4190.101

Discrete Mathematics

Chapter 3 Algorithms

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[Based on S. Srinivasa Rao's Lecture Notes]

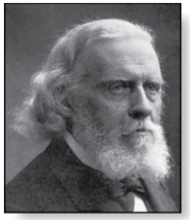
The Growth of Functions

Section 3.2

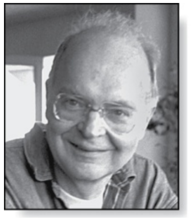
Section Summary

- Big-O Notation
- Big-O Estimates for Important Functions
- Big-Omega and Big-Theta Notation

Edmund Landau
(1877-1938)



Paul Gustav Heinrich Bachmann
(1837-1920)



Donald E. Knuth
(Born 1938)

The Growth of Functions

- In both computer science and in mathematics, there are many times when we care about how fast a function grows.
- In computer science, we want to understand how quickly an algorithm can solve a problem as the size of the input grows.
  - We can compare the efficiency of two different algorithms for solving the same problem.
  - We can also determine whether it is practical to use a particular algorithm as the input grows.
  - We'll study these questions in Section 3.3.
- Two of the areas of mathematics where questions about the growth of functions are studied are:
  - Number theory (covered in Chapter 4)
  - Combinatorics (covered in Chapters 6 and 8)

Big-*O* Notation

- **Definition**: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers. We say that *f*(*x*) is *O*(*g*(*x*)) if there are constants *C* and *k* such that

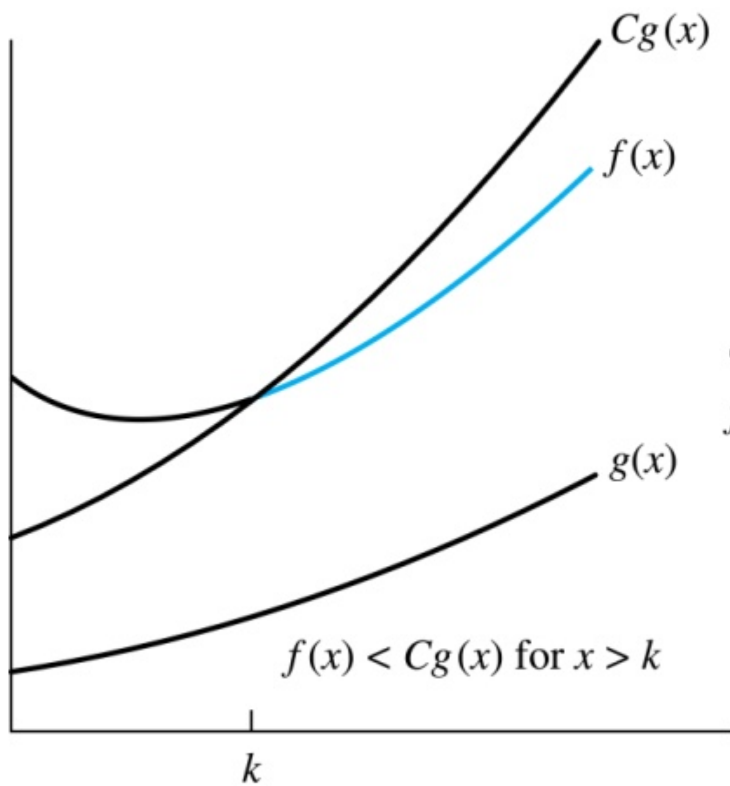
$$|f(x)| \leq C|g(x)|$$

whenever $x > k$ (illustration on next slide)

  – This is read as "*f*(*x*) is big-*O* of *g*(*x*)" or "*g* asymptotically dominates *f*."

- The constants C and k are called *witnesses* to the relationship *f*(*x*) is *O*(*g*(*x*)). Only one pair of witnesses is needed.

Illustration of Big-O Notation



$f(x)$ is $O(g(x))$

The part of the graph of $f(x)$ that satisfies $f(x) < Cg(x)$ is shown in color.

Some Important Points about Big-O Notation

- If one pair of witnesses is found, then there are infinitely many pairs. We can always make the k or the C larger and still maintain the inequality $|f(x)| \leq C|g(x)|$.
  - Any pair C' and k' where $C < C'$ and $k < k'$ is also a pair of witnesses since $|f(x)| \leq C|g(x)| \leq C'|g(x)|$ whenever $x > k' > k$.

