

Computer_Homework1

Generated by Doxygen 1.9.1

1 Computer Homework 1	1
1.1 Requirements	1
1.2 Installation	1
1.3 How To Use	1
1.4 Copyright	2
1.5 License	2
2 Finite difference Method	3
2.1 Complexity	4
2.2 Accuracy	4
2.3 Convergence	4
3 Theta	7
3.1 Complexity	7
3.2 Accuracy	7
3.3 Convergence	8
4 Trajectory	9
4.1 Trajectory Plot	9
4.2 Trajectory Error analysis	10
5 File Index	13
5.1 File List	13
6 File Documentation	15
6.1 hw1.hpp File Reference	15
6.1.1 Detailed Description	16
6.1.2 Function Documentation	16
6.1.2.1 HW1()	16
6.2 main.cpp File Reference	17
6.2.1 Detailed Description	17
Index	19

Chapter 1

Computer Homework 1

Solve Kepler problem via finite difference Method

1.1 Requirements

To install this program, you should have

- C++ compiler like g++
- gnu make or cmake

1.2 Installation

- gnu make
 - Type make, then we can see hw1 executable file in bin directory
- cmake
 1. make build directory
 2. go to build directory and type cmake .. -DPRECISION_LEVEL precision_level
 - precision_level 0: float
 - precision_level 1: double
 3. Type make then we can see hw1 executable in build directory

1.3 How To Use

Execute hw1 then, it will interactively read

- initial condition
- number of grid points to evaluate
- output file name

Then it computes and saves solution to file. You can plot the result using usual plotting software like gnuplot

1.4 Copyright

Copyright 2021 pistack (Junho Lee). All rights reserved.

1.5 License

This project is released under the GNU Lesser General Public License v3.0.

Chapter 2

Finite difference Method

To solve Kepler problem, we need to solve

$$\frac{d^2\zeta}{dt^2} = \frac{1}{\zeta^3} - \frac{1}{\zeta^2} \quad (2.1)$$

with initial condition

$$\begin{aligned}\zeta(t_0) &= \zeta_0 \\ \zeta'(t_0) &= \zeta'_0\end{aligned}$$

To solve above 2nd order ordinary differential equation (2.1) numerically, we need to approximate 2nd derivative as finite difference. Suppose that the solution $\zeta(t)$ has continuous 4th order derivative in the Domain $[t_0, t_f]$. then

$$\zeta(x+h) = \zeta(x) + \zeta'(x)h + \frac{1}{2!}\zeta''(x)h^2 + \frac{1}{3!}\zeta'''(x)h^3 + \frac{1}{4!}\zeta^{(4)}(x)h^4 \quad (2.2)$$

for some $\eta(x, h) \in (t_0, t_f)$. Using 4th order taylor approximation (2.2) , we can derive following equation

$$\zeta(x-h) - 2\zeta(x) + \zeta(x+h) = h^2\zeta''(x) + O(h^4) \quad (2.3)$$

Next uniformly divide the domain $[t_0, t_f]$ into n sub intervals. let x_i be the end points of the sub intervals then for $0 \leq i \leq n$,

$$x_i = t_0 + ih \quad (2.4)$$

, where $h = (t_f - t_0)/n$. Now for $0 \leq i \leq n$, define ζ_i as following

$$\zeta_i = \zeta(x_i) \quad (2.5)$$

Then we can rewrite finite difference equation (2.3) as following

$$\zeta_{i-1} - 2\zeta_i + \zeta_{i+1} = h^2\zeta''_i + O(h^4) \quad (2.6)$$

for $1 \leq i \leq n-1$. Plug this equation (2.6) into 2nd order ode (2.1) , the we have following recurrence relation

$$\zeta_{i-1} - 2\zeta_i + \zeta_{i+1} = h^2 \left(\frac{1}{\zeta_i^3} - \frac{1}{\zeta_i^2} \right) \quad (2.7)$$

In above equation (2.7) , we turncate $O(h^4)$ terms, so local turncation error is $O(h^4) = O(n^{-4})$. Therefore global turncation error can be roughly estimated to $O(n^{-3})$. To solve recurrence relation, we need to know both ζ_0 and ζ_1 . However only ζ_0 is explicitly given by the initial condition. To approximate ζ_1 with $O(n^{-3})$ error bound, I use 2nd order taylor expension.

