

# Lecture notes of Stochastic Process

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# Thank list

LeoSW, windker

# Stochastic Process

## Definition

A Stochastic process is a set of random variables  $\{X(t) | t \in T\}$  where  $T$  is a index ( time) set.

State Space: possible value of  $X(t)$  for each  $t$ , which is defined as subset of  $R$ .

# Markov Chain

## Definition

A Stochastic Process  $\mathbb{X}$  with state space  $S$  is a Markov Chain if

$\exists 0 \leq p_{ij} \leq 1 \quad \forall i, j \in S$  such that

$$(a) \quad \sum_{j \in S} p_{ij} = 1 \quad \forall i \in S$$

$$(b) \quad P(X(t+1) = j | X(0) = i_0, X(1) = i_1, \dots, X(t) = i) = p_{ij} \\ \forall t, i_0, i_1, \dots, i_{t-1}$$

$\mathbb{P}$  denotes the matrix form of  $p_{ij}$  with sum of any row is 1.

Lemma:  $P(X(n) = j | X(0) = i) = \mathbb{P}^n[i, j]$

# Proof of lemma

We know statement is true for  $(m + n) = 0$ . For  $(m + n) > 0$ :

$$\begin{aligned} & P(X(m + n) = j | X(0) = i) \\ &= \sum_{k \in S} P(X(m + n) = j \text{ and } X(m) = k | X(0) = i) \\ &= \sum_{k \in S} P(X(m + n) = j | X(m) = k \text{ and } X(0) = i) \cdot \\ & \quad P(X(m) = k | X(0) = i) \\ &= \sum_{k \in S} P(X(m + n) = j | X(m) = k) \cdot P(X(m) = k | X(0) = i) \\ &= \sum_{k \in S} P^n[k, j] \cdot P^m[i, k] \\ &= \sum_{k \in S} P^m[i, k] \cdot P^n[k, j] \\ &= \mathbb{P}^n[i, j] \end{aligned}$$

# Proof of lemma(cont)

- $\stackrel{\text{orange}}{=}$  : conditional on  $X(m)$
- $\stackrel{\text{blue}}{=}$  : definition of conditional probability
- $\stackrel{\text{green}}{=}$  : (see next page)
- $\stackrel{\text{red}}{=}$  : inductive hypothesis

## Proof of lemma(cont)

$$\begin{aligned} &P(X(m+n) = j | X(m) = k \text{ and } X(0) = i) \\ &= \sum_{r \in S} P(X(m+n) = j | \\ &\quad X(m+n-1) = r \text{ and } X(m) = k \text{ and } X(0) = i) \cdot \\ &\quad P(X(m+n-1) = r | X(m) = k \text{ and } X(0) = i) \\ &= \sum_{r \in S} P(X(m+n) = j | X(m+n-1) = r) \cdot \\ &\quad P(X(m+n-1) = r | X(m) = k) \\ &= P(X(m+n) = j | X(m) = k) \end{aligned}$$

$=$ : conditional on  $X(m+n-1)$

$=$ : first part by definition of Markov chain and second part by inductive hypothesis

# Absorbing State

Let  $\mathbb{A}$  be a set of accepting states. We would like to know the probability that  $\mathbb{X}$  has ever entered some state in  $\mathbb{A}$ . Technique: merge all state of  $\mathbb{A}$  into a new absorbing state  $a$ . Set matrix of  $\mathbb{X}$  by once enter  $a$ , then probability of  $a$  goes to  $a$  is 1.



# Recurrent & transient

## Definition

The *recurrent probability* of state  $i$  of Markov chain  $\mathbb{X}$  is

$$f_i = P(\text{there exists an index } t \geq 1 \text{ with } X(t) = i | X(0) = i)$$

- State  $i$  of  $\mathbb{X}$  is *recurrent* if  $f_i = 1$ .
- State  $i$  of  $\mathbb{X}$  is *transient* if  $f_i < 1$ .

## Recurrent & transient (cont.)

- If state  $i$  is recurrent, by the property of Markov chain, once it re-enter the state  $i$ , we can take it as starting from  $X(0)$  again. Hence we know that it will keep re-entering the state  $i$  again and again in the process.
- If state  $i$  is transient, in each period it start going from  $i$ , it may have probability  $1 - f_i$  that it won't come back anymore. Hence the probability that the process will be in state  $i$  for exactly  $n$  periods equals  $f_i^{n-1}(1 - f_i)$ ,  $n \geq 1$ , which is a geometric distribution.

## Recurrent & transient (cont.)

- From the preceding page, it follows that state  $i$  is recurrent if and only if, starting in state  $i$ , the expected number of steps that the process is in state  $i$  is infinite.
- We can also derive that, if the Markov chain has finite states, at least one state is recurrent.

# Expected number of visits

Let

$$I(n) = \begin{cases} 1 & \text{if } X(n) = i \\ 0 & \text{if } X(n) \neq i \end{cases}$$

we have  $\sum_{n=0}^{\infty} I(n)$  represents the number of steps that the process is in state  $i$ , and

$$\begin{aligned} E \left[ \sum_{n=0}^{\infty} I(n) | X(0) = i \right] &= \sum_{n=0}^{\infty} E[I(n) | X(0) = i] \\ &= \sum_{n=0}^{\infty} 1 \cdot P(X(n) = i | X(0) = i) \\ &= \sum_{n=0}^{\infty} P_{ii}^n \end{aligned}$$

We set  $T = \sum_{n=0}^{\infty} I(n)$

# Lemma 1

From the above statements, we prove the following

## Lemma

*State  $i$  is*

$$\textit{recurrent} \iff \sum_{n=0}^{\infty} P_{ii}^n = \infty,$$

$$\textit{transient} \iff \sum_{n=0}^{\infty} P_{ii}^n < \infty$$

# Proof of Lemma 1

( $\Rightarrow$ ):

( $\Leftarrow$ ):

Suppose state  $i$  is transient ( $f_i < 1$ ), consider  $P(T = k) = f_i^{k-1} \cdot (1 - f_i)$ . Since  $T$  is a geometric distribution, we have

$$\begin{aligned} E[T] &= \sum_{k=0}^{\infty} k \cdot f_i^{k-1} \cdot f_i \\ &= \frac{1}{1 - f_i} < \infty \end{aligned}$$

# Communicated states

## Definition

State  $i$  and  $j$  *communicate*, denoted  $i \leftrightarrow j$ , if there exist integers  $m \geq 0$  and  $n \geq 0$  such that

$$P_{ij}^m > 0 \text{ and } P_{ji}^n > 0$$

We say a Markov chain  $\mathbb{X}$  is irreducible if  $i \leftrightarrow j \quad \forall i, j \in S$

