4190.101
Discrete Mathematics

Chapter 9 Relations

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[Based on S. Srinivasa Rao's Lecture Notes]

Chapter Summary

- Relations and Their Properties
- *n*-ary Relations and Their Applications (*not currently included in overheads*)
- Representing Relations
- Closures of Relations (*not currently included in overheads*)
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

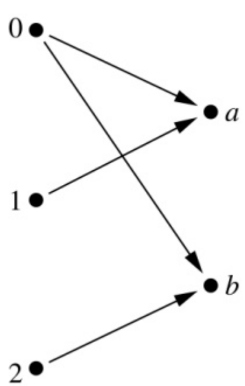
Section 9.1

Section Summary

- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Binary Relations

- **Definition:** A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.
- **Example**:
  - Let $A = \{0,1,2\}$ and $B = \{a,b\}$
  - $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from *A* to *B*.
  - We can represent relations from a set *A* to a set *B* graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where elements of *B* are related to each element of *A*.

Binary Relation on a Set

- **Definition:** A binary relation *R on a set A* is a subset of $A \times A$ or a relation from A to A.
- **Example**:
  - Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
  - Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a \text{ divides } b\}$ are

    (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Binary Relation on a Set (*cont.*)

- **Definition:** How many relations are there on a set A?
- **Solution**: Because a relation on A is the same thing as a subset of A $\times$ A, we count the subsets of A × A. Since A × A has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of A × A. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relation on a Set (*cont.*)

- **Example:** Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

- Which of these relations contain each of the pairs (1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?
- **Solution**: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6: (1,2) is in R_1 and R_6: (2,1) is in R_2, R_5, and R_6: (1, −1) is in R_2, R_3, and R_6 : (2,2) is in R_1, R_3, and R_4.

Reflexive Relations

- **Definition:** R is *reflexive* iff $(a, a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if $\forall x[x \in U \rightarrow (x, x) \in R]$

- **Example:** The following relations on the integers are reflexive:

  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
  $R_4 = \{(a,b) \mid a = b\}$.

  – The following relations are not reflexive:

  $R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),
  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),
  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is, the empty relation on an empty set is reflexive!

Symmetric Relations

- **Definition:** R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if $\forall x \forall y\ [(x,y) \in R \longrightarrow (y,x) \in R]$

- **Example:** The following relations on the integers are symmetric:

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
  $R_4 = \{(a,b) \mid a = b\}$,
  $R_6 = \{(a,b) \mid a + b \leq 3\}$.
  – The following are not symmetric:
  $R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \not\leq 3$),
  $R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \not> 4$),
  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

- **Definition:** A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if $\forall x \forall y\ [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$
- **Example:** The following relations on the integers are antisymmetric:

  $R_1 = \{(a,b) \mid a \leq b\}$,

  $R_2 = \{(a,b) \mid a > b\}$,

  $R_4 = \{(a,b) \mid a = b\}$,

  $R_5 = \{(a,b) \mid a = b + 1\}$.

  For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.

  - The following relations are not antisymmetric:

  $R_3 = \{(a,b) \mid a = b$ or $a = -b\}$ (note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

- **Definition:** A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if
  $\forall x \forall y \, \forall z[(x,y) \in R \wedge (y,z) \in \mathrm{R} \longrightarrow (x,z) \in R\,]$

- **Example:** The following relations on the integers are transitive:
  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_2 = \{(a,b) \mid a > b\}$,
  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
  $R_4 = \{(a,b) \mid a = b\}$.

  For every integer, $a \leq b$ and $b \leq c$, then $b \leq c$.

