

**4190.101**

# **Discrete Mathematics**

## Chapter 1 The Foundations: Logic and Proofs

Gunhee Kim

# Outline

- Propositional Logic
  - The Language of Propositions
  - Applications
  - Logical Equivalences
- Predicate Logic
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
- Proofs
  - Rules of Inference
  - Proof Methods
  - Proof Strategy

# Propositional Logic

## Section 1.1

# Section Summary

- Propositions
- Connectives
  - Negation
  - Conjunction
  - Disjunction
  - Implication: contrapositive, inverse, converse
  - Biconditional
- Truth Tables

# Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
  - a) The Moon is made of green cheese.
  - b) Trenton is the capital of New Jersey.
  - c) Toronto is the capital of Canada.
  - d)  $1 + 0 = 1$
  - e)  $0 + 0 = 2$
- Examples that are not propositions.
  - a) Sit down!
  - b) What time is it?
  - c)  $x + 1 = 2$
  - d)  $x + y = z$

# Propositional Logic

- Constructing propositions
  - Propositional Variables:  $p, q, r, s, \dots$
  - The proposition that is always true is denoted by **T**
  - The proposition that is always false is denoted by **F**
  - Compound propositions: constructed from logical connectives and other propositions
    - Negation  $\neg$
    - Conjunction  $\wedge$
    - Disjunction  $\vee$
    - Implication  $\rightarrow$
    - Biconditional  $\leftrightarrow$

# Compound Propositions: Negation

- The *negation* of a proposition  $p$  is denoted by  $\neg p$  and has this truth table:

$p$	$\neg p$
T	F
F	T

- Example: If  $p$  denotes “The earth is round.”, then  $\neg p$  denotes “It is not the case that the earth is round,” or more simply “The earth is not round.”

# Conjunction

- The *conjunction* of propositions  $p$  and  $q$  is denoted by  $p \wedge q$  and has this truth table:

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Example: If  $p$  denotes “I am at home.” and  $q$  denotes “It is raining.” then  $p \wedge q$  denotes “I am at home and it is raining.”



# Disjunction

- The *disjunction* of propositions  $p$  and  $q$  is denoted by  $p \vee q$  and has this truth table:

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Example:** If  $p$  denotes “I am at home.” and  $q$  denotes “It is raining.” then  $p \vee q$  denotes “I am at home or it is raining.”

# The Connective Or in English

- In English “or” has two distinct meanings.
  - **Inclusive Or** - “Students who have taken CS202 or Math120 can take this class”: Students need to have taken one of the prerequisites, but may have taken both.
  - This is the meaning of disjunction. For  $p \vee q$  to be true, either one or both of  $p$  and  $q$  must be true.
  - **Exclusive Or** - “Soup or salad comes with this entrée”: We do not expect to be able to get both soup and salad.
  - This is the meaning of Exclusive Or (Xor). In  $p \oplus q$ , one of  $p$  and  $q$  must be true, but not both. The truth table for  $\oplus$  is:

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

# Implication

- If  $p$  and  $q$  are propositions, then  $p \rightarrow q$  is a *conditional statement* or *implication* which is read as “if  $p$ , then  $q$ ” and has this truth table:

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Example: If  $p$  denotes “I am at home.” and  $q$  denotes “It is raining.” then  $p \rightarrow q$  denotes “If I am at home then it is raining.”
- In  $p \rightarrow q$ ,  $p$  is the *hypothesis* (*antecedent* or *premise*) and  $q$  is the *conclusion* (or *consequence*).

# Understanding Implication

- These implications ( $p:F \rightarrow q:F$ ) are perfectly fine, but would not be used in ordinary English.
  - “If the moon is made of green cheese, then I have more money than Bill Gates. ”
  - “If the moon is made of green cheese then I’m on welfare.”
  - “If  $1 + 1 = 3$ , then your grandma wears combat boots.”

# Understanding Implication (cont)

- One way to view the logical conditional is to think of an obligation or contract.
  - “If I am elected, then I will lower taxes.”
  - “If you get 100% on the final, then you will get an A.”
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge.
  - This corresponds to the case where  $p$  is true and  $q$  is false.