# Lemma 2

## Lemma

*If  $i \leftrightarrow j$ , then the following statements hold.*

- *State  $i$  is recurrent if and only if state  $j$  is recurrent.*
- *State  $i$  is transient if and only if state  $j$  is transient.*

*Corollary:  $\mathbb{X}$  is finite and irreducible  $\implies$  all states are recurrent.*

- *$\mathbb{X}$  is finite  $\implies \exists i \in S$  is recurrent (proof later)*
- *By Lemma 2, all states are recurrent*



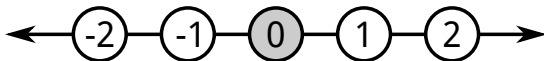
# Proof

Let  $m$  and  $n$  be nonnegative integers with  $P_{ij}^m, P_{ji}^n > 0$ . Suppose that state  $j$  is recurrent, i.e.,  $\sum_{t=0}^{\infty} P_{jj}^t = \infty$ . We have

$$\begin{aligned}\sum_{t=0}^{\infty} P_{ii}^t &\geq \sum_{t=0}^{\infty} P_{ii}^{m+t+n} \\ &\geq \sum_{t=0}^{\infty} P_{ij}^m \cdot P_{jj}^t \cdot P_{ji}^n \\ &= P_{ij}^m \cdot P_{ji}^n \cdot \sum_{t=0}^{\infty} P_{jj}^t = \infty\end{aligned}$$

Thus, state  $i$  is also recurrent.

# Infinite drunken man problem



Let the state space consist of all integers. Let  $X(0) = 0$  (i.e. at time 0 the drunken man is in state 0). The transition probabilities are such that

$$P_{i,(i+1)} = P_{i,(i-1)} = 0.5$$

holds for all states  $i$  of  $\mathbb{X}$ .

# Gambler's ruin

# Outline

- 1 Limiting probabilities
- 2 Stationary distribution
- 3 Long-run proportion
- 4 (Inverse of) Expected return time

# Limiting Probabilities

## Definition

Number  $\pi_j$  is the *limiting probability* of  $j$  if

$$\pi_j = \lim_{n \rightarrow \infty} P_{ij}^n$$

holds for all states  $i \in S$  ( $S \subseteq \mathbb{N}$  is the state space).

- $\pi_j$  is independent of  $i$ .

- $\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \pi \\ \pi \\ \vdots \end{pmatrix}$ , where  $\pi = (\pi_1, \pi_2, \dots)$

# Stationary Probability Distribution

## Definition

Non-negative row vector  $\pi = (\pi_1, \pi_2, \dots)$  is a *stationary probability distribution* of  $\mathbb{X}$  if  $\pi \times P = \pi$  holds and  $\sum_{i \in S} \pi_i = 1$

- $\pi$  is a normalized left eigenvector with eigenvalue  $= 1$ .
- If  $X(0)$  has distribution  $\pi$ , then  $X(t)$  has the same distribution  $\pi$  for all  $t \geq 1$ .  $\pi$  is also called as *steady-state distribution*.
- It doesn't mean that each  $X(t)$  become independent.  $\pi$  only means the distribution of  $X(t)$  when the previous random variable's value is unknown.

# Theorem 1

## Theorem

*Let  $\mathbb{X}$  be an irreducible, aperiodic, positive recurrent Markov chain, then*

- *The limiting probability  $\pi_j$  of each state  $j$  exists.*
  - *$\pi = (\pi_1, \pi_2, \dots)$  is the unique stationary probability distribution.*
- 
- The proof will be stated at page 38.

# Expected return time

## Definition

The *expected return time* of state  $i \in S$  is

$$\mu_i = \sum_{n \geq 1} n \cdot f_i^{(n)}$$

where

$$f_i^{(n)} = P(\min\{t : X(t) = i, t \geq 1\} = n | X(0) = i)$$

$$\blacksquare f_i = \sum_{n \geq 1} f_i^{(n)}$$



# Positive recurrent & null recurrent

## Definition

State  $i$  is *positive recurrent* if  $\mu_i < \infty$

## Definition

State  $i$  is *null recurrent* if  $\mu_i = \infty$

- Both are recurrent states, and are *class properties*, which means that if state  $i$  and  $j$  communicate, they will share this property.
- If  $\mathbb{X}$  is finite, then each recurrent state of  $\mathbb{X}$  is positive recurrent.  
Proof stated at page 63.

# Example of null recurrent

## Example

For a Markov chain with  $n$  states  $(1, \dots, n)$ , if

$$P(X(t+1) = i+1 | X(t) = i) = 1 - 1/n$$

and

$$P(X(t+1) = 1 | X(t) = i) = 1/n$$

According to geometric distribution (taking  $p = 1/n$ ), the expectation value of “steps taken for state 1 to come back” will be  $1/p = n$ , hence  $\lim_{n \rightarrow \infty} n = \infty$ .

# Period of a chain

## Definition

The *period* of state  $i$  is  $d$  if  $d$  is the largest integer such that

$$P_{ii}^n = 0$$

holds for all  $n$  which is not divisible by  $d$ .

## Definition

If each state of  $\mathbb{X}$  has period 1, then  $\mathbb{X}$  is called *aperiodic*.

- If  $P_{ii} > 0$  for all  $i \in S$ , then  $\mathbb{X}$  is aperiodic.
- Period can be seen as the gcd of all  $n$  that have  $P_{ii}^n > 0$ , note that  $P_{ii}^{\text{gcd}} > 0$  is not necessary.
- The period of drunken man problem is 2.

# Lemma 1

## Lemma

*If state  $j$  has period 1 and is positive recurrent, then*

$$\pi_{ij} \equiv \lim_{n \rightarrow \infty} P_{ij}^n$$

*exists and is positive for all states  $i \in S$ .*

- This can be proved by the Blackwell theorem in Renewal theory.
- It doesn't promise that  $\pi_{ij} = \pi_{i'j}$  for any  $i, i' \in S$ . But they will be the same if we add the irreducible property ( $i \leftrightarrow i'$ ).

# Property of lim

- The position of lim may not be switched arbitrarily in an equation.

## Example

$$1 = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{m}{m+n} \neq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{m}{m+n} = 0$$

- lim would not influence the inequality.