$$\zeta_1 \approx \zeta_0 + \zeta'_0 h + \frac{1}{2!}\zeta''_0 h^2 \quad (2.8)$$

ζ''_0 can be derived by 2nd order ode (2.1)

$$\zeta''_0 = \frac{1}{\zeta_0^3} - \frac{1}{\zeta_0^2} \quad (2.9)$$

2.1 Complexity

Clearly

$$O(n).$$

2.2 Accuracy

Global turncation error is roughly estimated by

$$O(n^{-3}).$$

2.3 Convergence

- Initial Condition

$$\zeta(0) = 0.9$$

$$\zeta'(0) = 0$$

- Initial time: 0
- Final time: 10

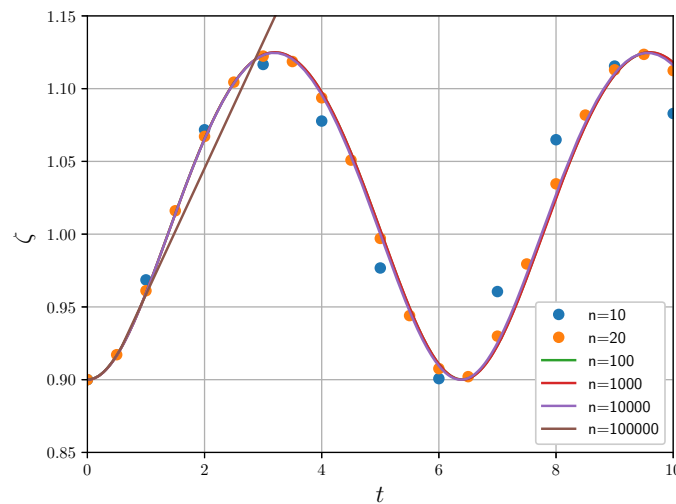


Figure 2.1 Convergence plot: single precision

Due to tuncation error it diverges at $n = 10^5$.

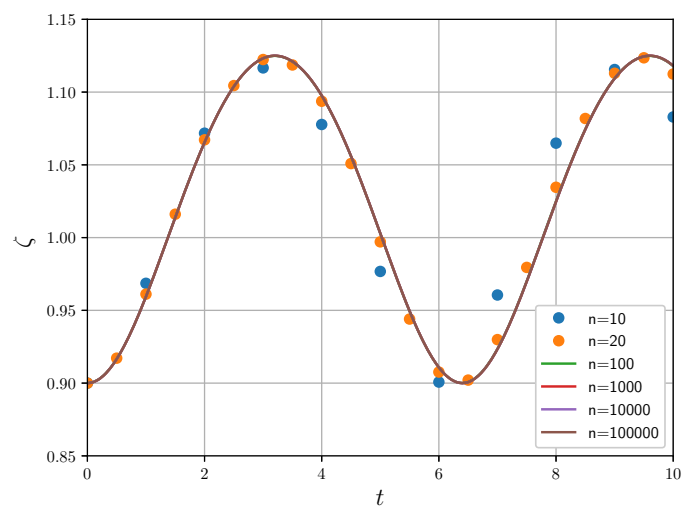


Figure 2.2 Convergence plot: double precision

However in double precision, it converges to exact solution.

Chapter 3

Theta

By conservation of angular momentum, Angle θ satisfies following relation

$$\frac{d\theta}{dt} = \frac{1}{\zeta^2} \quad (3.1)$$

Integrate both side then we can deduce

$$\theta(t) = \theta_0 + \int_{t_0}^t \frac{1}{\zeta^2} dt \quad (3.2)$$

Let $\theta_i = \theta(t_i)$ as in [Finite difference Method](#), then for $1 \leq i$,

$$\theta_i = \theta_{i-1} + \int_{t_{i-1}}^{t_i} \frac{1}{\zeta^2} dt \quad (3.3)$$

Next approximate the integral using trapezoidal rule then

$$\theta_i \approx \theta_{i-1} + \frac{t_i - t_{i-1}}{2} \left(\frac{1}{\zeta_{i-1}^2} + \frac{1}{\zeta_i^2} \right) \quad (3.4)$$

θ_i has $O(n^{-3})$ local turncation error for trapezoidal rule and additional $O(n^{-3})$ for the global turncation error of ζ (see [Finite difference Method Accuracy](#)). So the global turncation error of θ can be estimated to $O(n^{-2})$

