CHAPTER II
Vectors, Matrices and Determinants

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Definition 1

A **vector** in the plane is a directed line segment. The directed line segment $\overrightarrow{AB}$ has initial point A and terminal point B; its length is denoted by $\|\overrightarrow{AB}\|$. Two vectors are equal if they have the same length and direction.

Definition 2

If $\vec{v}$ is a two-dimensional vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) then the component form of $\vec{v}$ is

$$\vec{v} = (v_1, v_2)$$

If $\vec{v}$ is a three-dimensional vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) then the component form of $\vec{v}$ is

$$\vec{v} = (v_1, v_2, v_3).$$

The coordinates v_1, v_2 and v_3 are called the **components** of $\vec{v}$.

Definition 3

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in space and c a scalar.

1. $\vec{u} = \vec{v}$ if and only if $u_1 = v_1, u_2 = v_2$ and $u_3 = v_3$.
2. If $\vec{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$ then
$$\vec{v} = (v_1, v_2, v_3) = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$$
3. $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
4. Unit vector in the direction of $\vec{v}$ is $\dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\|\vec{v}\|}(v_1, v_2, v_3), \vec{v} \neq \vec{0}$.

Example 1

Find the component form and magnitude of the vector $\vec{v}$ having initial point $(-2, 3, 1)$ and terminal point $(0, -4, 4)$. Then find a unit vector in the direction of $\vec{v}$.

Solution

- $\vec{v} = (2, -7, 3)$

- $\|\vec{v}\| = \sqrt{4+49+9} = \sqrt{62}$

- $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{62}}(2, -7, 3).$

```python
import numpy as np
import numpy.linalg as la
v = np.array([2,-7,3])
print("v=",v)
v_norm = la.norm(v)
print("v_norm=",v_norm)
u = v/v_norm
print("u=",u)
```

```
v= [ 2 -7  3]
v_norm= 7.874007874011811
u= [ 0.25400025 -0.88900089  0.38100038]
```

Definition 4

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors and c a scalar.

1. $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$
2. $c\vec{u} = (cu_1, cu_2, cu_3)$
3. $-\vec{u} = (-1)\vec{u} = (-u_1, -u_2, -u_3)$
4. $\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$

Theorem 1

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and c, d be scalars.

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + \vec{0} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = \vec{0}$
5. $c(d\vec{u}) = (cd)\vec{u}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $1(\vec{u}) = \vec{u}, 0(\vec{u}) = \vec{0}$

Theorem 2

Let $\vec{u}$ be a vector and c be a scalar. Then

$$\|c\vec{u}\| = |c|\|\vec{u}\|$$

Theorem 3

Let $\vec{v}$ be a nonzero vector, then the vector

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

has length 1 and the same direction as $\vec{v}$. This vector $\vec{u}$ is called a **unit vector in the direction of** $\vec{v}$.

Definition 5

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel when there is some scalar c such that

$$\vec{u} = c\vec{v}$$

Example 2

Vector $\vec{w}$ has initial point $(2, -1, 3)$ and terminal point $(-4, 7, 5)$. Which of the following vectors is parallel to $\vec{w}$?

(a) $\vec{u} = (3, -4, -1)$. $\vec{w} = (-6, 8, 2) = -2(3, -4, -1) = -2\vec{u}$

(b) $\vec{v} = (12, -16, 4)$. $\vec{w} = \frac{-2}{-2}(-6, 8, 2) = -\frac{1}{2}(12, -16, -4)$

$\Rightarrow \vec{w} \parallel \vec{u}$ but $\vec{w} \not\parallel \vec{v}$.

Example 3

Determine whether the points $P(1, -2, 3), Q(2, 1, 0)$ and $R(4, 7, -6)$ are collinear.

. $\overrightarrow{PQ} = (1, 3, -3)$

. $\overrightarrow{PR} = (3, 9, -9) = 3(1, 3, -3) = 3\overrightarrow{PQ}$

So, P, Q, R are collinear.

Example 4

(a) Write the vector $\vec{v} = 4\vec{i} - 5\vec{k}$ in component form.

(b) Find the terminal point of the vector $\vec{v} = 7\vec{i} - \vec{j} + 3\vec{k}$, given that the initial point is $P(-2, 3, 5)$.

(c) Find the magnitude of the vector $\vec{v} = (-6, 2, -3)$. Then find a unit vector in the direction of $\vec{v}$.

Solution

(a) $\vec{V} = (4, 0, -5)$

(c). $\|\vec{v}\| = \sqrt{36+4+9} = 7 \Rightarrow \vec{u} = \frac{\vec{V}}{\|\vec{v}\|} = \frac{1}{7}(7,-1,3)$

(b) Let $Q(x,y,z)$ be the terminal point of $\vec{V} = \overrightarrow{PQ}$, then

$\vec{V} = \overrightarrow{PQ} = (x+2, y-3, z-5) = (7,-1,3) \Rightarrow \begin{cases} x+2=7 \\ y-3=-1 \\ z-5=3 \end{cases} \Rightarrow \begin{cases} x=5 \\ y=2 \\ z=8 \end{cases}$

Thus $Q(5,2,8)$.

