4190.101
Discrete Mathematics

Chapter 2 Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Gunhee Kim

[Based on S. Srinivasa Rao's Lecture Notes]

Cardinality of Sets

Section 2.5

Section Summary

- Cardinality
- Countable Sets
- Computability

Cardinality

- **Definition**: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

  if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.
- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality

- **Definition**: A set that is either finite or has the same cardinality as some subset of natural numbers (**N**) is called *countable*. A set that is not countable is *uncountable*.
  - The set of real numbers **R** is an uncountable set.
- When an infinite set is countable (*countably infinite*), its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that *S* has cardinality "aleph null."

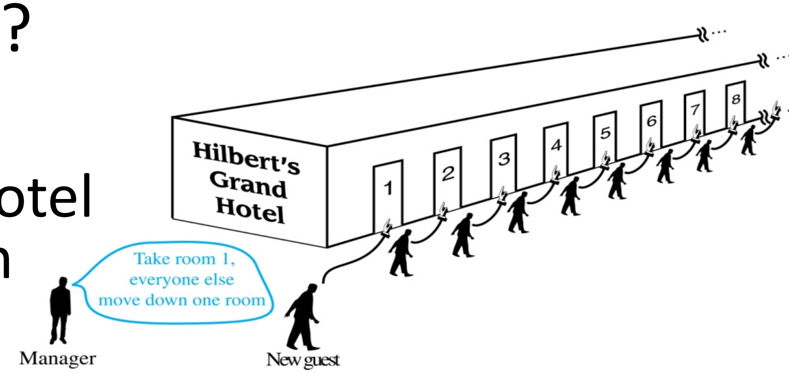
Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Hilbert’s Grand Hotel

David Hilbert

- The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?
- **Explanation**
  - Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on.



  - When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in R2 to R3, and in general the guest in Room n to Room $n + 1$, for all positive integers n.
  - This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.

Showing that a Set is Countable

- **Example 1:** Show that the set of positive even integers *E* is countable set.
- **Solution**: Let $f(x) = 2x$.

$$\begin{array}{ccccccc}
\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \ldots\ldots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{8} & \mathbf{10} & \mathbf{12} & \ldots\ldots
\end{array}$$

- Then f is a bijection from **N** to *E* since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

Showing that a Set is Countable

- **Example 2:** Show that the set of integers **Z** is countable.
- **Solution**: can list in a sequence:
  0, 1, − 1, 2, − 2, 3, − 3 ,...........
- Or can define a bijection from **N** to **Z**:
  - When n is even: $f(n) = n/2$
  - When n is odd: $f(\mathrm{n}) = -(n-1)/2$

The Positive Rational Numbers are Countable

- **Definition**: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - √2 is not a rational number.
- **Example 3**: Show that the positive rational numbers are countable.
- **Solution**: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1 , r_2 , r_3 ,\ldots$$

The next slide shows how this is done. $\rightarrow$

The Positive Rational Numbers are Countable

First row $q = 1$.
Second row $q = 2$.
etc.

