4190.101
Discrete Mathematics

Chapter 9 Relations

Gunhee Kim

[Based on S. Srinivasa Rao's Lecture Notes]

Chapter Summary

- Relations and Their Properties
- *n*-ary Relations and Their Applications (*not currently included in overheads*)
- Representing Relations
- Closures of Relations (*not currently included in overheads*)
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

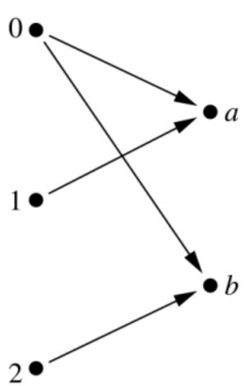
Section 9.1

Section Summary

- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Binary Relations

- **Definition:** A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.
- **Example**:
  - Let $A = \{0,1,2\}$ and $B = \{a,b\}$
  - $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from *A* to *B*.
  - We can represent relations from a set *A* to a set *B* graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where elements of *B* are related to each element of *A.*

Binary Relation on a Set

- **Definition:** A binary relation *R on a set A* is a subset of $A \times A$ or a relation from A to A.
- **Example**:
  - Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
  - Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a$ divides $b\}$ are

    (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Binary Relation on a Set (*cont.*)

- **Definition:** How many relations are there on a set A?
- **Solution**: Because a relation on A is the same thing as a subset of A $\times$ A, we count the subsets of A × A. Since A × A has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of A × A. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relation on a Set (*cont.*)

- **Example:** Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

- Which of these relations contain each of the pairs (1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?
- **Solution**: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6: (1,2) is in R_1 and R_6: (2,1) is in R_2, R_5, and R_6: (1, −1) is in R_2, R_3, and R_6 : (2,2) is in R_1, R_3, and R_4.

Reflexive Relations

- **Definition:** R is *reflexive* iff $(a, a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if $\forall x[x \in U \rightarrow (x, x) \in R]$

- **Example:** The following relations on the integers are reflexive:

  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
  $R_4 = \{(a,b) \mid a = b\}$.

  – The following relations are not reflexive:

  $R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),
  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),
  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is, the empty relation on an empty set is reflexive!

Symmetric Relations

- **Definition:** R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if $\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$

- **Example:** The following relations on the integers are symmetric:

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

  $R_4 = \{(a,b) \mid a = b\}$,

  $R_6 = \{(a,b) \mid a + b \leq 3\}$.

  – The following are not symmetric:

  $R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),

  $R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),

  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

- **Definition:** A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if $\forall x \forall y\, [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$

- **Example:** The following relations on the integers are antisymmetric:

  $R_1 = \{(a,b) \mid a \leq b\}$,

  $R_2 = \{(a,b) \mid a > b\}$,

  $R_4 = \{(a,b) \mid a = b\}$,

  $R_5 = \{(a,b) \mid a = b + 1\}$.

  For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.

  – The following relations are not antisymmetric:

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

  (note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

- **Definition:** A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if
  $\forall x \forall y \, \forall z[(x,y) \in R \land (y,z) \in \mathrm{R} \longrightarrow (x,z) \in R\,]$

- **Example:** The following relations on the integers are transitive:
  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_2 = \{(a,b) \mid a > b\}$,
  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
  $R_4 = \{(a,b) \mid a = b\}$.

  For every integer, $a \leq b$ and $b \leq c$, then $b \leq c$.

  - The following are not transitive:

  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (4,3) and (3,2) belong to R_5, but not (3,3)),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