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Definition 6

The **dot product** of two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is

Scalar product

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Theorem 4

Let $\vec{u}$ and $\vec{v}$ be vectors in the plane or in space and let c be a scalar.

1. $\vec{u}.\vec{v} = \vec{v}.\vec{u}$
2. $\vec{u}.(\vec{v} + \vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$
3. $c(\vec{u}.\vec{v}) = c\vec{u}.\vec{v} = \vec{u}.c\vec{v}$
4. $\vec{0}.\vec{u} = 0$
5. $\vec{u}.\vec{u} = \|\vec{u}\|^2$

Theorem 5

If θ is the angle between two nonzero vectors $\vec{u}$ and $\vec{v}$ where $0 \le \theta \le \pi$, then

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

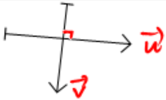
Example 5

Given that $\|\vec{u}\| = 10$ and $\|\vec{v}\| = 7$,and the angle between $\vec{u}$ and $\vec{v}$ is $\pi/4$, find $\vec{u}.\vec{v}$.

$\vec{u}.\vec{v} = \|\vec{u}\|.\|\vec{v}\| \cos\theta = (10)(7)\cos\frac{\pi}{4} = 70 \times \frac{\sqrt{2}}{2} = 35\sqrt{2}$

Definition 7

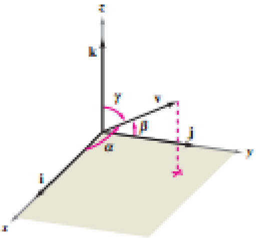
The vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u}.\vec{v} = 0$.



$\vec{u} = (1, 2, 3)$
$\vec{v} = (2, 1, 4)$
$\Rightarrow \vec{u}.\vec{v} = 2 + 2 + 12 = 16$

```python
import numpy as np
import numpy.linalg as la
u = np.array([1,2,3])
v = np.array([2,1,4])
w = u @ v # @ for dot product of two vectors
print("Dot product of u and v is",w)
```

```
Dot product of u and v is 16
```



Direction angles
Figure 11.26

Definition 8

The angles α, β and γ are the **direction angles of** $\vec{v}$, and $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the **direction cosines of** $\vec{v}$.

Example 6

Find the direction cosines and angles for the vector $\vec{u} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ and show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Solution $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$, $\vec{u} = (2,3,4)$

$\alpha = (\vec{i}, \vec{u})$, $\beta = (\vec{j}, \vec{u})$, $\gamma = (\vec{k}, \vec{u})$, then we have

- $\vec{i} \cdot \vec{u} = \|\vec{i}\| \cdot \|\vec{u}\| . \cos\alpha \Rightarrow \cos\alpha = \dfrac{\vec{i} \cdot \vec{u}}{\|\vec{i}\| \cdot \|\vec{u}\|} = \dfrac{2}{\sqrt{4+9+16}} = \dfrac{2}{\sqrt{29}}$

- $\cos\beta = \dfrac{\vec{j} \cdot \vec{u}}{\|\vec{j}\| \cdot \|\vec{u}\|} = \dfrac{3}{\sqrt{29}}$ and $\cos\gamma = \dfrac{\vec{k} \cdot \vec{u}}{\|\vec{k}\| \cdot \|\vec{u}\|} = \dfrac{4}{\sqrt{29}}$

Thus $\alpha = \arccos\left(\dfrac{2}{\sqrt{20}}\right)$, $\beta = \arccos\left(\dfrac{3}{\sqrt{29}}\right)$ and $\gamma = \arccos\left(\dfrac{4}{\sqrt{29}}\right)$

- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{4}{29} + \dfrac{9}{29} + \dfrac{16}{29} = \dfrac{29}{29} = 1$.

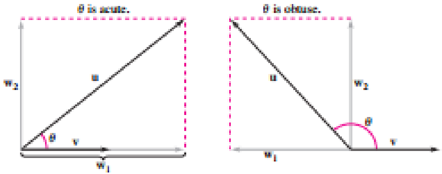
Definition 9

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors. Moreover, let

$$\vec{u} = \vec{w}_1 + \vec{w}_2$$

where $\vec{w}_1$ is parallel to $\vec{v}$ and $\vec{w}_2$ is orthogonal to $\vec{v}$.

1. $\vec{w}_1$ is called **projection of $\vec{u}$ onto $\vec{v}$** and is denoted by $\vec{w}_1 = \text{proj}_{\vec{v}}\vec{u}$.
2. $\vec{w}_2 = \vec{u} - \vec{w}_1$ is called the **vector component of $\vec{u}$ orthogonal to $\vec{v}$**.