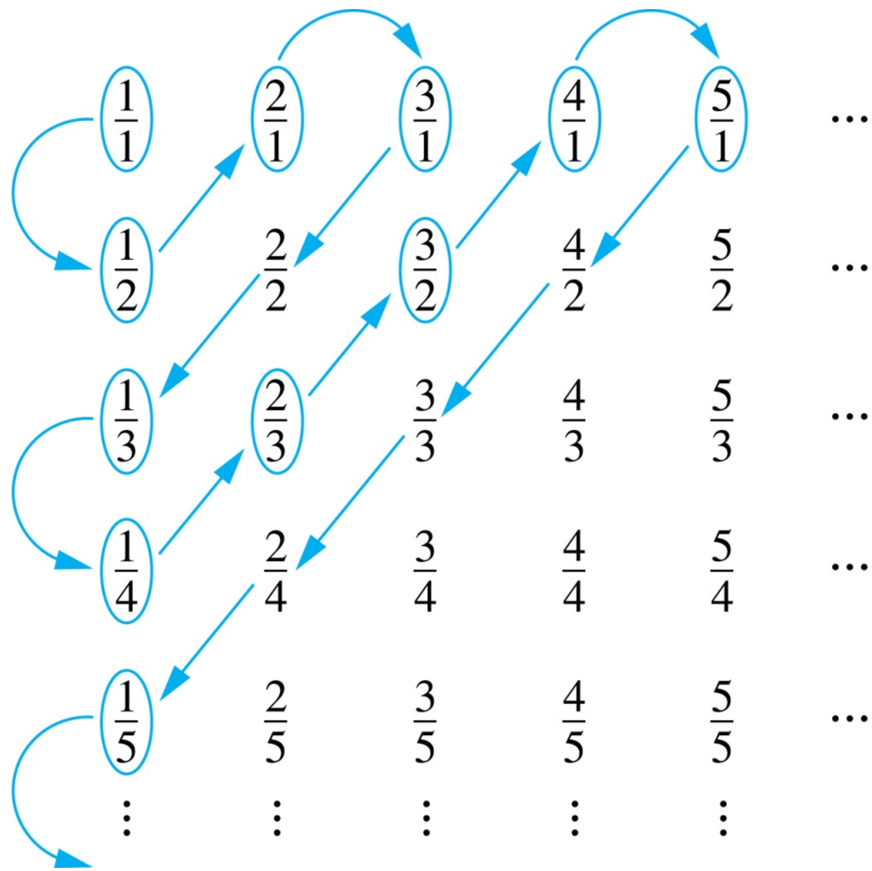
Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

1, ½, 2, 3, 1/3,1/4, 2/3, ….

Terms not circled are not listed because they repeat previously listed terms



Strings

- **Example 4**: Show that the set of finite strings *S* over the lowercase letters is countably infinite.
- **Solution**: Show that the strings can be listed in a sequence. First list
  1. All the strings of length 0 in alphabetical order.
  2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
  3. Then all the strings of length 2 in lexicographic order.
  4. And so on.
- This implies a bijection from **N** to *S* and hence it is a countably infinite set.

◀

The set of all Java programs is countable

- **Example 5**: Show that the set of all Java programs is countable.
- **Solution**: Let S be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:
  - Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program).
  - If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
  - We move on to the next string.
- In this way we construct an implied bijection from **N** to the set of Java programs. Hence, the set of Java programs is countable.

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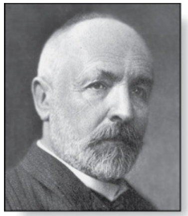
The Real Numbers are Uncountable



Georg Cantor
(1845-1918)

- Show that the set of real numbers is uncountable.
- **Solution**: The method is called the Cantor diagonalization argument, and is a proof by contradiction.
  1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the textbook).
  2. The real numbers btw 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots, r_n, \ldots$.
  3. Let the decimal representation of this listing be

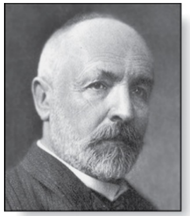
$$
\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\
r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\
&\ldots
\end{aligned}
$$

  4. However, given this list we can now construct a new real number r_q between 0 and 1 that does not appear on this list,

$$r_q = 0.d_{q1}d_{q2}d_{q3}\ldots d_{qn}\ldots$$

with $d_{q1} \neq d_{11}, d_{q2} \neq d_{22}, d_{q3} \neq d_{33}, \ldots$

The Real Numbers are Uncountable



Georg Cantor (1845-1918)

- Show that the set of real numbers is uncountable.
- **Solution**: The method is called the Cantor diagonalization argument, and is a proof by contradiction.

5. With this construction r_q differs from any real number on the list in at least one position.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable. ◀

Matrices

Section 2.6

Section Summary

- Definition of a Matrix
- Matrix Arithmetic
- Transposes and Powers of Arithmetic
- Zero-One matrices

Matrices

- Matrices are useful discrete structures that can be used in many ways. For example, they are used to:
  - Describe certain types of functions known as linear transformations.
  - Express which vertices of a graph are connected by edges (see Chapter 10).
- In later chapters, we will see matrices used to build models of:
  - Transportation systems.
  - Communication networks.
- Algorithms based on matrix models will be presented in later chapters.
- Here we cover the aspect of matrix arithmetic that will be needed later.

