions:

Q–1 [5+2+3 points] Let $A=\begin{bmatrix}1&1&2\\2&0&1\\0&-1&2\end{bmatrix}$ and $B=\begin{bmatrix}1&0&2\\2&0&3\\0&1&1\end{bmatrix}$.

a) Find B^{-1}.

$$[B|I_3]=\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\2&0&3&0&1&0\\0&1&1&0&0&1\end{array}\right]\xrightarrow{R_2-2R_1}\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&0&-1&-2&1&0\\0&1&1&0&0&1\end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&1&0&0&1\\0&0&-1&-2&1&0\end{array}\right]\xrightarrow{-R_3}\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&1&0&0&1\\0&0&1&2&-1&0\end{array}\right]$$

$$\xrightarrow[R_2-R_3]{R_1-2R_3}\left[\begin{array}{ccc|ccc}1&0&0&-3&2&0\\0&1&0&-2&1&1\\0&0&1&2&-1&0\end{array}\right]=[I_3|B^{-1}]$$

b) Compute $3A-2B$.

$$3A-2B=\begin{bmatrix}3&3&6\\6&0&3\\0&-3&6\end{bmatrix}-\begin{bmatrix}2&0&4\\4&0&6\\0&2&2\end{bmatrix}=\begin{bmatrix}1&3&2\\2&0&-3\\0&-5&4\end{bmatrix}.$$

c) Find the matrix X such that $XB=2A$.

$$XB=2A\Rightarrow X=2AB^{-1}=2\begin{bmatrix}1&1&2\\2&0&1\\0&-1&2\end{bmatrix}\begin{bmatrix}-3&2&0\\-2&1&1\\2&-1&0\end{bmatrix}=\begin{bmatrix}-2&2&2\\-8&6&0\\12&-6&-2\end{bmatrix}.$$

Q–2 [5+3+2 points] Let $A=\begin{bmatrix}1&1&1&1\\0&1&2&1\\0&0&1&1\\1&1&0&0\end{bmatrix}$ and $B=\begin{bmatrix}2\\1\\0\\2\end{bmatrix}$.

a) Find a matrix $[C|D]$ in reduced row echelon form that is row equivalent to the matrix $[A|B]$.

$$[A|B]=\left[\begin{array}{cccc|c}1&1&1&1&2\\0&1&2&1&1\\0&0&1&1&0\\1&1&0&0&2\end{array}\right]\xrightarrow{R_4-R_1}\left[\begin{array}{cccc|c}1&1&1&1&2\\0&1&2&1&1\\0&0&1&1&0\\0&0&-1&-1&0\end{array}\right]\xrightarrow{R_1-R_2}$$

$$\left[\begin{array}{cccc|c}1&0&-1&0&1\\0&1&2&1&1\\0&0&1&1&0\\0&0&-1&-1&0\end{array}\right]\xrightarrow[R_4+R_3]{\substack{R_1+R_3\\R_2-2R_3}}\left[\begin{array}{cccc|c}1&0&0&1&1\\0&1&0&-1&1\\0&0&1&1&0\\0&0&0&0&0\end{array}\right]=[C|D].$$

b) Show that the linear system $AX=B$ has infinitely many solutions and find its solution set.

$$S=\left\{\begin{bmatrix}1-t\\1+t\\-t\\t\end{bmatrix}\;\middle|\; t\in R\right\}.$$

c) Find a specific solution of $AX=B$.

$$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}\in S.$$

Q–3 [4+6 points]

a) Determine whether the following set is a subspace of $\mathbf{R}^2$:

$$W=\left\{\begin{bmatrix}a\\b\end{bmatrix}\;\middle|\; b=2a\right\}.$$

$$W=\left\{\begin{bmatrix}a\\2a\end{bmatrix}\;\middle|\; a\in R\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}\;\middle|\; a\in R\right\}=span\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}\Rightarrow \text{Subspace.}$$

b) Let $X_1=\begin{bmatrix}1\\0\\-2\end{bmatrix}$, $X_2=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $X_3=\begin{bmatrix}0\\2\\4\end{bmatrix}$. Show that $X=\begin{bmatrix}3\\4\\10\end{bmatrix}$ is a linear combination of X_1, X_2 and X_3.

$$X=c_1X_1+c_2X_2+c_3X_3\Rightarrow\left[\begin{array}{ccc|c}1&1&0&3\\0&-1&2&4\\-2&0&4&10\end{array}\right]\approx\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&3\end{array}\right]$$

$$\Rightarrow c_1=1,\; c_2=2,\; c_3=3.$$

Q–4 [7+3 points] Let $T:\mathbf{R}^3 \to \mathbf{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix}$.

a) Show that T is a linear transformation.