$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ = projection of u onto v = vector component of u along v
$\mathbf{w}_2$ = vector component of u orthogonal to v

Example 7

$\vec{w}_2 = \vec{u} - \vec{w}_1 = (5,10) - (8,6) = (-3,4)$

Find the vector component of $\vec{u} = (5, 10)$ that is orthogonal to $\vec{v} = (4, 3)$ given that $\vec{w}_1 = \text{proj}_{\vec{v}}\vec{u} = (8, 6)$ and $\vec{u} = (5, 10) = \vec{w}_1 + \vec{w}_2$.

Theorem 6

If $\vec{u}$ and $\vec{v}$ are nonzero vectors, then the projection of $\vec{u}$ onto $\vec{v}$ is

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}.\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}.$$

Example 8

Find the projection of $\vec{u}$ onto $\vec{v}$ and the vector component of $\vec{u}$ orthogonal to $\vec{v}$ for

$$\vec{u} = 3\vec{i} - 5\vec{j} + 2\vec{k}, \qquad \vec{v} = 7\vec{i} + \vec{j} - 2\vec{k}$$

Solution

- $\vec{u} \cdot \vec{v} = (3)(7) + (-5)(1) + (2)(-2) = 21 - 5 - 4 = 21 - 9 = 12$
- $\|\vec{v}\| = \sqrt{49 + 1 + 4} = \sqrt{54}$
- Then $\vec{w}_1 = \text{proj}_{\vec{v}} \vec{u} = \left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} = \left(\dfrac{12}{54} \right)(7, 1, -2) = \frac{2}{9}(7, 1, -2)$

and $\vec{w}_2 = \vec{u} - \vec{w}_1 = (3, -5, 2) - \frac{2}{9}(7, 1, -2) = \ldots ?$

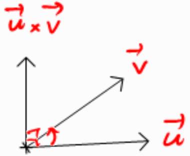
```
import numpy as np
import numpy.linalg as la
u = np.array([3,-5,2])
v = np.array([7,1,-2])
c = (u@v)/la.norm(v)**2
w1 = c*v
w2 = u - w1
print("c=",c)
print("w1=",w1)
print("w2=",w2)
```

```
c= 0.2222222222222222
w1= [ 1.55555556  0.22222222 -0.44444444]
w2= [ 1.44444444 -5.22222222  2.44444444]
```

Definition 10

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be vectors in space. The **cross product** of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$



$$\vec{u} \times \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Theorem 7

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in space, and let c be a scalar.

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
5. $\vec{u} \times \vec{u} = \vec{0}$
6. $\vec{u}.(\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}).\vec{w}$

$\vec{u} \times \vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \Longrightarrow$ np.cross(u,v) for $\vec{u} \times \vec{v}$
np.dot(u,v) for $\vec{u} . \vec{v}$

Example 9

For $\vec{u} = \vec{i} - 2\vec{j} + \vec{k}$ and $\vec{v} = 3\vec{i} + \vec{j} - 2\vec{k}$, find each of the following.

(a) $\vec{u} \times \vec{v} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ (b) $\vec{v} \times \vec{u} = \begin{pmatrix} -3 \\ -5 \\ -7 \end{pmatrix}$ (c) $\vec{v} \times \vec{v}. = \vec{0}$

Theorem 8

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in space, and let θ be the angle between $\vec{u}$ and $\vec{v}$.

1. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
2. $||\vec{u} \times \vec{v}|| = ||\vec{u}|| ||\vec{v}|| \sin\theta$
3. $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.
4. $||\vec{u} \times \vec{v}|| =$ area of parallelogram having $\vec{u}$ and $\vec{v}$ as adjacent sides.

Example 10

Find a unit vector that is orthogonal to both $\vec{u} = \vec{i} - 4\vec{j} + \vec{k}$ and $\vec{v} = 2\vec{i} + 3\vec{j}$.

$\vec{u} \times \vec{v} = \begin{pmatrix}1\\-4\\1\end{pmatrix} \times \begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}-3\\2\\11\end{pmatrix}$, $\|\vec{u} \times \vec{v}\| = \sqrt{9+4+121} = \sqrt{134}$

So $\dfrac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} = \ldots$ is the unit vector that $\perp \vec{u}$ and $\perp \vec{v}$.

Example 11

The vertices of a quadrilateral are listed below. Show that the quadrilateral is a parallelogram, and find its area.

$$A = (5,2,0), B = (2,6,1), C = (2,4,7), D = (5,0,6)$$

$\vec{AB} = (-3,4,1)$, $\vec{DC} = (-3,4,1)$. Since $\vec{AB} = \vec{DC} \Rightarrow ABCD$ is a parallelogram

Area $= \|\vec{AB} \times \vec{AD}\| = \ldots$?

Theorem 9

For $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}, \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$, the triple scalar product is

$$\vec{u}.(\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Theorem 10

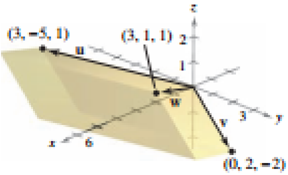
The volume V of a parallelepiped with vectors $\vec{u}, \vec{v}$ and $\vec{w}$ as adjacent edges is

$$V = |\vec{u}.(\vec{v} \times \vec{w})|.$$

Example 12

Find the volume of the parallelepiped shown in Figure having $\vec{u} = (3, -5, 1), \vec{v} = (0, 2, -2)$ and $\vec{w} = (3, 1, 1)$.

