Homework 1

Real Analysis: Preparation with Siv Meng

July 11, 2024

Exercise 1. Prove the following limits via ϵ-N definition:

(1). $\lim_{n\to\infty} \dfrac{n}{2n-1} = \dfrac{1}{2}$

(2). $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

Solution

(1). $\lim_{n\to\infty} \dfrac{n}{2n-1} = \dfrac{1}{2}$

Scratch work: Since we want to find an index N, given an arbitrary $\epsilon > 0$ such that the property: $\forall n \geq N : |x_n - x| < \epsilon$, holds. This calls for us to observe the inequality inside that property/statement. That is we work backward (and we are **justified** in doing so because it is implicitly assumed here that the given limit value is **correct**, if not then our next working will run into unforeseen trouble). We simplify as follows (a tip here is to simplify the L.H.S *only one n is found* so that it will help us choose N more easily):

$$\begin{aligned}
|x_n - x| &= \left|\frac{n}{2n-1} - \frac{1}{2}\right| \\
&= \left|\frac{2n-1-(2n-1)}{2(2n-1)}\right| \quad \text{(common denominator)} \\
&= \left|\frac{1}{2(2n-1)}\right| \quad \text{(simplifying the numerator)} \\
&< \left|\frac{1}{2n-1}\right| \quad \text{(further simplification)} \\
&= \frac{1}{2n-1} \quad \text{(since } n \text{ is a natural number)} \\
&\leq \frac{1}{2n-n} \quad \text{(since } 1 \leq n\text{)} \\
&= \frac{1}{n} < \epsilon
\end{aligned}$$

This implies that n $> \frac{1}{\epsilon}$

Now that our scratch work is over, we know how to choose N. We shall begin our proof as follows.

Proof:

Let $\epsilon > 0$ be arbitrary.

We choose N $= \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1 > \frac{1}{\epsilon}$

$$
\begin{aligned}
|x_n - x| &= \left| \frac{n}{2n-1} - \frac{1}{2} \right| \\
&= \left| \frac{2n-1-(2n-1)}{2(2n-1)} \right| \quad \text{(common denominator)} \\
&= \left| \frac{1}{2(2n-1)} \right| \quad \text{(simplifying the numerator)} \\
&< \left| \frac{1}{2n-1} \right| \quad \text{(further simplification)} \\
&= \frac{1}{2n-1} \quad \text{(since } n \text{ is a natural number)} \\
&\leq \frac{1}{2n-n} \quad \text{(since } 1 \leq n\text{)} \\
&= \frac{1}{n} \\
&\leq \frac{1}{N} \\
&< \epsilon
\end{aligned}
$$

□

(2). $\lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0$

Scratch work Our reasoning and justifications are the same as the previous exercise and we shall proceed with the observation that $\sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$

Then, we work with that last expression as below:

$$
\begin{aligned}
|x_n - x| &= \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} - 0 \right| \\
&= \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} \right| \quad \text{(common denominator)} \\
&< \left| \frac{1}{\sqrt{n} + \sqrt{n}} \right| \quad (n < n+1) \\
&= \frac{1}{2\sqrt{n}} \quad (n \geq 1) \\
&< \frac{1}{\sqrt{n}} \quad (2 > 1) \\
&< \epsilon
\end{aligned}
$$

$$\implies n > \frac{1}{\epsilon^2}$$

Proof: Let $\epsilon > 0$ be arbitrary. Choose a natural number $N = \left\lfloor \frac{1}{\epsilon^2} + 1 \right\rfloor > \frac{1}{\epsilon^2}$. Then, $\forall n \geq N$,

$$\begin{aligned}
|x_n - x| &= \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} - 0 \right| \\
&= \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} \right| \quad \text{(common denominator)} \\
&< \left| \frac{1}{\sqrt{n} + \sqrt{n}} \right| \quad (n < n+1) \\
&= \frac{1}{2\sqrt{n}} \quad (n \geq 1) \\
&< \frac{1}{\sqrt{n}} \quad (2 > 1) \\
&\leq \sqrt{\frac{1}{N}} \\
&< \sqrt{\epsilon^2} \\
&= \epsilon \quad (\epsilon > 0)
\end{aligned}$$

$\square$

Exercise 2. Suppose $(x_n) \to 1$. Using the ϵ-N definition, prove that

(1). $\lim\limits_{n \to \infty} \frac{x_n + 1}{2} = 1$

(2). $\lim\limits_{n \to \infty} \left(x_n^2 + 1 \right) = 2$

Solution

(1). $\lim\limits_{n \to \infty} \frac{x_n + 1}{2} = 1$

Scratch work:

$$\begin{aligned}
|a_n - a| &= \left| \frac{x_n + 1}{2} - 1 \right| \\
&= \left| \frac{x_n - 1}{2} \right| \quad \text{(common denominator)} \\
&= \frac{|x_n - 1|}{2} \quad \text{(Property of absolute value on quotient)}
\end{aligned}$$

Now observe that $(x_n) \to 1$, by definition for any arbitrary positive real-valued ϵ, $\exists N : \forall n \geq N : |x_n - 1| < \epsilon$.

