# Lecture notes of Stochastic Process

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# Thank list

LeoSW, windker

# Stochastic Process

## Definition

A Stochastic process is a set of random variables $\{X(t)|t \in T\}$ where T is a index ( time) set.
State Space: possible value of X(t) for each t, which is defined as subset of R.

# Markov Chain

## Definition

A Stochastic Process $\mathbb{X}$ with state space S is a Markov Chain if
$\exists 0 \leq p_{ij} \leq 1 \quad \forall i, j \in S$ such that

$$
\begin{aligned}
(a) \quad & \sum_{j \in S} p_{ij} = 1 \quad \forall i \in S \\
(b) \quad & P(X(t+1) = j | X(0) = i_0, X(1) = i_1, ..., X(t) = i) = p_{ij} \\
& \quad \forall t, i_0, i_1, ..., i_{t-1}
\end{aligned}
$$

$\mathbb{P}$ denotes the matrix form of $p_{ij}$ with sum of any row is 1.

Lemma: $P(X(n) = j | X(0) = i) = \mathbb{P}^n[i, j]$

# Proof of lemma

We know statement is true for $(m+n)=0$. For $(m+n)>0$:

$$
\begin{aligned}
& P(X(m+n)=j|X(0)=i) \\
&= \sum_{k\in S} P(X(m+n)=j \text{ and } X(m)=k|X(0)=i) \\
&= \sum_{k\in S} P(X(m+n)=j|X(m)=k \text{ and } X(0)=i)\cdot \\
&\quad P(X(m)=k|X(0)=i) \\
&= \sum_{k\in S} P(X(m+n)=j|X(m)=k)\cdot P(X(m)=k|X(0)=i) \\
&= \sum_{k\in S} P^n[k,j]\cdot P^m[i,k] \\
&= \sum_{k\in S} P^m[i,k]\cdot P^n[k,j] \\
&= \mathbb{P}^n[i,j]
\end{aligned}
$$

# Proof of lemma(cont)

$=$

: conditional on $X(m)$

$=$ : definition of conditional probability

$=$ : (see next page)

$=$ : inductive hypothesis

# Proof of lemma(cont)

$$
\begin{aligned}
&P(X(m+n) = j | X(m) = k \text{ and } X(0) = i) \\
&\quad = \sum_{r \in S} P(X(m+n) = j | \\
&\qquad X(m+n-1) = r \text{ and } X(m) = k \text{ and } X(0) = i) \cdot \\
&\qquad P(X(m+n-1) = r | X(m) = k \text{ and } X(0) = i) \\
&\quad = \sum_{r \in S} P(X(m+n) = j | X(m+n-1) = r) \cdot \\
&\qquad P(X(m+n-1) = r | X(m) = k) \\
&\quad = P(X(m+n) = j | X(m) = k)
\end{aligned}
$$

$=$: conditional on $X(m+n-1)$
$=$: first part by definition of Markov chain and second part by inductive hypothesis

# Absorbing State

Let $\mathbb{A}$ be a set of accepting states. We would like to know the probability that $\mathbb{X}$ has ever entered some state in $\mathbb{A}$. Technique: merge all state of $\mathbb{A}$ into a new absorbing state a. Set matrix of $\mathbb{X}$ by once enter a, then probability of a goes to a is 1.

# Recurrent & transient

**Definition**

The *recurrent probability* of state $i$ of Markov chain $\mathbb{X}$ is

$$f_i = P(\text{there exists an index } t \geq 1 \text{ with } X(t) = i | X(0) = i)$$

- State $i$ of $\mathbb{X}$ is *recurrent* if $f_i = 1$.
- State $i$ of $\mathbb{X}$ is *transient* if $f_i < 1$.

# Recurrent & transient (cont.)

- If state $i$ is recurrent, by the property of Markov chain, once it re-enter the state $i$, we can take it as starting from $X(0)$ again. Hence we know that it will keep re-entering the state $i$ again and again in the process.
- If state $i$ is transient, in each period it start going from $i$, it may have probability $1 - f_i$ that it won't come back anymore.
  Hence the probability that the process will be in state i for exactly $n$ periods equals ${f_i}^{n-1}(1 - f_i),\ n \geq 1$, which is a geometric distribution.

# Recurrent & transient (cont.)

- From the preceding page, it follows that state $i$ is recurrent if and only if, starting in state $i$, the expected number of steps that the process is in state $i$ is infinite.
- We can also derive that, if the Markov chain has finite states, at least one state is recurrent.

# Expected number of visits

Let

$$I(n) = \begin{cases} 1 & \text{if } X(n) = i \\ 0 & \text{if } X(n) \neq i \end{cases}$$

we have $\sum_{n=0}^{\infty} I(n)$ represents the number of steps that the process is in state $i$, and

$$\begin{aligned} E\left[\sum_{n=0}^{\infty} I(n)|X(0) = i\right] &= \sum_{n=0}^{\infty} E[I(n)|X(0) = i] \\ &= \sum_{n=0}^{\infty} 1 \cdot P(X(n) = i|X(0) = i) \\ &= \sum_{n=0}^{\infty} P_{ii}^{n} \end{aligned}$$

We set $T = \sum_{n=0}^{\infty} I(n)$

# Lemma 1

From the above statements, we prove the following

## Lemma

*State $i$ is*

$$\textit{recurrent} \iff \sum_{n=0}^{\infty} P_{ii}^{n} = \infty,$$

$$\textit{transient} \iff \sum_{n=0}^{\infty} P_{ii}^{n} < \infty$$

# Proof of Lemma 1

($\Rightarrow$:)
($\Leftarrow$:)
Suppose state i is transient($f_i < 1$), consider $P(T = k) = f_i^{k-1} \cdot (1 - f_i)$.
Since T is a geometric distribution, we have

$$E[T] = \sum_{k=0}^{\infty} k \cdot f_i^{k-1} \cdot f_i$$
$$= \frac{1}{1 - f_i} < \infty$$

## Communicated states

### Definition

State $i$ and $j$ *communicate*, denoted $i \leftrightarrow j$, if there exist integers $m \geq 0$ and $n \geq 0$ such that

$$P_{ij}^{m} > 0 \text{ and } P_{ji}^{n} > 0$$

We say a Markov chain $\mathbb{X}$ is irreducible if $i \leftrightarrow j \quad \forall i, j \in S$

# Lemma 2

**Lemma**

*If $i \leftrightarrow j$, then the following statements hold.*

- *State $i$ is recurrent if and only if state $j$ is recurrent.*
- *State $i$ is transient if and only if state $j$ is transient.*

*Corollary: $\mathbb{X}$ is finite and irreducible $\implies$ all states are recurrent.*

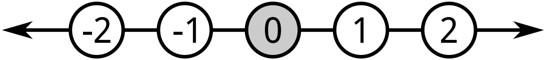
- *$\mathbb{X}$ is finite $\implies \exists i \in S$ is recurrent (proof later)*
- *By Lemma 2, all states are recurrent*

# Proof

Let $m$ and $n$ be nonnegative integers with $P_{ij}^m, P_{ji}^n > 0$. Suppose that state $j$ is recurrent, i.e., $\sum_{t=0}^{\infty} P_{jj}^t = \infty$. We have

$$\begin{aligned}
\sum_{t=0}^{\infty} P_{ii}^t &\geq \sum_{t=0}^{\infty} P_{ii}^{m+t+n} \\
&\geq \sum_{t=0}^{\infty} P_{ij}^m \cdot P_{jj}^t \cdot P_{ji}^n \\
&= P_{ij}^m \cdot P_{ji}^n \cdot \sum_{t=0}^{\infty} P_{jj}^t = \infty
\end{aligned}$$

Thus, state $i$ is also recurrent.




Let the state space consist of all integers. Let $X(0) = 0$ (i.e. at time 0 the drunken man is in state 0). The transition probabilities are such that

$$P_{i,(i+1)} = P_{i,(i-1)} = 0.5$$

holds for all states $i$ of $\mathbb{X}$.

# Gambler's ruin

# Outline

1. Limiting probabilities
2. Stationary distribution
3. Long-run proportion
4. (Inverse of) Expected return time

# Limiting Probabilities

## Definition

Number $\pi_j$ is the *limiting probability* of $j$ if

$$\pi_j = \lim_{n \to \infty} P_{ij}^n$$

holds for all states $i \in S$ ($S \subseteq \mathbb{N}$ is the state space).

- $\pi_j$ is independent of $i$.
- $\lim_{n \to \infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \end{pmatrix}$, where $\pi = (\pi_1, \pi_2, \ldots)$

# Stationary Probability Distribution

**Definition**

Non-negative row vector $\pi = (\pi_1, \pi_2, \ldots)$ is a *stationary probability distribution* of $\mathbb{X}$ if $\pi \times P = \pi$ holds and $\sum_{i \in S} \pi_i = 1$

- $\pi$ is a normalized left eigenvector with eigenvalue $= 1$.
- If $X(0)$ has distribution $\pi$, then $X(t)$ has the same distribution $\pi$ for all $t \geq 1$. $\pi$ is also called as *steady-state distribution*.
- It doesn't mean that each $X(t)$ become independent. $\pi$ only means the distribution of $X(t)$ when the previous random variable's value is unknown.

# Theorem 1

**Theorem**

*Let $\mathbb{X}$ be an irreducible, aperiodic, positive recurrent Markov chain, then*

- *The limiting probability $\pi_j$ of each state $j$ exists.*
- *$\pi = (\pi_1, \pi_2, \ldots)$ is the unique stationary probability distribution.*

- The proof will be stated at page 38.

# Expected return time

## Definition

The *expected return time* of state $i \in S$ is

$$\mu_i = \sum_{n \geq 1} n \cdot f_i^{(n)}$$

where

$$f_i^{(n)} = P(\min\{t : X(t) = i, t \geq 1\} = n | X(0) = i)$$

- $f_i = \sum_{n \geq 1} f_i^{(n)}$

# Positive recurrent & null recurrent

**Definition**

State $i$ is *positive recurrent* if $\mu_i < \infty$

**Definition**

State $i$ is *null recurrent* if $\mu_i = \infty$

- Both are recurrent states, and are *class properties*, which means that if state $i$ and $j$ communicate, they will share this property.
- If $\mathbb{X}$ is finite, then each recurrent state of $\mathbb{X}$ is positive recurrent. Proof stated at page 63.

# Example of null recurrent

**Example**

For a Markov chain with $n$ states $(1, \ldots, n)$, if

$$P(X(t+1) = i+1 | X(t) = i) = 1 - 1/n$$

and

$$P(X(t+1) = 1 | X(t) = i) = 1/n$$

According to geometric distribution (taking $p = 1/n$), the expectation value of "steps taken for state 1 to come back" will be $1/p = n$, hence $\lim_{n\to\infty} n = \infty$.