$V = |\vec{u}.(\vec{v} \times \vec{w})|$
$= 36$



The parallelepiped has a volume of 36.

```
import numpy as np
import numpy.linalg as la
u = np.array([3,-5,1])
v = np.array([0,2,-2])
w = np.array([3,1,1])
V = abs(np.dot(u,np.cross(v,w)))
print(V)
```

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Theorem 11

A line L parallel to the vector $\vec{v}(a, b, c)$ and passing through the point $P(x_0, y_0, z_0)$ is represented by the parametric equations

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

The vector $\vec{v}$ is called a **direction vector** for the line L, and a, b, c are **direction numbers**. If the direction numbers a, b, c are all nonzero, then you can eliminate the parameter to obtain **symmetric equations** of the line.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 13

Find parametric and symmetric equations of the line L that passes through the point $(1, -2, 4)$ and is parallel to $\vec{v} = (2, 4, -4)$.

Example 14

Find a set of parametric equations of the line that passes through the points $\underbrace{(-2,1,0)}_{P}$ and $\underbrace{(1,3,5)}_{Q}$.

$\vec{v} = \overrightarrow{PQ} = ?$

$(PQ): \begin{cases} x = \\ y = \\ z = \end{cases}$?

Theorem 12

The plane containing the point $M(x_0, y_0, z_0)$ and having normal vector $\vec{n}(a, b, c)$ can be represented by the **standard form** of the equation of a plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

By regrouping terms, we obtain the **general form** of the equation of a plane in space

$$ax + by + cz + d = 0.$$

Example 15

Find the general equation of the plane containing the points $\underbrace{(2,1,1)}_{A}$, $\underbrace{(0,4,1)}_{B}$ and $\underbrace{(-2,1,4)}_{C}$.

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, $A(2,1,0)$

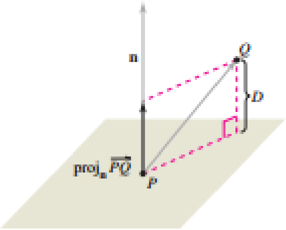
$(ABC): \ldots ?$

Theorem 13

The distance between a plane and a point Q (not in the plane) is

$$D = \|\text{proj}_{\vec{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ}.\vec{n}|}{\|\vec{n}\|}$$

where P is a point in the plane and $\vec{n}$ is normal to the plane.



$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\|$

The distance between a point and a plane

$\vec{n} \cdot \vec{PQ} = 3 - 5 - 14 = -16$

$\|\vec{n}\|^2 = 9 + 1 + 4 = 14$

$\vec{PQ} = (1, 5, -7)$

Example 16

Find the distance between the point $Q(1, 5, -4)$ and the plan $3x - y + 2z = 6$.

Let $P(0,0,3)$ be a point on the plane $3x-y+2z=6$

$\vec{n} = (3,-1,2)$ is the normal vector of plane

$\vec{w} = \text{proj}_{\vec{n}} \vec{PQ} = \left(\frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|^2}\right)\vec{n} = \left(\frac{-16}{14}\right)(3,-1,2) = -\frac{8}{7}(3,-1,2)$

Example 17

Show that the distance between the point $Q(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$D = \|\vec{w}\| = \ldots ?$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

(Homework)

Example 18

Find the distance between two parallel planes $3x - y + 2z - 6 = 0$ and $6x - 2y + 4z + 4 = 0$.

Let $Q(0,0,3)$ be a point on the plane $3x-y+2z-6=0$, then

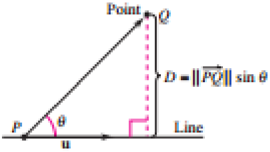
$d = \frac{|12+4|}{\sqrt{36+4+16}} = \frac{16}{\sqrt{56}}$

Theorem 14

The distance between a point Q and a line in space is

$$D = \frac{\|\overrightarrow{PQ} \times \vec{u}\|}{\|\vec{u}\|}$$

where $\vec{u}$ is a direction vector for the line and P is a point on the line.



The distance between a point and a line

Example 19 (Homework)

Find the distance between the point $Q(3, -1, 4)$ and the line $x = -2 + 3t, y = -2t, z = 1 + 4t$.

Definition 11

Let m and n are positive integers. An $m \times n$ (read "m by n") matrix is a rectangular array of numbers or functions arranged in m horizontal rows and n vertical columns. Matrices are usually denoted by upper case letters, such as A and B. The entries in the matrix are called the elements of the matrix.

Example 1

The following are examples of a 3×3 and a 4×2 matrix, respectively:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 3 & 6 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 3 \\ -3 & -7 \end{bmatrix}$$

Definition 12

Two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal if $a_{ij} = b_{ij}$ for each i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 13

A $1 \times n$ matrix is called a row $n-$vector. An $n \times 1$ matrix is called a column $n-$vector. The elements of a row or column $n-$vector are called the components of the vector.