- You may see "$f(x) = O(g(x))$" instead of "$f(x)$ is $O(g(x))$."
  - But this is an abuse of the equals sign since the meaning is that there is an inequality relating the values of f and g, for sufficiently large values of x.
  - It is ok to write $f(x) \in O(g(x))$, because $O(g(x))$ represents the set of functions that are $O(g(x))$.

- Usually, we will drop the absolute value sign since we will always deal with functions that take on positive values.

Using the Definition of Big-*O* Notation

- **Example**: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$.
- **Solution**: Since when $x > 1$, $x < x^2$ and $1 < x^2$

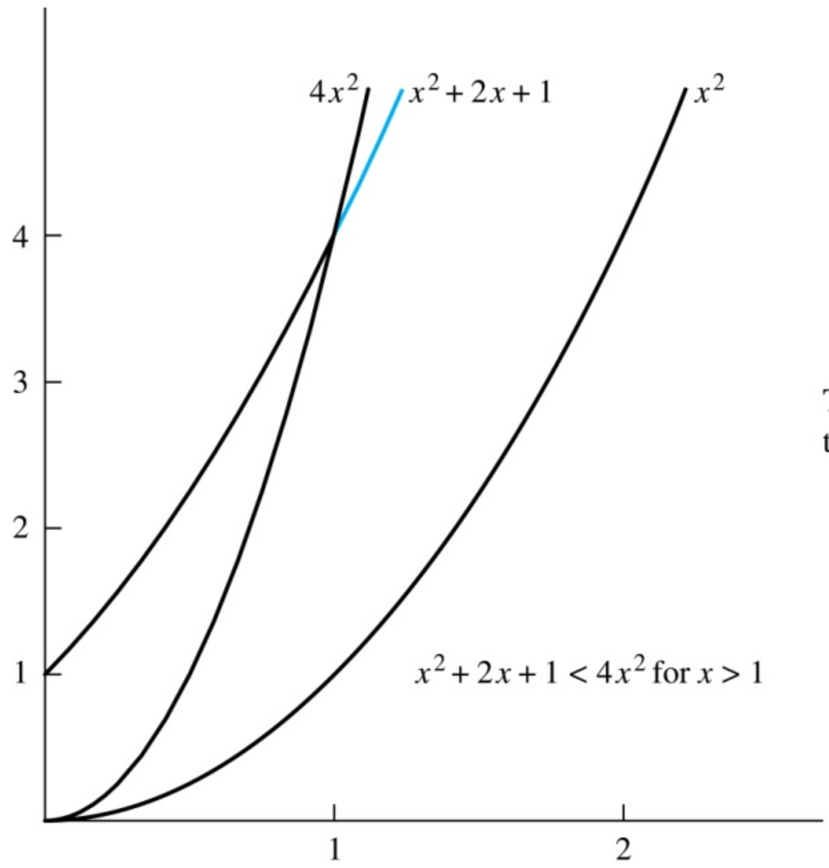
$$0 \leq x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 \leq 4x^2$$

  - Take $C = 4$ and $k = 1$ as witnesses to show that
  - (see graph on next slide)

- Alternatively, when $x > 2$, we have $2x \leq x^2$ and $1 < x^2$. Hence, $0 \leq x^2 + 2x + 1 \leq x^2 + x^2 + x^2 \leq 3x^2$ when $x > 2$.
  - Can take $C = 3$ and $k = 2$ as witnesses instead.

Illustration of Big-*O* Notation

$f(x) = x^2 + 2x + 1$ is $O(x^2)$



The part of the graph of $f(x) = x^2 + 2x + 1$ that satisfies $f(x) < 4x^2$ is shown in blue.

Big-*O* Notation

- Both $f(x) = x^2 + 2x + 1$ and $g(x) = x^2$ are such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.
  We say that the two functions are of the *same order*.
  (More on this later)
- If $f(x)$ is $O(g(x))$ and $h(x)$ is larger than $g(x)$ for all positive real numbers, then $f(x)$ is $O(h(x))$.
  - If $|f(x)| \leq C|g(x)|$ for $x > k$ and if $|h(x)| > |g(x)|$ for all x, then $|f(x)| \leq C|h(x)|$ if $x > k$. Hence, $f(x)$ is $O(h(x))$.
- For many applications, the goal is to select the function $g(x)$ in $O(g(x))$ as small as possible (up to multiplication by a constant, of course).