# Period of a chain

### Definition

The *period* of state $i$ is $d$ if $d$ is the largest integer such that

$$P_{ii}^{n} = 0$$

holds for all $n$ which is not divisible by $d$.

### Definition

If each state of $\mathbb{X}$ has period 1, then $\mathbb{X}$ is called *aperiodic*.

- If $P_{ii} > 0$ for all $i \in S$, then $\mathbb{X}$ is aperiodic.
- Period can be seen as the $\gcd$ of all $n$ that have $P_{ii}^{n} > 0$, note that $P_{ii}^{\gcd} > 0$ is not necessary.
- The period of drunken man problem is 2.

# Lemma 1

**Lemma**

*If state $j$ has period $1$ and is positive recurrent, then*

$$\pi_{ij} \equiv \lim_{n \to \infty} P_{ij}^n$$

*exists and is positive for all states $i \in S$.*

- This can be proved by the Blackwell theorem in Renewal theory.
- It doesn't promise that $\pi_{ij} = \pi_{i'j}$ for any $i, i' \in S$. But they will be the same if we add the irreducible property ($i \leftrightarrow i'$).

# Property of $\lim$

- The position of $\lim$ may not be switched arbitrarily in an equation.

**Example**

$$1 = \lim_{n\to\infty} \lim_{m\to\infty} \frac{m}{m+n} \neq \lim_{m\to\infty} \lim_{n\to\infty} \frac{m}{m+n} = 0$$

- $\lim$ would not influence the inequality.

**Example**

If $f(n) \geq g(n)$, then $\lim_{n\to\infty} f(n) \geq \lim_{n\to\infty} g(n)$

# Property of $\lim$ (cont.)

- $\lim$ is linear operator under finite number of functions.

**Example**

For $m < \infty$,

$$\sum_{i=1}^{m} \lim_{n\to\infty} f_i(n) = \lim_{n\to\infty} \sum_{i=1}^{m} f_i(n)$$

need an example of $m = \infty$

# Inequality 1

## Inequality

$$\sum_{j \in S} \pi_{ij} \leq 1 \quad \forall i \in S$$

# Proof

$$
\begin{aligned}
\lim_{m\to\infty}\sum_{j=1}^{m}\pi_{ij} &= \lim_{m\to\infty}\sum_{j=1}^{m}\lim_{n\to\infty}P_{ij}^{n}\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{j=1}^{m}P_{ij}^{n}\\
&\leq \lim_{m\to\infty}\lim_{n\to\infty}\sum_{j\in S}P_{ij}^{n} = 1
\end{aligned}
$$

- The last equation works since $\sum_{j\in S} P_{ij}^{n} = 1$.

# Inequality 2

## Inequality

*For state $j \in S$, we have*

$$\pi_{ij} \geq \sum_{k \in S} \pi_{ik} P_{kj}$$

# Proof

For $m \geq 1$ and $n \geq 1$,

$$P_{ij}^{n+1} = \sum_{k \in S} P_{ik}^{n} P_{kj} \geq \sum_{k=1}^{m} P_{ik}^{n} P_{kj}$$

then

$$\pi_{ij} = \lim_{n \to \infty} P_{ij}^{n+1} \geq \lim_{n \to \infty} \sum_{k=1}^{m} P_{ik}^{n} P_{kj} = \sum_{k=1}^{m} \lim_{n \to \infty} P_{ik}^{n} P_{kj} = \sum_{k=1}^{m} \pi_{ik} P_{kj}$$

hence, we know

$$\lim_{m \to \infty} \pi_{ij} = \pi_{ij} \geq \lim_{m \to \infty} \sum_{k=1}^{m} \pi_{ik} P_{kj} = \sum_{k \in S} \pi_{ik} P_{kj}$$

# Equality 1

## Equality

$$\pi_{ij} = \sum_{k \in S} \pi_{ik} P_{kj}$$

# Proof

Assume for contradiction $\pi_{ij} > \sum_{k\in S} \pi_{ik} P_{kj}$, then

$$\begin{aligned}
\lim_{m\to\infty} \sum_{j=1}^{m} > \lim_{m\to\infty} \sum_{j=1}^{m} \lim_{p\to\infty} \sum_{k=1}^{p} \pi_{ik} P_{kj} \\
&= \lim_{m\to\infty} \lim_{p\to\infty} \sum_{j=1}^{m} \sum_{k=1}^{p} \pi_{ik} P_{kj} \\
&= \lim_{m\to\infty} \lim_{p\to\infty} \sum_{k=1}^{p} \pi_{ik} \sum_{j=1}^{m} P_{kj} \\
&= \lim_{p\to\infty} \sum_{k=1}^{p} \pi_{ik} \lim_{m\to\infty} \sum_{j=1}^{m} P_{kj} \\
&= \lim_{p\to\infty} \sum_{k=1}^{p} \pi_{ik} \cdot 1 = \lim_{p\to\infty} \sum_{k=1}^{p} \pi_{ik}
\end{aligned}$$

# Proof (cont.)

- Since a value cannot be greater than itself, we got contradiction.
- In the 4th line, two $\lim$ can be switched because the value can only get larger when applying $\lim$ on it. not sure

# Proof of theorem 1

- **Step 0**: existence of limiting probability.
- **Step 1**: existence of stationary probability distribution.
- **Step 2**: uniqueness.

# 0. Existence of limiting probability

**Proof.**

By lemma 1, we know that there exists a $\pi_j$ for row $i$. Since the Markov chain is irreducible and all the states are positive recurrent, for any state $i'$ other than $i$, we know that $i'$ surely will visit $i$ in finite steps. Therefore, the $\pi_j$ value at row $i'$ will equal to the $\pi_j$ value at row $i$, which means that all the $\pi_j$ for column $j$ are the same, and is the limiting probability. $\square$

still not clear enough

# 1. Existence of stationary probability distribution

We want to prove that

**Target**

*There's a vector* $s = (s_1, s_2, \ldots)$ *such that*

1. $\sum_{i \in S} s_i = 1$
2. $s \times P = s$

**Proof.**

By lemma 1, we know that there exists a $\pi = (\pi_1, \pi_2, \ldots)$.
And by equality 1, we know that

$$(\pi_1, \pi_2, \ldots) \times P = (\pi_1, \pi_2, \ldots)$$

Hence $\pi$ can satisfy the 2nd part of our target.
Then, we take $k = \sum_{i \in S} \pi_i$. By inequality 1, we know that $k < \infty$, and can get

$$(\frac{\pi_1}{k}, \frac{\pi_2}{k}, \ldots) \times P = (\frac{\pi_1}{k}, \frac{\pi_2}{k}, \ldots)$$

where $\sum_{i \in S} \frac{\pi_i}{k} = 1$ also satisfy the 1st part of our target.
Therefore, this vector can be $s$, which means that it exists. $\square$

## 2. Uniqueness

**Target**

*If $s = (s_1, s_2, \ldots)$ is a stationary distribution of $\mathbb{X}$, then $s = \pi$.*

- We'll prove this by inequality 3 & 4.

# Inequality 3

## Inequality

$$s_j \geq \pi_j, \forall j \in S$$

## Proof.

Let the distribution of $X(0)$ be $s$, by the property of stationary distribution, we have

$$
\begin{aligned}
s_j &= P(X(n)=j) = \sum_{i\in S} P(X(n)=j|X(0)=i)P(X(0)=i) \\
&= \sum_{i\in S} P_{ij}^n \cdot s_i \\
&\geq \sum_{i=1}^{m} P_{ij}^n \cdot s_i \\
\Rightarrow s_j &= \lim_{m\to\infty}\lim_{n\to\infty} s_j \\
&\geq \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{m} P_{ij}^n \cdot s_i = \lim_{m\to\infty}\sum_{i=1}^{m} \pi_j \cdot s_i = \pi_j
\end{aligned}
$$

□

# Inequality 4

## Inequality

$$s_j \leq \pi_j, \forall j \in S$$

## Proof.

Similar in the proof above, $\forall m, n \geq 1$, we have

$$
\begin{aligned}
s_j &= \sum_{i \in S} P_{ij}^n \cdot s_i \\
&\leq \sum_{i=1}^{m} P_{ij}^n \cdot s_i + \sum_{i=m+1}^{\infty} s_i \\
\Rightarrow s_j &= \lim_{m \to \infty} \lim_{n \to \infty} s_j \\
&\leq \lim_{m \to \infty} \lim_{n \to \infty} \left( \sum_{i=1}^{m} P_{ij}^n \cdot s_i + \sum_{i=m+1}^{\infty} s_i \right) \\
&= \pi_j
\end{aligned}
$$

□

# An example Markov chain

## Example

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}, 0 < \alpha, \beta < 1$$

$$\pi = \left( \frac{\beta}{1+\beta-\alpha}, \frac{1-\alpha}{1+\beta-\alpha} \right)$$

## Real world example: Hardy-Weinberg Law

### Example

There're two kinds of allele:

- dominant: **A**
- recessive: **a**

And three kinds of senotype with population proportion as follow:

- AA: $p$
- aa: $q$
- Aa: $r = 1 - (p + q)$

## Example (cont.)

$$P = \begin{array}{c c} & \begin{array}{ccc} AA & aa & Aa \end{array} \\ \begin{array}{c} AA \\ aa \\ Aa \end{array} & \begin{pmatrix} p+\frac{r}{2} & 0 & q+\frac{r}{2} \\ 0 & q+\frac{r}{2} & p+\frac{r}{2} \\ \frac{p}{2}+\frac{r}{4} & \frac{p}{2}+\frac{r}{4} & \frac{p+q+r}{2} \end{pmatrix} \end{array}$$

we get $\pi = (p, q, r)$ when

- $p = \left(p + \frac{r}{2}\right)^2$
- $q = \left(q + \frac{r}{2}\right)^2$
- $r = 2\left(p + \frac{r}{2}\right)\left(q + \frac{r}{2}\right)$

# Long-run proportion

**Definition**

We say that $r_j$ is the *long-run proportion* of state $j \in S$ if

$$r_j = \lim_{n\to\infty} \frac{1}{n} \sum_{1\le t\le n} P_{ij}^t$$

holds for each state $i \in S$.

- It represents the average appearance times of state $j$ in the whole process.
- We will show that (in theorem 3) if $\mathbb{X}$ is irreducible, then the long-run proportion of all states exist.