Example 2

The matrix $u = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a row 3-vector and the matrix $v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a column 4-vector.

Definition 14

If we interchange the row vectors and column vectors in an $m \times n$ matrix A, we obtain an $n \times m$ matrix called the **transpose of** A. We denote this matrix by A^T . In index notation, the (i, j)-th element of A^T , denoted a^T_{ij}, is given by

$$a^T_{ij} = a_{ji}.$$

Example 3

If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 5 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix}$.

Definition 15

- An $n \times n$ matrix is called a **square matrix** if it has the same number of rows as columns. If A is a square matrix, then the elements $a_{ii}, 1 \leq i \leq n$, make up the **main diagonal**, or **leading diagonal**, of the matrix.
- The sum of the main diagonal elements of an $n \times n$ matrix A is called the **trace** of A and is denoted $tr(A)$. Thus,

$$tr(A) = a_{11} + a_{22} + \ldots + a_{nn}.$$

- An $n \times n$ matrix $A = [a_{ij}]$ is said to be **lower triangular** if $a_{ij} = 0$ whenever $i < j$ (zeros everywhere above (i.e., "northeast of") the main diagonal), and it is said to be **upper triangular** if $a_{ij} = 0$ whenever $i > j$ (zeros everywhere below (i.e., "southwest of") the main diagonal). An $n \times n$ matrix $D = [d_{ij}]$ is said to be a **diagonal matrix** if $d_{ij} = 0$ whenever $i \neq j$ (zeros everywhere off the main diagonal).

Example 4

The matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ are upper triangular and lower triangular matrix, respectively. The matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a diagonal matrix.

Remark 1

Note that we write $D = \text{diag}(d_1, d_2, \ldots, d_n)$, where d_i denotes the diagonal element d_{ii}.

Definition 16

- A square matrix A satisfying $A^T = A$ is called a **symmetric matrix**.
- If $A = [a_{ij}]$, then we let $-A$ denote the matrix with elements $-a_{ij}$. A square matrix A satisfying $A^T = -A$, is called a **skew-symmetric** (or anti-symmetric) matrix.

Remark 2

if A is a skew-symmetric matrix, then $a_{ij} = -a_{ji}$, which implies that when $i = j$, $a_{ii} = -a_{ii}$, so that $a_{ii} = 0$.

Example 5

The matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ are symmetric and skew-symmetric, respectively.

Definition 17

An $n \times n$ nonsingular matrix A is orthogonal if $A^T = A^{-1}$. In other words, A is orthogonal if $A^T.A = I$.

Remark 3

An $n \times n$ matrix A is orthogonal if and only if its columns $X_1, X_2, \dots, X_n$ form anorthonormal set.

Example 20

The matrix $A = \begin{pmatrix} 1/4 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$ is orthogonal.

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Definition 18

If A and B are both $m \times n$ matrices, then we define addition (or the sum) of A and B, denoted by $A + B$, to be the $m \times n$ matrix whose elements are obtained by adding corresponding elements of A and B. In index notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A + B = [a_{ij} + b_{ij}]$.

Theorem 15 (Properties of Matrix Addition)

If A and B are both $m \times n$ matrices, then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$.

Example 6

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1 & 3 & 6 \\ 6 & 3 & 6 \end{bmatrix}$.

Definition 19

If A is an $m \times n$ matrix and λ is a scalar, then we let λA denote the matrix obtained by multiplying every element of A by λ. This procedure is called **scalar multiplication**. In index notation, if $A = [a_{ij}]$, then $\lambda A = [\lambda a_{ij}]$.

Theorem 16 (Properties of Scalar Multiplication)

For any scalars s and t, and for any matrices A and B of the same size

1. $1A = A$
2. $s(A + B) = sA + sB$
3. $(s + t)A = sA + tA$
4. $s(tA) = (st)A = (ts)A = t(sA)$

Definition 20

If A and B are both $m \times n$ matrices, then we define subtraction of these two matrices by

$$A - B = A + (-1)B$$

. In index notation $A - B = [a_{ij} - b_{ij}]$. That is, we subtract corresponding elements.

Definition 21

The $m \times n$ zero matrix, denoted $0_{m\times n}$ (or simply 0, if the dimensions are clear), is the $m \times n$ matrix whose elements are all zeros. In the case of the $n \times n$ zero matrix, we may write 0_n.

Theorem 17 (Properties of the Zero Matrix)

For all matrices A and the zero matrix of the same size, we have

$$A + 0 = A, \qquad A - A = 0, \qquad 0A = 0.$$

Definition 22

If $A = [a_{ij}]$ is an $m \times n$ matrix, $B = [b_{ij}]$ is an $n \times p$ matrix, and $C = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq p.$$

This is called the index form of the matrix product.

Theorem 18

If A, B, and C have appropriate dimensions for the operations to be performed, then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$

Example 7

If $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 2 \\ 3 & -2 \end{bmatrix}$, find AB and BA.