## Example

$$\text{If } f(n) \geq g(n), \text{ then } \lim_{n \rightarrow \infty} f(n) \geq \lim_{n \rightarrow \infty} g(n)$$

# Property of $\lim$ (cont.)

- $\lim$  is linear operator under finite number of functions.

## Example

For  $m < \infty$ ,

$$\sum_{i=1}^m \lim_{n \rightarrow \infty} f_i(n) = \lim_{n \rightarrow \infty} \sum_{i=1}^m f_i(n)$$

need an example of  $m = \infty$

# Inequality 1

## Inequality

$$\sum_{j \in S} \pi_{ij} \leq 1 \quad \forall i \in S$$

$$\begin{aligned}\lim_{m \rightarrow \infty} \sum_{j=1}^m \pi_{ij} &= \lim_{m \rightarrow \infty} \sum_{j=1}^m \lim_{n \rightarrow \infty} P_{ij}^n \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j=1}^m P_{ij}^n \\ &\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{j \in S} P_{ij}^n = 1\end{aligned}$$

- The last equation works since  $\sum_{j \in S} P_{ij}^n = 1$ .



# Inequality 2

## Inequality

*For state  $j \in S$ , we have*

$$\pi_{ij} \geq \sum_{k \in S} \pi_{ik} P_{kj}$$

# Proof

For  $m \geq 1$  and  $n \geq 1$ ,

$$P_{ij}^{n+1} = \sum_{k \in S} P_{ik}^n P_{kj} \geq \sum_{k=1}^m P_{ik}^n P_{kj}$$

then

$$\pi_{ij} = \lim_{n \rightarrow \infty} P_{ij}^{n+1} \geq \lim_{n \rightarrow \infty} \sum_{k=1}^m P_{ik}^n P_{kj} = \sum_{k=1}^m \lim_{n \rightarrow \infty} P_{ik}^n P_{kj} = \sum_{k=1}^m \pi_{ik} P_{kj}$$

hence, we know

$$\lim_{m \rightarrow \infty} \pi_{ij} = \pi_{ij} \geq \lim_{m \rightarrow \infty} \sum_{k=1}^m \pi_{ik} P_{kj} = \sum_{k \in S} \pi_{ik} P_{kj}$$

# Equality 1

## Equality

$$\pi_{ij} = \sum_{k \in S} \pi_{ik} P_{kj}$$

# Proof

Assume for contradiction  $\pi_{ij} > \sum_{k \in S} \pi_{ik} P_{kj}$ , then

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{j=1}^m &> \lim_{m \rightarrow \infty} \sum_{j=1}^m \lim_{p \rightarrow \infty} \sum_{k=1}^p \pi_{ik} P_{kj} \\ &= \lim_{m \rightarrow \infty} \lim_{p \rightarrow \infty} \sum_{j=1}^m \sum_{k=1}^p \pi_{ik} P_{kj} \\ &= \lim_{m \rightarrow \infty} \lim_{p \rightarrow \infty} \sum_{k=1}^p \pi_{ik} \sum_{j=1}^m P_{kj} \\ &= \lim_{p \rightarrow \infty} \sum_{k=1}^p \pi_{ik} \lim_{m \rightarrow \infty} \sum_{j=1}^m P_{kj} \\ &= \lim_{p \rightarrow \infty} \sum_{k=1}^p \pi_{ik} \cdot 1 = \lim_{p \rightarrow \infty} \sum_{k=1}^p \pi_{ik} \end{aligned}$$

## Proof (cont.)

- Since a value cannot be greater than itself, we got contradiction.
- In the 4th line, two  $\lim$  can be switched because the value can only get larger when applying  $\lim$  on it. **not sure**

# Proof of theorem 1

- **Step 0:** existence of limiting probability.
- **Step 1:** existence of stationary probability distribution.
- **Step 2:** uniqueness.

## 0. Existence of limiting probability

### Proof.

By lemma 1, we know that there exists a  $\pi_j$  for row  $i$ . Since the Markov chain is irreducible and all the states are positive recurrent, for any state  $i'$  other than  $i$ , we know that  $i'$  surely will visit  $i$  in finite steps. Therefore, the  $\pi_j$  value at row  $i'$  will equal to the  $\pi_j$  value at row  $i$ , which means that all the  $\pi_j$  for column  $j$  are the same, and is the limiting probability.  $\square$

still not clear enough

# 1. Existence of stationary probability distribution

We want to prove that

Target

*There's a vector  $s = (s_1, s_2, \dots)$  such that*

1  $\sum_{i \in S} s_i = 1$

2  $s \times P = s$



## Proof.

By lemma 1, we know that there exists a  $\pi = (\pi_1, \pi_2, \dots)$ .

And by equality 1, we know that

$$(\pi_1, \pi_2, \dots) \times P = (\pi_1, \pi_2, \dots)$$

Hence  $\pi$  can satisfy the 2nd part of our target.

Then, we take  $k = \sum_{i \in S} \pi_i$ . By inequality 1, we know that  $k < \infty$ , and can get

$$\left(\frac{\pi_1}{k}, \frac{\pi_2}{k}, \dots\right) \times P = \left(\frac{\pi_1}{k}, \frac{\pi_2}{k}, \dots\right)$$

where  $\sum_{i \in S} \frac{\pi_i}{k} = 1$  also satisfy the 1st part of our target.

Therefore, this vector can be  $s$ , which means that it exists. □

## 2. Uniqueness

### Target

*If  $s = (s_1, s_2, \dots)$  is a stationary distribution of  $\mathbb{X}$ , then  $s = \pi$ .*

- We'll prove this by inequality 3 & 4.

# Inequality 3

Inequality

$$s_j \geq \pi_j, \forall j \in S$$

## Proof.

Let the distribution of  $X(0)$  be  $s$ , by the property of stationary distribution, we have

$$\begin{aligned}s_j &= P(X(n) = j) = \sum_{i \in S} P(X(n) = j | X(0) = i) P(X(0) = i) \\&= \sum_{i \in S} P_{ij}^n \cdot s_i \\&\geq \sum_{i=1}^m P_{ij}^n \cdot s_i \\ \Rightarrow s_j &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} s_j \\&\geq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^m P_{ij}^n \cdot s_i = \lim_{m \rightarrow \infty} \sum_{i=1}^m \pi_j \cdot s_i = \pi_j\end{aligned}$$