Using the Definition of Big-O Notation

- **Example**: Show that $7x^2$ is $O(x^3)$.
- **Solution**: When $x > 7$, $7x^2 < x^3$. Take $C = 1$ and $k = 7$ as witnesses to establish that $7x^2$ is $O(x^3)$.
- (Would $C = 7$ and $k = 1$ work?)

- **Example**: Show that n^2 is not $O(n)$.
- **Solution**: Proof by contradiction
  - Suppose there are constants C and k for which $n^2 \leq Cn$, whenever $n > k$.
  - If we divide both sides of $n^2 \leq Cn$ by n (>0), then $n \leq C$ must hold for all $n > k$. A contradiction!

Big-O Estimates for Polynomials

- **Example**: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \cdots, a_n$ are real numbers with $a_n \neq 0$. Then $f(x)$ is $O(x^n)$.
- **Proof**: $|f(\mathrm{x})| = |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_1|$

$$\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x^1 + |a_1|$$
$$= x^n (|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_1|/x^n)$$

Uses triangle inequality, an exercise in sec 1.8.

Assuming $x > 1$

$$\leq x^n (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_1|)$$

- Take $C = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_1|$ and $k = 1$. Then $f(x)$ is $O(x^n)$.
- The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-*O* Estimates for some Important Functions

- **Example**: Use big-*O* notation to estimate the sum of the first *n* positive integers.
- **Solution**: $1 + 2 + \cdots + n \ \leq \ n + n + \cdots n \ = \ n^2$

  $$1 + 2 + \ldots + n \text{ is } O(n^2) \text{ taking } C = 1 \text{ and } k = 1.$$

- **Example**: Use big-*O* notation to estimate the factorial function $f(n) = n! = 1 \times 2 \times \cdots \times n$.
- **Solution**: $n! = 1 \times 2 \times \cdots \times n \ \leq \ n \times n \times \cdots \times n \ = \ n^n$

  $$n! \text{ is } O(n^n) \text{ taking } C = 1 \text{ and } k = 1.$$

Big-O Estimates for some Important Functions

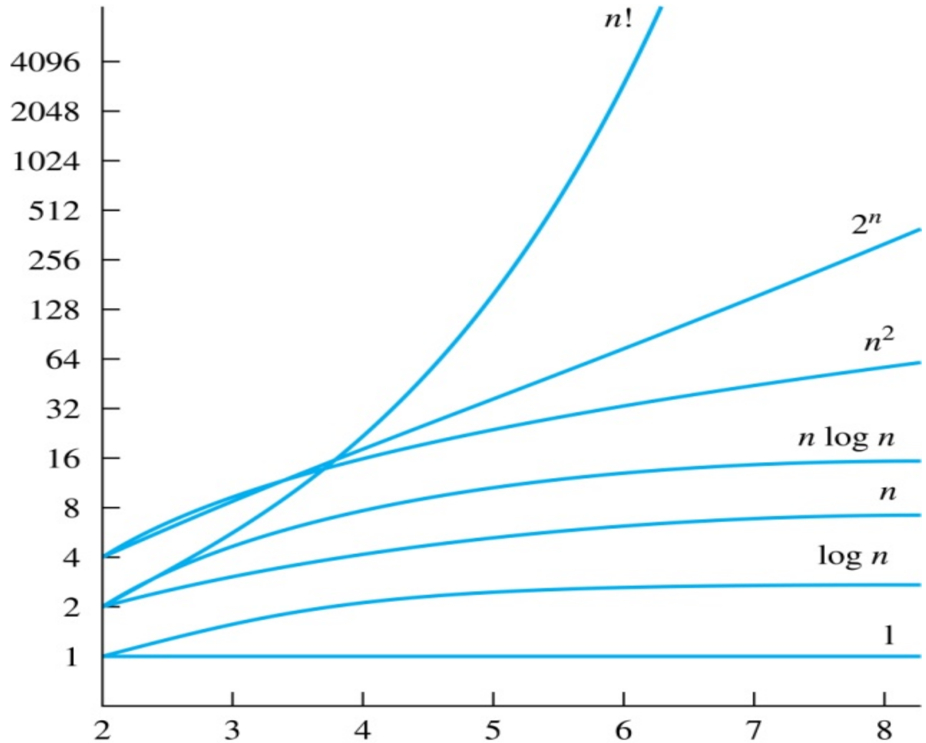
- **Example**: Use big-O notation to estimate log n!.
- **Solution**: Given that $n! \leq \; n^n$ (previous slide)

  then $\log(n!) \leq \; n \cdot \log(n)$.

