1 Prerequisites

1.1 Ordered Set

Partial Ordered Set

Formally, a partial order is a **homogeneous binary relation** that is **reflexive**, **antisymmetric**, and **transitive**. A **partially ordered set** (*poset* for short) is an *ordered pair* $P = (X, \leq)$ consisting of a set X (called the *ground set* of P) and a partial order $\leq$ on X. When the meaning is clear from context and there is no ambiguity about the partial order, the set X itself is sometimes called a poset.

A **reflexive, weak,** or **non-strict partial order**, commonly referred to simply as a **partial order**, is a **homogeneous relation** $\leq$ on a set P that is **reflexive**, **antisymmetric**, and **transitive**. That is, for all $a, b, c \in P$, it must satisfy:

1. **Reflexivity:** $a \leq a$, i.e. every element is related to itself.
2. **Antisymmetry:** if $a \leq b$ and $b \leq a$ then $a = b$, i.e. no two distinct elements precede each other.
3. **Transitivity:** if $a \leq b$ and $b \leq c$ then $a \leq c$.

A non-strict partial order is also known as an **antisymmetric preorder**.

Total Order

In mathematics, a total order or linear order is a partial order ***in which any two elements are comparable***. That is, a total order is a binary relation $\leq$ on some set X, which satisfies the following for all a, b and c in X:

1. $a \leq a$ (*reflexive*).
2. If $a \leq b$ and $b \leq c$ then $a \leq c$ (*transitive*).
3. If $a \leq b$ and $b \leq a$ then $a = b$ (*antisymmetric*).
4. $a \leq b$ or $b \leq a$ (*strongly connected*, formerly called *totality*).

1.2 Cartesian Product

In **mathematics**, specifically **set theory**, the **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all **ordered pairs** (a, b) where a is an element of A and b is an element of B. In terms of **set-builder notation**, that is

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Orders on the Cartesian product of totally ordered sets

There are several ways to take two totally ordered sets and extend to an order on the **Cartesian product**, though the resulting order may only be **partial**. Here are three of these possible orders, listed such that each order is stronger than the next:

- **Lexicographical order:** $(a, b) \leq (c, d)$ if and only if $a < c$ or ($a = c$ and $b \leq d$). This is a total order.
- $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$ (the **product order**). This is a partial order.
- $(a, b) \leq (c, d)$ if and only if ($a < c$ and $b < d$) or ($a = c$ and $b = d$) (the reflexive closure of the **direct product** of the corresponding strict total orders). This is also a partial order.

Relations as Subsets of Cartesian Products Mathematics Exchange

The set of ordered pairs drawn from the Cartesian product can be defined as

$$R = \{(x, y) \in A \times B |\ xRy\}$$

Thus we can say

$$R \subseteq A \times B$$

1.3 Pareto Set

We assume w.l.o.g that k objective functions,

$$\begin{aligned} f_i &: X^k \to \mathbb{R} \\ f(x) &= \big(f_1(x), f_2(x), \dots, f_k(x)\big) \in \mathbb{R}^k, \quad \text{where } \mathbf{x} \in X^k \end{aligned}$$

A solution $x \in X$ is said to *dominate* another solution $y \in X$ iff $\forall\ 1 \leq i \leq k : f_i(x) \leq f_i(y)$ and $\exists\ 1 \leq i \leq k : f_i(x) < f_i(y)$. This can denoted as $x \prec y$. A solution $x \in X$ *weakly dominates* a solution $y \in X$ iff $\forall\ 1 \leq i \leq k : f_i(x) \leq f_i(y)$. This can be denoted as $x \preceq y$.

Parento Optimal

A solution $x^* \in X$ is then called *Parento Optimal* iff there is no other solution in X that dominates x^*.

Pareto set approximations

Specific sets of solutions are the so-called *Pareto set approximations*, which are solution sets of pairwisely non-dominated solutions.

2 Contents

In single-objective optimization, every solution is mapped to a real value and solutions can always be pairwisely compared via the less or equal relation $\leq$ on $\mathbb{R}$. In another words, the total order (any elements are comparable) $\leq \subseteq \mathbb{R} \times \mathbb{R}$ induces via f order on the search space X that is a total preorder. In a multiobjective scenario, the $\leq$ relation is generalized to objective vectors, i.e., $\leq$ is a subset of $\mathbb{R}^k \times \mathbb{R}^k$. Here, the totality is not given due to vectors $a, b \in \mathbb{R}^k$ where $f_1(a) < f_1(b)$ but $f_2(a) > f_2(b)$—the relation $\leq$ on the set of objective vectors is only a partial order, i.e., reflexive, antisymmetric, and transitive. This means that *not any elements are comparable.*