# Inequality 4

## Inequality

$$s_j \leq \pi_j, \forall j \in S$$

## Proof.

Similar in the proof above,  $\forall m, n \geq 1$ , we have

$$\begin{aligned} s_j &= \sum_{i \in S} P_{ij}^n \cdot s_i \\ &\leq \sum_{i=1}^m P_{ij}^n \cdot s_i + \sum_{i=m+1}^{\infty} s_i \\ \Rightarrow s_j &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} s_j \\ &\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \left( \sum_{i=1}^m P_{ij}^n \cdot s_i + \sum_{i=m+1}^{\infty} s_i \right) \\ &= \pi_j \end{aligned}$$



# An example Markov chain

## Example

$$P = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix}, 0 < \alpha, \beta < 1$$

$$\pi = \left( \frac{\beta}{1 + \beta - \alpha}, \frac{1 - \alpha}{1 + \beta - \alpha} \right)$$

# Real world example: Hardy-Weinberg Law

## Example

There're two kinds of allele:

- dominant: **A**
- recessive: **a**

And three kinds of senotype with population proportion as follow:

- AA:  $p$
- aa:  $q$
- Aa:  $r = 1 - (p + q)$



## Example (cont.)

$$P = \begin{array}{cc} & \begin{array}{ccc} AA & aa & Aa \end{array} \\ \begin{array}{c} AA \\ aa \\ Aa \end{array} & \left( \begin{array}{ccc} p + \frac{r}{2} & 0 & q + \frac{r}{2} \\ 0 & q + \frac{r}{2} & p + \frac{r}{2} \\ \frac{p}{2} + \frac{r}{4} & \frac{p}{2} + \frac{r}{4} & \frac{p+q+r}{2} \end{array} \right) \end{array}$$

we get  $\pi = (p, q, r)$  when

- $p = \left(p + \frac{r}{2}\right)^2$
- $q = \left(q + \frac{r}{2}\right)^2$
- $r = 2 \left(p + \frac{r}{2}\right) \left(q + \frac{r}{2}\right)$

# Long-run proportion

## Definition

We say that  $r_j$  is the *long-run proportion* of state  $j \in S$  if

$$r_j = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq t \leq n} P_{ij}^t$$

holds for each state  $i \in S$ .

- It represents the average appearance times of state  $j$  in the whole process.
- We will show that (in theorem 3) if  $\mathbb{X}$  is irreducible, then the long-run proportion of all states exist.

# Theorem 2

## Theorem (type 1)

*If  $r_j$  exists for each  $j \in S$  and  $\sum_{j \in S} r_j > 0$ , then  $r = (r_1, r_2, \dots)$  is the unique stationary distribution of  $\mathbb{X}$ .*

or

## Theorem (type 2)

*If  $r_j$  exists for each  $j \in S$  and **a stationary distribution exists**, then  $r = (r_1, r_2, \dots)$  is the unique stationary distribution of  $\mathbb{X}$ .*

# Proof

## Existence of stationary distribution in type 1:

Let

$$R = \begin{pmatrix} r \\ r \\ \vdots \end{pmatrix} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq t \leq n} P^t$$

then

$$\begin{aligned} R \times P &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq t \leq n} P^{t+1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq t \leq n} P^t + \lim_{n \rightarrow \infty} \frac{1}{n} (P^{n+1} - P) \\ &= R \end{aligned}$$

As stated later,  $\sum_{j \in S} r_j \leq 1$ , hence by normalizing  $r$ , we prove that stationary distribution exist.

- $(\lim f(n)) \cdot g(n) = \lim f(n) \cdot g(n)?$
- can replace the proof on page 40?

## Proof (cont.)

### Uniqueness:

Let  $\pi$  be an arbitrary stationary distribution, then

$$\begin{aligned} r &= \pi \times R \\ &= \pi \times \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq t \leq n} P^t \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq t \leq n} \pi \times P^t \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{1 \leq t \leq n} \pi \\ &= \pi \end{aligned}$$

can replace the proof for page 42?

## Proof (cont.)

**Prove that**  $\sum_{j \in S} r_j \leq 1$ :

$$\begin{aligned}\sum_{j \in S} r_j &= \lim_{m \rightarrow \infty} \sum_{j=1}^m \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n P_{ij}^t \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \sum_{j=1}^m P_{ij}^t \\ &\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \sum_{j \in S} P_{ij}^t \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n 1 = 1\end{aligned}$$

## Example 1

On a highway, if we know the probability that

- A truck is followed by a truck:  $1/4$
- A truck is followed by a car:  $3/4$
- A car is followed by a truck:  $1/5$
- A car is followed by a car:  $4/5$

We can construct a matrix

$$\begin{array}{cc} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \left( \begin{array}{cc} 1/4 & 3/4 \\ 1/5 & 4/5 \end{array} \right) \end{array}$$

and get the portion of trucks and cars on the whole highway as the eigenvector  $(4/19, 15/19)$  (we will know that long-run proportion exists by Theorem 3).

## Example 2

For a system which has several good and bad states, we have a matrix  $P$ :

$$\begin{matrix} & g_1 & g_2 & \cdots & b_1 & b_2 & \cdots \\ \begin{matrix} g_1 \\ g_2 \\ \vdots \\ b_1 \\ b_2 \\ \vdots \end{matrix} & \left( \begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \right) \end{matrix}$$



## Example 2 (cont.)

**Q1:** Breakdown rate (breakdown times / total time)

The long-run frequency of going to a bad state from a good state is

$$\sum_{i \in g} \sum_{j \in b} r_i P_{ij}$$

## Example 2 (cont.)

**Q2:** The expected time  $\mu_G$  (resp.  $\mu_B$ ) of staying in good (resp. bad) states once we reach a good (resp. bad) state?

**Ans:**

For each  $t = 1, 2, \dots$ , let  $G_t$  (resp.  $B_t$ ) be the length of the  $t$ -th good (resp. bad) phase of consecutive good (resp. bad) states. By the strong law of large numbers,

$$P\left(\lim_{t \rightarrow \infty} \frac{G_1 + B_1 + G_2 + B_2 + \dots + G_t + B_t}{t} = \mu_G + \mu_B\right) = 1$$

Since the reciprocal of above is the breakdown rate, we get equation (1):

$$P\left(\sum_{i \in G} \sum_{j \in B} \pi_i P_{ij} = \frac{1}{\mu_G + \mu_B}\right) = 1$$

## Example 2 (cont.)

Also, with probability 1, we get equation (2):

$$P \left( \sum_{i \in G} r_i = \lim_{t \rightarrow \infty} \frac{G_1 + G_2 + \cdots + G_t}{G_1 + B_1 + \cdots + G_t + B_t} = \frac{\mu_G}{\mu_G + \mu_B} \right) = 1$$