# Theorem 2

**Theorem (type 1)**

*If $r_j$ exists for each $j \in S$ and $\sum_{j \in S} r_j > 0$, then $r = (r_1, r_2, \ldots)$ is the unique stationary distribution of $\mathbb{X}$.*

or

**Theorem (type 2)**

*If $r_j$ exists for each $j \in S$ and **a stationary distribution exists**, then $r = (r_1, r_2, \ldots)$ is the unique stationary distribution of $\mathbb{X}$.*

# Proof

**Existence of stationary distribution in type 1:**
Let

$$R = \begin{pmatrix} r \\ r \\ \vdots \end{pmatrix} = \lim_{n\to\infty} \frac{1}{n} \sum_{1\le t\le n} P^t$$

then

$$\begin{aligned} R \times P &= \lim_{n\to\infty} \frac{1}{n} \sum_{1\le t\le n} P^{t+1} \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{1\le t\le n} P^t + \lim_{n\to\infty} \frac{1}{n}(P^{n+1} - P) \\ &= R \end{aligned}$$

As stated later, $\sum_{j\in S} r_j \le 1$, hence by normalizing $r$, we prove that stationary distribution exist.

- $(\lim f(n)) \cdot g(n) = \lim f(n) \cdot g(n)$?
- can replace the proof on page 40?

# Proof (cont.)

**Uniqueness:**
Let $\pi$ be an arbitrary stationary distribution, then

$$
\begin{aligned}
r &= \pi \times R \\
&= \pi \times \lim_{n \to \infty} \frac{1}{n} \sum_{1 \le t \le n} P^t \\
&= \lim_{n \to \infty} \frac{1}{n} \sum_{1 \le t \le n} \pi \times P^t \\
&= \lim_{n \to \infty} \frac{1}{n} \sum_{1 \le t \le n} \pi \\
&= \pi
\end{aligned}
$$

can replace the proof for page 42?

# Proof (cont.)

**Prove that $\sum_{j\in S} r_j \le 1$:**

$$\begin{aligned}
\sum_{j\in S} r_j &= \lim_{m\to\infty}\sum_{j=1}^{m}\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}P_{ij}^{t}\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\sum_{j=1}^{m}P_{ij}^{t}\\
&\le \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\sum_{j\in S}P_{ij}^{t}\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}1 = 1
\end{aligned}$$

# Example 1

On a highway, if we know the probability that

- A truck is followed by a truck: $1/4$
- A truck is followed by a car: $3/4$
- A car is followed by a truck: $1/5$
- A car is followed by a car: $4/5$

We can construct a matrix

$$\begin{array}{cc} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{pmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{pmatrix} \end{array}$$

and get the portion of trucks and cars on the whole highway as the eigenvector $(4/19, 15/19)$ (we will know that long-run proportion exists by Theorem 3).

# Example 2

For a system which has several good and bad states, we have a matrix $P$:

$$
\begin{array}{c@{}c}
 & \begin{array}{cccccc} g_1 & g_2 & \cdots & b_1 & b_2 & \cdots \end{array} \\
\begin{array}{c} g_1 \\ g_2 \\ \vdots \\ b_1 \\ b_2 \\ \vdots \end{array} &
\left(\begin{array}{cccccc} \\ \\ \\ \\ \\ \\ \end{array}\right)
\end{array}
$$

# Example 2 (cont.)

**Q1:** Breakdown rate (breakdown times / total time)
The long-run frequency of going to a bad state from a good state is

$$\sum_{i \in g} \sum_{j \in b} r_i P_{ij}$$

# Example 2 (cont.)

**Q2:** The expected time $\mu_G$ (resp. $\mu_B$) of staying in good (resp. bad) states once we reach a good (resp. bad) state?

**Ans:**

For each $t = 1, 2, \ldots$, let $G_t$ (resp. $B_t$) be the length of the $t$-th good (resp. bad) phase of consecutive good (resp. bad) states. By the strong law of large numbers,

$$P\left(\lim_{t\to\infty} \frac{G_1 + B_1 + G_2 + B_2 + \cdots + G_t + B_t}{t} = \mu_G + \mu_B\right) = 1$$

Since the reciprocal of above is the breakdown rate, we get equation (1):

$$P\left(\sum_{i\in G}\sum_{j\in B} \pi_i P_{ij} = \frac{1}{\mu_G + \mu_B}\right) = 1$$

# Example 2 (cont.)

Also, with probability 1, we get equation (2):

$$P\left(\sum_{i\in G} r_i = \lim_{t\to\infty} \frac{G_1 + G_2 + \cdots + G_t}{G_1 + B_1 + \cdots + G_t + B_t} = \frac{\mu_G}{\mu_G + \mu_B}\right) = 1$$

Then, by $(2)/(1)$, we get that

$$P\left(\mu_G = \frac{\sum_{i\in G} r_i}{\sum_{i\in G}\sum_{j\in B} r_i P_{ij}}\right) = 1$$

- $\lim \frac{f(n)}{g(n)} = \frac{\lim f(n)}{\lim g(n)}$?

# Theorem 3

**Theorem**

*If $\mathbb{X}$ is irreducible, then the long-run proportion $r_i$ exists with probability $1$, moreover,*

1. *If state $i$ is positive recurrent (i.e. $0 < \mu_i < \infty$), then* $P(r_i = \frac{1}{\mu_i}) = 1$.
2. *If state $i$ is null recurrent (i.e. $\mu_i = \infty$) or transient, then* $P(r_i = 0) = 1$.

*where $\mu_i$ is the expected return time of state $i$*

# Proof

**Part 1:**
Suppose $X(0) = i$, $T_k$ is the number of steps required for the $k$-th $i$ goes to $(k+1)$-st $i$, then by the strong law of large number,

$$P\left(\lim_{k\to\infty}\frac{T_1+T_2+\cdots+T_k}{k}=\mu_i\right)=1$$
$$\Rightarrow P\left(r_i=\lim_{k\to\infty}\frac{k}{T_1+T_2+\cdots+T_k}=\frac{1}{\mu_i}\right)=1$$

- $\lim(A/B)=\frac{1}{\lim(B/A)}$?

# Proof (cont.)

**Part 2:**

1. If $i$ is transient, $i$ will only appear finite times in the long-run, hence

$$r_i = \frac{finite}{\infty} = 0$$

2. If $i$ is null recurrent, $\mu_i$ is $\infty$, then

$$P\left(\lim_{k\to\infty} \frac{T_1 + T_2 + \cdots + T_k}{k} = \infty\right) = 1$$

$$P\left(r_i = \lim_{k\to\infty} \frac{k}{T_1 + T_2 + \cdots + T_k} = 0\right) = 1$$

(The first equation is not promised by the strong law of large number. But if it's not $\infty$, we can say that $\mu_i$ is not $\infty$, which is a contradiction.)

# Example 1

**Example (type 1)**

If $\mathbb{X}$ is **irreducible** and finite, then $\mathbb{X}$ has no null recurrent states.

**Example (type 2)**

If $\mathbb{X}$ is finite, then $\mathbb{X}$ has no null recurrent states.

- Finite irreducible imply positive recurrent.

# Proof

- **Type 1:**
  If there's a state which is null recurrent, by irreducible, all the states will be null recurrent. Then, all states have $P(r_i = 0) = 1$. By changing the proof in page 54 into finite states version, we know that $\sum r_i = 1$. So it's impossible for finite $r_i$, which are all close to 0, to sum up to 1.
- **Type 2:**
  If it's not irreducible, the finite set of communicated null recurrent states still form an irreducible and finite Markov chain, which can fit the requirement of type 1.

# Example 2

**Example**

In the drunken man problem with infinite states, no state will be positive recurrent.

- Infinite drunken man imply no positive recurrent. Note that it doesn't mean all infinite irreducible Markov chain has no positive recurrent state.

# Proof

If all the states are positive recurrent, then by theorem 3, we know that all the $r_i > 0$ and is a finite value. Since each state of drunken man problem has the same structure, all the $r_i$ has same value. We then set $r = \epsilon \cdot r_i$ $(0 < \epsilon < 1)$ such that $r_i > r > 0, \forall i$. And get

$$\sum_{i \in S} r_i > \sum_{i \in S} r = \infty > 1$$

which is contradiction to page 54.

# Example 3: Poisson Hotel

## Example

There's a hotel, with $N$ representing the number of newly occupied rooms each day ($N$ is a poisson distribution with parameter $\lambda$). And the number of consecutive check-in days of each room is a geometric distribution with probability $p$ ($p$ is the probability of check-out). $X(t)$ is the number of occupied rooms in day $t$.

## Q1: $P_{ij} = ?$

We set $R_i$ as a binomial distribution with parameter $(i, 1-p)$, which represents the number of rooms which will remain occupied in the next day, then

$$\begin{aligned} P_{ij} &= P(R_i + N = j) \\ &= \sum_{k \geq 0} P(R_i + N = j | R_i = k) P(R_i = k) \\ &= \sum_{k \geq 0} P(N = j - k) P(R_i = k) \\ &= \sum_{0 \leq k \leq \min(i,j)} \frac{e^{-\lambda} \cdot \lambda^{j-k}}{(j-k)!} \binom{i}{k} (1-p)^k p^{1-k} \end{aligned}$$

# Q2: $r_i = ?$

We guess (by a dream?) there's a stationary distribution which is a poisson distribution with parameter $\lambda_0$. Setting $X(0)$ with this distribution. And let $R$ as the number of rooms in $X(0)$ which remain check-in in the next day ($R$ is a poisson distribution with parameter $\lambda_0(1-p)$). $X(1)$ will have distribution $R+N$, which is a poisson distribution with parameter $\lambda_0(1-p)+\lambda$. Then since $X(0)$ is a stationary distribution, it will have the same distribution with $X(1)$, which means that $\lambda_0 = \lambda_0(1-p)+\lambda$, and we get $\lambda_0 = \lambda/p$.
After getting $r_i$, we get that with probability 1,

$$\mu_i = \frac{1}{P(X(0)=i)} = \frac{i!}{e^{-\lambda/p} \cdot (\lambda/p)^i}$$

not clear enough

# Corollary of theorem 2 & 3

## Corollary

*If $\mathbb{X}$ is irreducible, then*

$$\mathbb{X} \text{ is positive recurrent } \iff \mathbb{X} \text{ admits a stationary distribution.}$$

# Moving to transient states

For transient states $i$ and $j$, we define the following:

1. Expected steps in a transient state:

**Definition**

$E$ is a matrix where $E_{ij}$ is the expected number of steps $t$ with $X(t) = j$ when $X(0) = i$.