  Hence, log(n!) is $O(n \cdot \log(n))$ taking C = 1 and k = 1.

Display of Growth of Functions

- Note the difference in behavior of functions as *n* gets larger



Useful Big-*O* Estimates Involving Logarithms, Powers, and Exponents

- If *d* > c > 1, then
  n^c is $O(n^d)$, but n^d is not $O(n^c)$.
- If b > 1 and *c* and *d* are positive, then
  $(\log_b n)^c$ is $O(n^d)$, but n^d is not $\mathrm{O}((\log_b n)^c)$.
- If b > 1 and *d* is positive, then
  n^d is $O(b^n)$, but b^n is not $O(n^d)$.
- If *c* > b > 1, then
  b^n is $O(c^n)$, but c^n is not $O(b^n)$.

Combinations of Functions

- If $f_1(x)$ is $O(\mathrm{g}_1(\mathrm{x}))$ and $f_2(x)$ is $O(\mathrm{g}_2(\mathrm{x}))$ then
  $(f_1 + f_2)(x)$ is $O(\max(|\mathrm{g}_1(\mathrm{x})|, |\mathrm{g}_2(\mathrm{x})|))$.
  - See next slide for proof
- If $f_1(x)$ and $f_2(x)$ are both $O(\mathrm{g}(x))$ then
  $(f_1 + f_2)(x)$ is $O(\mathrm{g}(x))$.
  - See text for argument
- If $f_1(x)$ is $O(\mathrm{g}_1(\mathrm{x}))$ and $f_2(x)$ is $O(\mathrm{g}_2(x))$ then
  $(f_1 f_2)(x)$ is $O(\mathrm{g}_1(\mathrm{x})\mathrm{g}_2(\mathrm{x}))$.
  - See text for argument

Combinations of Functions

- If $f_1(x)$ is $O(g_1(\mathrm{x}))$ and $f_2(x)$ is $O(g_2(\mathrm{x}))$ then

  $$(f_1 + f_2)(x) \text{ is } O(\max(|g_1(\mathrm{x})|,|g_2(\mathrm{x})|)).$$

  - By the definition of big-O notation, there are constants C_1, C_2, k_1, k_2 such that $|f_1(x)| \leq C_1|g_1(\mathrm{x})|$ when $\mathrm{x} > k_1$ and $f_2(x) \leq C_2|g_2(\mathrm{x})|$ when $x > k_2$.
  - $|(f_1 + f_2)(x)| = |f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)|$
    by the triangle inequality $|a + b| \leq |a| + |b|$
  - $|f_1(x)| + |f_2(x)| \leq C_1|g_1(\mathrm{x})| + C_2|g_2(\mathrm{x})| \leq C_1|g(\mathrm{x})| + C_2|g(\mathrm{x})|$
    where $g(x) = \max(|g_1(x)|,|g_2(x)|)$

    $= (C_1 + C_2)\ |g(x)| = C\ |g(x)|$ where $C = C_1 + C_2$
  - Therefore $|(f_1 + f_2)(x)| \leq C|g(x)|$ whenever $x > k$, where $k = \max(k_1,k_2)$.

Ordering Functions by Order of Growth

- Put the functions below in order so that each function is big-O of the next function on the list.
  - $f_1(n) = (1.5)^n$
  - $f_2(n) = 8n^3 + 17n^2 + 111$
  - $f_3(n) = (\log n)^2$
  - $f_4(n) = 2^n$
  - $f_5(n) = \log(\log n)$
  - $f_6(n) = n^2 (\log n)^3$
  - $f_7(n) = 2^n (n^2 + 1)$
  - $f_8(n) = n^3 + n(\log n)^2$
  - $f_9(n) = 10000$
  - $f_{10}(n) = n!$

Ordering Functions by Order of Growth

- We solve this exercise by successively finding the function that grows slowest among all those left on the list.
  - $f_9(n) = 10000$ (constant, does not increase with n)
  - $f_5(n) = \log(\log n)$ (grows slowest of all the others)
  - $f_3(n) = (\log n)^2$ (grows next slowest)
  - $f_6(n) = n^2 (\log n)^3$ (next largest, $(\log n)^3$ factor smaller than any power of n)
  - $f_2(n) = 8n^3+17n^2 +111$ (tied with the one below)
  - $f_8(n) = n^3+ n(\log n)^2$ (tied with the one above)
  - $f_1(n) = (1.5)^n$ (next largest, an exponential function)
  - $f_4(n) = 2^n$ (grows faster than one above since 2 > 1.5)
  - $f_7(n) = 2^n (n^2 +1)$ (grows faster than above because of the n^2 +1 factor)
  - $f_{10}(n) = n!$ ($n!$ grows faster than c^n for every c)

Big-Omega Notation

Ω is the upper case of the lower Greek letter ω.