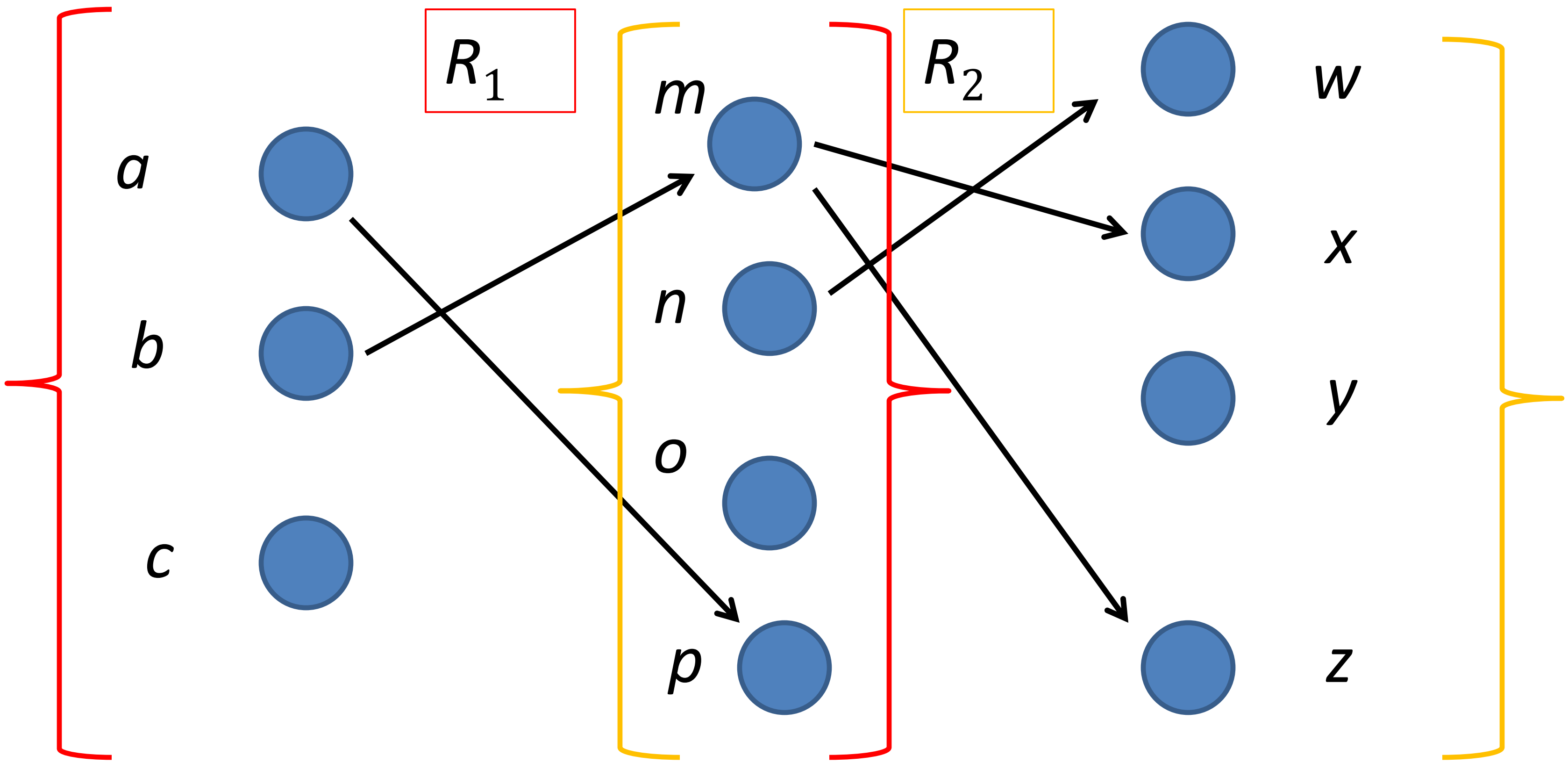
$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

- **Definition:** Suppose
  - R_1 is a relation from a set A to a set B.
  - R_2 is a relation from B to a set C.
- Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where
  - If (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b,x),(b,z)\}$$

Powers of a Relation

- **Definition:** Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:
  – Basis Step: $R^1 = R$
  – Inductive Step: $R^{n+1} = R^n \circ R$

  (*see the slides for Section* 9.3 *for further insights*)

- The powers of a transitive relation are subsets of the relation. This is established by the following theorem:
- **Theorem 1:** The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots$.