2. Probability of reaching a transient state:

**Definition**

$F$ is a matrix where

$$F_{ij} = P(X(t) = j \text{ for some } t \geq 1 | X(0) = i)$$

# Computing $E$ & $F$

**Theorem**

*For a Markov chain $\mathbb{X}$ consisting finite transient states,*

$$E = (I - T)^{-1}$$

*where $I$ is an identity matrix, $T$ is the induced matrix of $P$ by all the transient states in $P$. Moreover,*

$$F_{ij} = \frac{E_{ij} - \delta_{ij}}{E_{jj}} \text{ ,where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

# Proof

Conditioned on $X(1)$, we have

$$E_{ij} = \underbrace{\delta_{ij}}_{\text{step}=0} + \underbrace{\sum_k P_{ik} \cdot E_{kj}}_{\text{step}\geq 1} = \delta_{ij} + \sum_k T_{ik} \cdot E_{kj}$$

The 2nd equation works since the process will not go back to transient state once it enter a recurrent state. Then, we have

$$\begin{aligned} & I \times E = E = I + T \times E \\ \Longrightarrow\ & (I - T) \times E = I \\ \Longrightarrow\ & E = (I - T)^{-1} \end{aligned}$$

# Proof (cont.)

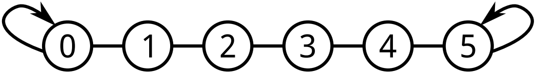
Conditioned on whether or not $X(t) = j$ holds for some $t \geq 1$, we have

$$E_{ij} = \underbrace{\delta_{ij}}_{\text{step}=0} + \underbrace{F_{ij} \cdot E_{jj}}_{\text{steps} \geq \text{the first } j}$$

therefore,

$$F_{ij} = \frac{E_{ij} - \delta_{ij}}{E_{jj}}$$

# Example: Gambler's ruin




$$T = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 \end{pmatrix} \end{array} \quad E = \begin{pmatrix} 1.6 & 1.2 & 0.8 & 0.4 \\ 1.2 & 2.4 & 1.6 & 0.8 \\ 0.8 & 1.6 & 2.4 & 1.2 \\ 0.4 & 0.8 & 1.2 & 1.6 \end{pmatrix}$$

$$F = \begin{pmatrix} 0.375 & 0.5 & 1/3 & 0.25 \\ 0.75 & 1.75/3 & 1.9/3 & 0.5 \\ 0.5 & 1.9/3 & 1.75/3 & 0.75 \\ 0.25 & 1/3 & 0.5 & 0.375 \end{pmatrix}$$

# Branching process

In the beginning, there're $X(0)$ life forms, each life form has probability $p_i$ of becoming $i$ life forms in the next step.

- state 0 is recurrent (absorbing).
- if $p_0 > 0$, all other states $(1, 2, \ldots)$ are transient since $P(X(t+1) = 0 | X(t) = i) = {p_0}^i > 0$

We'll show that

$$E[X(n)] = \mu^n \cdot X(0)$$

where

$$\mu = \sum_{j \geq 1} j \cdot p_j = E[Z_k]$$

and $Z_k$ is the number of offspring of the $k$-th life form, all $Z_k$ are i.i.d.

# Proof

$$
\begin{aligned}
E[X(n)] &= E[E[X(n)|X(n-1)]] \\
&= E\left[E\left[\sum_{k=1}^{X(n-1)} Z_k|X(n-1)\right]\right] \\
&= E[X(n-1)\cdot\mu] \\
&= \mu\cdot E[X(n-1)] \\
&= \mu^n\cdot X(0)
\end{aligned}
$$

# Probability of extinction

## Definition

$e_i$ is the probability of extinction when $X(0) = i$.

**Case 1:** $\mu < 1$

$$
\begin{aligned}
1 - e_i &= \lim_{n\to\infty} P(X(n) \geq 1 | X(0) = i) \\
&= \lim_{n\to\infty} \sum_{j\geq 1} P(X(n) = j | X(0) = i) \\
&\leq \lim_{n\to\infty} \sum_{j\geq 1} j \cdot P(X(n) = j | X(0) = i) \\
&= \lim_{n\to\infty} E[X(n) | X(0) = i] \\
&= \lim_{n\to\infty} \mu^n \cdot i = 0
\end{aligned}
$$

# Probability of extinction (cont.)

**Case 2:** $\mu \geq 1$

$$e_2 = {e_1}^2, \quad e_3 = e_2 \cdot e_1, \quad \ldots$$

$$\begin{aligned} e_1 &= P(\text{extinct} | X(0) = 1) \\ &= \sum_{j \geq 0} P(\text{extinct} | X(1) = j) \cdot P_{1j} \\ &= \sum_{j \geq 0} e_j \cdot p_j \\ &= \sum_{j \geq 0} {e_1}^j \cdot p_j \end{aligned}$$

We then solve the above equation to get $e_1$.

# Example

$$
\begin{aligned}
& p_0 = p_1 = 0.25, \quad p_2 = 0.5 \\
\implies & \mu = 1 \cdot 0.25 + 2 \cdot 0.5 > 1 \\
\implies & e_1 = {e_1}^0 \cdot 0.25 + {e_1}^1 \cdot 0.25 + {e_1}^2 \cdot 0.5 \\
\implies & e_1 = \{1/2,\ 1\}
\end{aligned}
$$

Since $\mu > 1$, we know $\lim_{n\to\infty} E[X(n)] = \infty$.
But if $e_1 = 1$, we have $\lim_{n\to\infty} P(X(n) = 0) = 1$, which would not make $\lim_{n\to\infty} E[X(n)] = \infty$, hence $e_1 \neq 1$.

# Reversed Markov chain

## Definition

Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a Markov chain with matrix $P$ (resp. $Q$).
We say that $\mathbb{Y}$ is the *reversed chain* of $\mathbb{X}$ if there exists a stationary distribution $\pi$ of $\mathbb{X}$ such that

$$\pi_i \cdot Q_{ij} = \pi_j \cdot P_{ji}$$

holds for all states $i,\ j \in S$.

# Observation 1

**Observation**

*The reversed sequence $\mathbb{Y}$ of $\mathbb{X}$ is a Markov chain.*

# Proof

$$
\begin{aligned}
&P(Y(n) = i_0 | Y(n-1) = i_1, Y(n-2) = i_2, \ldots, Y(n-k) = i_k) \\
=&P(X(n) = i_0 | X(n+1) = i_1, X(n+2) = i_2, \ldots, X(n+k) = i_k) \\
=&\frac{P(X(n) = i_0, X(n+1) = i_1, \ldots, X(n+k) = i_k)}{P(X(n+1) = i_1, \ldots, X(n+k) = i_k)} \\
=&\frac{P(X(n) = i_0) \cdot P(X(n+1) = i_1 | X(n) = i_0) \cdot P_{i_1 i_2} \cdots P_{i_{k-1} i_k}}{P(X(n+1) = i_1) \cdot P_{i_1 i_2} \cdots P_{i_{k-1} i_k}} \\
=&\frac{P(X(n) = i_0, X(n+1) = i_1)}{P(X(n+1) = i_1)} \\
=&P(X(n) = i_0 | X(n+1) = i_1) \\
=&P(Y(n) = i_0 | Y(n-1) = i_1)
\end{aligned}
$$

# Observation 2

## Observation

*If* $\mathbb{Y}$ *is the reversed sequence of Markov chain* $\mathbb{X}$ *and* $\pi$ *is a stationary distribution of* $\mathbb{X}$*, then*

$$\pi_i \cdot Q_{ij} = \pi_j \cdot P_{ji}$$

*holds for all* $i, j \in S$*, where* $Q$ *is the transition matrix of* $\mathbb{Y}$*.*

# Proof

Let $\mathbb{X}$ and $\mathbb{Y}$ have distribution $\pi$

$$
\begin{aligned}
\pi_i \cdot Q_{ij} &= P(Y(n-1) = i) \cdot P(Y(n) = j | Y(n-1) = i) \\
&= P(Y(n-1) = i, Y(n) = j) \\
&= P(Y(n-1) = i | Y(n) = j) \cdot P(Y(n) = j) \\
&= P(X(n+1) = i | X(n) = j) \cdot P(X(n) = j) = \pi_j \cdot P_{ji}
\end{aligned}
$$

# Observation 3

## Observation

*Let $P$ (resp. $Q$) be the transition matrix of $\mathbb{X}$ (resp. $\mathbb{Y}$), if vector $\pi$ satisfy the following*

- $\sum_{i \in S} \pi_i = 1$
- $\pi_i \geq 0 \quad \forall i \in S$
- $\pi_i \cdot Q_{ij} = \pi_j \cdot P_{ji} \quad \forall i, j \in S$

*then $\mathbb{Y}$ is the reversed sequence of $\mathbb{X}$.*

- The long-run proportion of $i \to j$ in the sequence of $\mathbb{Y}$ is equal to the long-run proportion of $j \to i$ in the sequence of $\mathbb{X}$.
- Reversed Markov chain is the reversed sequence.

# Proof

From the third property, we have

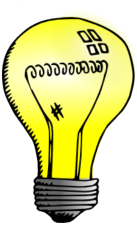
$$\sum_{j \in S} \pi_i \cdot Q_{ij} = \pi_i = \sum_{j \in S} \pi_j \cdot P_{ji} \quad \forall i \in S$$

From the 2nd equation, we know that $\pi \times P = \pi$, hence $\pi$ is a stationary distribution of $\mathbb{X}$.
Then by observation 2, we know that for any $\pi$, there's a reversed sequence $\mathbb{Y}'$, whose transition matrix $Q'$ satisfy

$$\pi_i \cdot Q'_{ij} = \pi_j \cdot P_{ji} \quad \forall i, j \in S$$

hence $\mathbb{Y} = \mathbb{Y}'$, which is a reversed sequence of $\mathbb{X}$.

# Example: Bulb's life



There's a room which need to be lighted by one bulb, when the bulb in use fails, it will be replaced by a new one on next day.

- $X(n) = i$ if the bulb in use on day $n$ is in its $i$th day of use.
- $L$ is a random variable representing the lifetime of a bulb.

We want to know the stationary probability $\pi_i$ of state $i$.

# Example: Bulb's life (cont.)

$\mathbb{X}$ is a irreducible, positive recurrent, aperiodic Markov chain which has the sequence like this:

$$1, 2, 3, 1, 2, 3, 4, 5, 1, 1, 2, 1, 2, 3, 4, \ldots$$

We know that

$$P_{i1} = P(\text{buld, on its } i\text{th day of use, fails}) = \frac{P(L = i)}{P(L \geq i)} = 1 - P_{i(i+1)}$$

And the expected return time of state $1$ is $E[L]$, which means that the long-run proportion of state $1$ is $1/E[L]$ by page 60.

# Example: Bulb's life (cont.)

Take $\mathbb{Y}$ (with matrix $Q$) as the reversed chain of $\mathbb{X}$, we know that for all $i \in S$,

- $Q_{(i+1)i} = 1$
- $Q_{1i} = P(L = i)$
- $\pi_1 \cdot Q_{1i} = \pi_i \cdot P_{i1}$

Hence,

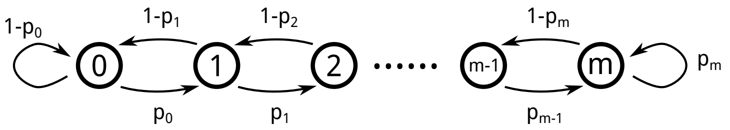
$$\pi_i = \frac{\pi_1 \cdot Q_{1i}}{P_{i1}} = \frac{P(L = i) \cdot P(L \geq i)}{E[L] \cdot P(L = i)} = \frac{P(L \geq i)}{E[L]}$$

# Time-reversible

**Definition**

$\mathbb{X}$ is *time-reversible* if $\mathbb{X}$ is the reversed chain of $\mathbb{X}$.