3.1 Complexity

Clearly

$$O(n) \quad (3.5)$$

3.2 Accuracy

The global turncation error of θ is roughltly estimated to

$$O(n^{-2}) \quad (3.6)$$

3.3 Convergence

- Initial Condition

$$\zeta(0) = 0.9$$

$$\zeta'(0) = 0$$

$$\theta(0) = 0$$

- Initial time: 0
- Final time: 10

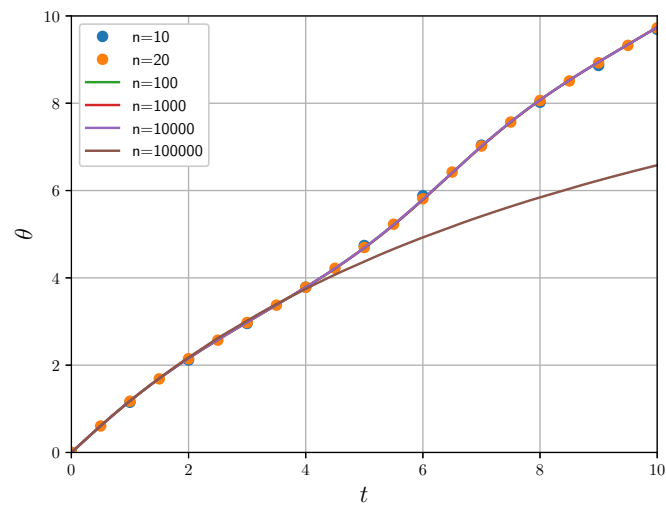


Figure 3.1 Convergence plot: single precision

Due to tuncation error it diverges at $n = 10^5$.

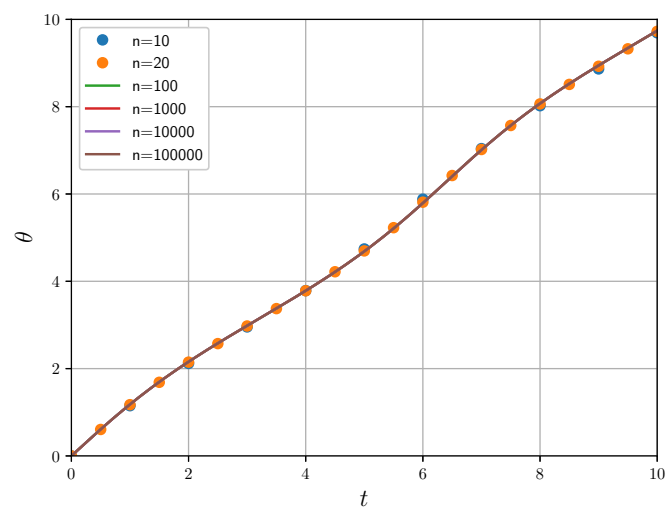


Figure 3.2 Convergence plot: double precision

However in double precision, it converges to exact solution.

Chapter 4

Trajectory

We know that

$$\begin{aligned}x(t) &= \zeta(t) \cos \theta(t) \\y(t) &= \zeta(t) \sin \theta(t)\end{aligned}\tag{4.1}$$

Using above relation (4.1) , we can draw trajectory plot.

4.1 Trajectory Plot

- Initial Condition

$$\begin{aligned}\zeta(0) &= 0.9 \\ \zeta'(0) &= 0 \\ \theta(0) &= 0\end{aligned}$$

- Initial time: 0
- Final time: 10

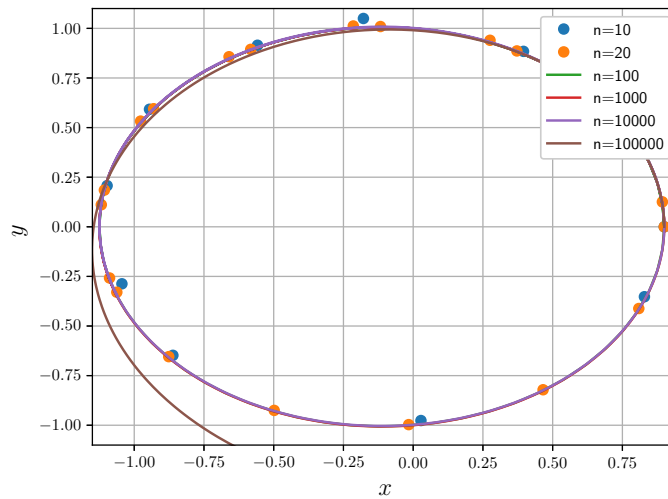


Figure 4.1 Convergence plot: single precision

Due to floating point tuncation error, it diverges at $n = 10^5$ (i.e. $\Delta t = 10^{-4}$).