$$\bullet\ T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ x_1+x_2+y_1+y_2 \\ x_1+x_2+y_1+y_2+z_1+z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_1+y_1 \\ x_1+y_1+z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2+y_2 \\ x_2+y_2+z_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right),$$

$$\bullet\ T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx \\ cx+cy \\ cx+cy+cz \end{bmatrix} = c\begin{bmatrix} x \\ x+y \\ x+y+x \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right).$$

b) Show that the null space $N(T)$ is trivial.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x=0, y=0, z=0.$$

Q–5 [4+3+3 points] Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A.

$$\lambda I_2 - A = \begin{bmatrix} \lambda-1 & -2 \\ 0 & \lambda-3 \end{bmatrix}.\ |\lambda I_2 - A| = 0 \Rightarrow \lambda = 1,3.$$

b) If $T:\mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation defined by $T(X) = AX$, $X \in \mathbf{R}^2$, find $T\left(2\begin{bmatrix} a \\ b \end{bmatrix}\right)$.

$$T\left(2\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2a \\ 2b \end{bmatrix} = \begin{bmatrix} 2a+4b \\ 6b \end{bmatrix}.$$

c) Are the columns of A linearly independent? Explain.

Yes, as $|A| = 3 \neq 0$.

Q–1 [5×2 points] Answer each of the following as True or False (Justify your answer):

a) If A and B are 2×2 matrices, then $(A + B)(A - B) = A^2 - B^2$.

False: $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix},\ (A+B)(A-B)\neq A^2-B^2.$

b) If A and B are square matrices, then $|A - B| = |A| - |B|$.

False: $A=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},\ |A|=3,\ |B|=2, |A-B|=0.$

c) If $X_1 = 2X_2$, then X_1, X_2 and X_3 are linearly dependent.

True: $X_1-2X_2+0X_3=O\Rightarrow$ Vectors are linearly dependent

d) The set $W=\left\{\begin{bmatrix}x\\ y\end{bmatrix}\ \middle|\ x^2+y^2=0\right\}$ is a subspace of R^2.

True: $W=\{O\}$, so subspace.

e) The transformation $T:R^3\to R$ defined by $T\left(\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right)=x+y+z$, is a linear transformation.

True: $T(cX + Y) = cT(X) + Y$ for all X and Y in R^3.

Q–2 [6+4 points] Let $A=\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$, $B=\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$ and $C=\begin{bmatrix}1 & 4\\ 0 & 2\end{bmatrix}$.

a) Show that A, B are nonsingular matrices and find $(A - B)^{-1}$.

$|A|=\begin{vmatrix}1 & 2\\ 3 & 7\end{vmatrix}=1, |B|=\begin{vmatrix}2 & 3\\ 3 & 5\end{vmatrix}=1\Rightarrow A$ and B nonsingular.

$A-B=\begin{bmatrix}-1 & -1\\ 0 & -2\end{bmatrix}.\ (A-B)^{-1}=\frac{1}{2}\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}.$

b) Find the matrix X such that $AXB = C^T$.

$X=A^{-1}C^TB^{-1}=\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 4 & 2\end{bmatrix}\begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}=\begin{bmatrix}7 & -5\\ -1 & 1\end{bmatrix}.$

Q–3 [6+4 points] Let $W = \left\{ \begin{bmatrix} a+2b \\ 0 \\ a-2b \end{bmatrix} \middle| \; a,b \in R \right\}.$

a) Show that W is a subspace of R^3 and find two independent vectors that span W.

$$\begin{bmatrix} a+2b \\ 0 \\ a-2b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 2b \\ 0 \\ -2b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \Rightarrow W = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\} \Rightarrow W \text{ is a subspace.}$$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ are independent vectors that span W.

b) Determine whether or not the vector $V = \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}$ in W.

$$\begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2b \\ 0 \\ a-2b \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 0 & 0 \\ 1 & -2 & 3 \end{bmatrix} \approx\approx \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow a = 5, b = 1$$

$$\Rightarrow \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix} \in W.$$

Q–4 [4+3+3 points] Let $V_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ and $V_4 = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$ be vectors in R^3.

a) Determine whether or not $T = \{V_1, V_2, V_4\}$ is linearly independent.

$$c_1V_1 + c_2V_2 + c_3V_4 = O \Rightarrow \begin{vmatrix} 1 & 2 & -2 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 0 \Rightarrow \text{Linearly dependent.}$$

b) Does the set T span R^3? Justify your answer.

No since V_1, V_2, V_4 are linearly dependent.

c) Is V_3 in span(T)? Justify you answer.

$$\text{No, since } V_3 = c_1V_1 + c_2V_2 + c_4V_4 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ -2 & -1 & -2 & -2 \\ 1 & 1 & 0 & -2 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right].$$

Q–5 [5+2+3 points] Define $T : R^4 \to R^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$.

a) Show that T is a linear transformation.

