4190.101

Discrete Mathematics

Chapter 6 Counting

Gunhee Kim

[Based on S. Srinivasa Rao's Lecture Notes]

Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

- **Definition**: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.
- **Example**: Let $S = \{1,2,3\}$.
  - The ordered arrangement 3,1,2 is a permutation of S.
  - The ordered arrangement 3,2 is a 2-permutation of S.
- The number of r-permutations of a set with n elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

Permutations

- **Theorem 1**: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

  $$P(n, r) = n(n-1)(n-2) \cdots (n-r+1)$$

  r-permutations of a set with n distinct elements
- **Proof**: Use the product rule. The first element can be chosen in n ways. The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.
- Note that $P(n,0) = 1$, since there is only one way to order zero elements.
- **Corollary 1**: If n and r are integers with $1 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

- **Example**: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?
- **Solution**:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems by Counting Permutations (*continued*)

- **Example**: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
- **Solution**: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

  $$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$$

  If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

- **Example**: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?
- **Solution**: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

- **Definition**: An *r-combination* of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.
- The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section* 6.4.)
- **Example**: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.
- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

- **Theorem 2**: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

- **Proof**: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

- **Example**: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?
- **Solution**: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

- **Corollary 2**: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.
- **Proof**: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} \ .$$

Hence, $C(n, r) = C(n, n - r)$.

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

- **Definition 1**: A *combinatorial proof* of an identity is a proof that uses one of the following methods.
  - A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
  - A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements. ◀
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. ◀

Combinations

- **Example**: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.
- **Solution**: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

- **Example**: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?
- **Solution**: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775\ .$$

Binomial Coefficients and Identities

Section 6.4

Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle
- Other Identities Involving Binomial Coefficients (*not currently included in overheads*)

Powers of Binomial Expressions

- **Definition**: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)
- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
  - $(x + y)\,(x + y)\,(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.

Powers of Binomial Expressions

- Terms of the form x^3, x^2y, $x\ y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two. There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

- **Binomial Theorem**: Let *x* and *y* be variables, and *n* is a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- **Proof**: We use combinatorial reasoning. The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ x's from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◀

Using the Binomial Theorem

- **Example**: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?
- **Solution**: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

- Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

- **Corollary 1**: With $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$
- **Proof** (*using binomial theorem*): With x = 1 and y = 1, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

◀

- **Proof** (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements. Therefore the total is

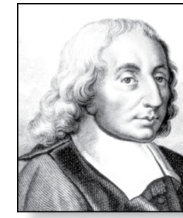
$$\sum_{k=0}^{n} \binom{n}{k}.$$

- Since, we know that a set with n elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

◀

Pascal's Identity



Blaise Pascal (1623-1662)

