4190.101
Discrete Mathematics

Chapter 3 Algorithms

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[Based on S. Srinivasa Rao’s Lecture Notes]

Complexity of Algorithms

Section 3.3

Section Summary

- Time Complexity
- Worst-Case Complexity
- Algorithmic Paradigms
- Understanding the Complexity of Algorithms

The Complexity of Algorithms

- Given an algorithm, how efficient is this algorithm for solving a problem given input of a particular size? To answer this question, we ask:
  - How much time does this algorithm use to solve a problem?
  - How much computer memory does this algorithm use to solve a problem?
- When we analyze the time the algorithm uses to solve the problem given input of a particular size, we are studying the *time complexity* of the algorithm.
- When we analyze the computer memory the algorithm uses to solve the problem given input of a particular size, we are studying the *space complexity* of the algorithm.

The Complexity of Algorithms

- In this course, we focus on time complexity. The space complexity of algorithms is studied in later courses.
- We will measure time complexity in terms of the number of operations an algorithm uses and we will use big-*O* and big-Theta notation to estimate the time complexity.
- We can use this analysis to see whether it is practical to use this algorithm to solve problems with input of a particular size. We can also compare the efficiency of different algorithms for solving the same problem.
- We ignore implementation details (including the data structures used and both the hardware and software platforms) because it is extremely complicated to consider them.

Time Complexity

- To analyze the time complexity of algorithms, we determine the number of operations, such as comparisons and arithmetic operations (addition, multiplication, etc.). We can estimate the time a computer may actually use to solve a problem using the amount of time required to do basic operations.
- We ignore minor details, such as the *housekeeping* aspects of the algorithm.
- We will focus on the *worst-case time* complexity of an algorithm. This provides an upper bound on the number of operations an algorithm uses to solve a problem with input of a particular size.
- It is usually much more difficult to determine the *average case time complexity* of an algorithm. This is the average number of operations an algorithm uses to solve a problem over all inputs of a particular size.

Complexity Analysis of Algorithms

- **Example**: Describe the time complexity of the algorithm for finding the maximum element in a finite sequence.

```
procedure max(a_1, a_2, …., a_n: integers)
   max := a_1
   for i := 2 to n
      if max < a_i then max := a_i
   return max{max is the largest element}
```

- **Solution**: Count the number of comparisons.
  - The $max < a_i$ comparison is made $n - 1$ times.
  - Each time i is incremented, a test is made to see if $i \leq n$.
  - One last comparison determines that $i > n$ (i.e. $i=n+1$).
  - Exactly $2(n - 1) + 1 = 2n - 1$ comparisons are made. Hence, the time complexity of the algorithm is $\Theta(n)$.

Worst-Case Complexity of Linear Search

- **Example**: Determine the time complexity of the linear search algorithm.

> **procedure** *linear search*(x:integer, a_1, a_2, …,a_n: distinct integers)
> $i := 1$
> **while** ($i \leq n$ and $x \neq a_i$)
> $i := i + 1$
> **if** $i \leq n$ **then** *location* $:= i$
> **else** *location* $:= 0$
> **return** *location* {*location* is the subscript of the term that equals x, or is 0 if x is not found}

- **Solution**: Count the number of comparisons.
  - At each step two comparisons are made; $i \leq n$ and $x \neq a_i$.
  - After the loop, one more $i \leq n$ comparison is made.
  - If x is on the list, $2i + 1$ comparisons are used.
  - If not, $2n + 1$ comparisons are made and then an additional comparison $i \leq n$ is used to exit the loop. So, in the worst case $2n + 2$ comparisons are made. Hence, the complexity is $\Theta(n)$.

Average-Case Complexity of Linear Search

- **Example**: Describe the average case performance of the linear search algorithm. (Although usually it is very difficult to determine average-case complexity, it is easy for linear search).
- **Solution**: Assume the element is in the list and that the possible positions are equally likely.
  - By the argument on the previous slide, if $x = a_i$, the number of comparisons is $2i + 1$. Thus, the average comparison is

$$\frac{3+5+7+\cdots+(2n+1)}{n} = \frac{2(1+2+3+\cdots+n)+n}{n} = \frac{2[\frac{n(n+1)}{2}]}{n} + 1 = n+2$$

- Hence, the average-case complexity of linear search is $\Theta(n)$.

