Computer_Homework2

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Chapter 1

Computer Homework 2

Solve Kepler problem via numerical integration

1.1 Requirements

To install this program, you should have

- C++ compiler like g++
- gnu make or cmake

1.2 Installation

- gnu make
  - Type make, then you can see hw2 executable file in bin directory
- cmake
  1. make build directory
  2. go to build directory and type cmake .. -DPRECISION_LEVEL precision_level
     - precision_level 0: float
     - precision_level 1: double
  3. Type make then you can see hw2 executable in build directory

1.3 How To Use

Execute hw2 then, it will interactively read

- initial condition
- number of gird points to evaluate
- output file name

Then it computes and saves solution to file. You can plot the result using usual plotting software like gnuplot

1.4 Copyright

1.5 License

Chapter 2

Numerical Integration

By conservation of energy, we can derive following integral equation.

$$t - t_0 = \int_{\zeta_{min}}^{\zeta(t)} \frac{\zeta'}{\sqrt{-a(\zeta' - \zeta_{min})(\zeta_{max} - \zeta')}} \mathrm{d}\zeta' \tag{2.1}$$

,where

- ζ_{min} is periapsis (minimum value of ζ),
- ζ_{max} is apoapsis (maximum value of ζ)
- α is parameter defined by following relation

$$\begin{aligned} \alpha &= \frac{1}{\zeta_{min}^2} - \frac{2}{\zeta_{min}} \\ &= \frac{1}{\zeta_{max}^2} - \frac{2}{\zeta_{max}} \end{aligned}$$

Using Vieta's Formula, we could simplify above relation.

$$\zeta_{max} = \frac{\zeta_{min}}{2\zeta_{min} - 1} \tag{2.2}$$

$$\alpha = -\frac{1}{\zeta_{min}\zeta_{max}} \tag{2.3}$$

To solve above integral equation (2.1) we need to view time t as a function of ζ with domain $D = [\zeta_{min}, \zeta_{max}]$. Now uniformly divide the domain D into n sub intervals. Let ζ_i be end points of the sub intervals then for $0 \leq \zeta \leq n$,

$$\zeta_i = \zeta_{min} + i\frac{\zeta_{max} - \zeta_{min}}{n} \tag{2.4}$$

Define $t_i = t(\zeta_i)$ then we have following recurrence relation for $i \geq 1$,

$$t_i = t_{i-1} + \int_{\zeta_{i-1}}^{\zeta_i} \frac{\zeta'}{\sqrt{-\alpha(\zeta' - \zeta_{min})(\zeta_{max} - \zeta')}} \mathrm{d}\zeta' \tag{2.5}$$

However, due to the divergence feature of the integrand at $\zeta_0 = \zeta_{min}$ and $\zeta_n = \zeta_{max}$, It is hard to approximate such integral directly. To remove singularity, first consider the following equation.

$$\begin{aligned} \frac{\zeta'}{\sqrt{-\alpha(\zeta' - \zeta_{min})(\zeta_{max} - \zeta')}} &= \frac{\zeta_{min}}{\sqrt{-\alpha}(\zeta_{max} - \zeta_{min})} \frac{\sqrt{\zeta_{max} - \zeta'}}{\sqrt{\zeta' - \zeta_{min}}} \\ &+ \frac{\zeta_{max}}{\sqrt{-\alpha}(\zeta_{max} - \zeta_{min})} \frac{\sqrt{\zeta' - \zeta_{min}}}{\sqrt{\zeta_{max} - \zeta'}} \end{aligned} \tag{2.6}$$

Then we can seperate integral into two parts. Substitute $u = \sqrt{\zeta' - \zeta_{min}}$ and $v = \sqrt{\zeta_{max} - \zeta'}$ to the first and second part of integral respectively, then

$$\mathrm{integral}_i = \frac{2\zeta_{min}}{\sqrt{-\alpha}(\zeta_{max} - \zeta_{min})} \int_{\sqrt{\zeta_{i-1}-\zeta_{min}}}^{\sqrt{\zeta_i-\zeta_{min}}} \sqrt{\zeta_{max} - \zeta_{min} - u^2} \mathrm{d}u \tag{2.7}$$
$$+ \frac{2\zeta_{max}}{\sqrt{-\alpha}(\zeta_{max} - \zeta_{min})} \int_{\sqrt{\zeta_{max}-\zeta_i}}^{\sqrt{\zeta_{max}-\zeta_{i-1}}} \sqrt{\zeta_{max} - \zeta_{min} - v^2} \mathrm{d}v \tag{2.8}$$

by above equation (2.7) , We can deduce

$$\begin{aligned} t_f - t_0 &= \pi \frac{\zeta_{min} + \zeta_{max}}{\sqrt{-\alpha}} \\ &= \pi a^{3/2} \end{aligned} \tag{2.9}$$

,where $a = (\zeta_{min} + \zeta_{max})/2$.

