4190.101

Discrete Mathematics

Chapter 1 The Foundations: Logic and Proofs

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[Based on S. Srinivasa Rao's Lecture Notes]

Outline

- Propositional Logic
  - The Language of Propositions
  - Applications
  - Logical Equivalences
- Predicate Logic
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
- Proofs
  - Rules of Inference
  - Proof Methods
  - Proof Strategy

Propositional Logic

Section 1.1

Section Summary

- Propositions
- Connectives
  - Negation
  - Conjunction
  - Disjunction
  - Implication: contrapositive, inverse, converse
  - Biconditional
- Truth Tables

Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
  a) The Moon is made of green cheese.
  b) Trenton is the capital of New Jersey.
  c) Toronto is the capital of Canada.
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

Propositional Logic

- Constructing Propositions
  - Propositional Variables: *p, q, r, s*, …
  - The proposition that is always true is denoted by **T**
  - The proposition that is always false is denoted by **F**
  - Compound Propositions: constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\land$
    - Disjunction $\lor$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

Compound Propositions: Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

- Example: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Conjunction

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Example: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

The Connective Or in English

- In English "or" has two distinct meanings.
  - **Inclusive Or** - "Students who have taken CS202 or Math120 can take this class": Students need to have taken one of the prerequisites, but may have taken both.
  - This is the meaning of disjunction. For $p \vee q$ to be true, either one or both of p and q must be true.
  - **Exclusive Or** - "Soup or salad comes with this entrée": We do not expect to be able to get both soup and salad.
  - This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Example: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."
- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

- These implications ($p{:}F \rightarrow q{:}F$) are perfectly fine, but would not be used in ordinary English.
  - “If the moon is made of green cheese, then I have more money than Bill Gates. ”
  - “If the moon is made of green cheese then I’m on welfare.”
  - “If 1 + 1 = 3, then your grandma wears combat boots.”

Understanding Implication (cont)

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."

- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge.
  - This corresponds to the case where p is true and q is false.

Different Ways of Expressing $p \rightarrow q$

- **if** p, **then** q
- **if** p, q
- p **implies** q
- p **only if** q
- q **if** p
- q **unless** $\neg$p
- q **when** p
- q **whenever** p
- p **is sufficient for** q
- q **follows from** p
- q **is necessary for** p
- **A necessary condition for** p **is** q
- **A sufficient condition for** q **is** p

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$
- Example: Find the converse, inverse, and contrapositive of "Raining is a sufficient condition for my not going to town."
- Solution:
  - **Converse**: If I do not go to town, then it is raining.
  - **Inverse**: If it is not raining, then I will go to town.
  - **Contrapositive**: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q."
- The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "*p* if and only if *q*" is expressed in English:
  - *p* **is necessary and sufficient for** *q*
  - **if** *p* **then** *q* , **and conversely**
  - *p* **iff** *q*

Truth Tables for Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Include the atomic propositions
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value.
- Example: Show using a truth table that the *implication* is equivalent to its *contrapositive*.
- Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using Truth Table for Non-Equivalence

- Example: Show using truth tables that neither the converse nor inverse of an implication is equivalent to the implication.
- Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

- How many rows are there in a truth table with *n* propositional variables?

- Solution: 2^n (will see how to do this in Chapter 6)

- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

- $p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
- If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Applications of Propositional Logic

Section 1.2

Applications of Propositional Logic

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Puzzles
- Logic Circuits
- AI Diagnosis Method (Optional)

Translating English Sentences

- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives
- "If I go to Harry's or to the country, I will not go shopping."
  - *p*: I go to Harry's
  - q: I go to the country.
  - *r*: I will go shopping.

If *p* or *q* then not *r*.

$$(p \vee q) \rightarrow \neg r$$

Example

- Problem: Translate the following sentence into propositional logic:
  - “You can access the Internet from campus only if you are a computer science major or you are not a freshman.”
- One Solution
  - a : “You can access the internet from campus”
  - c : “You are a computer science major”
  - f : “You are a freshman.”

$$a \rightarrow (c \vee \neg f)$$

System Specifications

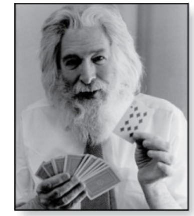
- System and Software engineers take requirements in English and express them in a precise specification language based on logic.
- Example: Express in propositional logic:
  - “The automated reply cannot be sent when the file system is full”
- One possible solution:
  - p denotes “The automated reply can be sent”
  - q denotes “The file system is full.”