  (*see the text for a proof via mathematical induction*)

Representing Relations

Section 9.3

Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose *R* is a relation from *A* = $\{a_1, a_2, \ldots, a_m\}$ to *B* = $\{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When *A* = *B*, we use the same ordering.
- The relation *R* is represented by the matrix $m_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- The matrix representing *R* has a 1 as its (*i*,*j*) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

- **Example 1**: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?
- **Solution:** Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Examples of Representing Relations Using Matrices

- **Example 2**: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

- **Solution:** Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

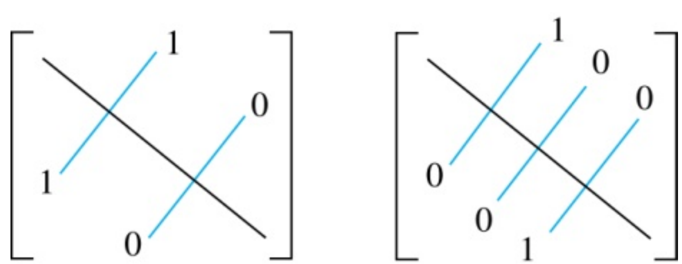
$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$

Matrices of Relations on Sets

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$.
- R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

- **Example 3**: Suppose that the relation *R* on a set is represented by the matrix

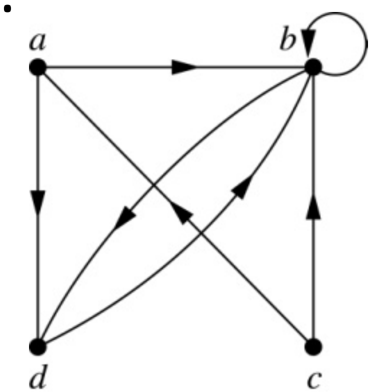
$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

  is *R* reflexive, symmetric, and/or antisymmetric?
- **Solution**: Because all the diagonal elements are equal to 1, *R* is reflexive. Because M_R is symmetric, *R* is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

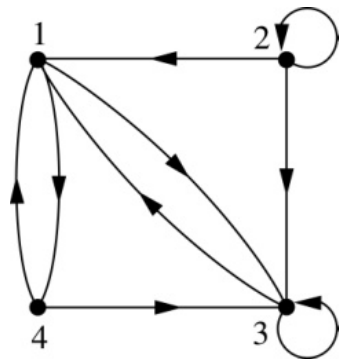
- **Definition**: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.
  - An edge of the form (a,a) is called a *loop*.

- **Example 7**: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



Examples of Digraphs Representing Relations

- **Example 8**: What are the ordered pairs in the relation represented by this directed graph?



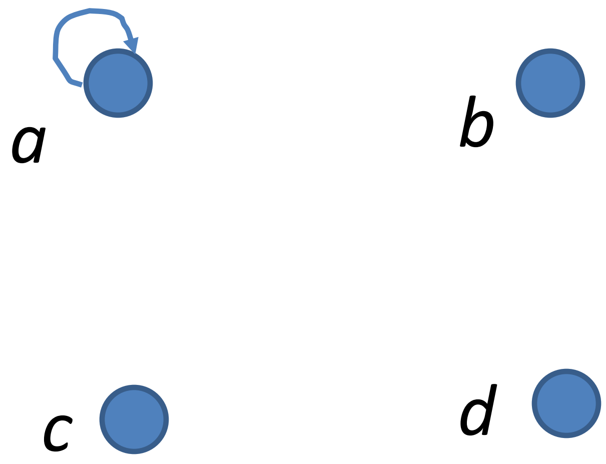
- **Example 7**: The ordered pairs in the relation are (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3).