2.1 Approximation

Let $f(u) = \sqrt{\zeta_{max} - \zeta_{min} - u^2}$ and define u_i and v_i as following

$$u_i = \sqrt{\zeta_i - \zeta_{min}} \tag{2.10}$$
$$v_i = \sqrt{\zeta_{max} - \zeta_i} \tag{2.11}$$

then we have following relation

$$u_i = f(v_i) \tag{2.12}$$
$$v_i = f(u_i) \tag{2.13}$$
$$v_{n-i} = u_i \tag{2.14}$$

To exploit above relations (2.12)–(2.14) I use Trapezoidal rule with unequivalently spaced interval. Then,

$$\mathrm{integral}_i = c_1 \frac{u_i - u_{i-1}}{2} (v_i + v_{i-1}) + c_2 \frac{v_i - v_{i-1}}{2} (u_i + u_{i-1}) \tag{2.15}$$

where

$$c_1 = \frac{2\zeta_{min}}{\sqrt{-\alpha}(\zeta_{max} - \zeta_{min})}$$
$$c_2 = \frac{2\zeta_{max}}{\sqrt{-\alpha}(\zeta_{max} - \zeta_{min})}$$

2.2 Complexity

Complex is clearly

$$O(n)$$

2.3 Accuracy

Error bound is given by

$$
\begin{aligned}
\text{Error bound} &\leq M \sum_{i=1}^{n} (u_i - u_{i-1})^3 \\
&= M \left(\frac{\zeta_{max} - \zeta_{min}}{n} \right)^{3/2} \sum_{i=1}^{n} \left(\frac{1}{\sqrt{i} + \sqrt{i-1}} \right)^3 \\
&< MC \left(\frac{\zeta_{max} - \zeta_{min}}{n} \right)^{3/2}
\end{aligned}
$$

So, the error bound is

$$O(n^{-3/2})$$

2.4 Convergence

initial condition

$$
\begin{aligned}
t_0 &= 0 \\
\zeta_{min} &= 0.9
\end{aligned}
$$

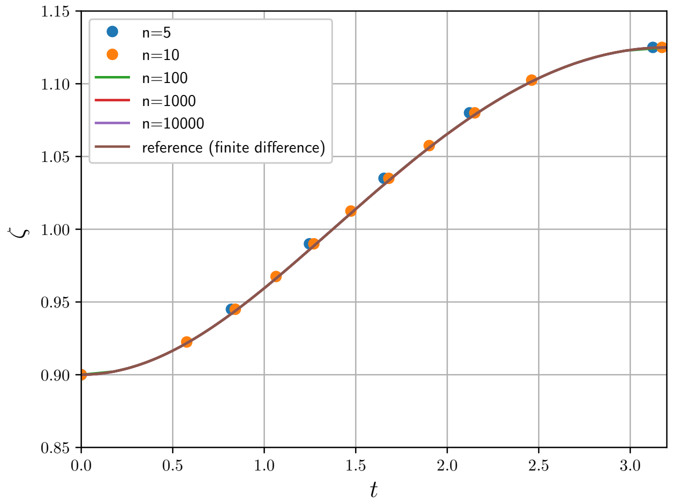


Figure 2.1 Convergence plot: single precision

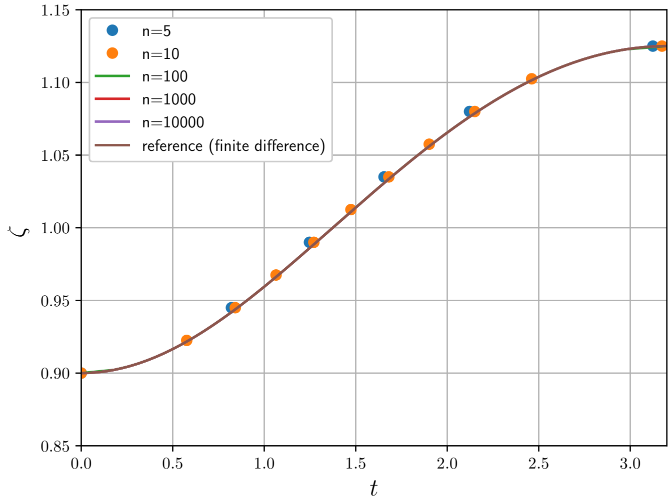


Figure 2.2 Convergence plot: double precision

2.5 Practical Error Bound

We know that the exact value of t_f is $\pi a^{3/2}$. Where $a = (\zeta_{min} + \zeta_{max})/2.0$. So, error can be estimated by