  - The following are not transitive:
  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (4,3) and (3,2) belong to R_5, but not (3,3)),
  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

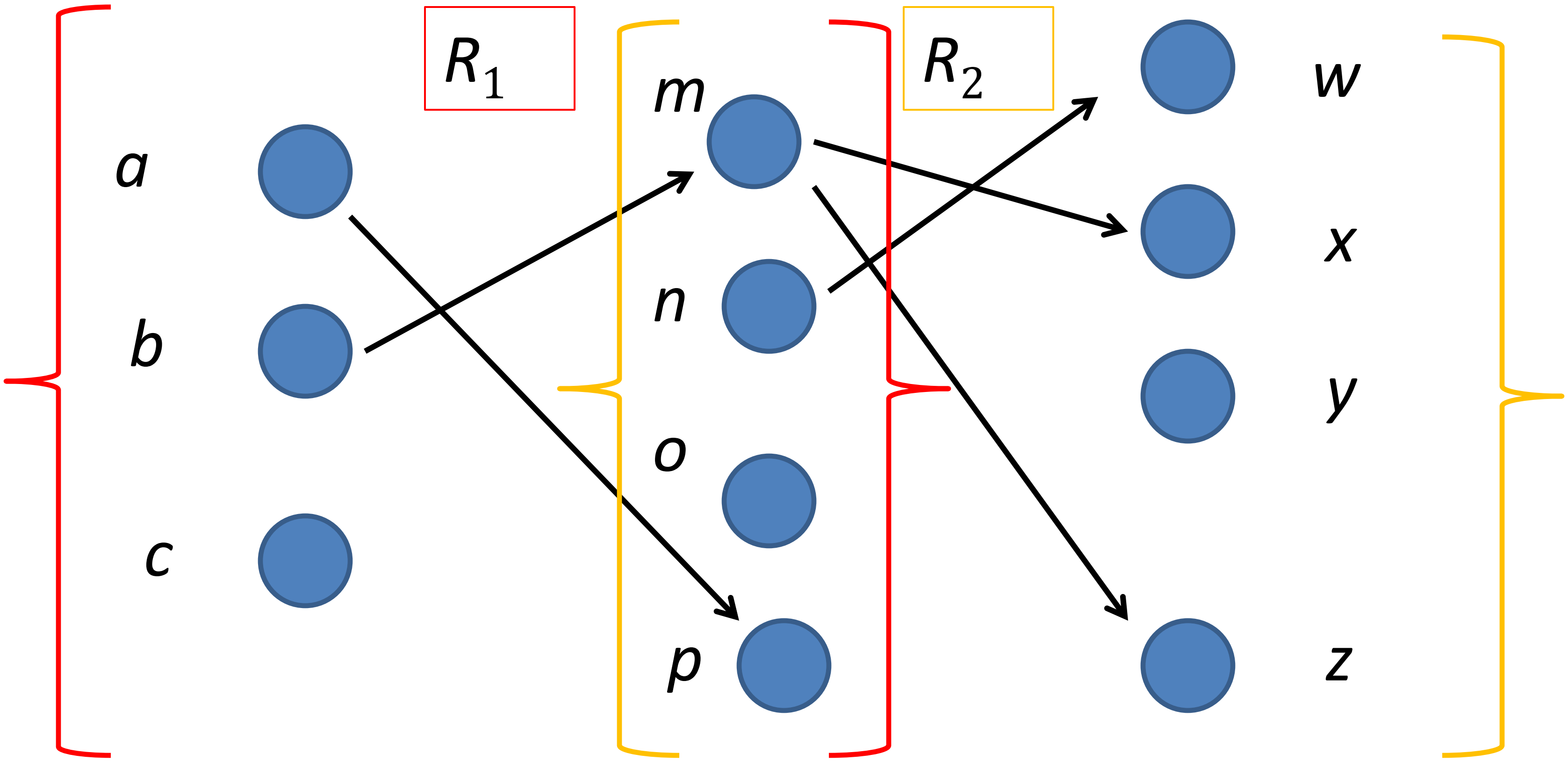
$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

- **Definition:** Suppose
  - R_1 is a relation from a set A to a set B.
  - R_2 is a relation from B to a set C.
- Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where
  - If (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b,x),(b,z)\}$$

Powers of a Relation

- **Definition:** Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:
  - Basis Step: $R^1 = R$
  - Inductive Step: $R^{n+1} = R^n \circ R$

  (*see the slides for Section* 9.3 *for further insights*)

- The powers of a transitive relation are subsets of the relation. This is established by the following theorem:
- **Theorem 1:** The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots$.