# Different Ways of Expressing $p \rightarrow q$

- if  $p$ , then  $q$
- if  $p$ ,  $q$
- $p$  implies  $q$
- $p$  only if  $q$
- $q$  if  $p$
- $q$  unless  $\neg p$
- $q$  when  $p$
- $q$  whenever  $p$
- $p$  is sufficient for  $q$
- $q$  follows from  $p$
- $q$  is necessary for  $p$
- A necessary condition for  $p$  is  $q$
- A sufficient condition for  $q$  is  $p$

# Converse, Contrapositive, and Inverse

- From  $p \rightarrow q$  we can form new conditional statements .
  - $q \rightarrow p$  is the **converse** of  $p \rightarrow q$
  - $\neg q \rightarrow \neg p$  is the **contrapositive** of  $p \rightarrow q$
  - $\neg p \rightarrow \neg q$  is the **inverse** of  $p \rightarrow q$
- Example: Find the converse, inverse, and contrapositive of “Raining is a sufficient condition for my not going to town.”
- Solution:
  - **Converse**: If I do not go to town, then it is raining.
  - **Inverse**: If it is not raining, then I will go to town.
  - **Contrapositive**: If I go to town, then it is not raining.

# Biconditional

- If  $p$  and  $q$  are propositions, then we can form the *biconditional* proposition  $p \leftrightarrow q$ , read as “ $p$  if and only if  $q$ .”
- The biconditional  $p \leftrightarrow q$  denotes the proposition with this truth table:

$p$	$q$	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If  $p$  denotes “I am at home.” and  $q$  denotes “It is raining.” then  $p \leftrightarrow q$  denotes “I am at home if and only if it is raining.”



# Expressing the Biconditional

- Some alternative ways “ $p$  if and only if  $q$ ” is expressed in English:
  - $p$  is **necessary and sufficient** for  $q$
  - **if  $p$  then  $q$**  , and **conversely**
  - $p$  **iff**  $q$

# Truth Tables for Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Include the atomic propositions
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up

# Example Truth Table

- Construct a truth table for  $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

# Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value.
- Example: Show using a truth table that the *implication* is equivalent to its *contrapositive*.
- Solution:

$p$	$q$	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

# Using Truth Table for Non-Equivalence

- Example: Show using truth tables that neither the converse nor inverse of an implication is equivalent to the implication.
- Solution:

$p$	$q$	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

# Problem

- How many rows are there in a truth table with  $n$  propositional variables?
- Solution:  $2^n$  (will see how to do this in Chapter 6)
- Note that this means that with  $n$  propositional variables, we can construct  $2^n$  distinct (i.e., not equivalent) propositions.

# Precedence of Logical Operators

- $p \vee q \rightarrow \neg r$  is equivalent to  $(p \vee q) \rightarrow \neg r$
- If the intended meaning is  $p \vee (q \rightarrow \neg r)$  then parentheses must be used.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

# Applications of Propositional Logic

Section 1.2



# Applications of Propositional Logic

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Puzzles
- Logic Circuits
- AI Diagnosis Method (Optional)

# Translating English Sentences

- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives
- “If I go to Harry’s or to the country, I will not go shopping.”
  - $p$ : I go to Harry’s
  - $q$ : I go to the country.
  - $r$ : I will go shopping.

If  $p$  or  $q$  then not  $r$ .

$$(p \vee q) \rightarrow \neg r$$

# Example

- Problem: Translate the following sentence into propositional logic:
  - “You can access the Internet from campus only if you are a computer science major or you are not a freshman.”
- One Solution
  - $a$  : “You can access the internet from campus”
  - $c$  : “You are a computer science major”
  - $f$  : “You are a freshman.”

$$a \rightarrow (c \vee \neg f)$$

# System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.
- Example: Express in propositional logic:
  - “The automated reply cannot be sent when the file system is full”
- One possible solution:
  - $p$  denotes “The automated reply can be sent”
  - $q$  denotes “The file system is full.”

$$q \rightarrow \neg p$$

# Consistent System Specifications

- Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.
- Exercise: Are these specifications consistent?
  - “The diagnostic message is stored in the buffer or it is retransmitted.”
  - “The diagnostic message is not stored in the buffer.”
  - “If the diagnostic message is stored in the buffer, then it is retransmitted.”

# Consistent System Specifications

- Solution:
  - p: “The diagnostic message is stored in the buffer.”
  - q: “The diagnostic message is retransmitted”
  - The specification can be written as:  $p \vee q, \neg p, p \rightarrow q$
  - When p is false and q is true all three statements are true.  
So the specification is consistent.
- What if “The diagnostic message is not retransmitted” is added.”
  - Now we are adding  $\neg q$  and there is no satisfying assignment.
  - The specification is not consistent.