Definition 23

The **identity matrix**, I_n (or just I if the dimensions are obvious), is the $n \times n$ matrix with ones on the main diagonal and zeros elsewhere.

Theorem 19 (Properties of the Identity Matrix)

1. $A_{m\times n} I_n = A_{m\times n}$
2. $I_m A_{m\times p} = A_{m\times p}$.

Theorem 20 (Properties of the Transpose)

Let A and C be $m \times n$ matrices, and let B be an $n \times p$ matrix. Then

1. $(A^T)^T = A$
2. $(A + C)^T = A^T + C^T$
3. $(AB)^T = B^T A^T$.

Definition 24

An $m \times n$ matrix is called a row-echelon matrix if it satisfies the following three conditions:

1. If there are any rows consisting entirely of zeros, they are grouped together at the bottom of the matrix.
2. The first nonzero element in any nonzero row is a 1 (called a **leading 1**).
3. The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

Example 8

Examples of row-echelon matrices are

$$\begin{bmatrix} 1 & -8 & -3 & 7 \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -3 & -6 & 5 & 7 \\ 0 & 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 25

The following notation will be used to describe elementary row operations performed on a matrix A.

1. $R_i \leftrightarrow R_j$: Permute the ith and j th rows of A.
2. $R_i \longrightarrow kR_i$: Multiply every element of the ith row of A by a nonzero scalar k.
3. $R_i \longrightarrow R_i + kR_j$:Add to the elements of the i th row of A the scalar k times the corresponding elements of the jth row of A.

Definition 26

Let A be an $m \times n$ matrix. Any matrix B obtained from A by a finite sequence of elementary row operations is said to be **row-equivalent** to A and we write $A \sim B$.

Theorem 21

Every matrix is row-equivalent to a row-echelon matrix.

When a matrix A has been reduced to a row-echelon matrix, we say that it has been reduced to **row-echelon form** and refer to the resulting matrix as a row-echelon form of A.

Algorithm for Reducing an $m \times n$ Matrix A to Row-Echelon Form

1. Start with an $m \times n$ matrix A. If $A = 0$, go to (7).
2. Determine the leftmost nonzero column (this is called a **pivot column** and the topmost position in this column is called a **pivot position**).
3. Use elementary row operations to put a 1 in the pivot position.
4. Use elementary row operations to put zeros below the pivot position.
5. If there are no more nonzero rows below the pivot position go to (7), otherwise go to (6).
6. Apply (2)–(5) to the submatrix consisting of the rows that lie below the pivot position.
7. The matrix is a row-echelon matrix.

Example 9

Use elementary row operations to reduce the following matrices to row-echelon form.

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{bmatrix}$$

Ans: $A \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Theorem 22

Let A be an $m \times n$ matrix. All row-echelon matrices that are row-equivalent to A have the same number of nonzero rows.

Definition 27

The number of nonzero rows in any row-echelon form of a matrix A is called the **rank of** A and is denoted $\text{rank}(A)$.

Example 10

Determine $\text{rank}(A)$ if $A = \begin{bmatrix} 3 & -1 & 4 & 2 \\ 1 & -1 & 2 & 3 \\ 7 & -1 & 8 & 0 \end{bmatrix}$.

Ans: $\text{rank}(A)=2$

Definition 28

An $m \times n$ matrix is called a **reduced row-echelon matrix** if it satisfies the following conditions:

1. It is a row-echelon matrix.
2. Any column that contains a leading 1 has zeros everywhere else.

Example 11

The following are examples of reduced row-echelon matrices:

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \textbf{and} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 23

An $m \times n$ matrix is row-equivalent to a unique reduced row-echelon matrix.

Example 12

Determine the reduced row-echelon form of $A = \begin{bmatrix} 3 & -2 & -1 & 17 \\ 2 & 2 & -4 & 8 \\ -1 & 4 & -3 & 1 \end{bmatrix}$.

Ans: $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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Definition 29

An equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b_n,$$

where $a_1, a_2, ..., a_n,$ and b_n are real numbers, is a **linear equation** in the n variables $x_1, x_2, ..., x_n$.

Definition 30

A system of m linear equations in n variables, or unknowns, denoted by (S) has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where the system coefficients a_{ij} and the system constants b_j are given scalars and $x_1, x_2, \dots, x_n$ denote the unknowns in the system. If $b_i = 0$ for all i, then the system is called **homogeneous**; otherwise it is called **nonhomogeneous**.

Definition 31

A solution of a linear system (S) is a set of n numbers $x_1, x_2, \dots, x_n$ that satisfies each equation in the system.

Definition 32

A system of linear equations that has at least one solution is said to be **consistent**, whereas a system that has no solution is called **inconsistent**.

Remark 4

If we let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \text{and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

then the linear system (S) can be written in vector equation $AX = b$, where A is the matrix of coefficients.

$$(A|b) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

is called **augmented matrix**.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to reduce the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Theorem 24

Consider the $m \times n$ linear system $Ax = b$. Let r denote the rank of A, and let r^* denote the rank of the augmented matrix of the system. Then

1. if $r < r^*$, the system is inconsistent.
2. If $r = r^*$, the system is consistent and
   (a) There exists a unique solution if and only if $r^* = n$.
   (b) There exists an infinite number of solutions if and only if $r^* < n$.