Matrices

- **Definition**: A *matrix* is a rectangular array of numbers. A matrix with *m* rows and *n* columns is called an $m \times n$ matrix.
  - The plural of matrix is *matrices*.
  - A matrix with the same number of rows as columns is called *square*.
  - Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

3×2 matrix $\quad \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$

Notation

- Let *m* and *n* be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The *i*-th row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$. The *j*-th column of **A** is the $m \times 1$ *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}$$

- The (*i,j*)-th *element* or *entry* of **A** is the element a_{ij}. We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (*i,j*)-th element equal to a_{ij}.

Matrix Arithmetic: Addition

- **Definition**: Let **A** = [a_{ij}] and **B** = [b_{ij}] be $m \times n$ matrices. The sum of **A** and **B**, denoted by **A** + **B**, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)-th element. In other words, **A** + **B** = [$a_{ij} + b_{ij}$].
- **Example**:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

- Note that matrices of different sizes can not be added.

Matrix Multiplication

- **Definition**: Let **A** be an $n \times k$ matrix and **B** be a $k \times n$ matrix. The *product* of **A** and **B**, denoted by **AB**, is the $\times$ m n matrix that has its (i,j)-th element equal to the sum of the products of the corresponding elements from the i-th row of **A** and the j-th column of **B**. In other words, if **AB** = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1\mathrm{j}} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.
- **Example**:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

- The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.

Illustration of Matrix Multiplication

- The Product of **A** = [a$_{ij}$] and **B** = [b$_{ij}$]

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & a_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & \cdot & & \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & c_{ij} & \cdot \\ \cdot & \cdot & & \cdot \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix Multiplication is not Commutative

- **Example**: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

  Does **AB** = **BA**?

- **Solution:**

$$\mathbf{AB} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{AB} \neq \mathbf{BA}$$

Identity Matrix and Powers of Matrices

- **Definition**: The *identity matrix of order n* is the $m \times n$ matrix $\mathbf{I}_n = [I_{ij}]$, where $I_{ij} = 1$ if $i = j$ and $I_{ij} = 0$ if $i \neq j$.

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$
when $\mathbf{A}$ is an $m \times n$ matrix

- Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n; \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{\text{r times}}$$

Transpose of Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an $m \times n$ matrix. The *transpose* of **A**, denoted by $\mathbf{A}^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of **A**.

- If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1,2,\ldots,n$ and $j = 1,2, \ldots,m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Transposes of Matrices

- **Definition**: A square matrix **A** is called symmetric if **A** = $\mathbf{A}^t$. Thus **A** = $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

$$\text{The matrix } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is square.}$$

  – Symmetric matrices do not change when their rows and columns are interchanged.

Zero-One Matrices

- **Definition**: A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*. (These will be used in Chapters 9 and 10).
- Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

Zero-One Matrices

- **Definition**: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be an $m \times n$ zero-one matrices.
  - The *join* of **A** and **B** is the zero-one matrix with (i,j)-th entry $a_{ij} \vee b_{ij}$. The *join* of **A** and **B** is denoted by **A** $\vee$ **B**.
  - The meet of of **A** and **B** is the zero-one matrix with (i,j)-th entry $a_{ij} \wedge b_{ij}$. The *meet* of **A** and **B** is denoted by **A** $\wedge$ **B**.

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Joins and Meets of Zero-One Matrices

- **Example**: Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

- **Solution**: The join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an $m \times k$ zero-one matrix and **B** = $[b_{ij}]$ be a $k \times n$ zero-one matrix. The *Boolean product* of **A** and **B**, denoted by **A** $\odot$ **B**, is the $m \times n$ zero-one matrix with(i,j)-th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

- **Example**: Find the Boolean product of **A** and **B,** where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Continued on next slide →

Boolean Product of Zero-One Matrices

- **Solution**: The Boolean product **A** $\odot$ **B** is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices

- **Definition**: Let **A** be a square zero-one matrix and let *r* be a positive integer. The *r*-th Boolean power of **A** is the Boolean product of *r* factors of **A**, denoted by $\mathbf{A}^{[r]}$. Hence,

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{r \text{ times}}.$$

- We define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

(The Boolean product is well defined because the Boolean product of matrices is associative.)

Boolean Product of Zero-One Matrices

- **Example**: Let

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

  Find $\mathbf{A}^{[n]}$ for all positive integers n.

- **Solution**:

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[\mathbf{n}]} = \mathbf{A}^{\mathbf{5}} \quad \text{for all positive integers } n \text{ with } n \geq 5.$$