# Example: Reversed drunken man




- $0 < p_0 \leq 1$
- $0 \leq p_m < 1$
- $0 < p_i < 1 \quad \forall i = 1, \ldots, m-1$

The long-run proportion of transition $i \to i+1$ and $i+1 \to i$ are the same, since one must go back to $i$ from $i+1$ in order to go to $i+1$ from $i$.

Hence the drunken man problem is time-reversible.

# Example: Reversed drunken man (cont.)

$$\pi_0 \cdot p_0 = \pi_1 \cdot (1 - p_1)$$
$$\pi_1 \cdot p_1 = \pi_2 \cdot (1 - p_2)$$
$$\vdots$$
$$\pi_{m-1} \cdot p_{m-1} = \pi_m \cdot (1 - p_m)$$

Thus,

$$\pi_1 = \pi_0 \cdot p_0 / (1 - p_1)$$
$$\pi_2 = \pi_1 \cdot p_1 / (1 - p_2)$$
$$\vdots$$
$$\pi_m = \pi_{m-1} \cdot p_{m-1} / (1 - p_m)$$

# Example: Reversed drunken man (cont.)

$$\pi_i = \underbrace{\frac{\prod_{j=0}^{i-1} p_j}{\prod_{j=1}^{i}(1-p_j)}}_{q_i} \cdot \pi_0 \quad \forall i = 1, \dots m$$

$$\Longrightarrow \pi_0 + \sum_{i=1}^{m} \pi_i = 1 = \pi_0 + \sum_{i=1}^{m} q_i \cdot \pi_0$$

$$\Longrightarrow \pi_0 = \frac{1}{1 + \sum_{i=1}^{m} q_i}$$

$$\Longrightarrow \pi_k = \frac{q_k}{1 + \sum_{i=1}^{m} q_i} \quad \forall k = 0, 1, \dots m$$

# Example: Two bukkits of balls

There're two bukkits contain total $m$ balls.
In each step, we randomly choose one ball and put it in another bukkit.
Let $X(n)$ represent the number of balls in the first bukkit, it's the Markov chain of previous example with

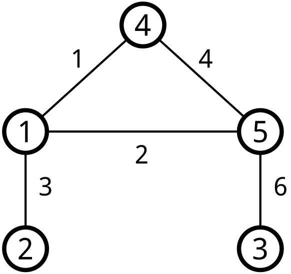
$$p_0 = 1,\ p_m = 0,\ p_i = \frac{m-i}{m} \quad \forall i = 1, \ldots, m-1$$

We can get that

$$q_i = \frac{\prod_{j=0}^{i-1} \frac{m-j}{m}}{\prod_{j=1}^{i} \frac{j}{m}} = \frac{\prod_{j=0}^{i-1} m-j}{\prod_{j=1}^{i} j} = \binom{m}{i} \quad \forall i = 1, \ldots m$$

$$\implies \pi_0 = \frac{1}{1 + \sum_{i=1}^{m} \binom{m}{i}} = \frac{1}{2^m} \implies \pi_k = \frac{\binom{m}{k}}{2^m} \quad \forall k = 0, 1, \ldots m$$

## Example: A random walk




$$P_{ij} = \frac{w(i,j)}{\sum_k w(i,k)}$$

where $w(a,b)$ is the weight of edge $(a,b)$.
To make it as a time-reversible chain, we let

$$\pi_i = \frac{\sum_k w(i,k)}{\sum_\ell \sum_k w(\ell,k)} \quad \forall i$$

We can see that

$$\pi_i \cdot P_{ij} = \pi_j \cdot P_{ji} \quad \forall i,j$$

# Hastings-Metropolis sampling algorithm

Design an irreducible Markov chain $\mathbb{X}$ such that the unique stationary distribution of $\mathbb{X}$ is the distribution of random variable $Y$.
Since the long-run proportion of state $i$ is $P(Y = i)$,

$$\lim_{n\to\infty} \frac{X(1) + X(2) + \ldots + X(n)}{n} = \sum_{i \in S} i \cdot P(Y = i) = E[Y] = \mu$$

While computing $\mu$ by the law of large number is difficult (hard to sample on $Y$), we use this alternative method to compute $\mu$ by generating a sequence of $\mathbb{X}$, which is sometime easier.

# Hastings-Metropolis sampling algorithm (cont.)

There's a random variable $Y$ such that

$$P(Y = i) = \frac{b_i}{C}$$

for some unknown (or intractable) $C = \sum_{i \in S} b_i$.
We then design a Markov chain $\mathbb{X}$ that

- $P_{ii} = Q_{ii} + \sum_{k \in S, k \neq i} Q_{ik} \cdot (1 - q_{ik})$
- $P_{ij} = Q_{ij} \cdot q_{ij} \quad \forall j \neq i$

where

- $Q$ is the transition matrix of an arbitrary irreducible Markov chain $\mathbb{X}$ which has the same state space as $Y$.
- $q$ is a matrix to be determined later.

# Hastings-Metropolis sampling algorithm (cont.)

For $n = 0, 1, \ldots,$

1. If $X(n) = i$, set $Z$ such that $P(Z = j) = Q_{ij} \quad \forall j \in S$.
2. If $Z = j$, set $X(n+1)$ such that
   - $P(X(n+1) = j) = q_{ij}$
   - $P(X(n+1) = i) = 1 - q_{ij}$

One can see that this satisfies the requirement on previous page.

# Hastings-Metropolis sampling algorithm (cont.)

Then, we let

$$
\begin{aligned}
q_{ij} &= \min\left(\frac{b_j \cdot Q_{ji}}{b_i \cdot Q_{ij}}, 1\right) \\
\Longrightarrow b_i \cdot Q_{ij} \cdot q_{ij} &= b_j \cdot Q_{ji} \cdot q_{ji} \\
\Longrightarrow \frac{b_i}{C} \cdot P_{ij} &= \frac{b_j}{C} \cdot P_{ji}
\end{aligned}
$$

By observation 3 on page 86, we know that $(b_1 / C, b_2 / C, \ldots)$ is the stationary distribution of $\mathbb{X}$.

# Example: Space of permutations

**Example**

Let $S$ consist of all the permutations $(x_1, x_2, \ldots, x_n)$ of $\{1, 2, \ldots, n\}$ that

$$\sum_{k=1}^{n} k \cdot x_k \geq \frac{n^3}{4}$$

- This is same as $Y$ in page 98 with $C = |S|$ and $b_i = 1\ \forall i$.
- $S$ is hard to compute.
- We need to design a matrix $Q$ such that when given a permutation $x$, it's efficient to compute the value of $Q_{xy}\ \forall y \in S$.

# Example: Space of permutations (cont.)

We let

$$Q_{xy} = \frac{1}{N(x)} \quad \text{, if } y \text{ can be obtained from } x \text{ by one swap}$$

where $N(x)$ is the number of permutations that can be obtained from $x$ by one swap. For example:

$$\underbrace{(1,2,3,4,5)}_{y} \leftrightarrow \underbrace{(1,3,2,4,5)}_{x} \leftrightarrow \underbrace{(1,3,4,2,5)}_{y}$$

This chain is irreducible since each $x \in S$ can go to $(x_1, x_2, \ldots, x_n)$, where $x_1 \leq x_2 \leq \ldots \leq x_n$, by several swaps.
Also, given a $x$, finding all the obtainable $y$ can be done efficiently.

# Counting process

**Definition**

A collection $\mathbb{N}$ of random variables is a *counting process* if $N(t)$ denotes the total number of events that occur by time $t$.

- $N(t)$ is a nonnegative integer.
- The value of $N(t)$ is increasing as $t$ increase.
- $N(t) - N(s)$ is the number of events that occur between time index $s$ and $t$, where $t > s$.

# Two properties

**Independent increments**:

### Definition

A counting process is *independent increments* if the number of events in two non-overlapping time intervals are independent.

- For example, $N(s) - N(0)$ and $N(s+t) - N(s)$ are independent.

**Stationary increments**:

### Definition

A counting process is *stationary increments* if the number of events in any time interval depends only on the length of the interval.

- For example,
  $P(N(s_1 + t) - N(s_1) = k) = P(N(s_2 + t) - N(s_2) = k)$.

# Poisson process

## Definition

A *Poisson process* with rate $\lambda$ is a counting process with independent increments and stationary increments such that

$$P(N(s+t) - N(s) = n) = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}$$

holds for all nonnegative integers.

- $N(s+t) - N(s)$ is Poisson distributed with parameter $\lambda t$.
- The average number of events that occur in an unit time interval $(t = 1)$ is $\lambda$ (since the expectation value of Poisson distribution with parameter $\lambda$ is $\lambda$.)

# An operational definition

**Theorem**

*Let $\mathbb{N}$ be a counting process with independent increments and stationary increments. Then $\mathbb{N}$ is a Poisson process if and only if the following two conditions hold:*

1. $P(N(t) = 1) = \lambda \cdot t + o(t)$
2. $P(N(t) \geq 2) = o(t)$

- We say that $f(t) = o(t)$ if

$$\lim_{t \to 0} \frac{f(t)}{t} = 0$$

# Proof

$(\Longrightarrow)$:
Since $N(t)$ is Poisson distributed with parameter $\lambda t$,

$$\begin{aligned} P(N(t)=1) = \frac{(\lambda t)\cdot e^{-\lambda t}}{1!} &= \lambda t \cdot \left(1 - \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} - \cdots\right) \\ &= \lambda t - \lambda^2 t^2 + \cdots \\ &= \lambda t + o(t) \end{aligned}$$

$$\begin{aligned} P(N(t)=2) = \frac{(\lambda t)^2\cdot e^{-\lambda t}}{2!} &= \frac{(\lambda t)^2}{2!} \cdot \left(1 - \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} - \cdots\right) \\ &= o(t) \end{aligned}$$

One can prove that $P(N(t)=k) = o(t)$ for all $k \geq 2$,
hence $P(N(t) \geq 2) = o(t)$.

