1 Prerequisites

1.1 Ordered Set

Partial Ordered Set

Formally, a partial order is a **homogeneous binary relation** that is **reflexive**, **antisymmetric**, and **transitive**. A **partially ordered set** (*poset* for short) is an *ordered pair* $P = (X, \leq)$ consisting of a set X (called the *ground set* of P) and a partial order $\leq$ on X. When the meaning is clear from context and there is no ambiguity about the partial order, the set X itself is sometimes called a poset.

A **reflexive, weak,** or **non-strict partial order**, commonly referred to simply as a **partial order**, is a **homogeneous relation** $\leq$ on a set P that is **reflexive**, **antisymmetric**, and **transitive**. That is, for all $a, b, c \in P$, it must satisfy:

1. **Reflexivity:** $a \leq a$, i.e. every element is related to itself.
2. **Antisymmetry:** if $a \leq b$ and $b \leq a$ then $a = b$, i.e. no two distinct elements precede each other.
3. **Transitivity:** if $a \leq b$ and $b \leq c$ then $a \leq c$.

A non-strict partial order is also known as an **antisymmetric preorder**.

Total Order

In mathematics, a total order or linear order is a partial order ***in which any two elements are comparable***. That is, a total order is a binary relation $\leq$ on some set X, which satisfies the following for all a, b and c in X:

1. $a \leq a$ (*reflexive*).
2. If $a \leq b$ and $b \leq c$ then $a \leq c$ (*transitive*).
3. If $a \leq b$ and $b \leq a$ then $a = b$ (*antisymmetric*).
4. $a \leq b$ or $b \leq a$ (*strongly connected*, formerly called *totality*).

1.2 Cartesian Product

In **mathematics**, specifically **set theory**, the **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all **ordered pairs** (a, b) where a is an element of A and b is an element of B. In terms of **set-builder notation**, that is

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Orders on the Cartesian product of totally ordered sets

There are several ways to take two totally ordered sets and extend to an order on the **Cartesian product**, though the resulting order may only be **partial**. Here are three of these possible orders, listed such that each order is stronger than the next:

- **Lexicographical order:** $(a, b) \leq (c, d)$ if and only if $a < c$ or ($a = c$ and $b \leq d$). This is a total order.
- $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$ (the **product order**). This is a partial order.
- $(a, b) \leq (c, d)$ if and only if ($a < c$ and $b < d$) or ($a = c$ and $b = d$) (the reflexive closure of the **direct product** of the corresponding strict total orders). This is also a partial order.

Relations as Subsets of Cartesian Products Mathematics Exchange

The set of ordered pairs drawn from the Cartesian product can be defined as

$$R = \{(x, y) \in A \times B |\ xRy\}$$

Thus we can say

$$R \subseteq A \times B$$

1.3 Pareto Set

We assume w.l.o.g that k objective functions,

$$f_i : X^k \to \mathbb{R}$$
$$f(x) = \big(f_1(x), f_2(x), \ldots, f_k(x)\big) \in \mathbb{R}^k, \quad \text{where } \mathbf{x} \in X^k$$

A solution $x \in X$ is said to *dominate* another solution $y \in X$ iff $\forall\ 1 \le i \le k : f_i(x) \le f_i(y)$ and $\exists\ 1 \le i \le k : f_i(x) < f_i(y)$. This can denoted as $x \prec y$. A solution $x \in X$ *weakly dominates* a solution $y \in X$ iff $\forall\ 1 \le i \le k : f_i(x) \le f_i(y)$. This can be denoted as $x \preceq y$.

Parento Optimal

A solution $x^* \in X$ is then called *Parento Optimal* iff there is no other solution in X that dominates x^*.

Pareto set approximations

Specific sets of solutions are the so-called *Pareto set approximations*, which are solution sets of pairwisely non-dominated solutions.

2 Deteriorative Cycles

In single-objective optimization, every solution is mapped to a real value and solutions can always be pairwisely compared via the less or equal relation $\le$ on $\mathbb{R}$. In another words, the total order (any elements are comparable) $\le \subseteq \mathbb{R} \times \mathbb{R}$ induces via f order on the search space X that is a total preorder. In a multiobjective scenario, the $\le$ relation is generalized to objective vectors, i.e., $\le$ is a subset of $\mathbb{R}^k \times \mathbb{R}^k$. Here, the totality is not given due to vectors $a, b \in \mathbb{R}^k$ where $f_1(a) < f_1(b)$ but $f_2(a) > f_2(b)$—the relation $\le$ on the set of objective vectors is only a partial order, i.e., reflexive, antisymmetric, and transitive. This means that *not any elements are comparable.* Brockhoff 2009

2.1 Downfall of The Evolutionary Algorithm Multiobjective Optimisation

Among the well-established ones, NSGA-II [Deb et al., 2002] and SPEA2 [Zitzler et al., 2002] have to be mentioned here. Both use the Pareto dominance concept as the main selection criterion in an elitist manner where non-dominated solutions are favored over dominated ones. In addition, a second selection criterion establishes diversity among the solutions. However, experimental studies have shown that both algorithms do not scale well if the number of objectives increases and that a cyclic behavior can be observed. This means that—although non-dominated solutions are preferred over dominated ones—over time, previously dominated solutions enter the population again, resulting in an oscillating distance to the Pareto front. Brockhoff 2009

2.2

References

Brockhoff, Dimo (Sept. 2009). *Theoretical Aspects of Evolutionary Multiobjective Optimization—A Review*. Research Report 7030. Inria Research Report RR-7030. France: Inria.