Corollary 9.1

The homogeneous linear system $Ax = 0$ is consistent for any coefficient matrix A, with a solution given by $x = 0$.

Corollary 9.2

A homogeneous system ofm linear equations in n unknowns, with $m < n$, has an infinite number of solutions.

Example 13

Use Gaussian elimination to determine the solution set to the given system.

$$(a)\begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 3 \\ x_1 - x_2 - 2x_3 = -6 \end{cases} \qquad (b)\begin{cases} 4x_1 + 8x_2 - 12x_3 = 44 \\ 3x_1 + 6x_2 - 8x_3 = 32 \\ -2x_1 - x_2 = -7 \end{cases}$$

Example 14

Determine all values of the constant k for which the following system has (a) no solution, (b) an infinite number of solutions, and (c) a unique solution.

$$(a)\begin{cases} x_1 + 2x_2 - x_3 = 3 \\ 2x_1 + 5x_2 + x_3 = 7 \\ x_1 + x_2 - k^2 x_3 = -k. \end{cases} \qquad (b)\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 0 \\ 4x_1 + 2x_2 - x_3 + x_4 = 0 \\ 3x_1 - x_2 + x_3 + kx_4 = 0. \end{cases}$$

Definition 33

An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is the (multiplicative) inverse of A. A matrix that does not have an inverse is **noninvertible** (or **singular**).

Theorem 25 (Uniqueness of an Inverse Matrix)

If A is an invertible matrix, then its inverse is **unique**. The inverse of A is denoted by A^{-1}.

Example 15

Show that B is the inverse of A, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of A on the left and the $n \times n$ identity matrix I on the right to obtain $[A|I]$. This process is called **adjoining** matrix I to matrix A.
2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A|I]$. The result will be the matrix $[I|A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Example 16

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$.

Example 17

Show that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$ has no inverse.

Theorem 26 (Properties of Inverse Matrices)

If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then $A^{-1}1, A^k, cA,$ and A^T are invertible and the statements below are true.

$$1.(A^{-1})^{-1} = A \qquad 2.(A^k)^{-1} = A^{-1} \dots A^{-1} = (A^{-1})^k$$
$$3.(cA)^{-1} = \frac{1}{c}A^{-1} \quad 4.(A^T)^{-1} = (A^{-1})^T.$$

Theorem 27 (The Inverse of a Product)

If A and B are invertible matrices of order n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Corollary 10.1

Let $A_1, A_2, \ldots, A_k$ be invertible $n \times n$ matrices. Then $A_1 A_2 \ldots A_k$ is invertible, and

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \ldots A_1^{-1}.$$

Theorem 28 (Cancellation Properties)

If C is an invertible matrix, then the properties below are true.

1. If $AC = BC$, then $A = B$ (Right cancellation property)
2. If $CA = CB$, then $A = B$ (Left cancellation property)

Theorem 29 (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following conditions on A are equivalent:

(a) A is invertible.

(b) The equation $Ax = b$ has a unique solution is $x = A^{-1}b$ for every b in $\mathbb{R}^n$.

(c) The equation $Ax = 0$ has only the trivial solution $x = 0$.

(d) $rank(A) = n$.

(e) A can be expressed as a product of elementary matrices.

(f) A is row-equivalent to I_n.

Definition 34

Let A be a square matrix of order n and k be a positive integer. We define

$$A^0 = I_n, A^1 = A, A^2 = A.A, A^3 = A^2.A, \ldots, A^k = A^{k-1}.A.$$

Definition 35

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k$ be a polynomial of degree k and A be a square matrix of order n. Then
$p(A) = a_0 I_n + a_1 A + a_2 A^2 + \ldots + a_k A^k$.

Example 18

Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. Compute A^n and B^n for $n \in \mathbb{N}$.

Example 19

Let $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ and $n \in \mathbb{N}, n \geq 2$.

(a) Find the rest of the division of x^n by $x^2 - 3x + 2$.

(b) Compute A^n in term of A, I and n.

Example 20

Let $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

(a) Compute $A^2 - 3A + 2I_3$.

(b) Deduce that A is invertible and find A^{-1}.

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Definition 36

Suppose A is an $n \times n$ matrix. Associated with A is a number called the **determinant** of A and is denoted by det A. Symbolically, we distinguish a matrix A from the determinant of A by replacing the parentheses by vertical bars:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots a_{nn} \end{pmatrix} \qquad \text{and} \qquad \det A = \begin{vmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots a_{nn} \end{vmatrix}$$

Definition 37 (Determinant of a 2×2 Matrix)

The determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is the number

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Definition 38 (Determinant of a 3×3 Matrix)

The determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the number

$$\begin{aligned} \det A = & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

Example 21

Evaluate the determinant of $A = \begin{pmatrix} 6 & 5 & 0 \\ -1 & 8 & -7 \\ -2 & 4 & 0 \end{pmatrix}$.