# Logic Puzzles



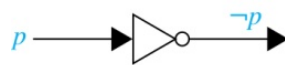
Raymond  
Smullyan  
(Born 1919)

- An island has two kinds of inhabitants, *knight*s, who always tell the truth, and *knave*s, who always lie.
- You go to the island and meet A and B.
  - A says “B is a knight.”
  - B says “The two of us are of opposite types.”
- Example: What are the types of A and B?
- Solution
  - Let  $p$ : A is a knight,  $q$  : B is a knight.
  - Then  $\neg p$ : A is a knave,  $\neg q$ : B is a knave.
  - If A is a knight, then  $p$  is true. Since knights tell the truth,  $q$  must also be true. Then  $(p \wedge \neg q) \vee (\neg p \wedge q)$  that B says would have to be true, but it is not. So, A is not a knight and therefore  $\neg p$  must be true.
  - If A is a knave, then B must not be a knight since knaves always lie. So, then both  $p$  and  $\neg q$  hold since both are knaves.

# Logic Circuits

## (Studied in depth in Chapter 12)

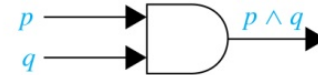
- Electronic circuits; each input/output signal can be viewed as a 0 (False) or 1 (True).
- Circuits are constructed from three basic circuits called gates.



Inverter

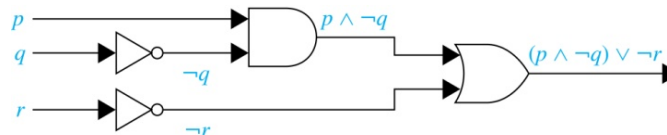


OR gate



AND gate

- The inverter (**NOT gate**) takes an input bit and produces its negation.
  - The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
  - The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.
- More complicated digital circuits can be constructed by combining these basic circuits to produce the desired output For example:





# Propositional Equivalences

## Section 1.3

# Section Summary

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- Normal Forms (*optional, covered in exercises in text*)
  - Disjunctive Normal Form
  - Conjunctive Normal Form
- Propositional Satisfiability
  - Sudoku Example

# Tautologies, Contradictions, and Contingencies

- A *tautology* is a proposition which is always true.
  - Example:  $p \vee \neg p$
- A *contradiction* is a proposition which is always false.
  - Example:  $p \wedge \neg p$
- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as  $p$

$p$	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

# Logically Equivalent

- Two compound propositions  $p$  and  $q$  are logically equivalent if  $p \leftrightarrow q$  is a tautology.
- We write this as  $p \Leftrightarrow q$  or as  $p \equiv q$  where  $p$  and  $q$  are compound propositions.
- Two compound propositions  $p$  and  $q$  are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table shows  $\neg p \vee q$  is equivalent to  $p \rightarrow q$ .

$p$	$q$	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

# De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$



Augustus De Morgan

1806-1871

- This truth table shows that De Morgan's Second Law holds.

$p$	$q$	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

# Key Logical Equivalences

- Identity Laws:  $p \wedge T \equiv p$  ,  $p \vee F \equiv p$
- Domination Laws:  $p \vee T \equiv T$  ,  $p \wedge F \equiv F$
- Idempotent laws:  $p \vee p \equiv p$  ,  $p \wedge p \equiv p$
- Double Negation Law:  $\neg(\neg p) \equiv p$
- Negation Laws:  $p \vee \neg p \equiv T$  ,  $p \wedge \neg p \equiv F$

# Key Logical Equivalences (*cont*)

- Commutative Laws:  $p \vee q \equiv q \vee p$  ,  $p \wedge q \equiv q \wedge p$
- Associative Laws:  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$   
 $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- Distributive Laws:  $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$   
 $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
- Absorption Laws:  $p \vee (p \wedge q) \equiv p$     $p \wedge (p \vee q) \equiv p$

# More Logical Equivalences

**TABLE 7** Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

**TABLE 8** Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$



# Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.
- To prove that  $A \equiv B$ , we produce a series of equivalences beginning with  $A$  and ending with  $B$ .

$$\begin{array}{c} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{array}$$

# Equivalence Proofs

- Example: Show that  $\neg(p \vee (\neg p \wedge q))$   
is logically equivalent to  $\neg p \wedge \neg q$

- Solution:

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	$\equiv$	$\neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	$\equiv$	$F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
	$\equiv$	$(\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
	$\equiv$	$(\neg p \wedge \neg q)$	by the identity law for <b>F</b>

# Equivalence Proofs

- Example: Show that  $(p \wedge q) \rightarrow (p \vee q)$  is a tautology.