  (*see the text for a proof via mathematical induction*)

Representing Relations

Section 9.3

Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose *R* is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When *A* = *B*, we use the same ordering.
- The relation *R* is represented by the matrix $m_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- The matrix representing *R* has a 1 as its (*i*,*j*) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

- **Example 1**: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?
- **Solution:** Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Examples of Representing Relations Using Matrices

- **Example 2**: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

- **Solution:** Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

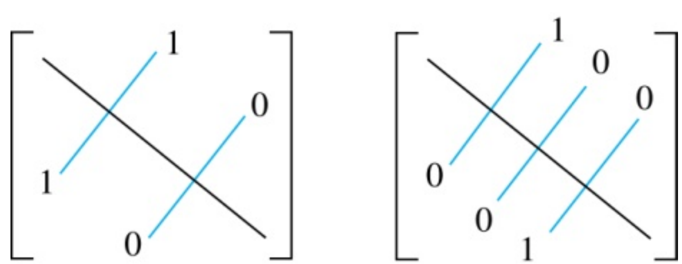
$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$

Matrices of Relations on Sets

- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

- R is a symmetric relation, if and only if m_{ij} = 1 whenever m_{ji} = 1.
- R is an antisymmetric relation, if and only if m_{ij} = 0 or m_{ji} = 0 when $i \neq j$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

- **Example 3**: Suppose that the relation R on a set is represented by the matrix

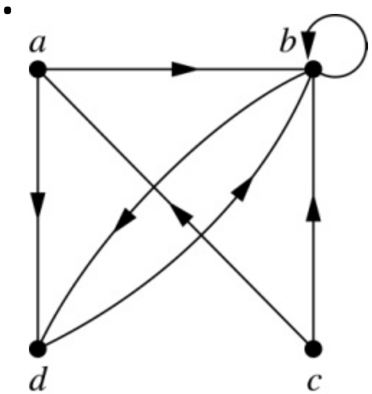
$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

  is R reflexive, symmetric, and/or antisymmetric?
- **Solution**: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

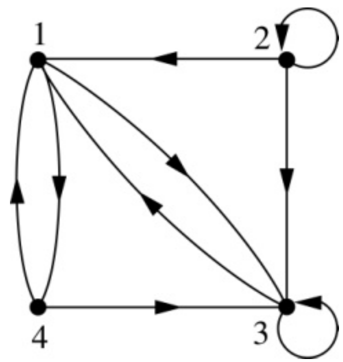
- **Definition**: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.
  - An edge of the form (a,a) is called a *loop*.

- **Example 7**: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), $(c,$ a$)$, (c, b), and (d, b) is shown here.



Examples of Digraphs Representing Relations

- **Example 8**: What are the ordered pairs in the relation represented by this directed graph?



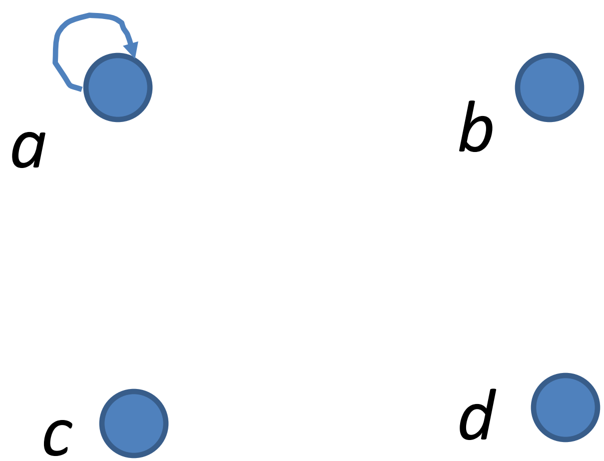
- **Example 7**: The ordered pairs in the relation are (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3).