Then, by (2)/(1), we get that

$$P \left( \mu_G = \frac{\sum_{i \in G} r_i}{\sum_{i \in G} \sum_{j \in B} r_i P_{ij}} \right) = 1$$

■  $\lim \frac{f(n)}{g(n)} = \frac{\lim f(n)}{\lim g(n)}?$

# Theorem 3

## Theorem

*If  $\mathbb{X}$  is irreducible, then the long-run proportion  $r_i$  exists with probability 1, moreover,*

- 1** *If state  $i$  is positive recurrent (i.e.  $0 < \mu_i < \infty$ ), then  $P(r_i = \frac{1}{\mu_i}) = 1$ .*
- 2** *If state  $i$  is null recurrent (i.e.  $\mu_i = \infty$ ) or transient, then  $P(r_i = 0) = 1$ .*

*where  $\mu_i$  is the expected return time of state  $i$*

## Part 1:

Suppose  $X(0) = i$ ,  $T_k$  is the number of steps required for the  $k$ -th  $i$  goes to  $(k+1)$ -st  $i$ , then by the strong law of large number,

$$P\left(\lim_{k \rightarrow \infty} \frac{T_1 + T_2 + \cdots + T_k}{k} = \mu_i\right) = 1$$
$$\Rightarrow P\left(r_i = \lim_{k \rightarrow \infty} \frac{k}{T_1 + T_2 + \cdots + T_k} = \frac{1}{\mu_i}\right) = 1$$

■  $\lim(A/B) = \frac{1}{\lim(B/A)}$ ?

# Proof (cont.)

## Part 2:

- 1 If  $i$  is transient,  $i$  will only appear finite times in the long-run, hence

$$r_i = \frac{\text{finite}}{\infty} = 0$$

- 2 If  $i$  is null recurrent,  $\mu_i$  is  $\infty$ , then

$$P\left(\lim_{k \rightarrow \infty} \frac{T_1 + T_2 + \cdots + T_k}{k} = \infty\right) = 1$$

$$P\left(r_i = \lim_{k \rightarrow \infty} \frac{k}{T_1 + T_2 + \cdots + T_k} = 0\right) = 1$$

(The first equation is not promised by the strong law of large number. But if it's not  $\infty$ , we can say that  $\mu_i$  is not  $\infty$ , which is a contradiction.)

# Example 1

## Example (type 1)

If  $\mathbb{X}$  is **irreducible** and finite, then  $\mathbb{X}$  has no null recurrent states.

## Example (type 2)

If  $\mathbb{X}$  is finite, then  $\mathbb{X}$  has no null recurrent states.

- Finite irreducible imply positive recurrent.

## ■ Type 1:

If there's a state which is null recurrent, by irreducible, all the states will be null recurrent. Then, all states have  $P(r_i = 0) = 1$ . By changing the proof in page 54 into finite states version, we know that  $\sum r_i = 1$ . So it's impossible for finite  $r_i$ , which are all close to 0, to sum up to 1.

## ■ Type 2:

If it's not irreducible, the finite set of communicated null recurrent states still form an irreducible and finite Markov chain, which can fit the requirement of type 1.



## Example 2

### Example

In the drunken man problem with infinite states, no state will be positive recurrent.

- Infinite drunken man imply no positive recurrent. Note that it doesn't mean all infinite irreducible Markov chain has no positive recurrent state.

# Proof

If all the states are positive recurrent, then by theorem 3, we know that all the  $r_i > 0$  and is a finite value. Since each state of drunken man problem has the same structure, all the  $r_i$  has same value. We then set  $r = \epsilon \cdot r_i$  ( $0 < \epsilon < 1$ ) such that  $r_i > r > 0, \forall i$ . And get

$$\sum_{i \in S} r_i > \sum_{i \in S} r = \infty > 1$$

which is contradiction to page 54.

## Example 3: Poisson Hotel

### Example

There's a hotel, with  $N$  representing the number of newly occupied rooms each day ( $N$  is a poisson distribution with parameter  $\lambda$ ). And the number of consecutive check-in days of each room is a geometric distribution with probability  $p$  ( $p$  is the probability of check-out).  $X(t)$  is the number of occupied rooms in day  $t$ .

## Q1: $P_{ij} = ?$

We set  $R_i$  as a binomial distribution with parameter  $(i, 1 - p)$ , which represents the number of rooms which will remain occupied in the next day, then

$$\begin{aligned} P_{ij} &= P(R_i + N = j) \\ &= \sum_{k \geq 0} P(R_i + N = j | R_i = k) P(R_i = k) \\ &= \sum_{k \geq 0} P(N = j - k) P(R_i = k) \\ &= \sum_{0 \leq k \leq \min(i, j)} \frac{e^{-\lambda} \cdot \lambda^{j-k}}{(j-k)!} \binom{i}{k} (1-p)^k p^{1-k} \end{aligned}$$

## Q2: $r_i = ?$

We guess (by a dream?) there's a stationary distribution which is a poisson distribution with parameter  $\lambda_0$ . Setting  $X(0)$  with this distribution. And let  $R$  as the number of rooms in  $X(0)$  which remain check-in in the next day ( $R$  is a poisson distribution with parameter  $\lambda_0(1 - p)$ ).  $X(1)$  will have distribution  $R + N$ , which is a poisson distribution with parameter  $\lambda_0(1 - p) + \lambda$ . Then since  $X(0)$  is a stationary distribution, it will have the same distribution with  $X(1)$ , which means that  $\lambda_0 = \lambda_0(1 - p) + \lambda$ , and we get  $\lambda_0 = \lambda/p$ . After getting  $r_i$ , we get that with probability 1,

$$\mu_i = \frac{1}{P(X(0) = i)} = \frac{i!}{e^{-\lambda/p} \cdot (\lambda/p)^i}$$

not clear enough

# Corollary of theorem 2 & 3

## Corollary

*If  $\mathbb{X}$  is irreducible, then*

*$\mathbb{X}$  is positive recurrent  $\iff \mathbb{X}$  admits a stationary distribution.*

# Moving to transient states

For transient states  $i$  and  $j$ , we define the following:

- 1 Expected steps in a transient state:

## Definition

$E$  is a matrix where  $E_{ij}$  is the expected number of steps  $t$  with  $X(t) = j$  when  $X(0) = i$ .