- Solution:

$$\begin{aligned}(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\ &\equiv (\neg p \vee p) \vee (\cancel{\neg p} \vee \neg q) && \text{by associative and commutative laws} \\ &&& \text{laws for disjunction} \\ &\equiv T \vee T && \text{by truth tables} \\ &\equiv T && \text{by the domination law}\end{aligned}$$

# Disjunctive Normal Form (*optional*)

- A propositional formula is in *disjunctive normal form* if it consists of a disjunction of  $(1, \dots, n)$  disjuncts where each disjunct consists of a conjunction of  $(1, \dots, m)$  atomic formulas or the negation of an atomic formula.
  - Yes  $(p \wedge \neg q) \vee (\neg p \wedge q)$
  - No  $p \wedge (p \vee q)$
- Every compound proposition can be put in disjunctive normal form

# Disjunctive Normal Form (*optional*)

- Example: Find the Disjunctive Normal Form (DNF) of false.

$$(p \vee q) \rightarrow \neg r$$

- Solution: This proposition is true when  $r$  is false or when both  $p$  and  $q$  are false.

$$\neg(p \vee q) \vee \neg r$$

$$(\neg p \wedge \neg q) \vee \neg r$$

# Conjunctive Normal Form (*optional*)

- A compound proposition is in *Conjunctive Normal Form* (CNF) if it is a conjunction of disjunctions.
- Every proposition can be put in an equivalent CNF.
- Conjunctive Normal Form (CNF) can be obtained by eliminating implications, moving negation inwards and using the distributive and associative laws.
- Important in resolution theorem proving used in artificial Intelligence (AI).
- A compound proposition can be put in conjunctive normal form through repeated application of the logical equivalences covered earlier.

# Disjunctive Normal Form (*optional*)

- Example: Put the following into CNF:

$$\neg(p \rightarrow q) \vee (r \rightarrow p)$$

- Solution:

1. Eliminate implication signs:

$$\neg(\neg p \vee q) \vee (\neg r \vee p)$$

2. Move negation inwards; eliminate double negation:

$$(p \wedge \neg q) \vee (\neg r \vee p)$$

3. Convert to CNF using associative/distributive laws

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

# Propositional Satisfiability

- A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that make it true.
- When no such assignments exist, the compound proposition is *unsatisfiable*.
- A compound proposition is unsatisfiable if and only if its negation is a tautology.



# Example on Propositional Satisfiability

- Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

- Solution: Satisfiable. Assign **T** to  $p$ ,  $q$ , and  $r$ .

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

- Solution: Satisfiable. Assign **T** to  $p$  and **F** to  $q$ .

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

- Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

# Notation

$\bigvee_{j=1}^n p_j$  is used for  $p_1 \vee p_2 \vee \dots \vee p_n$

$\bigwedge_{j=1}^n p_j$  is used for  $p_1 \wedge p_2 \wedge \dots \wedge p_n$

# Sudoku

- A **Sudoku puzzle** is represented by a  $9 \times 9$  grid made up of nine  $3 \times 3$  subgrids, known as **blocks**. Some of the 81 cells of the puzzle are assigned one of the numbers 1,2, ..., 9.
- The puzzle is solved by assigning numbers to each blank cell so that every row, column and block contains each of the nine possible numbers.
- Example

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

# Encoding as a Satisfiability Problem

- Let  $p(i,j,n)$  denote the proposition that is true when the number  $n$  is in the cell in the  $i$ -th row and the  $j$ -th column.
- There are  $9 \times 9 \times 9 = 729$  such propositions.
- In the sample puzzle  $p(5,1,6)$  is true, but  $p(5,j,6)$  is false for  $j = 2,3,\dots,9$

# Encoding (cont)

- For each cell with a given value, assert  $p(i,j,n)$ , when the cell in row  $i$  and column  $j$  has the given value.
- Assert that every row contains every number.

$$\bigwedge_{i=1}^9 \bigwedge_{n=1}^9 \bigvee_{j=1}^9 p(i, j, n)$$

- Assert that every column contains every number.

$$\bigwedge_{j=1}^9 \bigwedge_{n=1}^9 \bigvee_{i=1}^9 p(i, j, n)$$

# Encoding (cont)

- Assert that each of the 3 x 3 blocks contain every number.

$$\bigwedge_{r=0}^2 \bigwedge_{s=0}^2 \bigwedge_{n=1}^9 \bigwedge_{i=1}^3 \bigvee_{j=1}^3 p(3r + i, 3s + j, n)$$

(this is tricky - ideas from chapter 4 help)

- Assert that no cell contains more than one number. Take the conjunction over all values of  $n, n', i$ , and  $j$ , where each variable ranges from 1 to 9 and  $n \neq n'$ ,

$$p(i, j, n) \rightarrow \neg p(i, j, n')$$

# Solving Satisfiability Problems

- To solve a Sudoku puzzle, we need to find an assignment of truth values to the 729 variables of the form  $p(i,j,n)$  that makes the conjunction of the assertions true. Those variables that are assigned T yield a solution to the puzzle.
- A truth table can always be used to determine the satisfiability of a compound proposition. But this is too complex even for modern computers for large problems.
- There has been much work on developing efficient methods for solving satisfiability problems as many practical problems can be translated into satisfiability problems.