Determining which Properties a Relation has from its Digraph

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

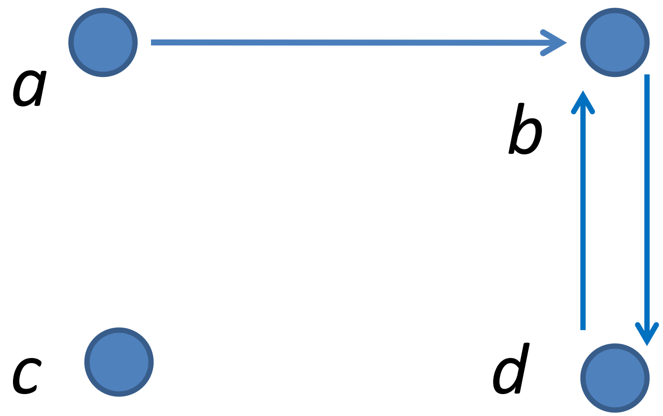
Determining which Properties a Relation has from its Digraph – Example 1

- *Reflexivity*? No, not every vertex has a loop
- *Symmetric?* Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric?* Yes (trivially), there is no edge from one vertex to another
- *Transitive?* Yes, (trivially) since there is no edge from one vertex to another



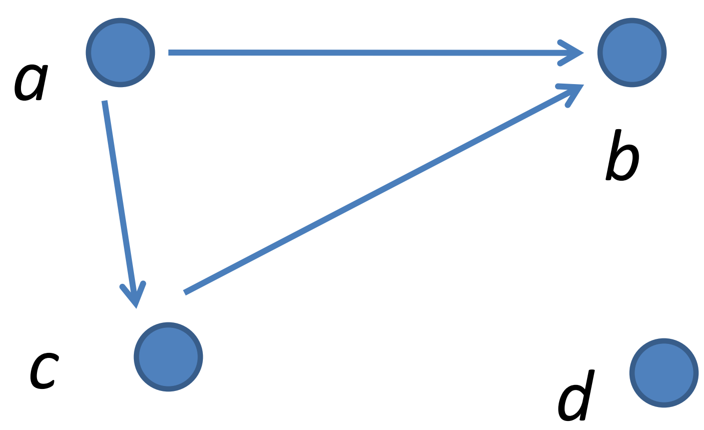
Determining which Properties a Relation has from its Digraph – Example 2

- *Reflexivity*? No, there are no loops
- *Symmetric?* No, there is an edge from *a* to *b*, but not from *b* to *a*
- *Antisymmetric?* No, there is an edge from *d* to *b* and *b* to *d*
- *Transitive?* No, there are edges from *a* to *c* and from *c* to *b*, but there is no edge from *a* to *d*



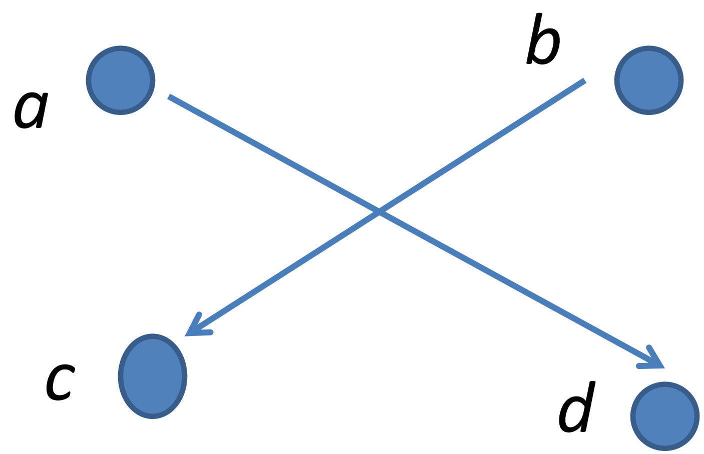
Determining which Properties a Relation has from its Digraph – Example 3

- *Reflexivity*? No, there are no loops
- *Symmetric?* No, for example, there is no edge from *c* to *a*
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive? Yes*.