$\therefore \frac{|x_n - 1|}{2} < \frac{\epsilon}{2}$ $(*)$

However, because we are given the assumption that $(x_n) \to 1$, we can make ϵ as arbitrarily small

as we want to, despite the definition saying that the absolute value of the difference is less than ϵ, what it really means is exactly what we had just said:
We can estimate $|a_n - a| < \epsilon, \forall n \geq N$ however we would like (making it either as big or as small as is necessary) $\iff$ ϵ can appear in any form, necessary, or given such an ϵ, taking multiples of it, square it, ... won't matter because there will always be an index N such that $|a_n - a| < \epsilon$. This is why we are justified in choosing $c \cdot \epsilon$ so that $|x_n - 1| < c \cdot \epsilon$, where c is a real number. Then, how should we pick c? We pick it in such a way that the R.H.S of inequality $(*)$ becomes ϵ, thus $c = 2$.

Proof: Let $\epsilon > 0$ be arbitrary. Choose a natural number $N = \lfloor 2\epsilon + 1 \rfloor > 2\epsilon$. Then, $\forall n \geq N$,

$$\begin{aligned} |a_n - a| &= \left| \frac{x_n + 1}{2} - 1 \right| \\ &= \left| \frac{x_n - 1}{2} \right| \\ &= \frac{|x_n - 1|}{2} \\ &< \frac{2\epsilon}{2} \\ &= \epsilon \end{aligned}$$

$\square$

(2). $\lim_{n\to\infty} \left(x_n^2 + 1\right) = 2$

Scratch work:

$$\begin{aligned} |a_n - a| &= \left|\left(x_n^2 + 1\right) - 2\right| \\ &= \left|x_n^2 - 1\right| \\ &< |x_n - 1|\,|x_n + 1|\,(\star) \end{aligned}$$

Because $(x_n) \to 1$, we can estimate it however we like. The remaining problem is to estimate $|x_n+1|$. We want to somehow form some sort of a connection out of that factor from the first factor, $|x_n-1|$. It is then apparent, then, to try the following algebraic manipulation: $|x_n+1| = |x_n-1+2|$ and using the triangle inequality, $|x_n - 1 + 2| \leq |x_n - 1| + 2$ $(*)$

Voila! Now, because we know that $(x_n) \to 1$, we can estimate $(*)$ however we like – meaning we can pick any $\epsilon > 0$, there will always exist a natural number $N_1, \forall n \geq N_1 : |x_n - 1| < \epsilon$ and so $|x_n - 1 + 2| \leq \epsilon + 2$ also holds for the same N_1.

Back to $(\star)$, we can write

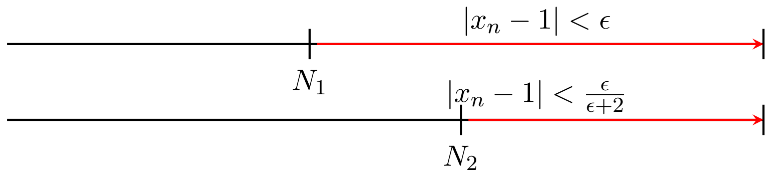
$$|a_n - a| < |x_n - 1||x_n + 1| < |x_n - 1| \cdot (\epsilon + 2)$$

As mentioned, we can pick another appropriate index N_2 (large enough) so that the L.H.S of the above expression is simplified into just ϵ, that appropriate choice would be $\frac{\epsilon}{\epsilon + 2}$.

Proof: Let $\epsilon > 0$ be arbitrary. Simplify:

$$\begin{aligned}
|a_n - a| &= \left|\left(x_n^2 + 1\right) - 2\right| \\
&= \left|x_n^2 - 1\right| \\
&< |x_n - 1|\,|x_n + 1| \\
&= |x_n - 1||(x_n - 1) + 2| \\
&\leq |x_n - 1|(|x_n - 1| + 2)
\end{aligned}$$

Since $(x_n) \to 1$,
$\exists N_1 \in \mathbb{N} : \forall n \geq N_1 : |x_n - 1| < \epsilon$
$\exists N_2 \in \mathbb{N} : \forall n \geq N_2 : |x_n - 1| < \dfrac{\epsilon}{\epsilon + 2}$



or vice versa.
Choose N $= \max\{N_1, N_2\}$ so that both inequalities hold simultaneously. Then, $\forall n \geq N :$

$$\begin{aligned}
|a_n - a| &= \left|\left(x_n^2 + 1\right) - 2\right| \\
&\leq |x_n - 1|(|x_n - 1| + 2) \\
&< (\epsilon + 2) \cdot \frac{\epsilon}{\epsilon + 2} \\
&= \epsilon
\end{aligned}$$

□