Worst-Case Complexity of Binary Search

- **Example**: Describe the time complexity of binary search in terms of the number of comparisons used.

```
procedure binary search(x: integer, a_1,a_2,…, a_n: increasing integers)
 i := 1 {i is the left endpoint of interval}
 j := n {j is right endpoint of interval}
 while i < j
       m := ⌊(i + j)/2⌋
       if x > a_m then i := m + 1
       else j := m
 if x = a_i then location := i
 else location := 0
 return location{location is the subscript i of the term a_i  equal to x, or 0 if x is not found}
```

- **Solution**: Assume (for simplicity) $n = 2^k$ elements. Note that $k = \log n$.
  - Two comparisons are made at each stage; $i < j$, and $x > a_m$.
  - At the first iteration the size of the list is 2^k and after the first iteration it is 2^{k-1}. Then 2^{k-2} and so on until the size of the list is $2^1 = 2$.
  - At the last step, a comparison tells us that the size of the list is the size is $2^0 = 1$ and x is compared with the single remaining element a_i.
  - Hence, at most $2k + 2 = 2 \log n + 2$ comparisons are made.
  - Therefore, the time complexity is $\Theta(\log n)$, better than linear search.

Worst-Case Complexity of Bubble Sort

- **Example**: What is the worst-case complexity of bubble sort in terms of the number of comparisons made?

> **procedure** *bubblesort*($a_1,\ldots,a_n$: real numbers with $n \geq 2$)
>   **for** i := 1 to $n-1$
>   **for** j := 1 to $n-i$
>   **if** $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}
> {$a_1,\ldots, a_n$ is now in increasing order}

- **Solution**: A sequence of n−1 passes is made through the list. On each pass n − i comparisons are made.

$$(n-1)+(n-2)+\cdots+2+1=\frac{n(n-1)}{2}$$

- The worst-case complexity of bubble sort is $\Theta(n^2)$ since

$$\frac{n(n-1)}{2}=\frac{1}{2}n^2-\frac{1}{2}n \quad .$$

Worst-Case Complexity of Insertion Sort

- **Example**: What is the worst-case complexity of insertion sort in terms of the number of comparisons made?

- **Solution**: The total number of comparisons are:.

$$2 + 3 + \cdots + n = \frac{n(n-1)}{2} - 1$$

- Therefore the complexity is $\Theta(n^2)$.

```
procedure insertion sort(a_1,…,a_n:
        real numbers with n ≥ 2)
    for j := 2 to n
       i := 1
       while a_j > a_i
           i := i + 1
        m := a_j
        for k := 0 to j − i − 1
            a_{j-k} := a_{j-k-1}
         a_i := m
```

Matrix Multiplication Algorithm

- The definition for matrix multiplication can be expressed as an algorithm; **C** = **A B** where **C** is an $m \times n$ matrix that is the product of the $m \times k$ matrix **A** and the $k \times n$ matrix **B**.
- This algorithm carries out matrix multiplication based on its definition.

```
procedure matrix multiplication(A,B: matrices)
  for i := 1 to m
    for j := 1 to n
       c_ij := 0
        for q := 1 to k
           c_ij := c_ij + a_iq b_qj
return C{C = [c_ij] is the product of A and B}
```

$\mathbf{A} = [a_{ij}]$ is a $m \times k$ matrix

$\mathbf{B} = [b_{ij}]$ is a $k \times n$ matrix

Complexity of Matrix Multiplication

- **Example**: How many additions of integers and multiplications of integers are used by the matrix multiplication algorithm to multiply two $n \times n$ matrices.
- **Solution**: There are n^2 entries in the product. Finding each entry requires n multiplications and $n - 1$ additions. Hence, n^3 multiplications and $n^2(n - 1)$ additions are used.
- Hence, the complexity of matrix multiplication is $O(n^3)$.
- c.f. The fastest algorithm is $O(n^{\sqrt{7}})$ ($\sqrt{7} = 2.64575 \ldots$)

Boolean Product Algorithm

- The definition of Boolean product of zero-one matrices can also be converted to an algorithm.