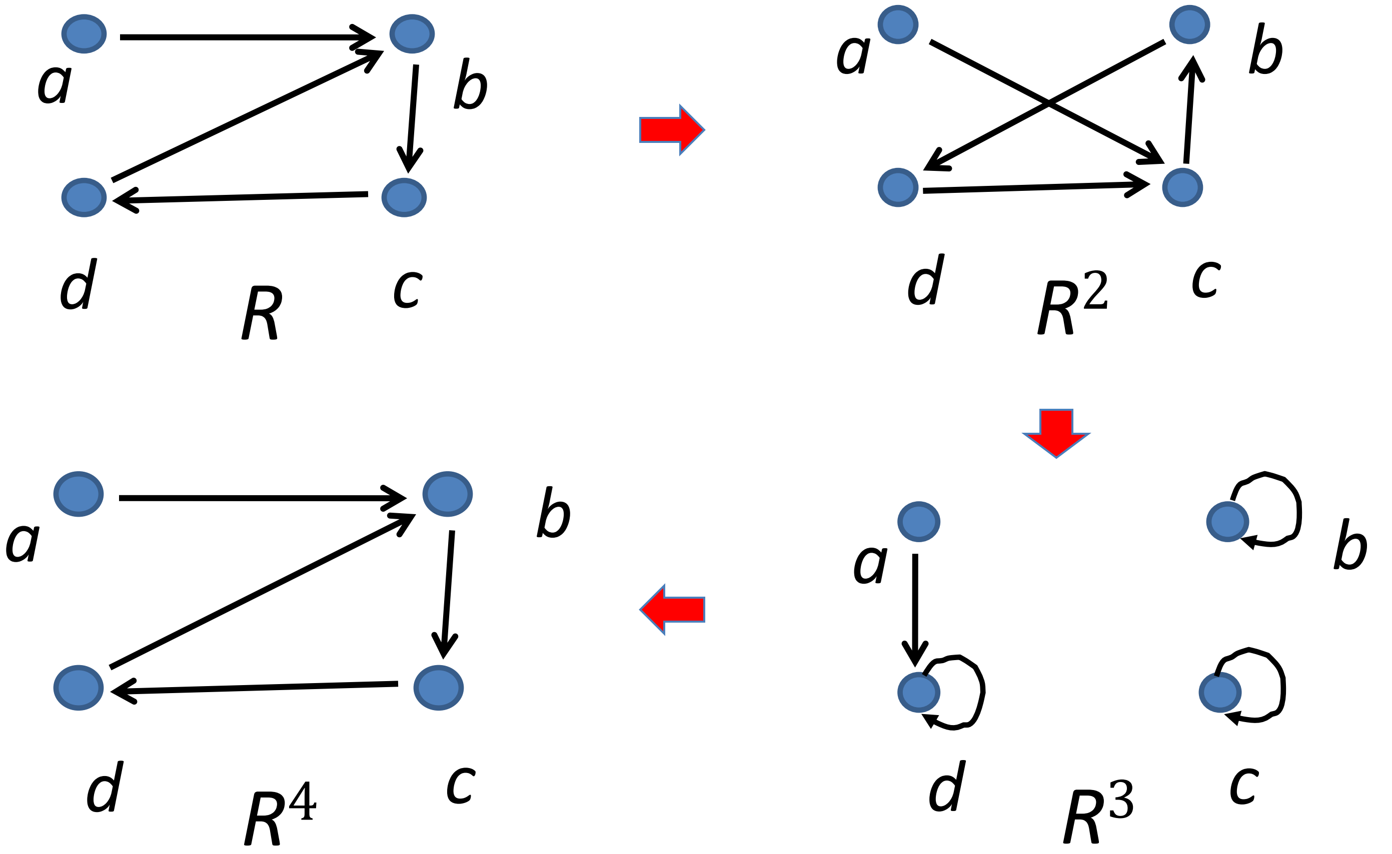


Determining which Properties a Relation has from its Digraph – Example 4

- *Reflexivity*? No, there are no loops
- *Symmetric?* No, for example, there is no edge from *d* to *a*
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?* Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins



Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length *n* from *x* to *y* in *R* (following the direction of the arrows).

Equivalence Relations

Section 9.5

Section Summary

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions

Equivalence Relations

- **Definition 1**: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.
- **Definition 2**: Two elements a and b that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

- **Example**: Suppose that R is the relation on the set of strings of English letters such that $a\ R\ b$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?
- **Solution**: Show that all of the properties of an equivalence relation hold.
  - *Reflexivity*: Because $l(a) = l(a)$, it follows that $a\ R\ a$ for all strings a.
  - *Symmetry*: Suppose that $a\ R\ b$. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and $b\ R\ a$.
  - *Transitivity*: Suppose that a R b and $b\ R\ c$. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(c)$ also holds and $a\ R\ c$.

