Limit of sequence

Exercise 1. Prove the following limit via ϵ-N definition:

(1). $\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$.

(2). $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

Exercise 2. Suppose that $(x_n) \to 1$. Using ϵ-N definition, prove that

(1). $\lim_{n\to\infty} \frac{x_n+1}{2} = 1$.

(2). $\lim_{n\to\infty} (x_n^2 + 1) = 2$.

Solution

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Solution.

(1). Observe that $\frac{n}{2n-1} - \frac{1}{2} = \frac{1}{2(2n-1)}$. Here we encounter $2n-1$ in the denominator, which is a bit trickier to simplify. For me I like to reduce it by playing around with the fact that $n \geq 1$. In this particular problem, it turns out that

$$\left|\frac{n}{2n-1} - \frac{1}{2}\right| = \frac{1}{2(2n-1)} = \frac{1}{2(n+n-1)} \leq \frac{1}{2n}.$$

Now we can proceed the proof as follows: For every $\epsilon > 0$, we choose $N = 1 + \lfloor \frac{1}{2\epsilon} \rfloor$. Thus $N \in \mathbb{N}$ and $N > \frac{1}{2\epsilon}$. Therefore for any integer $n \geq N$ we get

$$\left|\frac{n}{2n-1} - \frac{1}{2}\right| \leq \frac{1}{2n} \leq \frac{1}{2N} < \epsilon.$$

This shows that $\lim \frac{n}{2n-1} = \frac{1}{2}$.

(2). [Hint] This comes easy once you write $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

□

Exercise 2. Suppose that $(x_n) \to 1$. Using ϵ-N definition, prove that

(1). $\lim_{n\to\infty} \dfrac{x_n + 1}{2} = 1$.

(2). $\lim_{n\to\infty} (x_n^2 + 1) = 2$.

Solution. This exercise is to prepare you the upcoming theorem — the Algebraic limit theorem. Understanding the ideas in this exercise shows that you have mastered the ϵ-N definition quite skillfully.

(1). First observe that

$$\left|\frac{x_n+1}{2} - 1\right| = \frac{1}{2} \cdot |x_n - 1|.$$

We want to make the left hand side less than ϵ. However since $x_n \to 1$, thus we can make $|x_n - 1|$ as small as we like, so why not make it so that $|x_n - 1| < 2\epsilon$. This is completely valid. Even though the ϵ-N definition says that $|a_n - a| < \epsilon$, but what it really means is: if $a_n \to a$ then we can make $|a_n - a| <$ as small as we like.

The proof is as follows: Let $\epsilon > 0$ be arbitrary. Since $x_n \to 1$, thus there exists an $N \in \mathbb{N}$ so that $|x_n - 1| < 2\epsilon$ for all $n \geq N$. Therefore for all $n \geq N$, we get

$$\left| \frac{x_n + 1}{2} - 1 \right| = \frac{1}{2}|x_n - 1| < \frac{1}{2} \cdot 2\epsilon = \epsilon.$$

Hence $(x_n + 1)/2 \to 1$ as expected.

(2). The idea in this second problem is quite crucial and is worth investigating. Again we start by trying to estimate $|a_n - a|$: Observe that

$$\left|(x_n^2 + 1) - 2\right| = \left|x_n^2 - 1\right| = |x_n + 1| \cdot |x_n - 1|.$$

The term $|x_n - 1|$ can be made as small as we like since $x_n \to 1$. The question becomes how do we estimate the term $|x_n + 1|$? We use the following neat trick: Using triangle inequality we get

$$|x_n + 1| = |(x_n - 1) + 2| \leq |x_n - 1| + 2$$

Since $x_n \to 1$, we can estimate $|x_n - 1|$ as much as we like, and why not make it so that $|x_n - 1| < 1$. Formally written, there is an $N_1 \in \mathbb{N}$ so that $|x_n - 1| < 1$ and thus $|x_n + 1| < 1 + 2 = 3$ for all $n \geq N_1$. Looking back

$$\left|(x_n^2 + 1) - 2\right| = \underbrace{|x_n + 1|}_{<3} \cdot |x_n - 1| < 3|x_n - 1|$$

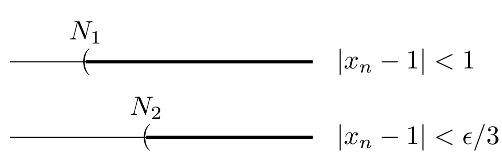
Now we can pick N_2 large enough so that $|x_n - 1| < \epsilon/3$ for all $n \geq N_2$. This might look quite confusing at first, and believe me it does. I am going to write it again, but this time with a formal proof.

Let $\epsilon > 0$ be arbitrary. By triangle inequality we get

$$\begin{aligned} \left|(x_n^2 + 1) - 2\right| &= |x_n + 1| \cdot |x_n - 1| \\ &= |(x_n - 1) + 2| \cdot |x_n - 1| \\ &\leq (|x_n - 1| + 2) \cdot |x_n - 1| \end{aligned}$$

Since $x_n \to 1$, then

- There is an $N_1 \in \mathbb{N}$ so that for all $n \geq N_1$, $|x_n - 1| < 1$.
- There is an $N_2 \in \mathbb{N}$ so that for all $n \geq N_2$, $|x_n - 1| < \frac{\epsilon}{3}$.



To satisfy both inequalities, we must choose from the largest between N_1 and N_2. Thus if we let $N = \max\{N_1, N_2\}$ then for all $n \geq N$ both inequalities work, and therefore

$$\begin{aligned} \left|(x_n^2 + 1) - 2\right| &\leq (|x_n - 1| + 2) \cdot |x_n - 1| \\ &< (1 + 2) \cdot \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Hence $(x_n^2 + 1) \to 2$ as desired.

$\square$