procedure *Boolean product*(**A**,**B**: zero-one matrices)
    for $i := 1$ to m
        for $j := 1$ to n
            $c_{ij} := 0$
            for $q := 1$ to k
                $c_{ij} := c_{ij} \vee (a_{iq} \wedge b_{qj})$
return C{**C = [**c_{ij}**]** is the Boolean product of **A** and **B**}

Complexity of Boolean Product Algorithm

- **Example**: How many bit operations are used to find $\mathbf{A} \odot \mathbf{B}$, where A and B are $n \times n$ zero-one matrices?
- **Solution**: There are n^2 entries in the $\mathbf{A} \odot \mathbf{B}$. A total of n ORs and n ANDs are used to find each entry. Hence, each entry takes $2n$ bit operations. A total of $2n^3$ operations are used.
- Therefore the complexity is $O(n^3)$

Matrix-Chain Multiplication

- How should the *matrix-chain* $\mathbf{A}_1\mathbf{A}_2\cdot\cdot\cdot\mathbf{A}_n$ be computed using the fewest multiplications of integers, where $\mathbf{A}_1$, $\mathbf{A}_2,\cdot\cdot\cdot, \mathbf{A}_n$ are $m_1\times m_2$, $m_2\times m_3$, $\cdot\cdot\cdot$ $m_n\times m_{n+1}$ integer matrices. Matrix multiplication is associative (exercise in Section 2.6).
- **Example**: In which order should the integer matrices $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ - where $\mathbf{A}_1$ is 30×20 , $\mathbf{A}_2$ 20×40, $\mathbf{A}_3$ 40×10 - be multiplied to use the least number of multiplications.

Matrix-Chain Multiplication

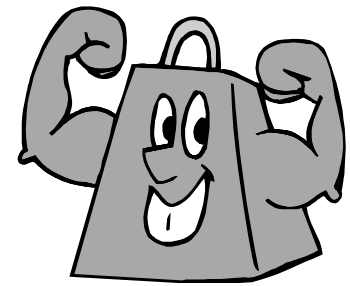
- **Example**: In which order should the integer matrices $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ - where $\mathbf{A}_1$ is 30×20 , $\mathbf{A}_2$ 20×40, $\mathbf{A}_3$ 40×10 - be multiplied to use the least number of multiplications.
- **Solution**: There are two possible ways to compute $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$.
  - $\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3)$: $\mathbf{A}_2\mathbf{A}_3$ takes $20\cdot 40\cdot 10 = 8{,}000$ multiplications. Then multiplying $\mathbf{A}_1$ by the 20×10 matrix $\mathbf{A}_2\mathbf{A}_3$ takes $30\cdot 20\cdot 10 = 6{,}000$ multiplications. So the total number is $8000 + 6000 = 14{,}000$.
  - $(\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_3$: $\mathbf{A}_1\mathbf{A}_2$ takes $30\cdot 20\cdot 40 = 24{,}000$ multiplications. Then multiplying the 30×40 matrix $\mathbf{A}_1\mathbf{A}_2$ by $\mathbf{A}_3$ takes $30\cdot 40\cdot 10 = 12{,}000$ multiplications. So the total number is $24{,}000 + 12{,}000 = 36{,}000$.
- So the first method is better.

An efficient algorithm for finding the best order for matrix-chain multiplication can be based on the algorithmic paradigm known as *dynamic programming*. (see Ex. 57 in Section 8.1)

Algorithmic Paradigms

- An *algorithmic paradigm* is a general approach based on a particular concept for constructing algorithms to solve a variety of problems.
  - Greedy algorithms were introduced in Section 3.1.
  - We discuss brute-force algorithms in this section.
  - We will see divide-and-conquer algorithms (Chapter 8), dynamic programming (Chapter 8), backtracking (Chapter 11), and probabilistic algorithms (Chapter 7). There are many other paradigms that you may see in later courses.

Brute-Force Algorithms



- A *brute-force* algorithm is solved in the most straightforward manner, without taking advantage of any ideas that can make the algorithm more efficient.
- Brute-force algorithms we have previously seen are sequential search, bubble sort, and insertion sort.