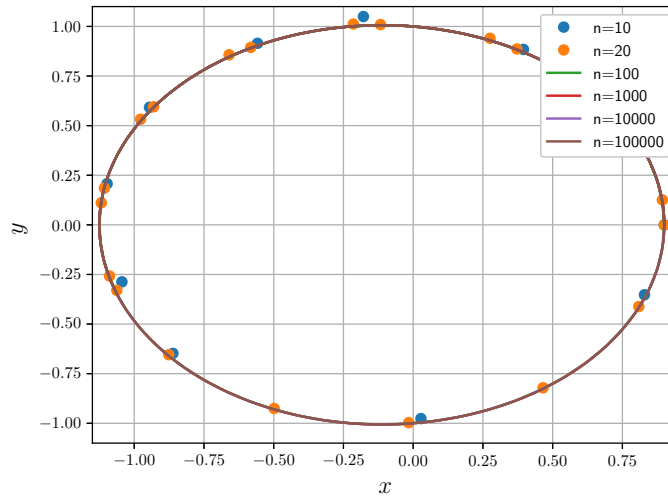


Figure 4.2 Convergence plot: double precision

However, in double precision, it converges to exact epllipse trajectory.

4.2 Trajectory Error analysis

Since trajectory approximated by finite difference method is converges to exact epllipse trajectory, when double precision, I assume $n = 10^5$ with double precision result as exact path. Then we can estimate error at the selected points ($t = 1, 2, \dots, 10$).

$$\text{error} = \sqrt{(x(t) - x_{ref}(t))^2 + (y(t) - y_{ref}(t))^2} \quad (4.2)$$

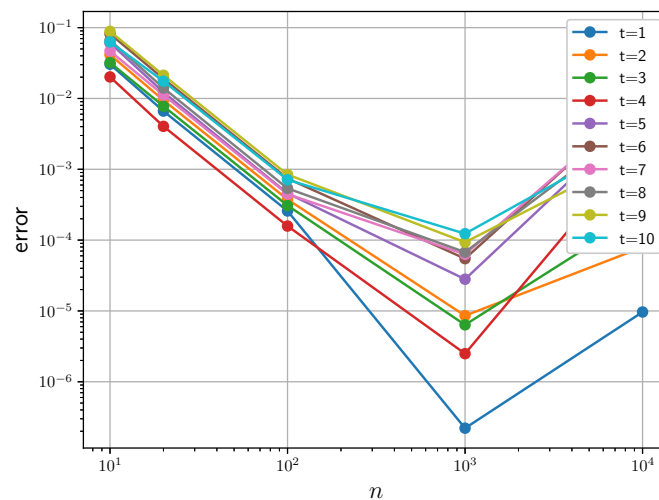


Figure 4.3 Error Analysis: single precision

As you can see due to tuncation error, accuracy is worsen when $n > 10^3$.

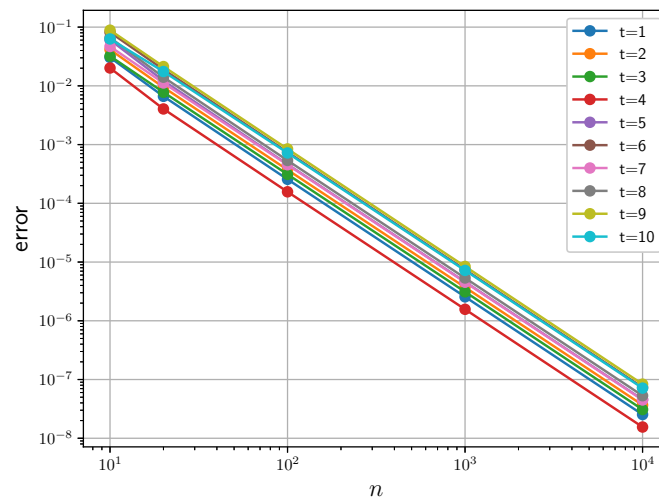


Figure 4.4 Error Analysis: double precision

However, when double precision, accuracy is better and better as n increases. Therefore we can estimate the order of error using above plot. Order of error is estimated to

$$O(n^{-2}) \quad (4.3)$$

Practical error bound is same as estimated error bound $O(n^{-2})$.