$$q \rightarrow \neg p$$

Consistent System Specifications

- Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.
- Exercise: Are these specifications consistent?
  - "The diagnostic message is stored in the buffer or it is retransmitted."
  - "The diagnostic message is not stored in the buffer."
  - "If the diagnostic message is stored in the buffer, then it is retransmitted."
- Solution:
  - p: "The diagnostic message is not stored in the buffer."
  - q: "The diagnostic message is retransmitted"
  - The specification can be written as: $p \vee q$, $\neg p$, $p \rightarrow q$
  - When p is false and q is true all three statements are true. So the specification is consistent.
- What if "The diagnostic message is not retransmitted" is added."
  - Now we are adding ¬q and there is no satisfying assignment.
  - The specification is not consistent.

Logic Puzzles

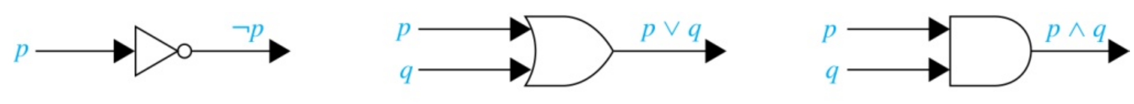


Raymond Smullyan (Born 1919)

- An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."
- Example: What are the types of A and B?
- Solution
  - Let *p:* A is a knight, *q* : B is a knight.
  - Then $\neg p$: A is a knave, $\neg q$: B is a knave.
  - If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ that B says would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
  - If A is a knave, then B must not be a knight since knaves always lie. So, then both p and $\neg$q hold since both are knaves.

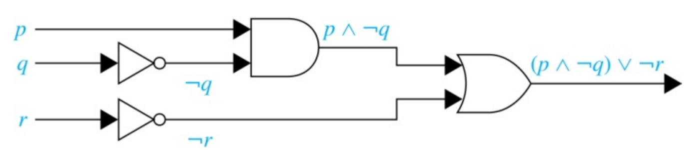
Logic Circuits
(Studied in depth in Chapter 12)

- Electronic circuits; each input/output signal can be viewed as a 0 (False) or 1 (True).
- Circuits are constructed from three basic circuits called gates.



Inverter | OR gate | AND gate

  - The inverter (**NOT gate**) takes an input bit and produces its negation.
  - The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
  - The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.
- More complicated digital circuits can be constructed by combining these basic circuits to produce the desired output For example:



Propositional Equivalences

Section 1.3

Section Summary

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- Normal Forms (*optional, covered in exercises in text*)
  - Disjunctive Normal Form
  - Conjunctive Normal Form
- Propositional Satisfiability
  - Sudoku Example

Tautologies, Contradictions, and Contingencies

- A *tautology* is a proposition which is always true.
  - Example: $p \lor \neg p$
- A *contradiction* is a proposition which is always false.
  - Example: $p \land \neg p$
- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table shows $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

- This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

- Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$
- Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$
- Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$
- Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$

Key Logical Equivalences (*cont*)

- Commutative Laws: $p \lor q \equiv q \lor p$, $p \land q \equiv q \land p$

- Associative Laws: $(p \land q) \land r \equiv p \land (q \land r)$
  $(p \lor q) \lor r \equiv p \lor (q \lor r)$
- Distributive Laws: $(p \lor (q \land r) \equiv (p \lor q)) \land (p \lor r)$

  $(p \land (q \lor r)) \equiv (p \land q) \lor (p \land r)$

- Absorption Laws: $p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

- Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs

- Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

- Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law} \\
& && \text{for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Equivalence Proofs

- Example: Show that $(p \land q) \to (p \lor q)$ is a tautology.
- Solution:

$$
\begin{array}{rcll}
(p \land q) \to (p \lor q) & \equiv & \neg(p \land q) \lor (p \lor q) & \text{by truth table for } \to \\
& \equiv & (\neg p \lor \neg q) \lor (p \lor q) & \text{by the first De Morgan law} \\
& \equiv & (\neg p \lor p) \lor (\neg \underset{q}{\cancel{p}} \lor \neg q) & \text{by associative and} \\
& & & \text{commutative laws} \\
& & & \text{laws for disjunction} \\
& \equiv & T \lor T & \text{by truth tables} \\
& \equiv & T & \text{by the domination law}
\end{array}
$$