Computing the Closest Pair of Points by Brute-Force

- **Example**: Construct a brute-force algorithm for finding the closest pair of points in a set of *n* points in the plane and provide a worst-case estimate of the number of arithmetic operations.
- **Solution**: Recall that the distance between $(x_i, y_i)$and (x_j, y_j) is $\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$. A brute-force algorithm simply computes the distance between all pairs of points and picks the pair with the smallest distance.
- **Note**: There is no need to compute the square root, since the square of the distance between two points is smallest when the distance is smallest.

Continued →

Computing the Closest Pair of Points by Brute-Force

- Algorithm for finding the closest pair in a set of n points.

procedure *closest pair*$((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$: x_i, y_i real numbers)
$min = \infty$
  for $i := 1$ to n
    for $j := 1$ to i
      if $(x_j - x_i)^2 + (y_j - y_i)^2 < min$
        then min $:= (x_j - x_i)^2 + (y_j - y_i)^2$
          closest pair $:= (x_i, y_i), (x_j, y_j)$
return *closest pair*

- The algorithm loops through $n(n-1)/2$ pairs of points, computes the value $(x_j - x_i)^2 + (y_j - y_i)^2$ and compares it with the minimum, etc. So, the algorithm uses $\Theta(n^2)$ arithmetic and comparison operations.
- We will develop an algorithm with $O(n \log n)$ worst-case complexity in Section 8.3.

Understanding the Complexity of Algorithms

TABLE 1 Commonly Used Terminology for the Complexity of Algorithms.

Complexity	*Terminology*
$\Theta(1)$	Constant complexity
$\Theta(\log n)$	Logarithmic complexity
$\Theta(n)$	Linear complexity
$\Theta(n \log n)$	Linearithmic complexity
$\Theta(n^b)$	Polynomial complexity
$\Theta(b^n)$, where $b > 1$	Exponential complexity
$\Theta(n!)$	Factorial complexity

Understanding the Complexity of Algorithms

TABLE 2 **The Computer Time Used by Algorithms.**

Problem Size	*Bit Operations Used*					
n	$\log n$	n	$n \log n$	n^2	2^n	$n!$
10	3×10^{-11} s	10^{-10} s	3×10^{-10} s	10^{-9} s	10^{-8} s	3×10^{-7} s
10^2	7×10^{-11} s	10^{-9} s	7×10^{-9} s	10^{-7} s	4×10^{11} yr	*
10^3	1.0×10^{-10} s	10^{-8} s	1×10^{-7} s	10^{-5} s	*	*
10^4	1.3×10^{-10} s	10^{-7} s	1×10^{-6} s	10^{-3} s	*	*
10^5	1.7×10^{-10} s	10^{-6} s	2×10^{-5} s	0.1 s	*	*
10^6	2×10^{-10} s	10^{-5} s	2×10^{-4} s	0.17 min	*	*

Times of more than 10^{100} years are indicated with an *.

Complexity of Problems

- *Tractable Problem*: There exists a polynomial time algorithm to solve this problem. These problems are said to belong to the *Class P*.
- *Intractable Problem*: There does not exist a polynomial time algorithm to solve this problem
- *Unsolvable Problem* : No algorithm exists to solve this problem, e.g., halting problem.
- *Class NP*: Solution can be checked in polynomial time.
- *NP Hard class:* A decision problem H is NP-hard when for every problem L in class NP, there is a polynomial-time reduction from L to H
- *NP Complete Class*: If a problem is both in NP and NP-hard

P Versus NP Problem

Stephen Cook
(Born 1939)

- The *P versus NP problem* asks whether the class P = NP? Are there problems whose solutions can be checked in polynomial time, and can be solved in polynomial time?
  - Just because no one has found a polynomial time algorithm is different from showing that the problem cannot be solved by a polynomial time algorithm.
- If a polynomial time algorithm for any of the problems in the NP complete class were found, then that algorithm could be used to obtain a polynomial time algorithm for every problem in the NP complete class.
  - Satisfiability (in Section 1.3) is an NP complete problem.
- It is generally believed that P≠NP since no one has been able to find a polynomial time algorithm for any of the problems in the NP complete class.
- The problem of P versus NP remains one of the most famous unsolved problems in mathematics (including theoretical computer science). The Clay Mathematics Institute has offered a prize of $1,000,000 for a solution.