Monotone Convergence and Divergent Sequence

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1 Monotone Convergence Theorem

So far we have seen properties of convergent sequence. There are uniqueness, boundedness, algebraic properties, and ordering properties. Now we turn our attention and ask the question: Can we find *all* convergent sequences? The answer is *yes*. However, it's not very clear how to find them.

Our initial step in answering the question is to first ask: What are the conditions on (a_n) so that it converges? First it must be *bounded*, this is because every convergent sequence is bounded. It turns out if we require (a_n) to be increasing (or decreasing), then it is true that (a_n) converges. Such a sequence is called *monotonic*. Thus now we have a following important theorem.

Theorem 1. *If a sequence is bounded and monotonic, then it converges.*

For the proof of this theorem, please refer to it from the book. Now let's see some applications of this theorem in the following examples.

Example 1. Show that the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

converges and find its limit.

Solution. First let a_n to be the n^{th} term of this sequence, therefore we have the following identity

$$\begin{cases} a_1 = \sqrt{2} \\ a_{n+1} = \sqrt{2a_n}, \quad \text{for } n = 1, 2, 3, \ldots \end{cases}$$

We will prove that $a_n \in [1, 2]$ for all n. Clearly this statement is true for $n = 1$, since $1 \leq \sqrt{2} \leq 2$. Now suppose for a moment that $a_n \in [1, 2]$ for some $n \in \mathbb{N}$ then

$$a_{n+1} = \sqrt{2a_n} \geq \sqrt{2} > 1$$

and similarly

$$a_{n+1} = \sqrt{2a_n} \leq \sqrt{2 \cdot 2} = 2.$$

Hence $a_{n+1} \in [1, 2]$. Thus from this induction argument, we have showed that $a_n \in [1, 2]$ for *all* $n \in \mathbb{N}$. In other words, (a_n) is bounded. Next, we will show that (a_n) is also increasing. To see this, observe that

$$a_{n+1} = \sqrt{2a_n} \geq \sqrt{a_n \cdot a_n} = a_n.$$

Using Monotone Convergence Theorem, we conclude that (a_n) converges. Therefore we can safely denote the limit of (a_n) by L, i.e. $L = \lim a_n$. To find L, we would love to *put* the limit from both sides of the identity

$$a_{n+1} = \sqrt{2a_n}.$$

However before we can do that, we need to make sure that both (a_{n+1}) and $(\sqrt{2a_n})$ converge. Observe the following:

- Since $(a_n) \to L$, then we also have $(a_{n+1}) \to L$. (Although this fact is not immediate for us, I encourage you to use the definition of convergence and try to convince yourself.)
- Since $a_n > 0$, then $\sqrt{2a_n} \to \sqrt{2L}$. (See Exercise 2.3.1, page 54)

Now we obtain that

$$\begin{aligned} a_{n+1} &= \sqrt{2a_n} \\ \lim a_{n+1} &= \lim \sqrt{2a_n} \\ L &= \sqrt{2L} \\ L &= 0 \quad \text{or} \quad L = 2. \end{aligned}$$

However, only one of these two values is correct. Observe that $1 \leq a_n \leq 2$, this implies that $1 \leq L \leq 2$. Therefore $\boxed{L = 2}$ is the limit of (a_n). $\square$

2 Divergent

Intuitively speaking, a sequence (a_n) is said to be *divergent* if it fails to converge to any real number $a \in \mathbb{R}$. In other words, for any $a \in \mathbb{R}$, $(a_n) \not\to a$. Now ask yourself: What does it mean to have $a_n \not\to a$? By negating the definition of convergent, $(a_n) \not\to a$ means:

$$\exists \epsilon > 0, \ \forall N \in \mathbb{N}, \ \exists n \geq N \text{ such that } |a_n - a| \geq \epsilon.$$

At first it might seem hard to make sense out of this statement, but try to understand it geometrically. Now carefully read the following:

- $(a_n) \to a$ means: for any strip around a, there is a *tail* of the sequence that is completely inside the strip.
- $(a_n) \not\to a$ means: we can find a strip around a, for which any *tails* of the sequence are not completely inside that strip.

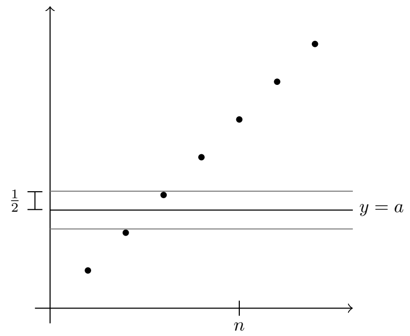
Below, we give a formal definition of divergent.

Definition 1. A sequence (a_n) is said to *diverge* provided that for any $a \in \mathbb{R}$, there exist $\epsilon > 0$ such that for any $N \in \mathbb{N}$, $|a_n - a| \geq \epsilon$ for some $n \geq N$.

Let's take a look at the following examples of divergent sequences.

Example 2. Prove that the sequence $a_n = n$ diverges.

Proof. Fix $a \in \mathbb{R}$. Our goal is to show that $a_n \not\to a$. To achieve this, we must produce an $\epsilon > 0$ so that any choice of N won't work; in other words we try to produce a strip so that any *tails* won't fit in that strip.



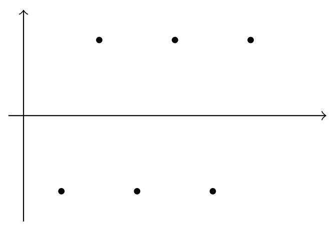
The above figure suggests us to choose $\epsilon = \frac{1}{2}$. Now for any $N \in \mathbb{N}$, we can choose $n \geq N$ such that $n > a + 1$ (how?). Thus for that particular $n \geq N$ we have

$$|a_n - a| = |n - a| > a + 1 - a > \epsilon.$$

This shows that $a_n \not\to a$ for any $a \in \mathbb{R}$. Therefore $\boxed{(a_n) \text{ diverges.}}$ $\square$

Example 3. Prove that the sequence $a_n = (-1)^n$ diverges.

Proof. Again our goal is to show that $a_n \not\to a$ for any $a \in \mathbb{R}$. We single out into cases.



- Case $a \geq 0$: Choose $\epsilon = 1/2$. For any $N \in \mathbb{N}$, pick $n = 2N + 1$ then
- Case $a < 0$: Choose $\epsilon = 1/2$. For any $N \in \mathbb{N}$, pick $n = 2N$ then

Now convince yourself that for this choice of n, we have $|a_n - a| \geq \epsilon$. $\square$