Determining which Properties a Relation has from its Digraph

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

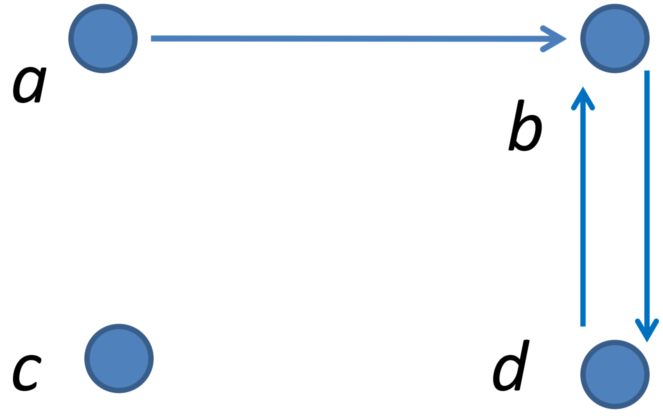
Determining which Properties a Relation has from its Digraph – Example 1

- *Reflexivity*? No, not every vertex has a loop
- *Symmetric?* Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric?* Yes (trivially), there is no edge from one vertex to another
- *Transitive?* Yes, (trivially) since there is no edge from one vertex to another



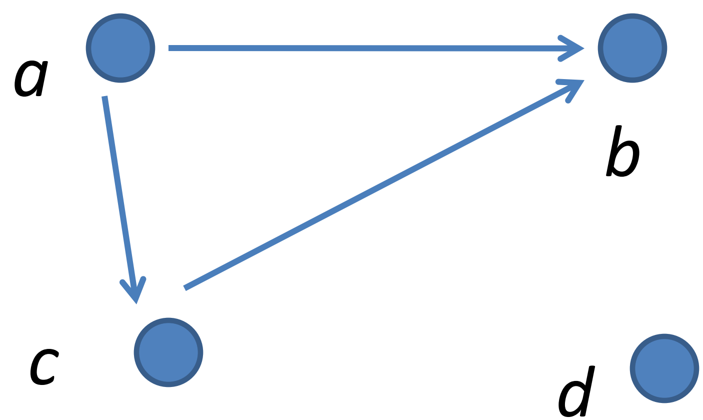
Determining which Properties a Relation has from its Digraph – Example 2

- *Reflexivity*? No, there are no loops
- *Symmetric?* No, there is an edge from *a* to *b*, but not from *b* to *a*
- *Antisymmetric?* No, there is an edge from *d* to *b* and *b* to *d*
- *Transitive?* No, there are edges from *a* to *c* and from *c* to *b*, but there is no edge from *a* to *d*



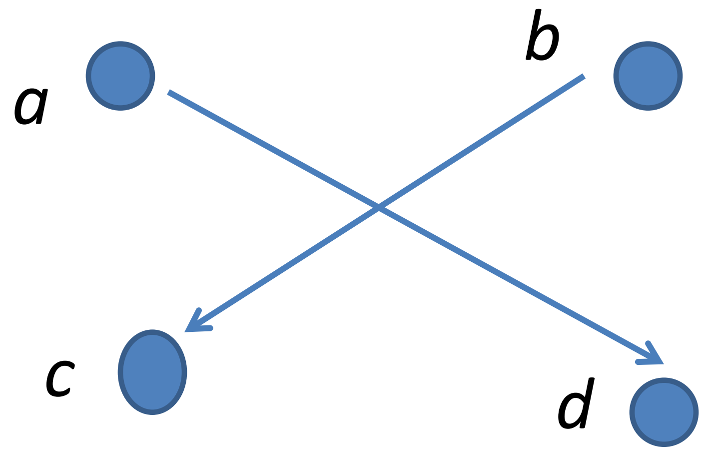
Determining which Properties a Relation has from its Digraph – Example 3

- *Reflexivity*? No, there are no loops
- *Symmetric?* No, for example, there is no edge from *c* to *a*
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive? Yes*.