$$\text{error} = \left| t_n - \pi a^{3/2} \right| \tag{2.16}$$

Error Analysis Table: single precision

n	Error
5	7.7061e-2
10	2.6981e-2
10^2	8.5121e-4
10^3	5.1212e-5
10^4	9.8788e-5

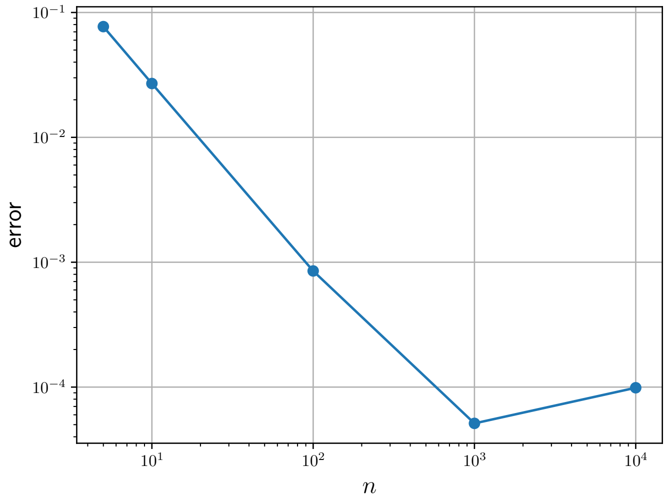


Figure 2.3 Error Analysis Plot: single precision

In single precision, difference of sucessive ζ reaches to 100 times of machine epsilon, roughly 10^{-5} at $n > 10^3$. So, numerical error is dominant or comparable to turncation error. Thus, at $n > 10^3$, due to the turncation error, accuracy is worsen as n increases. In double precision, the machine epsilon, roughly 10^{-16}, is far less than difference of sucessive ζ, so we can neglect numerical error. Moreover we could estimate numerical error of single precision result as

$$\text{error}_{num} = |t_{n,\text{float}} - t_{n,\text{double}}| \tag{2.17}$$

Numerical Error: single precision

n	Error
5	1.3735e-6
10	3.6656e-6
10^2	3.4982e-6
10^3	2.4420e-5
10^4	9.9635e-5

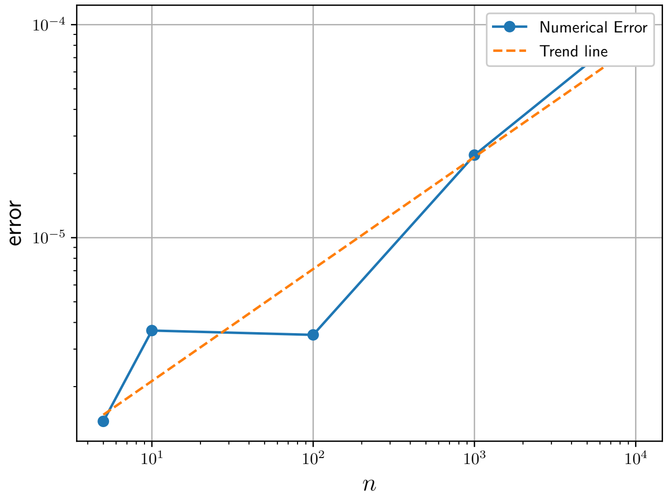


Figure 2.4 Numerical Error: single precision

The slope of trend line is 0.5. So the order of numerical error is $O(\sqrt{n})$. Combine this results with single precision numerical error table, error bound for the numerical error is

$$\mathrm{error}_{num} \leq 10^{-6}\sqrt{n} \tag{2.18}$$

Now estimates the error of double precision results. In double precision, we can neglect numerical error, so error is equal to the turncation error.

Error Analysis Table: double precision

n	Error
5	7.7060e-2
10	2.6985e-2
10^2	8.4771e-4
10^3	2.6792e-5
10^4	8.4719e-7

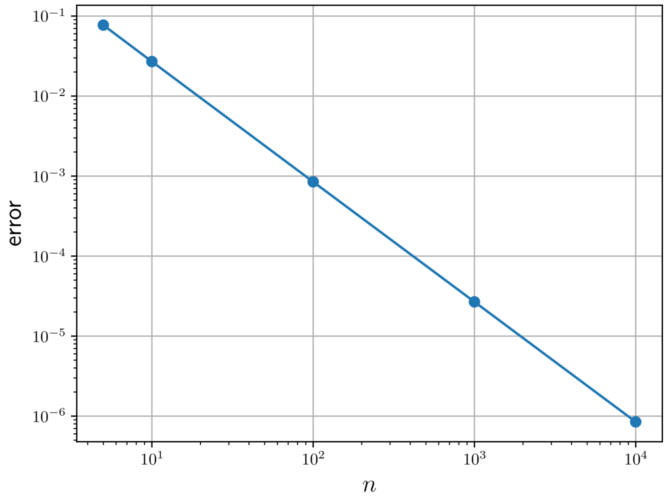


Figure 2.5 Error Analysis Plot: double precision

By above Error Analysis Plot, order of error is estimated as $O(n^{-1.5})$. It is same as the order of the theoretical error bound. Also, prefactor $C \leq 1$ by Error Analysis Table. Therefore error bound for the double precision error or turncation error is estimated to