- **Definition**: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers.
  We say that $f(x)$ is $\Omega(g(x))$ if there are constants *C* and *k* such that $|f(x)| \geq C|g(x)|$ when x > k.
  - We say that "*f*(*x*) is big-Omega of *g*(*x*)."
- Big-*O* gives an upper bound on the growth of a function, while Big-Omega gives a lower bound. Big-Omega tells us that a function grows at least as fast as another.
- $f(x)$ is $\Omega(g(x))$ if and only if $g(x)$ is $O(f(x))$. This follows from the definitions.

Big-Omega Notation

- **Example**: Show that $f(x) = 8x^3 + 5x^2 + 7$ is $\Omega(g(x))$ where $g(x) = x^3$.
- **Solution**: $f(x) = 8x^3 + 5x^2 + 7 \geq 8x^3$ for all positive real numbers *x*.
  - Is it also the case that $g(x) = x^3$ is $O(8x^3 + 5x^2 + 7)$?

Big-Theta Notation

Θ is the upper case of the lower Greek letter θ.

- **Definition**: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers. The function $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$.

- We say that "f is big-Theta of *g*(*x*)" and also that "*f*(*x*) is of *order g*(*x*)" and also that "*f*(*x*) and *g*(*x*) are of the *same order*."
- $f(x)$ is $\Theta(g(x))$ if and only if there exists constants C_1 , C_2 and *k* such that $C_1|g(x)| < |f(x)| < C_2|g(x)|$ if $x > k$. This follows from the definitions of big-*O* and big-Omega.

Big-Theta Notation

- **Example**: Show that the sum of the first n positive integers is $\Theta(n^2)$.
- **Solution**: Let $f(n) = 1 + 2 + \cdots + n$.
  - We have already shown that $f(n)$ is $O(n^2)$.
  - To show that $f(n)$ is $\Omega(n^2)$, we need a positive constant C such that $f(n) > Cn^2$ for sufficiently large n. Summing only the terms greater than $n/2$ we obtain the inequality

$$\begin{aligned} 1 + 2 + \cdots + n &\geq \lceil n/2 \rceil + (\lceil n/2 \rceil + 1) + \cdots + n \\ &\geq \lceil n/2 \rceil + \lceil n/2 \rceil + \cdots + \lceil n/2 \rceil \\ &= (\mathrm{n} - \lceil n/2 \rceil + 1)\lceil n/2 \rceil \\ &\geq (\mathrm{n}/2)(\mathrm{n}/2) = n^2/4 \end{aligned}$$

  - Taking $C = ¼$, $f(n) > Cn^2$ for all positive integers n. Hence, $f(n)$ is $\Omega(n^2)$, and we can conclude that $f(n)$ is $\Theta(n^2)$.

Big-Theta Notation

- **Example**: Show that $f(x) = 3x^2 + 8x \log x$ is $\Theta(x^2)$.
- **Solution**:
  - $3x^2 + 8x \log x \leq 11x^2$ for $x > 1$, since $0 \leq 8x \log x \leq 8x^2$.
  - Hence, $3x^2 + 8x \log x$ is $O(x^2)$.
  - x^2 is clearly $O(3x^2 + 8x \log x)$
  - Hence, $3x^2 + 8x \log x$ is $\Theta(x^2)$.

Big-Theta Notation

- When $f(x)$ is $\Theta(g(x))$, it must also be the case that $g(x)$ is $\Theta(f(x))$.
- Note that $f(x)$ is $\Theta(g(x))$ if and only if it is the case that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.
- Sometimes writers are careless and write as if big-*O* notation has the same meaning as big-Theta.

Big-Theta Estimates for Polynomials

- **Theorem**: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \cdots, a_n$ are real numbers with $a_n \neq 0$. Then $f(x)$ is of order x^n(or $\Theta(x^n)$).

- **Example**:
  - The polynomial $f(x) = 8x^5 + 5x^2 + 10$ is order of x^5 (or $\Theta(x^5)$).
  - The polynomial $f(x) = 8x^{199} + 7x^{100} + x^{99} + 5x^2 + 25$ is order of x^{199} (or $\Theta(x^{199})$).