- 2 Probability of reaching a transient state:

## Definition

$F$  is a matrix where

$$F_{ij} = P(X(t) = j \text{ for some } t \geq 1 | X(0) = i)$$

# Computing $E$ & $F$

## Theorem

*For a Markov chain  $\mathbb{X}$  consisting finite transient states,*

$$E = (I - T)^{-1}$$

*where  $I$  is an identity matrix,  $T$  is the induced matrix of  $P$  by all the transient states in  $P$ . Moreover,*

$$F_{ij} = \frac{E_{ij} - \delta_{ij}}{E_{jj}}, \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



Conditioned on  $X(1)$ , we have

$$E_{ij} = \underbrace{\delta_{ij}}_{\text{step}=0} + \underbrace{\sum_k P_{ik} \cdot E_{kj}}_{\text{step} \geq 1} = \delta_{ij} + \sum_k T_{ik} \cdot E_{kj}$$

The 2nd equation works since the process will not go back to transient state once it enter a recurrent state. Then, we have

$$\begin{aligned} I \times E &= E = I + T \times E \\ \implies (I - T) \times E &= I \\ \implies E &= (I - T)^{-1} \end{aligned}$$

## Proof (cont.)

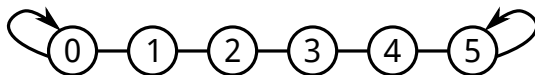
Conditioned on whether or not  $X(t) = j$  holds for some  $t \geq 1$ , we have

$$E_{ij} = \underbrace{\delta_{ij}}_{\text{step}=0} + \underbrace{F_{ij} \cdot E_{jj}}_{\text{steps} \geq \text{the first } j}$$

therefore,

$$F_{ij} = \frac{E_{ij} - \delta_{ij}}{E_{jj}}$$

## Example: Gambler's ruin



$$T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 \end{pmatrix} \end{matrix} \quad E = \begin{pmatrix} 1.6 & 1.2 & 0.8 & 0.4 \\ 1.2 & 2.4 & 1.6 & 0.8 \\ 0.8 & 1.6 & 2.4 & 1.2 \\ 0.4 & 0.8 & 1.2 & 1.6 \end{pmatrix}$$

$$F = \begin{pmatrix} 0.375 & 0.5 & 1/3 & 0.25 \\ 0.75 & 1.75/3 & 1.9/3 & 0.5 \\ 0.5 & 1.9/3 & 1.75/3 & 0.75 \\ 0.25 & 1/3 & 0.5 & 0.375 \end{pmatrix}$$

# Branching process

In the beginning, there're  $X(0)$  life forms, each life form has probability  $p_i$  of becoming  $i$  life forms in the next step.

- state 0 is recurrent (absorbing).
- if  $p_0 > 0$ , all other states  $(1, 2, \dots)$  are transient since
$$P(X(t+1) = 0 | X(t) = i) = p_0^i > 0$$

We'll show that

$$E[X(n)] = \mu^n \cdot X(0)$$

where

$$\mu = \sum_{j \geq 1} j \cdot p_j = E[Z_k]$$

and  $Z_k$  is the number of offspring of the  $k$ -th life form, all  $Z_k$  are i.i.d.

$$\begin{aligned} E[X(n)] &= E[E[X(n)|X(n-1)]] \\ &= E \left[ E \left[ \sum_{k=1}^{X(n-1)} Z_k | X(n-1) \right] \right] \\ &= E[X(n-1) \cdot \mu] \\ &= \mu \cdot E[X(n-1)] \\ &= \mu^n \cdot X(0) \end{aligned}$$

# Probability of extinction

## Definition

$e_i$  is the probability of extinction when  $X(0) = i$ .

**Case 1:**  $\mu < 1$

$$\begin{aligned} 1 - e_i &= \lim_{n \rightarrow \infty} P(X(n) \geq 1 | X(0) = i) \\ &= \lim_{n \rightarrow \infty} \sum_{j \geq 1} P(X(n) = j | X(0) = i) \\ &\leq \lim_{n \rightarrow \infty} \sum_{j \geq 1} j \cdot P(X(n) = j | X(0) = i) \\ &= \lim_{n \rightarrow \infty} E[X(n) | X(0) = i] \\ &= \lim_{n \rightarrow \infty} \mu^n \cdot i = 0 \end{aligned}$$

# Probability of extinction (cont.)

**Case 2:**  $\mu \geq 1$

$$e_2 = e_1^2, \quad e_3 = e_2 \cdot e_1, \quad \dots$$

$$\begin{aligned} e_1 &= P(\text{extinct} | X(0) = 1) \\ &= \sum_{j \geq 0} P(\text{extinct} | X(1) = j) \cdot P_{1j} \\ &= \sum_{j \geq 0} e_j \cdot p_j \\ &= \sum_{j \geq 0} e_1^j \cdot p_j \end{aligned}$$

We then solve the above equation to get  $e_1$ .

# Example

$$\begin{aligned}p_0 &= p_1 = 0.25, \quad p_2 = 0.5 \\ \implies \mu &= 1 \cdot 0.25 + 2 \cdot 0.5 > 1 \\ \implies e_1 &= e_1^0 \cdot 0.25 + e_1^1 \cdot 0.25 + e_1^2 \cdot 0.5 \\ \implies e_1 &= \{1/2, 1\}\end{aligned}$$

Since  $\mu > 1$ , we know  $\lim_{n \rightarrow \infty} E[X(n)] = \infty$ .

But if  $e_1 = 1$ , we have  $\lim_{n \rightarrow \infty} P(X(n) = 0) = 1$ , which would not make  $\lim_{n \rightarrow \infty} E[X(n)] = \infty$ , hence  $e_1 \neq 1$ .



# Reversed Markov chain

## Definition

Let  $\mathbb{X}$  (resp.  $\mathbb{Y}$ ) be a Markov chain with matrix  $P$  (resp.  $Q$ ). We say that  $\mathbb{Y}$  is the *reversed chain* of  $\mathbb{X}$  if there exists a stationary distribution  $\pi$  of  $\mathbb{X}$  such that

$$\pi_i \cdot Q_{ij} = \pi_j \cdot P_{ji}$$

holds for all states  $i, j \in S$ .