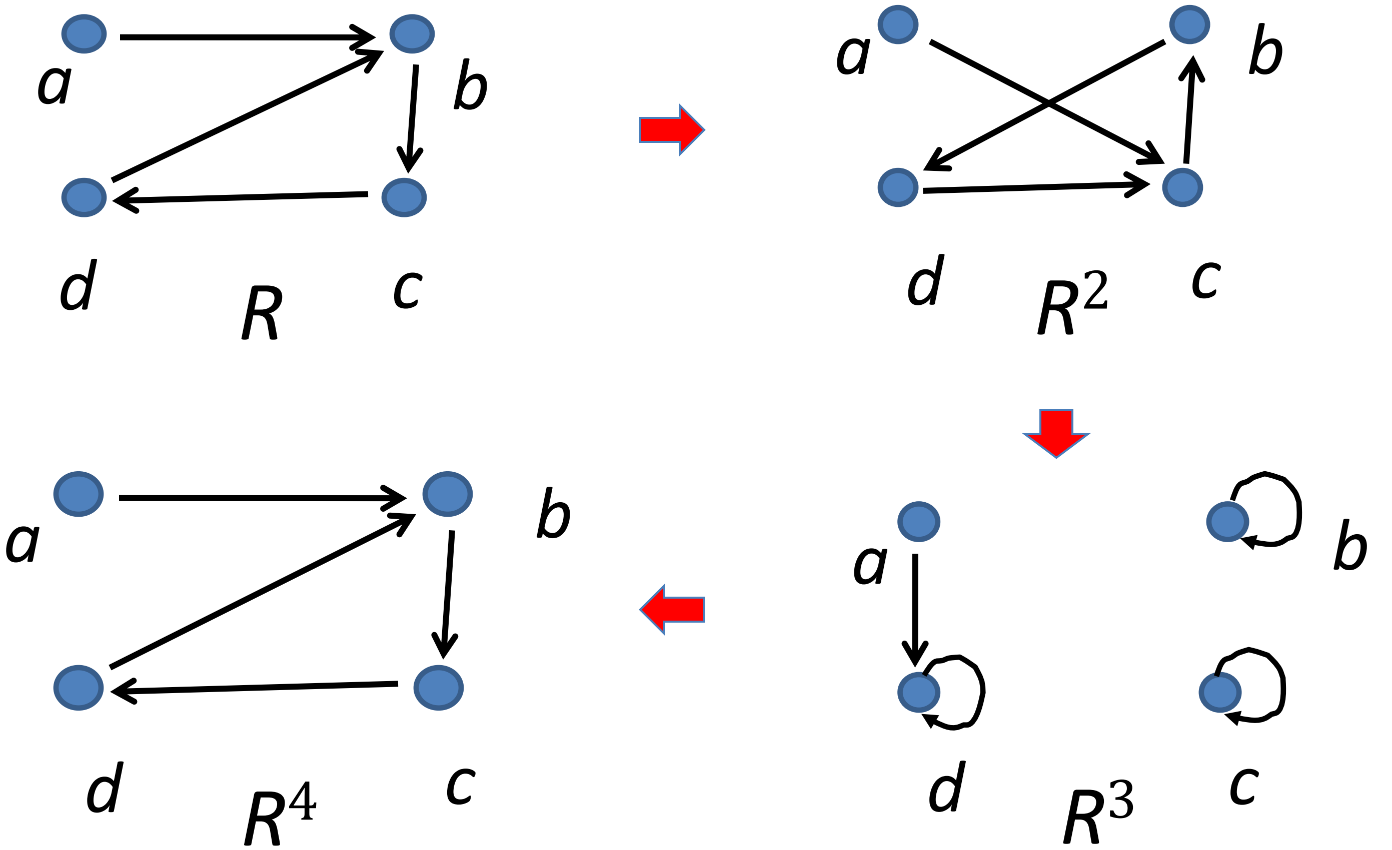


Determining which Properties a Relation has from its Digraph – Example 4

- *Reflexivity*? No, there are no loops
- *Symmetric?* No, for example, there is no edge from *d* to *a*
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?* Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins



Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length *n* from *x* to *y* in *R* (following the direction of the arrows).