$$\text{error}_{turn} \leq \frac{1}{n^{1.5}} \tag{2.19}$$

Hence for single precision, total error bound is estimated to

$$\text{error}_{single} \leq 10^{-6}\sqrt{n} + \frac{1}{n^{1.5}} \tag{2.20}$$

By Generalized Arithmetic Mean and Geometric Mean inequality,

$$\begin{aligned} 3 \cdot \frac{10^{-6}}{3}\sqrt{n} + \frac{1}{n^{1.5}} &\geq \frac{4}{3^{3/4}} 10^{-4.5} \\ &\approx 5 \cdot 10^{-5} \end{aligned}$$

Hence, in single precision, the optimal absolute error is $5 \cdot 10^{-5}$.

Chapter 3

File Index

3.1 File List

Here is a list of all documented files with brief descriptions:

Chapter 4

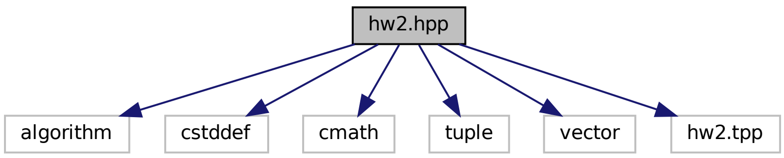
File Documentation

4.1 hw2.hpp File Reference

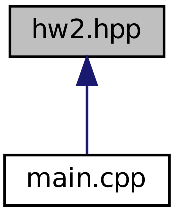
headerfile for homework2 of Computer1 class in Yonsei University Use numerical integration to solve Kepler problem

```
#include <algorithm>
#include <cstddef>
#include <cmath>
#include <tuple>
#include <vector>
#include "hw2.tpp"
```

Include dependency graph for hw2.hpp:



This graph shows which files directly or indirectly include this file:



Functions

- template< typename T >
  std::tuple< std::vector< T >, std::vector< T > > HW2 (T zeta_min, T t0, std::size_t n)
  HW2: Solve Kepler problem via numerical integration from zeta_min to zeta_max If type T is not equal to one of float, double, long double then behavior of HW2 is undefined.

4.1.1 Detailed Description

headerfile for homework2 of Computer1 class in Yonsei University Use numerical integration to solve Kepler problem

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pistack (Junho Lee)

Date

2021. 11. 10.

4.1.2 Function Documentation

4.1.2.1 HW2()

```
template<typename T >
std::tuple<std::vector<T>, std::vector<T> > HW2 (
            T zeta_min,
            T t0,
            std::size_t n )
```

HW2: Solve Kepler problem via numerical integration from zeta_min to zeta_max If type T is not equal to one of float, double, long double then behavior of HW2 is undefined.

Parameters

zeta_min	minimum value of zeta, for constraint motion 0.5 < zeta_min < 1
t0	initial time
n	number of points to evaluate

Returns

tuple of time and zeta

See also

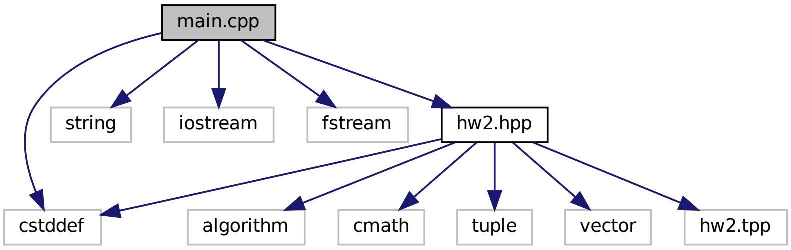
Numerical Integration

4.2 main.cpp File Reference

main program for homework2 of Computer1 class in Yonsei University Interactively reads inital condition, number of gird points to evaluate and output file name then computes and saves solution.

```
#include <cstddef>
#include <string>
#include <iostream>
#include <fstream>
#include "hw2.hpp"
```

Include dependency graph for main.cpp:



Macros

- #define **PRECISION** float
- #define **DIGITS** 6

Functions

- int **main** (void)

4.2.1 Detailed Description

main program for homework2 of Computer1 class in Yonsei University Interactively reads inital condition, number of gird points to evaluate and output file name then computes and saves solution.

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