# Observation 1

## Observation

*The reversed sequence  $\mathbb{Y}$  of  $\mathbb{X}$  is a Markov chain.*

$$\begin{aligned}
 & P(Y(n) = i_0 | Y(n-1) = i_1, Y(n-2) = i_2, \dots, Y(n-k) = i_k) \\
 &= P(X(n) = i_0 | X(n+1) = i_1, X(n+2) = i_2, \dots, X(n+k) = i_k) \\
 &= \frac{P(X(n) = i_0, X(n+1) = i_1, \dots, X(n+k) = i_k)}{P(X(n+1) = i_1, \dots, X(n+k) = i_k)} \\
 &= \frac{P(X(n) = i_0) \cdot P(X(n+1) = i_1 | X(n) = i_0) \cdot P_{i_1 i_2} \cdots P_{i_{k-1} i_k}}{P(X(n+1) = i_1) \cdot P_{i_1 i_2} \cdots P_{i_{k-1} i_k}} \\
 &= \frac{P(X(n) = i_0, X(n+1) = i_1)}{P(X(n+1) = i_1)} \\
 &= P(X(n) = i_0 | X(n+1) = i_1) \\
 &= P(Y(n) = i_0 | Y(n-1) = i_1)
 \end{aligned}$$

## Observation 2

### Observation

*If  $\mathbb{Y}$  is the reversed sequence of Markov chain  $\mathbb{X}$  and  $\pi$  is a stationary distribution of  $\mathbb{X}$ , then*

$$\pi_i \cdot Q_{ij} = \pi_j \cdot P_{ji}$$

*holds for all  $i, j \in S$ , where  $Q$  is the transition matrix of  $\mathbb{Y}$ .*

Let  $\mathbb{X}$  and  $\mathbb{Y}$  have distribution  $\pi$

$$\begin{aligned}\pi_i \cdot Q_{ij} &= P(Y(n-1) = i) \cdot P(Y(n) = j | Y(n-1) = i) \\ &= P(Y(n-1) = i, Y(n) = j) \\ &= P(Y(n-1) = i | Y(n) = j) \cdot P(Y(n) = j) \\ &= P(X(n+1) = i | X(n) = j) \cdot P(X(n) = j) = \pi_j \cdot P_{ji}\end{aligned}$$

# Observation 3

## Observation

*Let  $P$  (resp.  $Q$ ) be the transition matrix of  $\mathbb{X}$  (resp.  $\mathbb{Y}$ ), if vector  $\pi$  satisfy the following*

- $\sum_{i \in S} \pi_i = 1$
- $\pi_i \geq 0 \quad \forall i \in S$
- $\pi_i \cdot Q_{ij} = \pi_j \cdot P_{ji} \quad \forall i, j \in S$

*then  $\mathbb{Y}$  is the reversed sequence of  $\mathbb{X}$ .*

- The long-run proportion of  $i \rightarrow j$  in the sequence of  $\mathbb{Y}$  is equal to the long-run proportion of  $j \rightarrow i$  in the sequence of  $\mathbb{X}$ .
- Reversed Markov chain is the reversed sequence.

From the third property, we have

$$\sum_{j \in S} \pi_i \cdot Q_{ij} = \pi_i = \sum_{j \in S} \pi_j \cdot P_{ji} \quad \forall i \in S$$

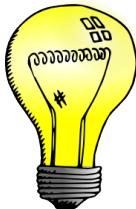
From the 2nd equation, we know that  $\pi \times P = \pi$ , hence  $\pi$  is a stationary distribution of  $\mathbb{X}$ .

Then by observation 2, we know that for any  $\pi$ , there's a reversed sequence  $\mathbb{Y}'$ , whose transition matrix  $Q'$  satisfy

$$\pi_i \cdot Q'_{ij} = \pi_j \cdot P_{ji} \quad \forall i, j \in S$$

hence  $\mathbb{Y} = \mathbb{Y}'$ , which is a reversed sequence of  $\mathbb{X}$ .

## Example: Bulb's life



There's a room which need to be lighted by one bulb, when the bulb in use fails, it will be replaced by a new one on next day.

- $X(n) = i$  if the bulb in use on day  $n$  is in its  $i$ th day of use.
- $L$  is a random variable representing the lifetime of a bulb.

We want to know the stationary probability  $\pi_i$  of state  $i$ .



## Example: Bulb's life (cont.)

$\mathbb{X}$  is a irreducible, positive recurrent, aperiodic Markov chain which has the sequence like this:

$$1, 2, 3, 1, 2, 3, 4, 5, 1, 1, 2, 1, 2, 3, 4, \dots$$

We know that

$$P_{i1} = P(\text{bulb, on its } i\text{th day of use, fails}) = \frac{P(L = i)}{P(L \geq i)} = 1 - P_{i(i+1)}$$

And the expected return time of state 1 is  $E[L]$ , which means that the long-run proportion of state 1 is  $1/E[L]$  by page 60.

## Example: Bulb's life (cont.)

Take  $\mathbb{Y}$  (with matrix  $Q$ ) as the reversed chain of  $\mathbb{X}$ , we know that for all  $i \in S$ ,

- $Q_{(i+1)i} = 1$
- $Q_{1i} = P(L = i)$
- $\pi_1 \cdot Q_{1i} = \pi_i \cdot P_{i1}$

Hence,

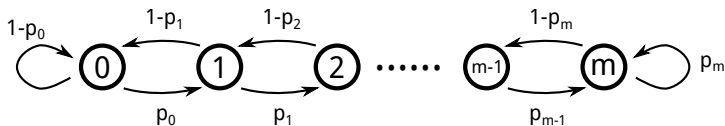
$$\pi_i = \frac{\pi_1 \cdot Q_{1i}}{P_{i1}} = \frac{P(L = i) \cdot P(L \geq i)}{E[L] \cdot P(L = i)} = \frac{P(L \geq i)}{E[L]}$$

# Time-reversible

## Definition

$\mathbb{X}$  is *time-reversible* if  $\mathbb{X}$  is the reversed chain of  $\mathbb{X}$ .

## Example: Reversed drunken man



- $0 < p_0 \leq 1$
- $0 \leq p_m < 1$
- $0 < p_i < 1 \quad \forall i = 1, \dots, m-1$

The long-run proportion of transition  $i \rightarrow i+1$  and  $i+1 \rightarrow i$  are the same, since one must go back to  $i$  from  $i+1$  in order to go to  $i+1$  from  $i$ . Hence the drunken man problem is time-reversible.