Congruence Modulo m

- **Example**: Let m be an integer with $m>1$. Show that the relation

  $R = \{(a,b) \mid a \equiv b \pmod{m}\}$

  is an equivalence relation on the set of integers.
- **Solution**: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a-b$.
  - *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
  - *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (- k) m$, so $b \equiv a \pmod{m}$.
  - *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

    $$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

    Therefore, $a \equiv c \pmod{m}$.

Divides

- **Example**: Show that the “divides” relation on the set of positive integers is not an equivalence relation.
- **Solution**: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, “divides” is not an equivalence relation.
  - *Reflexivity*: $a|a$ for all a.
  - *Not Symmetric*: For example, $2|4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
  - *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

- **Definition 3**: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.
- Note that $[a]_R = \{s \mid (a,s) \in R\}$.
- When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.
- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$. For example,

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

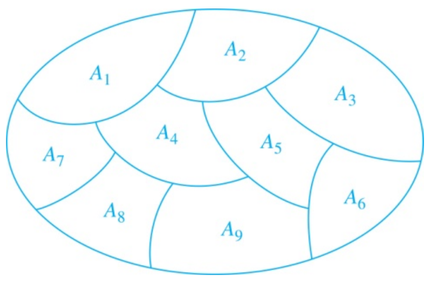
Equivalence Classes and Partitions

- **Theorem 1**: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:
  - (i) $a\ R\ b$
  - (ii) $[a] = [b]$
  - (iii) $[a] \cap [b] \neq \emptyset$
- **Proof:** We show that (i) implies (ii). Assume that $a\ R\ b$. Now suppose that $c \in [a]$. Then $a\ R\ c$. Because $a\ R\ b$ and R is symmetric, $b\ R\ a$. Because R is transitive and $b\ R\ a$ and $a\ R\ c$, it follows that $b\ R\ c$. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

(*see text for proof that* (ii) *implies* (iii) *and* (iii) *implies* (i))

Partition of a Set

- **Definition**: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if
  - $A_i \neq \emptyset$ for $i \in I$,
  - $A_i \cap A_j = \emptyset$ when $i \neq j$,
  - and $\bigcup_{i \in I} A_i = S$



A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set (*continued*)

- **Theorem 2**: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.
- **Proof**: We have already shown the first part of the theorem.
- For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.
  - *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
  - *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
  - *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

Partial Orderings

Section 9.6

Section Summary

- Partial Orderings and Partially-ordered Sets
- Lexicographic Orderings
- Hasse Diagrams
- Lattices (*not currently in overheads*)
- Topological Sorting (*not currently in overheads*)

Partial Orderings

- **Definition 1:** A relation *R* on a set S is called a *partial ordering,* or *partial order,* if it is reflexive, antisymmetric, and transitive.
  - A set together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by (*S*, *R*). Members of *S* are called *elements* of the poset.

Partial Orderings (*continued*)

- **Example 1:** Show that the “greater than or equal” relation ($\geq$) is a partial ordering on the set of integers.
  - *Reflexivity*: $a \geq a$ for every integer a.
  - *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
  - *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

- **Example 2:** Show that the the divisibility relation (|) is a partial ordering on the set of integers.
  - *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
  - *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
  - *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.
- $(\boldsymbol{Z}^+, \mid)$ is a poset.

Partial Orderings (*continued*)

- **Example 3:** Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.
  - *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
  - *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

- **Definition 2:** The elements a and b of a poset $(S,\preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called i*ncomparable*.

> The symbol $\preccurlyeq$ is used to denote the relation in any poset.

- **Definition 3:** If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain.*
- **Definition 4:** $(S,\preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Lexicographic Order

- **Definition:** Given two posets $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1,b_2), that is,

$$(a_1, a_2) \prec (b_1,b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

- **Example:** Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.
  - *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
  - *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.