Disjunctive Normal Form (*optional*)

- A propositional formula is in *disjunctive normal form* if it consists of a disjunction of (1, … ,*n*) disjuncts where each disjunct consists of a conjunction of (1, …, *m*) atomic formulas or the negation of an atomic formula.
  - Yes $(p \wedge \neg q) \vee (\neg p \vee q)$
  - No $p \wedge (p \vee q)$

- Every compound proposition can be put in disjunctive normal form

Disjunctive Normal Form (*optional*)

- Example: Find the Disjunctive Normal Form (DNF) of false.

$$(p \vee q) \rightarrow \neg r$$

- Solution: This proposition is true when *r* is false or when both *p* and *q* are false.

$$(\neg p \wedge \neg q) \vee \neg r$$

Conjunctive Normal Form (optional)

- A compound proposition is in *Conjunctive Normal Form* (CNF) if it is a conjunction of disjunctions.
- Every proposition can be put in an equivalent CNF.
- Conjunctive Normal Form (CNF) can be obtained by eliminating implications, moving negation inwards and using the distributive and associative laws.
- Important in resolution theorem proving used in artificial Intelligence (AI).
- A compound proposition can be put in conjunctive normal form through repeated application of the logical equivalences covered earlier.

Disjunctive Normal Form (*optional*)

- Example: Put the following into CNF:

$$\neg(p \to q) \vee (r \to p)$$

- Solution:
  1. Eliminate implication signs:
  $$\neg(\neg p \vee q) \vee (\neg r \vee p)$$
  2. Move negation inwards; eliminate double negation:
  $$(p \wedge \neg q) \vee (\neg r \vee p)$$
  3. Convert to CNF using associative/distributive laws
  $$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

Propositional Satisfiability

- A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that make it true.
- When no such assignments exist, the compound proposition is *unsatisfiable*.
- A compound proposition is unsatisfiable if and only if its negation is a tautology.

Example on Propositional Satisfiability

- Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

  – Solution: Satisfiable. Assign **T** to p, q, and r.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

  – Solution: Satisfiable. Assign **T** to p and $\boldsymbol{F}$ to q.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

  – Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

Notation

$\bigvee_{j=1}^{n} p_j$ is used for $p_1 \vee p_2 \vee \ldots \vee p_n$

$\bigwedge_{j=1}^{n} p_j$ is used for $p_1 \wedge p_2 \wedge \ldots \wedge p_n$

Sudoku

- A **Sudoku puzzle** is represented by a 9×9 grid made up of nine 3×3 subgrids, known as **blocks**. Some of the 81 cells of the puzzle are assigned one of the numbers 1,2, …, 9.
- The puzzle is solved by assigning numbers to each blank cell so that every row, column and block contains each of the nine possible numbers.
- Example

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Encoding as a Satisfiability Problem

- Let $p(i,j,n)$ denote the proposition that is true when the number n is in the cell in the ith row and the jth column.
- There are $9\times9 \times 9 = 729$ such propositions.
- In the sample puzzle $p(5,1,6)$ is true, but $p(5,j,6)$ is false for $j = 2,3,\ldots 9$

Encoding (cont)

- For each cell with a given value, assert *p*(*i*,*j*,*n*), when the cell in row *i* and column *j* has the given value.
- Assert that every row contains every number.

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$$

- Assert that every column contains every number.

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$$

Encoding (cont)

- Assert that each of the 3 x 3 blocks contain every number.

$$\bigwedge_{r=0}^{2}\bigwedge_{s=0}^{2}\bigwedge_{n=1}^{9}\bigwedge_{i=1}^{3}\bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

 (this is tricky - ideas from chapter 4 help)

- Assert that no cell contains more than one number. Take the conjunction over all values of *n*, *n'*, *i*, and j, where each variable ranges from 1 to 9 and $n \neq n'$,

$$p(i,j,n) \rightarrow \neg p(i,j,n')$$

Solving Satisfiability Problems

- To solve a Sudoku puzzle, we need to find an assignment of truth values to the 729 variables of the form *p(i,j,n)* that makes the conjunction of the assertions true. Those variables that are assigned T yield a solution to the puzzle.
- A truth table can always be used to determine the satisfiability of a compound proposition. But this is too complex even for modern computers for large problems.
- There has been much work on developing efficient methods for solving satisfiability problems as many practical problems can be translated into satisfiability problems.