# Proof (cont.)

($\Longleftarrow$):
The Laplace transform of a random variable $X$ is

$$\phi(u) = E[e^{-u \cdot X}]$$

We say that two random variables have the same distribution if their Laplace transform are the same.
And if $X$ is Poisson distributed with parameter $\lambda t$, then

$$E[e^{-u \cdot X}] = e^{(e^{-u}-1) \cdot \lambda t}$$

# Proof (cont.)

We define $\phi_u(t) = E[e^{-u \cdot N(t)}]$, then we know that

$$\begin{aligned}
\phi_u(s+t) &= E[e^{-u \cdot N(s+t)}] \\
&= E[e^{-u \cdot (N(s)-N(0))} e^{-u \cdot (N(s+t)-N(s))}] \\
&= E[e^{-u \cdot N(s)}] \cdot E[e^{-u \cdot (N(s+t)-N(s))}] \\
&= E[e^{-u \cdot N(s)}] \cdot E[e^{-u \cdot N(t)}] \\
&= \phi_u(s) \cdot \phi_u(t)
\end{aligned}$$

The 3rd equation is because two independent random variables $X$ and $Y$ will make

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

# Proof (cont.)

By the two conditions in page 106, we know

$$P(N(t) = 0) = 1 - \lambda t + o(t)$$

Therefore,

$$\begin{aligned}
\phi_u(t) &= E[e^{-u \cdot N(t)}] \\
&= e^{-u \cdot 0} \cdot (1 - \lambda t + o(t)) + e^{-u \cdot 1} \cdot (\lambda t + o(t)) \\
&\quad + (e^{-u \cdot 2} + e^{-u \cdot 3} + \cdots) \cdot o(t) \\
&= 1 - \lambda t + e^{-u} \cdot \lambda t + o(t) \\
&= 1 + (e^{-u} - 1) \cdot \lambda t + o(t)
\end{aligned}$$

And

$$\phi_u(s+t) = \phi_u(s) \cdot \phi_u(t) = \phi_u(s) \cdot (1 + (e^{-u} - 1) \cdot \lambda t + o(t))$$

# Proof (cont.)

Differentiate on $\phi_u(s)$, we can get

$$\begin{aligned} {\phi_u}'(s) &= \lim_{t\to 0} \frac{\phi_u(s+t) - \phi_u(s)}{t} = \lim_{t\to 0}(\phi_u(s)\cdot(e^{-u}-1)\cdot\lambda + o(t)) \\ &= \phi_u(s)\cdot(e^{-u}-1)\cdot\lambda \end{aligned}$$

By $\frac{{\phi_u}'(s)}{\phi_u(s)} = (e^{-u}-1)\cdot\lambda$, we have

$$\ln\phi_u(s) = \int (e^{-u}-1)\cdot\lambda\ ds = (e^{-u}-1)\cdot\lambda s + C$$

By $\phi_u(0) = 1$ and $\ln 1 = 0$, we know $C = 0$, hence

$$\phi_u(s) = e^{(e^{-u}-1)\cdot\lambda s} \quad \forall s, u$$

which means that $N(s)$ is Poisson distributed for all $s$.

# Inter-arrival time

**Definition**

The $k$th inter-arrival time $T_k$ of $\mathbb{N}$ is the time interval between the $(k+1)$st and $k$th events.
$\mathbb{T} = T_1, T_2, \ldots$ is the sequence of inter-arrival times of $\mathbb{N}$.

- 0th event arrives at time $0$.

# Observation 1: Independent & exponential distributed

**Observation**

*If $\mathbb{N}$ is a Poisson process with rate $\lambda$, then each $T_k$ is an independent exponential distribution with parameter $\lambda$.*

**Proof**:
The cumulative distribution function of $T_1$ is

$$\begin{aligned} F_1(s) &= P(T_1 \leq s) \\ &= 1 - P(T_1 > s) \\ &= 1 - P(N(s) = 0) \\ &= 1 - e^{-\lambda s} \end{aligned}$$

The 3rd equation is because $T_1 > s \iff N(s) = 0$.
We can observe that $T_1$ is expenential distributed.

# Proof (cont.)

$$
\begin{aligned}
P(T_2 > t | T_1 = s) &= P(N(T_1 + t) - N(T_1) = 0 | T_1 = s) \\
&= P(N(T_1 + t) - N(T_1) = 0) \\
&= P(N(t) = 0) \\
&= e^{-\lambda t}
\end{aligned}
$$

The equations are derived by stationary increments.

Thus, $T_2$ is also exponential distributed with parameter $\lambda$.
And $T_1$, $T_2$ are independent.
One can prove for $T_k$ with $k \geq 3$ by the same approach.

# Observation 2: Waiting time is gamma distributed

**Observation**

*The waiting time $S_k = T_1 + T_2 + \cdots + T_k$ of the $k$th event is gamma distributed with parameter $(k, \lambda)$. check gamma distribution*

- The probability density function is

$$f(t) = \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{k-1}}{(k-1)!}$$

- It's also called Erlang distribution since $k \in \mathbb{Z}^+$.

# Verification

$$P(S_k \le t) = P(N(t) \ge k) = \sum_{i \ge k} P(N(t) = i) = \sum_{i \ge k} \frac{(\lambda t)^i \cdot e^{-\lambda t}}{i!}$$

So,

$$\begin{aligned}
\frac{dP(S_k \le t)}{dt} &= \sum_{i \ge k} \frac{\lambda \cdot (\lambda t)^{i-1} \cdot e^{-\lambda t}}{(i-1)!} - \sum_{i \ge k} \frac{(\lambda t)^i \cdot e^{-\lambda t} \cdot \lambda}{i!} \\
&= \frac{\lambda \cdot (\lambda t)^{k-1} \cdot e^{-\lambda t}}{(k-1)!}
\end{aligned}$$

# Property 1 (Different types of events)

**Property**

*Let $\mathbb{N}$ be a Poisson process with rate $\lambda$, each event is classified as type 1 with probability $p$ or type 2 with probability $1-p$.*
*Then the arrival of type 1 and type 2 events are both Poisson processes with rate $p \cdot \lambda$ and $(1-p) \cdot \lambda$. And the two processes are independent.*

- Let $\mathbb{N}_1$ be the process of type 1 event, $N_1(k)$ is the number of type 1 events that occur by time $k$. (same for $\mathbb{N}_2$)
- $\mathbb{N}_1$ and $\mathbb{N}_2$ are said to be independent if $N_1(s_1+t_1) - N_1(s_1)$ and $N_2(s_2+t_2) - N(s_2)$ are independent for all $s_1, t_1, s_2, t_2$.

# Proof

Here we prove that $\mathbb{N}_1$ is a Poisson process with rate $\lambda p$.

**Stationary increments**:

$$P(N_1(s+t) - N_1(s) = k_1 | N(s+t) - N(s) = k) = \binom{k}{k_1} \cdot p^{k_1} \cdot (1-p)^{k-k_1}$$

Therefore,

$$P(N_1(s+t) - N_1(s) = k_1) = \sum_{k \geq 0} \binom{k}{k_1} \cdot p^{k_1} \cdot (1-p)^{k-k_1} \cdot \frac{(\lambda t)^k \cdot e^{\lambda t}}{k!}$$

which has nothing to do with $s$.

# Proof (cont.)

**Independent increments**:
Let $(s, s+t)$ and $(u, u+v)$ be two non-overlapping time intervals,

$$
\begin{aligned}
& P(N_1(s+t) - N_1(s) = k_1, N_1(u+v) - N_1(u) = \ell_1) \\
&= \sum_{k \geq 0} \sum_{\ell \geq 0} P(N_1(s+t) - N_1(s) = k_1, N_1(u+v) - N_1(u) = \ell_1 \\
&\qquad |N(s+t) - N(s) = k, N(u+v) - N(u) = \ell) \\
&\qquad \cdot P(N(s+t) - N(s) = k, N(u+v) - N(u) = \ell) \\
&= \sum_{k \geq 0} \sum_{\ell \geq 0} P(N_1(s+t) - N_1(s) = k_1 | N(s+t) - N(s) = k) \\
&\qquad \cdot P(N_1(u+v) - N_1(u) = \ell_1 | N(u+v) - N(u) = \ell) \\
&\qquad \cdot P(N(s+t) - N(s) = k) \cdot P(N(u+v) - N(u) = \ell) \\
&= P(N_1(s+t) - N_1(s) = k_1) \cdot P(N_1(u+v) - N_1(u) = \ell_1)
\end{aligned}
$$

# Proof (cont.)

**Conditions on page 106**:

**1.** $P(N_1(t) \geq 2) \leq P(N(t) \geq 2) = o(t)$

**2.** 
$$\begin{aligned} P(N_1(t) = 1) &= P(N_1(t) = 1 | N(t) = 1) \cdot P(N(t) = 1) \\ &\quad + P(N_1(t) = 1 | N(t) \geq 2) \cdot P(N(t) \geq 2) \\ &= p \cdot (\lambda t + o(t)) + o(t) \\ &= p\lambda t + o(t) \end{aligned}$$

Hence we know that $\mathbb{N}_1$ is a Poisson process with rate $\lambda p$.
Seems like we can also derive this from the result of next page and omit this page's proof? (By Example 3.23 on textbook?)

# Proof (cont.)

$\mathbb{N}_1$ **and** $\mathbb{N}_2$ **are independent**:

$$
\begin{aligned}
&P(N_1(t) = i, N_2(t) = j) \\
&= P(N_1(t) = i, N_2(t) = j | N(t) = i + j) \cdot P(N(t) = i + j) \\
&= \binom{i+j}{i} \cdot p^i \cdot (1-p)^j \cdot \frac{e^{-\lambda t} \cdot (\lambda t)^{i+j}}{(i+j)!} \\
&= \frac{e^{-\lambda p t} \cdot (\lambda p t)^i}{i!} \cdot \frac{e^{-\lambda(1-p)t} \cdot (\lambda(1-p)t)^j}{j!} \\
&= P(N_1(t) = i) \cdot P(N_2(t) = j)
\end{aligned}
$$

We only prove for two intervals that have the same length here.
This also prove that $N_1(t)$ and $N_2(t)$ are Poisson distributed over $t$ (Example 3.23 on textbook).

# Example 1

$\mathbb{N}$ has rate $10$ and $p = 1/12$,

$$P(N_1(4) = 0) = \frac{e^{-\frac{40}{12}} \cdot (\frac{40}{12})^0}{0!} = e^{-\frac{10}{3}}$$

# Example 2: Type transitions

There're $r$ classes of particles.

- $Y_i(k)$ is the number of class $i$ particles at time $k$.
- The time is discrete in this case.
- $Y_i(0)$ is Poisson distributed with parameter $\lambda_i$.
- $P_{ij}$ is the transition probability for a class $i$ particle to class $j$.

We prove that $Y_j(n)$ is Poisson distributed with parameter $\sum_{i=1}^{r} P_{ij}^{n} \cdot \lambda_i$.

# Proof

Take class $i$ for example, we consider a Poisson process $\mathbb{N}$ with rate $\lambda_i$, where each event is classified as type $k$ with probability $P_{ik}^n$.

For an arbitrary unit time interval, the number of events that occur in this interval is Poisson distributed with parameter $\lambda_i$. We take this Poisson distributed number as the value of $Y_i(0)$.

By property 1, we know that the number of type $k$ events in this interval is Poisson distributed with parameter $P_{ik}^n \cdot \lambda_i$, which also means that the number of class $i$ particles that eventually become class $k$ at time $n$, which is denoted as $C_{ik}^n$, is also Poisson distributed with parameter $P_{ik}^n \cdot \lambda_i$.