Definition 39 (Minors and Cofactors of a Square Matrix)

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Example 22

Find all the minors and cofactors of $A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$.

Theorem 30 (Cofactor Expansion of a Determinant)

Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix. For each $1 \leq i \leq n$, the cofactor expansion of $\det A$ along the ith row is

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

For each $1 \leq j \leq n$, the cofactor expansion of $\det A$ along the jth column is

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

Example 23

Evaluate the determinant of the matrix

$$A = \begin{pmatrix} 5 & 1 & 2 & 4 \\ -1 & 0 & 2 & 3 \\ 1 & 1 & 6 & 1 \\ 1 & 0 & 0 & -4 \end{pmatrix}.$$

Theorem 31 (Determinant of a Transpose)

If A^T is the transpose of the $n \times n$ matrix A, then $|A^t| = |A|$.

Theorem 32 (Two Identical Rows)

If any two rows (columns) of an $n \times n$ matrix A are the same, then $|A| = 0$.

Theorem 33 (Zero Row or Column)

If all the entries in a row (column) of an $n \times n$ matrix A are zero, then $|A| = 0$.

Theorem 34 (Interchanging Rows)

If B is the matrix obtained by interchanging any two rows (columns) of an $n \times n$ matrix A, then $|B| = -|A|$.

Theorem 35 (Constant Multiple of a Row)

If B is the matrix obtained from an $n \times n$ matrix A by multiplying a row (column) by a nonzero real number k, then $|B| = k|A|$.

Theorem 36

If A is an $n \times n$ matrix and c is a scalar, then $|cA| = c^n|A|$.

Theorem 37 (Determinant of a Matrix Product)

If A and B are both $n \times n$ matrices, then $|AB| = |A|.|B|$.

Theorem 38 (Determinant Is Unchanged)

Suppose B is the matrix obtained from an $n \times n$ matrix A by multiplying the entries in a row (column) by a nonzero real number k and adding the result to the corresponding entries in another row (column). Then $|B| = |A|$.

Theorem 39 (Determinant of a Triangular Matrix)

Suppose A is an $n \times n$ triangular matrix (upper or lower). Then

$$|A| = a_{11}a_{22}...a_{nn},$$

where $a_{11}, a_{22}, ..., a_{nn}$ are the entries on the main diagonal of A.

Theorem 40

A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$.

Theorem 41

If A is an $n \times n$ invertible matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

Example 24

$$\begin{vmatrix} -6 & 4 & 9 & -2 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & -5 & 6 \\ 0 & 0 & 0 & -3 \end{vmatrix} = (-6)(2)(-5)(-3) = -180.$$

Example 25

Evaluate $\begin{vmatrix} 3 & 4 & -1 & 5 \\ 1 & 2 & -1 & 3 \\ -2 & -2 & 2 & -7 \\ -4 & -3 & -2 & -8 \end{vmatrix}$. Ans: -34

Example 26

Evaluate $\begin{vmatrix} 2 & 1 & 8 & 6 \\ 1 & 4 & 1 & 3 \\ -1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 2 \end{vmatrix}$. Ans: 90

Example 27

Suppose that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $|A| = 7$. Compute

$$(a)|3A| \quad (b)|A^{-1}| \quad (c)|2A^{-1}| \quad (d)|(2A)^{-1}| \quad (e)\begin{vmatrix} a & 2g & d \\ b & 2h & e \\ c & 2i & f \end{vmatrix}$$

Example 28

Let A and B be two square matrices of order 3 such that $|A| = -2, |B| = 5$ and $D = \text{diag}(-2, 1, 3)$. Compute

(a) $\left|B^{-1}A^t\right|$

(b) $|2B|$

(c) $\left|\left(D^2A^{-1}B\right)^2\right|$

Definition 40 (Adjoint Matrix)

Let A be an $n \times n$ matrix. The matrix that is the transpose of the matrix of cofactors corresponding to the entries of A:

$$\begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & & \vdots & \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & & \vdots & \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}.$$

is called the **adjoint** of A and is denoted by adj$A$.

Example 29

Determine adj(A) if $A = \begin{bmatrix} 6 & -1 & 0 \\ 2 & -2 & 1 \\ 3 & 0 & -3 \end{bmatrix}$.

Theorem 42 (The Adjoint Method for Computing A^{-1})

Let A be an $n \times n$ matrix. If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A.$$

Example 30

Find the inverse of A if

(a) $A = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$

Theorem 43 (Cramer's Rule)

If a system of n linear equations in n variables has a coefficient matrix A with $|A| \neq 0$, then the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \ldots, x_1 = \frac{|A_n|}{|A|}$$

where the ith column of A_i is the column of constants in the system of equations.

Example 31

Use Cramer's rule to solve the system

$$\begin{cases} 3x_1 + 2x_2 + x_3 & = 7 \\ x_1 - x_2 + 3x_3 & = 3 \\ 5x_1 + 4x_2 - 2x_3 & = 1. \end{cases}$$