## Example: Reversed drunken man (cont.)

$$\pi_0 \cdot p_0 = \pi_1 \cdot (1 - p_1)$$

$$\pi_1 \cdot p_1 = \pi_2 \cdot (1 - p_2)$$

$$\vdots$$

$$\pi_{m-1} \cdot p_{m-1} = \pi_m \cdot (1 - p_m)$$

Thus,

$$\pi_1 = \pi_0 \cdot p_0 / (1 - p_1)$$

$$\pi_2 = \pi_1 \cdot p_1 / (1 - p_2)$$

$$\vdots$$

$$\pi_m = \pi_{m-1} \cdot p_{m-1} / (1 - p_m)$$

## Example: Reversed drunken man (cont.)

$$\pi_i = \frac{\prod_{j=0}^{i-1} p_j}{\underbrace{\prod_{j=1}^i (1 - p_j)}_{q_i}} \cdot \pi_0 \quad \forall i = 1, \dots, m$$

$$\implies \pi_0 + \sum_{i=1}^m \pi_i = 1 = \pi_0 + \sum_{i=1}^m q_i \cdot \pi_0$$

$$\implies \pi_0 = \frac{1}{1 + \sum_{i=1}^m q_i}$$

$$\implies \pi_k = \frac{q_k}{1 + \sum_{i=1}^m q_i} \quad \forall k = 0, 1, \dots, m$$

## Example: Two bukkits of balls

There're two bukkits contain total  $m$  balls.

In each step, we randomly choose one ball and put it in another bukket.

Let  $X(n)$  represent the number of balls in the first bukket, it's the Markov chain of previous example with

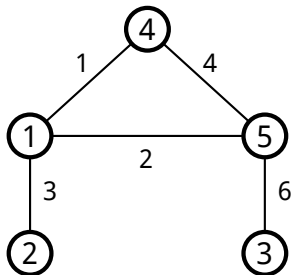
$$p_0 = 1, \quad p_m = 0, \quad p_i = \frac{m-i}{m} \quad \forall i = 1, \dots, m-1$$

We can get that

$$q_i = \frac{\prod_{j=0}^{i-1} \frac{m-j}{m}}{\prod_{j=1}^i \frac{j}{m}} = \frac{\prod_{j=0}^{i-1} m-j}{\prod_{j=1}^i j} = \binom{m}{i} \quad \forall i = 1, \dots, m$$

$$\implies \pi_0 = \frac{1}{1 + \sum_{i=1}^m \binom{m}{i}} = \frac{1}{2^m} \implies \pi_k = \frac{\binom{m}{k}}{2^m} \quad \forall k = 0, 1, \dots, m$$

## Example: A random walk



$$P_{ij} = \frac{w(i, j)}{\sum_k w(i, k)}$$

where  $w(a, b)$  is the weight of edge  $(a, b)$ .  
To make it as a time-reversible chain, we let

$$\pi_i = \frac{\sum_k w(i, k)}{\sum_\ell \sum_k w(\ell, k)}$$

We can see that

$$\pi_i \cdot P_{ij} = \pi_j \cdot P_{ji}$$



# Hastings-Metropolis sampling algorithm

Design an irreducible Markov chain  $\mathbb{X}$  such that the unique stationary distribution of  $\mathbb{X}$  is the distribution of random variable  $Y$ .

Since the long-run proportion of state  $i$  is  $P(Y = i)$ ,

$$\lim_{n \rightarrow \infty} \frac{X(1) + X(2) + \dots + X(n)}{n} = \sum_{i \in S} i \cdot P(Y = i) = E[Y] = \mu$$

While computing  $\mu$  by the law of large number is difficult (hard to sample on  $Y$ ), we use this alternative method to compute  $\mu$  by generating a sequence of  $\mathbb{X}$ , which is sometime easier.

# Hastings-Metropolis sampling algorithm (cont.)

There's a random variable  $Y$  such that

$$P(Y = i) = \frac{b_i}{C}$$

for some unknown (or intractable)  $C = \sum_{i \in S} b_i$ .

We then design a Markov chain  $\mathbb{X}$  that

- $P_{ii} = Q_{ii} + \sum_{k \in S, k \neq i} Q_{ik} \cdot (1 - q_{ik})$
- $P_{ij} = Q_{ij} \cdot q_{ij} \quad \forall j \neq i$

where

- $Q$  is the transition matrix of an arbitrary irreducible Markov chain  $\mathbb{X}$  which has the same state space as  $Y$ .
- $q$  is a matrix to be determined later.

# Hastings-Metropolis sampling algorithm (cont.)

For  $n = 0, 1, \dots$ ,

- 1 If  $X(n) = i$ , set  $Z$  such that  $P(Z = j) = Q_{ij} \quad \forall j \in S$ .
- 2 If  $Z = j$ , set  $X(n+1)$  such that
  - $P(X(n+1) = j) = q_{ij}$
  - $P(X(n+1) = i) = 1 - q_{ij}$

One can see that this satisfies the requirement on previous page.

## Hastings-Metropolis sampling algorithm (cont.)

Then, we let

$$\begin{aligned} q_{ij} &= \min \left( \frac{b_j \cdot Q_{ji}}{b_i \cdot Q_{ij}}, 1 \right) \\ \implies b_i \cdot Q_{ij} \cdot q_{ij} &= b_j \cdot Q_{ji} \cdot q_{ji} \\ \implies \frac{b_i}{C} \cdot P_{ij} &= \frac{b_j}{C} \cdot P_{ji} \end{aligned}$$

By observation 3 on page 86, we know that  $(b_1/C, b_2/C, \dots)$  is the stationary distribution of  $\mathbb{X}$ .

# Example: Space of permutations

## Example

Let  $S$  consist of all the permutations  $(x_1, x_2, \dots, x_n)$  of  $\{1, 2, \dots, n\}$  that

$$\sum_{i=1}^n i \cdot x_i \geq \frac{n^3}{4}$$

- This is same as  $Y$  in page 98 with  $C = |S|$  and  $b_i = 1 \forall i$ .
- $S$  is hard to compute.
- We need to design a matrix  $Q$  such that when given a permutation  $x$ , it's efficient to compute the value of  $Q_{xy} \forall y \in S$ .

## Example: Space of permutations (cont.)

We let

$$Q_{xy} = \frac{1}{N(x)} \quad , \text{ if } y \text{ can be obtained from } x \text{ by one swap}$$

where  $N(x)$  is the number of permutations that can be obtained from  $x$  by one swap. For example:

$$\underbrace{(1, 2, 3, 4, 5)}_y \leftrightarrow \underbrace{(1, 3, 2, 4, 5)}_x \leftrightarrow \underbrace{(1, 3, 4, 2, 5)}_y$$

This chain is irreducible since each  $x \in S$  can go to  $(x_1, x_2, \dots, x_n)$ , where  $x_1 \leq x_2 \leq \dots \leq x_n$ , by several swaps.

Also, given a  $x$ , finding all the obtainable  $y$  can be done efficiently.