$Y_j(n) = \sum_i C_{ij}^n$, which is Poisson distributed with parameter $\sum_{i=1}^r P_{ij}^n \cdot \lambda_i$. (since the Poisson parameter can be summed up.)

# Example 3: Selling a product

Consider a Poisson process with rate $\lambda$, where each event is an offer that has density function $f(x)$.
A product is sold if an offer with value higher than the price $y$ comes.
Assume the accepted offer comes at time $t$, then the storage cost is $c \cdot t$, where $c$ is a constant decided by the product.

We want to know the expected profit, which is $E[f(x) - ct | f(x) \geq y]$.

# Solution

The probability for each offer being accepted is

$$p(y) = P(X \geq y) = \int_y^\infty f(x)\ dx$$

The expectation of storage time $t$ is $1/(\lambda \cdot p(y))$.
Hence,

$$\begin{aligned}
& E[f(x)|f(x) \geq y] - E[ct|f(x) \geq y] \\
&= \int_0^\infty x \cdot f_{X|X\geq y}(x)\ dx - \frac{c}{\lambda \cdot p(y)} \\
&= \int_y^\infty x \cdot \frac{f_X(x)}{P(X \geq y)}\ dx - \frac{c}{\lambda \cdot p(y)} \\
&= \frac{1}{p(y)} \left( \int_y^\infty x \cdot f(x)\ dx - \frac{c}{\lambda} \right)
\end{aligned}$$

# Example 4: Coupon collection


There are $r$ types of coupons, and $p_i$ is the probability for a collected coupon being type $i$.

We want to know the expectation of $N$, where $N$ is the number of collected coupons so that all $r$ types of coupons are collected.

# Solution: First attempt

Let $N_i$ be the number of coupons collected to receive the first type $i$ coupon. We know that

$$E[N] = E[\max(N_1, N_2, \dots, N_r)]$$

And

$$P(N \leq n) = P(N_1 \leq n, N_2 \leq n, \cdots, N_r \leq n)$$

But since each $N_i$ are not independent, we can't go even further from here. For example, given that $N_1 = 1$, $P(N_2 = 1 | N_1 = 1) = 0$.

# Solution: Second attempt

Without loss of generality, we assume that the coupons arrive as a Poisson process $\mathbb{N}$ with rate $1$.
$\mathbb{N}_i$ is the process of type $i$ coupons, which has rate $p_i$.
$X_i$ is the time that the first type $i$ coupon appears, and

$$X = \max(X_1, X_2, \ldots, X_r)$$

$X_1, X_2, \ldots, X_r$ are independent since $\mathbb{N}_1, \mathbb{N}_2, \ldots, \mathbb{N}_r$ are independent.

# Solution: Second attempt (cont.)

We can see that

$$\begin{aligned} P(X \le t) &= P(X_1 \le t, X_2 \le t, \cdot, X_r \le t) \\ &= \prod_{i=1}^{r} P(X_i \le t) = \prod_{i=1}^{r} (1 - e^{-t \cdot p_i}) \end{aligned}$$

And

$$E[X] = \int_0^\infty P(X > t)\ dt = \int_0^\infty 1 - \prod_{i=1}^{r} (1 - e^{-t \cdot p_i})\ dt$$

The first equation is from the property of probability.
Surprisingly, $E[X] = E[N]$ as explained below.

# Solution: Second attempt (cont.)

$$X = T_1 + T_2 + \cdots + T_N$$

where $T_j$ is the $j$th inter-arrival time of $\mathbb{N}$.
Each $T_j$ are i.i.d. and are exponential distributed with rate $1$.
Also, $N$ is independent with each $T_j$.

$$E[X|N = n] = E[T_1 + T_2 + \cdots + T_n] = n \cdot E[T_1] = n$$

$$E[X|N] = N \cdot E[T_1] = N$$

Hence,

$$E[X] = E[E[X|N]] = E[N]$$

# Property 2-simple (Distribution of one event)

**Property**

*Given that exactly one event of a Poisson process arrives in the interval* $[0, t]$*, this arrival time is uniformly distributed over* $[0, t]$*.*

**Proof**:

$$
\begin{aligned}
P(T_1 \leq s | N(t) = 1) &= \frac{P(T_1 \leq s, N(t) = 1)}{P(N(t) = 1)} \\
&= \frac{P(N(s) = 1) \cdot P(N(t) - N(s) = 0)}{e^{-\lambda t} \cdot (\lambda t)^1 / 1!} \\
&= \frac{e^{-\lambda s} \cdot \lambda s \cdot e^{-\lambda (t-s)}}{e^{-\lambda t} \cdot \lambda t} = \frac{s}{t}
\end{aligned}
$$

# Property 2-advanced (Distribution of several events)

## Property

*Given that exactly $n$ events of a Poisson process arrive in the interval $[0,t]$, each with arrival time $X_1, X_2, \ldots, X_n$.*
*The order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ of these random variables have the joint density function*

$$f_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x_1, x_2, \ldots, x_n | N(t) = n) = \begin{cases} \frac{n!}{t^n} & \text{if } 0 < x_1 < x_2 < \cdots < x_n < t \\ 0 & \text{otherwise} \end{cases}$$

- This implies that $X_1, X_2, \ldots, X_n$ are i.i.d. and each is uniformly distributed over $[0,t]$.

# Proof

For any $0 < x_1 < x_2 < \ldots < x_n < t$,

$$\begin{aligned}
&f_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x_1, x_2, \ldots, x_n | N(t) = n) \\
&= \frac{f_{X_{(1)},X_{(2)},\ldots,X_{(n)},N(t)}(x_1, x_2, \ldots, x_n, n)}{P(N(t) = n)} \\
&= \frac{f_{T_1,T_2,\ldots,T_n}(x_1, x_2 - x_1, \ldots, x_n - x_{n-1}) \cdot P(T_{n+1} > t - x_n)}{P(N(t) = n)} \\
&= \frac{\lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda(x_2 - x_1)} \ldots \lambda e^{-\lambda(x_n - x_{n-1})} \cdot e^{\lambda(t - x_n)}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \\
&= \frac{n!}{t}
\end{aligned}$$

$T_i$ are the inter-arrival times, which are exponential distributed.

# Corollary of property 2 (simple version)

## Corollary

*Consider a Poisson process with rate $\lambda$.*
*Each event is classified into a type $i$, where there are $r$ types of event.*
*Suppose that $p_i(\cdot)$ is a sampling function over interval $[0, t]$ such that each arrived event at time $x$ has probability $p_i(x)$ to be classified as type $i$.*
*Then the number $N_i(t)$ of type $i$ events in $[0, t]$ is Poisson distributed with parameter*

$$\lambda \int_0^t p_i(x)\ dx = \lambda \cdot t \cdot R_i \quad \text{,where } R_i = \frac{1}{t} \int_0^t p_i(x)\ dx$$

*And each $N_i(t)$ for all $i$ are independent.*

- $N_i(\cdot)$ does not form a Poisson process here.
  It doesn't satisfy stationary distribution because of $p_i(\cdot)$.

# Proof

Assume that $N(t) = n$.
Let $n = n_1 + n_2 + \cdots + n_r$, where $n_i \geq 0 \ \forall i = 1, \ldots, r$.
These $n$ events arrive independent and uniformly at random over $[0, t]$.
If an event arrives at time $x \in [0, t]$, then with probability $p_i(x)$ it becomes type $i$. Therefore, each event is of type $i$ with probability

$$\begin{aligned}
&\int_0^t P(\text{type } i|\text{arrives at time } x) \cdot \frac{1}{t} \ dx \\
&= \frac{1}{t} \int_0^t p_i(x) \ dx \\
&= R_i
\end{aligned}$$

which can be seen as the average of $p_i(\cdot)$ over $[0, t]$.

# Proof (cont.)

$$
\begin{aligned}
P\Big(\bigwedge_{1\le i\le r} N_i(t)=n_i\Big) &= P\Big(\bigwedge_{1\le i\le r} N_i(t)=n_i \,|\, N(t)=n\Big)\cdot P(N(t)=n) \\
&= \left(\frac{n!}{n_1!n_2!\cdots n_r!}\cdot \prod_{1\le i\le r} {R_i}^{n_i}\right)\cdot \frac{e^{-\lambda t}\cdot(\lambda t)^n}{n!} \\
&= \prod_{1\le i\le r} \frac{e^{-\lambda t R_i}(\lambda t R_i)^{n_i}}{n_i!} \\
&= \prod_{1\le i\le r} P(N_i(t)=n_i)
\end{aligned}
$$

Hence we know that each $N_i(t)$ is Poisson distributed with parameter $\lambda t R_i$ and are independent. (Check Example 3.23 on textbook.)

# Corollary of property 2 (full version)

## Corollary

*Consider a Poisson process with rate $\lambda$.*
*Each event is classified into a type $i$, where there are $r$ types of event.*
*Suppose that $p_i(\cdot)$ is a sampling function over interval $[s, s+t]$ such that each arrived event at time $x$ has probability $p_i(x)$ to be classified as type $i$.*
*Then the number $N_i(s+t) - N_i(s)$ of type $i$ events in $[s, s+t]$ is Poisson distributed with parameter*

$$\lambda \int_s^{s+t} p_i(x)\ dx$$

*And each $N_i(s+t) - N_i(s)$ for all $i$ are independent.*

# Proof

Regarding $[s, s+t]$ as $[0, t]$, the sampling function becomes

$$p'_i(x) = p_i(x+s)$$

From the simple version, we know that $N_i(s+t) - N_i(s)$ is Poisson distributed with parameter

$$\lambda \int_0^t p'_i(x)\ dx = \lambda \int_0^t p_i(x+s)\ dx = \lambda \int_s^{s+t} p_i(x)\ dx$$

# Example 1: Infinite server queue

Suppose that jobs arrive at a Poisson rate $\lambda$, and we have infinite number of servers. The running time of each job is independent and distributed with function $T(\cdot)$. We want to know

1. The distribution of the number $X(t)$ of completed jobs by time $t$.
2. The distribution of the number $Y(t)$ of running jobs by time $t$.
3. The joint distribtuion of $Y(t_1)$ and $Y(t_2)$, where $t_1 < t_2$.

# Solution of question 1 & 2

We classify the jobs into two types:

- **type 1**: completed by time $t$.
- **type 2**: not completed by time $t$.