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 + y_2 \\ z_1 + z_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ z_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ z_2 \\ x_2 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix}\right),$$

$$T\left(c\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \\ cw \end{bmatrix}\right) = \begin{bmatrix} cy \\ cz \\ cx \end{bmatrix} = c\begin{bmatrix} y \\ z \\ x \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right).$$

b) Find $T\left(5\begin{bmatrix} a+b \\ b \\ c+a \\ 0 \end{bmatrix}\right)$.

$$T\left(5\begin{bmatrix} a+b \\ b \\ c+a \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 5a+5b \\ 5b \\ 5c+5a \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5b \\ 5c+5a \\ 5a+5b \end{bmatrix}.$$

c) Find the null space $N(T)$.

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = y = z = 0,\ w = t\ ; t \in R \Rightarrow N(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix} \middle|\ t \in R \right\}.$$

Q–1 [5×2 points] Answer each of the following as True or False (Justify your answer):

a) The matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is singular.

False: $|A| = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 \Rightarrow A$ is nonsingular.

b) The linear system $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is inconsistent.

True: The last equation is $0x + 0y + 0z = 3 \Rightarrow 0 = 3$.

c) If A is a nonsingular matrix such that $A^2 = A^T$, then $|A| = 1$.

True: $|A^2| = |A^T| \Rightarrow |A|^2 = |A| \Rightarrow |A|(|A| - 1) = 0 \Rightarrow |A| = 1$ since A is nonsingular.

d) The vector $X = \begin{bmatrix} 4 \\ -5 \\ -4 \end{bmatrix}$ is a linear combination of $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

True:

$$X = aX_1 + bX_2 \Rightarrow \begin{bmatrix} a+2b \\ a-b \\ -a-2b \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -4 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & -1 & -5 \\ -1 & -2 & -4 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -9 \\ 0 & 0 & 0 \end{array}\right] \approx$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow X = -2X_1 + 3X_2.$$

e) The set $W = \left\{ \begin{bmatrix} 3x \\ 4y \\ 1-x-y \end{bmatrix} : x, y \in R \right\}$ is a subspace of R^3.

False: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$.

Q–2 [6+4 points] Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$.

a) Compute $A^{-1} + 2B^T$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow[r_2 - r_3]{r_1 - r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

$$A^{-1} + 2B^T = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & 6 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 1 & 7 \\ 2 & 1 & 0 \end{bmatrix}.$$

b) Find the matrix X such that $AXB^{-1} = I_3$.

$$X = A^{-1}B = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -3 & 1 & 1 \\ 3 & 2 & -1 \end{bmatrix}.$$

Q–3 [5+5 points]

a) Find all values of a for which the resulting system has
(i) no solution, (ii) a unique solution, (iii) infinitely many solutions,

$$x + 9y = a$$
$$x + a^2 y = 3$$

$$\left[\begin{array}{cc|c} 1 & 9 & a \\ 1 & a^2 & 3 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|c} 1 & 9 & a \\ 0 & a^2 - 9 & 3 - a \end{array}\right]$$

(i) No solution if $a^2 - 9 = 0$ and $3 - a \neq 0 \Rightarrow a = -3$.

(ii) A unique solution if $a^2 - 9 \neq 0 \Rightarrow a \in R - \{\pm 3\}$.

(iii) Infinitely many solutions if $a^2 - 9 = 0$ and $3 - a = 0 \Rightarrow a = 3$.

b) Determine whether or not the vectors $X_1 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$ are linearly dependent.

$$c_1X_1 + c_2X_2 + c_3X_3 = O \Rightarrow \begin{vmatrix} 1 & 0 & 1 \\ 4 & 1 & 1 \\ 2 & 2 & -4 \end{vmatrix} = 0 \Rightarrow \text{Linearly dependent.}$$

Q–4 [4+6 points]

a) let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x = 2y \right\}$. Show that W is a subspace of R^2 and find a vector X so that each vector in W can be written as a linear combination of this vector.

$$W = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \middle| \, x \in R \right\}. \begin{bmatrix} x \\ 2x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow W = Span\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \Rightarrow W \text{ is a subspace of } R^2.$$

b) Find a basis for the subspace span(S),

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ -2 & -4 & -1 & -3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 + 2r_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{3}r_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[r_3 - r_2]{r_1 + r_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \text{ form a basis for span } S.$$

Q–5 [6+4 points] Let $T: R^3 \to R^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y - z \\ 0 \end{bmatrix}$.

a) Show that T is a linear transformation.

b) Find a basis for the null space $N(T)$.

$$a)\, T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - z_1 - z_2 \\ y_1 + y_2 - z_1 - z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - z_1 \\ y_1 - z_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 - z_2 \\ 0 \end{bmatrix} =$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right),$$

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ cy - cz \\ 0 \end{bmatrix} = c\begin{bmatrix} x - z \\ y - z \\ 0 \end{bmatrix} = cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right).$$

$$b)\, T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x - z = 0, y - z = 0 \Rightarrow z = t, x = t, y = t, t \in R.$$

$$N(T) = span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

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