Chapter 5

File Index

5.1 File List

Here is a list of all documented files with brief descriptions:

main.cpp	Main program for homework1 of Computer1 class in Yonsei University Interactively reads initial condition, number of grid points to evaluate and output file name then computes and saves solution	17
hw1.hpp	Header file for homework1 of Computer1 class in Yonsei University Use finite difference method to solve Kepler problem	15

Chapter 6

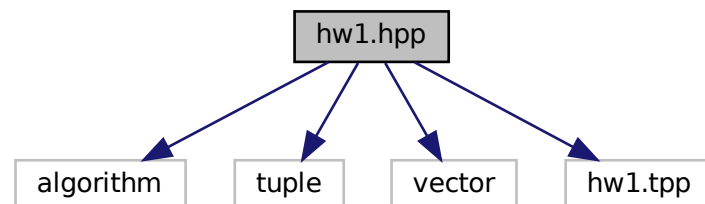
File Documentation

6.1 hw1.hpp File Reference

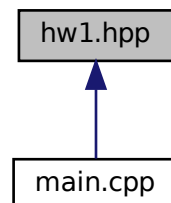
Header file for homework1 of Computer1 class in Yonsei University Use finite difference method to solve Kepler problem.

```
#include <algorithm>
#include <tuple>
#include <vector>
#include "hw1.hpp"
```

Include dependency graph for hw1.hpp:



This graph shows which files directly or indirectly include this file:



Functions

- `template<typename T >`
`std::tuple< std::vector< T >, std::vector< T >, std::vector< T > > HW1 (T t0, T t1, int n, T y0, T y0p, T theta0)`

HW1: Solve Kepler problem via finite difference Method Behavior of HW1 is undefined when type T is not equal to one of float, double, long double.

6.1.1 Detailed Description

Header file for homework1 of Computer1 class in Yonsei University Use finite difference method to solve Kepler problem.

Author

pistack (Junho Lee)

Date

2021. 11. 3.

6.1.2 Function Documentation

6.1.2.1 HW1()

```
template<typename T >
std::tuple<std::vector<T>, std::vector<T>, std::vector<T> > HW1 (
    T t0,
    T t1,
    int n,
    T y0,
    T y0p,
    T theta0 )
```

HW1: Solve Kepler problem via finite difference Method Behavior of HW1 is undefined when type T is not equal to one of float, double, long double.

Parameters

<i>t0</i>	initial time
<i>t1</i>	final time
<i>n</i>	number of gird points to evaluate
<i>y0</i>	initial condition for zeta
<i>y0p</i>	intial condition for derivative of zeta
<i>theta0</i>	initial condition for theta

Returns

tuple of time, zeta and theta

See also

[Finite difference Method](#)

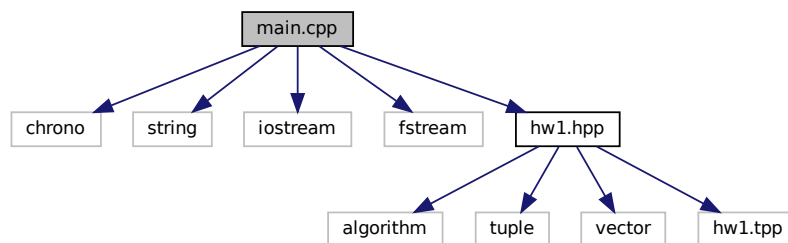
[Theta](#)

6.2 main.cpp File Reference

main program for homework1 of Computer1 class in Yonsei University Interactively reads initial condition, number of grid points to evaluate and output file name then computes and saves solution.

```
#include <chrono>
#include <string>
#include <iostream>
#include <fstream>
#include "hw1.hpp"
```

Include dependency graph for main.cpp:

**Macros**

- #define **PRECISION** float
- #define **DIGITS** 6

Functions

- int **main** (void)

6.2.1 Detailed Description

main program for homework1 of Computer1 class in Yonsei University Interactively reads initial condition, number of grid points to evaluate and output file name then computes and saves solution.

Author

pistack (Junho Lee)

Date

2021. 11. 3.

Index

HW1
 hw1.hpp, [16](#)
hw1.hpp, [15](#)
 HW1, [16](#)

main.cpp, [17](#)