- **Pascal's Identity**: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

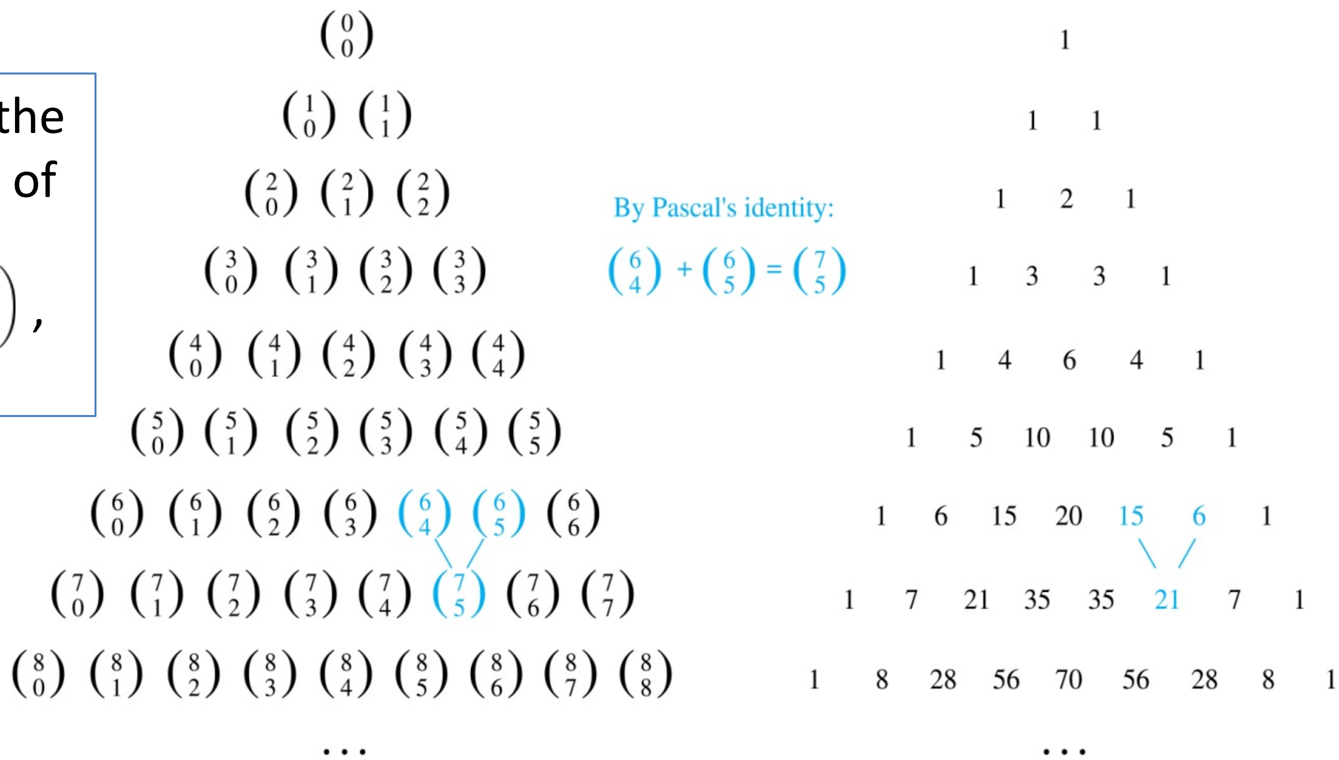
- **Proof** (*combinatorial*): Let T be a set where $|T| = n + 1$, a $\in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:
  - contains a with $k - 1$ other elements, or
  - contains k elements of S and not a.
- There are
  - $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
  - $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.
- Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \blacktriangleleft$$

See Exercise 19 *for an algebraic proof.*

Pascal's Triangle

The *n*-th row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0, 1, \ldots, n$.



(a) (b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Generalized Permutations and Combinations

Section 6.5

Section Summary

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes

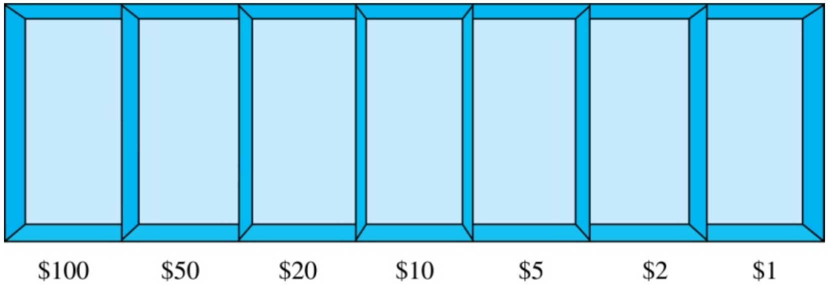
Permutations with Repetition

- **Theorem 1**: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.
- **Proof**: There are *n* ways to select an element of the set for each of the *r* positions in the *r*-permutation when repetition is allowed. Hence, by the product rule there are n^r *r*-permutations with repetition. ◀

- **Example**: How many strings of length *r* can be formed from the uppercase letters of the English alphabet?
- **Solution**: The number of such strings is 26^r, which is the number of *r*-permutations of a set with 26 elements.

Combinations with Repetition

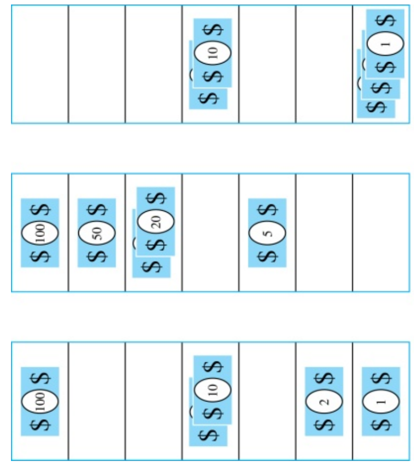
- **Example**: How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?
- **Solution**: Place the selected bills in the appropriate position of a cash box illustrated below:



continued →

Combinations with Repetition

- Some possible ways of placing the five bills:



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- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

- **Theorem 2**: The number of *r*-combinations from a set with *n* elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

- **Proof**: Each *r*-combination of a set with *n* elements with repetition allowed can be represented by a list of $n - 1$ bars and *r* stars. The bars mark the *n* cells containing a star for each time the *i*-th element of the set occurs in the combination.
- The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the *r* positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars. This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars. ◀

Combinations with Repetition

- **Example**: How many solutions does the equation

  $x_1 + x_2 + x_3 = 11$

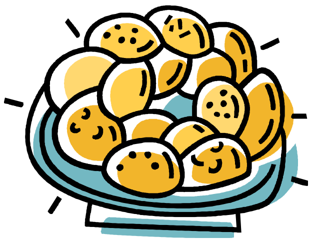
  have, where x_1, x_2 and x_3 are nonnegative integers?
- **Solution**: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.
- By Theorem 2 it follows that there are $C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$ solutions.

Combinations with Repetition

- **Example**: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?
- **Solution**: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.



Permutations and Combinations

- Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 **Combinations and Permutations With and Without Repetition.**

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

- **Example**: How many different strings can be made by reordering the letters of the word *SUCCESS*.
- **Solution**: There are seven possible positions for the three Ss, two Cs, one U, and one E.
  - The three Ss can be placed in *C*(7,3) different ways, leaving four positions free.
  - The two Cs can be placed in *C*(4,2) different ways, leaving two positions free.
  - The U can be placed in *C*(2,1) different ways, leaving one position free.
  - The E can be placed in *C*(1,1) way.

  By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem. →

Permutations with Indistinguishable Objects

- **Theorem 3**: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:
$$\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$
- **Proof**: By the product rule the total number of permutations is:
  $C(n, n_1)\ C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:
  - The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
  - Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
  - Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\,\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\ \cdots\ \frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}\ . \qquad ◀$$

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes

- *Distinguishable objects* and *distinguishable boxes*.
  - **Example**: How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?
  - **Solution**: 1st player: C(52,5) ways, 2nd player: C(47,5), 3rd player: C(42,5), 4th player: C(37,5)

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!\,5!} \cdot \frac{47!}{42!\,5!} \cdot \frac{42!}{37!\,5!} \cdot \frac{37!}{32!\,5!}$$
$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}$$

  - The number of ways to distribute *n* distinguishable objects into *k* distinguishable boxes.

$$\frac{n!}{n_1!n_2!\cdots n_k!} \; .$$

Distributing Objects into Boxes

- *Indistinguishable objects* and *distinguishable boxes*.
  - **Example**: How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?
  - **Solution**: It equals the number of 10-combinations from a set with eight elements when repetition is allowed.
  - Use Theorem 2*:* r-combination with n elements with repetition C(n + r – 1, r) = C(n + r – 1, n – 1).

$$C(8 + 10 - 1, 10) = C(17, 10) = \frac{17!}{10!7!} = 19{,}448. \quad ◀$$

Distributing Objects into Boxes

- *Distinguishable objects* and *indistinguishable boxes*.
  - **Example**: How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?
  - **Solution**: There is no simple closed formula for the number of ways to distribute *n* distinguishable objects into *j* indistinguishable boxes.
  - Call the four employees as A,B,C,D. There are five different groups: (i) all four to one office, (ii) three to one and the other to one, (iii) two to one, the other two to one, (iv) two to one, the other two to two for each.
  - The number of cases for each group is (i) 1 (because the office is indistinguishable), (ii) 4 ({{A,B,C},{D}}, {{A,B,D},{C}}, {{A,C,D},{B}} , {{B,C,D},{A}}}, (iii) 3, (iv) 6 ways (see the text).
  - So, the total number is 14.
  - See the text for a formula involving *Stirling numbers of the second kind*.

Distributing Objects into Boxes

- *Indistinguishable objects* and *indistinguishable boxes*.
  - **Example**: How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?
  - **Solution**: No simple closed formula exists for this number.
  - The ways we can pack the books are
    (i) 6, (ii) 5, 1, (iii) 4, 2, (iv) 4, 1, 1, (v) 3, 3, (vi) 3, 2, 1
    (vii) 3, 1, 1, 1, (viii) 2, 2, 2, (ix) 2, 2, 1, 1.
  - There are nine ways to pack the books. ◀