If a job arrives at time $x$, then the sampling function is

- $p_1(x) = T(t-x)$
- $p_2(x) = 1 - T(t-x)$

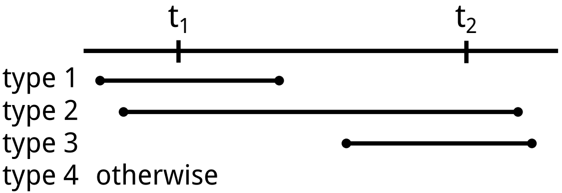
Thus, $X(t) = N_1(t)$ is Poisson distributed with parameter

$$\lambda \int_0^t T(t-x)\ dx$$

And $Y(t) = N_2(t)$ is Poisson distributed with parameter

$$\lambda \int_0^t (1 - T(t-x))\ dx = \lambda t - \lambda \int_0^t T(t-x)\ dx$$

# Solution of question 3




$$p_1(x) = T(t_2 - x) - T(t_1 - x) \quad \text{if } 0 < x < t_1$$

$$p_2(x) = 1 - T(t_2 - x) \quad \text{if } 0 < x < t_1$$

$$p_3(x) = 1 - T(t_2 - x) \quad \text{if } t_1 < x < t_2$$

# Solution of question 3 (cont.)

We know that $Y(t_1) = N_1(t_2) + N_2(t_2)$ and $Y(t_2) = N_2(t_2) + N_3(t_2)$, hence $Y(t_1)$ and $Y(t_2)$ are not independent. Therefore, we use the following method:

$$
\begin{aligned}
& P(Y(t_1) = m_1, Y(t_2) = m_2) \\
&= \sum_{n_2=0}^{\infty} P(N_1(t_2) = m_1 - n_2, N_2(t_2) = n_2, N_3(t_2) = m_2 - n_2) \\
&= \sum_{n_2=0}^{\infty} P(N_1(t_2) = m_1 - n_2) \cdot P(N_2(t_2) = n_2) \cdot P(N_3(t_2) = m_2 - n_2) \\
&= \cdots
\end{aligned}
$$

The $\infty$ in $\sum$ can be replaced by $\min(m_1, m_2)$.

# Example 2: Encounters on a highway

Cars enter a distance-$d$ highway in a Poisson rate $\lambda$.
The fixed speed of each car is i.i.d. with function $F_S(\cdot)$.
Suppose our car enters the highway and moves at a fixed speed $s$, what's the distribution of the number of encountering with other cars?

# Solution

Suppose we enter at time $t_1$ and leave at $t_2 = t_1 + d/s$.
Each car choose a fixed speed $S$ according to $F_S$, its travel time $T = d/S$.
The distribution function of $T$ is

$$F_T(t) = P(T \leq t) = P(S \geq \frac{d}{t}) = 1 - F_S(\frac{d}{t})$$

We classify the cars into three types:

- **type a** (overtaken by us): $0 < t < t_1$, $t + T > t_2$.
- **type b** (overtake us): $t_1 < t < t_2$, $t + T < t_2$.
- **type c**: otherwise

# Solution (cont.)

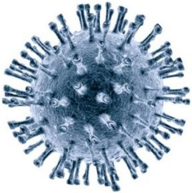
$$p_a(t) = P(T > t_2 - t) = 1 - F_T(t_2 - t) \quad \text{if } 0 < t < t_1$$
$$p_b(t) = P(T < t_2 - t) = F_T(t_2 - t) \quad \text{if } t_1 < t < t_2$$

Since the Poisson parameters can be summed up, $N_a(t_2) + N_b(t_2)$ is Poisson distributed with parameter

$$\lambda \int_0^{t_1} (1 - F_T(t_2 - t))\ dt + \lambda \int_{t_1}^{t_2} F_T(t_2 - t)\ dt$$

which is the distribution we want.

# Example 3: HIV infection



People are infected with an unknown Poisson rate $\lambda$.
The incubation time for each infected person has distribution function $F$.
At time $t$, we know the number $n_1$ of people who already have the AIDS symptoms (finished incubations).
We want to estimate the value of $\lambda$, and the number $n_2$ of the incubating people at time $t$.

# Solution

We classify the infected people into two types:

- **type 1**: have symptoms appear by time $t$.
- **type 2**: still in incubation by time $t$.

Then

$$E[N_1(t)] = \lambda \int_0^t F(t-x)\ dx = \lambda \int_0^t F(x)\ dx$$

$$E[N_2(t)] = \lambda \int_0^t (1 - F(t-x))\ dx = \lambda \int_0^t (1 - F(x))\ dx$$

$$n_1 \approx E[N_1(t)] \implies \hat{\lambda} = \frac{n_1}{\int_0^t F(x)\ dx}$$

$$n_2 \approx \hat{\lambda} \int_0^t (1 - F(x))\ dx = \frac{n_1 \int_0^t (1 - F(x))\ dx}{\int_0^t F(x)\ dx}$$

# Example 4: Hidden bugs

Suppose the errors come as a Poisson process.
Each error belongs to one of the $m$ bugs in the program.
For all $i = 1, \ldots, m$, $\mathbb{N}_i$ denotes the Poisson process of errors caused by bug $i$, which has an unknown rate $\lambda_i$.
At time $t$, we know the value of $M_j(t)$, which is the number of bugs causing exactly $j$ errors.
And a bug is still *hidden* if it hasn't cause any error.
We want to know the expected error rate of hidden bugs, which is the expectation of the summation of all hidden bugs' $\lambda_i$.

# Solution

Let

$$H_i(t) = \begin{cases} 1 & \text{if the } i\text{th bug is still hidden by time } t \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) = \sum_{i=1}^{m} H_i(t) \cdot \lambda_i$$

Then

$$P(H_i(t) = 1) = P(N_i(t) = 0) = \frac{(\lambda_i t)^0 \cdot e^{-\lambda_i t}}{0!} = e^{-\lambda_i t}$$

$$E[\Lambda(t)] = \sum_{i=1}^{m} \lambda_i \cdot E[H_i(t)] = \sum_{i=1}^{m} \lambda_i \cdot P(H_i(t) = 1) = \sum_{i=1}^{m} \lambda_i \cdot e^{-\lambda_i t}$$

$$= \frac{1}{t} \sum_{i=1}^{m} \lambda_i \cdot t \cdot e^{-\lambda_i t} = \frac{1}{t} \sum_{i=1}^{m} P(N_i(t) = 1) = \frac{1}{t} E[M_1(t)]$$

# Non-homogeneous Poisson process

**Definition**

$\mathbb{N}$ is a non-homogeneous Poisson process with intensity function $\lambda(\cdot)$ if $\mathbb{N}$ is a counting process such that the following four conditions hold:

1. $N(0) = 0$
2. $\mathbb{N}$ satisfies independent increments.
3. $P(N(t+h) - N(t) \geq 2) = o(h)$
4. $P(N(t+h) - N(t) = 1) = \lambda(t) \cdot h + o(h)$

- It's a Poisson process without stationary increments.
- If $\lambda(t) = \lambda$, it becomes the homogeneous (normal) Poisson process.

# Proposition 1

**Part 1**:
For a Poisson process $\mathbb{N}$ with rate $\lambda$, suppose that each arrived event has sampling function $p_i(t)$, then the counting process $\mathbb{N}_i$ describing the arrival of type-$i$ events is a non-homogeneous Poisson process with intensity function

$$\lambda_i(t) = \lambda \cdot p_i(t)$$

**Part 2**:
All non-homogeneous Poisson process with bounded $\lambda(\cdot)$ can be obtained in the above way.
**Proof of part 2**: Since there's a $\lambda$ that $\lambda(t) \leq \lambda$ for all $t$, we just let $p_i(t) = \lambda(t)/\lambda$ for a Poisson process with rate $\lambda$.

## Proof of part 1

**Condition 1**: trivial.
**Condition 3**:

$$P(N_i(t+h) - N_i(t) \geq 2) \leq P(N(t+h) - N(t) \geq 2) = o(h)$$

**Condition 4**:

$$\begin{aligned}
&P(N_i(t+h) - N_i(t) = 1) \\
=& P(N_i(t+h) - N_i(t) = 1 | N(t+h) - N(t) = 1) \\
&\quad \times P(N(t+h) - N(t) = 1) \\
&\quad + P(N_i(t+h) - N_i(t) = 1 | N(t+h) - N(t) \geq 2) \\
&\quad \times P(N(t+h) - N(t) \geq 2) \\
=& (p_i(t) + o(h)) \cdot (\lambda h + o(h)) + o(h) \\
=& p_i(t) \cdot \lambda h + o(h)
\end{aligned}$$

# Proof of part 1 (cont.)

The yellow part holds because

$$\lim_{h \to 0} P(N_i(t+h) - N_i(t) = 1 | N(t+h) - N(t) = 1) = p_i(t)$$
$$\Longrightarrow P(N_i(t+h) - N_i(t) = 1 | N(t+h) - N(t) = 1) = p_i(t) + o(h)$$

**Condition 2**:
Consider non-overlapped intervals $[s, s+t]$ and $[u, u+v]$, the distribution in each interval is decided by Poisson parameter $\lambda \int_s^{s+t} p_i(x)\, dx$ and $\lambda \int_u^{u+v} p_i(x)\, dx$, which doesn't influence each other.

# Distribution of non-homogeneous Poisson process

According to part 2 of proposition 1, for a non-homogeneous Poisson process with intensity function $\lambda(\cdot)$, we can observe that its distribution in $[s, s+t]$ is Poisson distributed with parameter

$$\lambda \int_{s}^{s+t} p_i(x)\ dx = \int_{s}^{s+t} \lambda(x)\ dx$$

# Example 1: Poisson tea shop

Suppose that customers come to tea shop as a non-homogeneous Poisson process with $\lambda(t)$ as follow:

$$\lambda(t) = \begin{cases} 0 & 0 \leq t \leq 8 \\ 5 + 5 \cdot (t - 8) & 8 < t \leq 11 \\ 20 & 11 < t < 13 \\ 20 - 2 \cdot (t - 13) & 13 < t \leq 17 \\ 0 & 17 < t < 24 \end{cases}$$

We want to know the probability that no one comes in $[8.5, 9.5]$.

# Solution

The distribution in $[8.5, 9.5]$ is a Poisson with parameter

$$\int_{8.5}^{9.5} (5 + 5 \cdot (t - 8))\ dt = \int_{0.5}^{1.5} (5 + 5t)\ dt = 10$$

Hence the probability is $e^{-10}$.

# Example 2: Infinite server queue again

Suppose that jobs arrive at a Poisson rate $\lambda$, and we have infinite number of servers. The running time of each job is independent and distributed with function $T(\cdot)$.
We want to prove that the process of jobs departing (finishing) is a non-homogeneous Poisson process.

# Proof

**Condition 1**: trivial.

**Condition 2**:
Consider two non-overlapping intervals $[s, s+t]$ and $[u, u+v]$, where $u > s$. Let the jobs finishing in $[s, s+t]$ be type 1, and those finishing in $[u, u+v]$ be type 2.
According to corollary of property 2, we know that the distribution of type 1 and type 2 jobs are independent in $[0, u+v]$, which means that the number of jobs finishing in $[s, s+t]$ and $[u, u+v]$ are independent.