4190.101
Discrete Mathematics

Chapter 2 Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

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[Based on S. Srinivasa Rao's Lecture Notes]

Functions

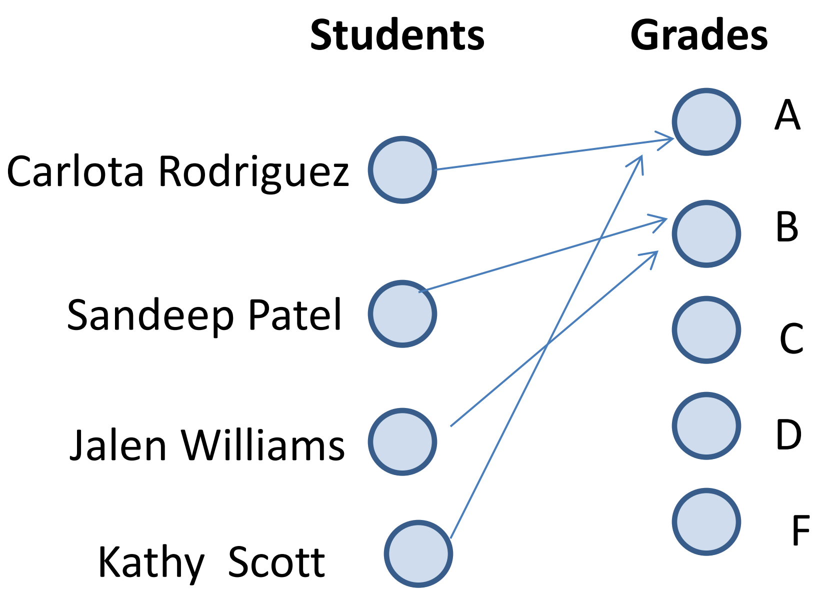
Section 2.3

Section Summary

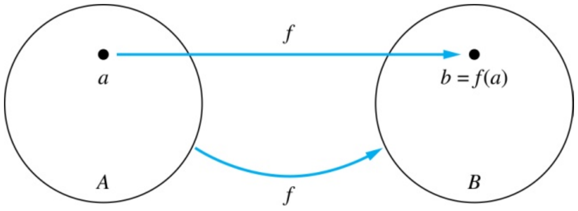
- Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (optional)

Functions

- **Definition**: Let A and B be nonempty sets. A *function f* from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- Functions are sometimes called *mappings* or *transformations*.



Functions



- Given a function f: $A \to B$:
- We say f *maps* A to B *or* f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in A under f. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

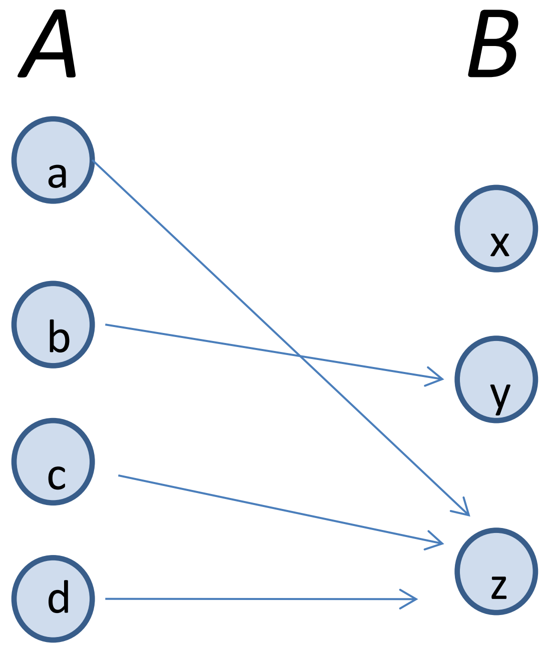
    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

- f(a) = ? z
- The image of d is ? z
- The domain of f is ? A
- The codomain of f is ? B
- The preimage of y is ? b
- The preimage(s) of z is (are) ?

  $\{a,c,d\}$



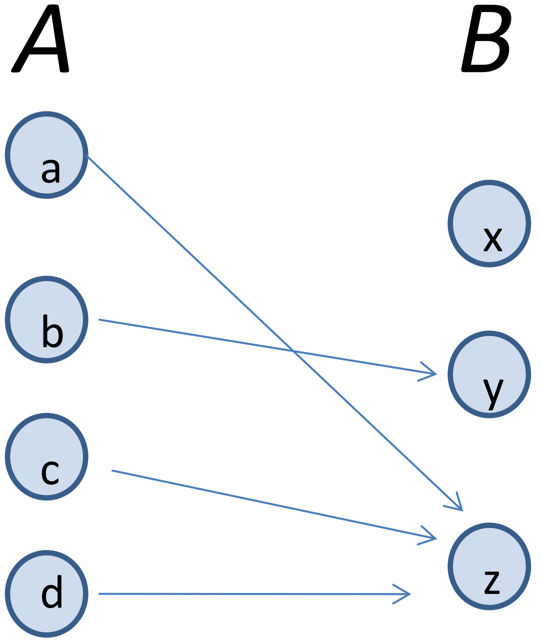
Question on Functions and Sets

- If f: $A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

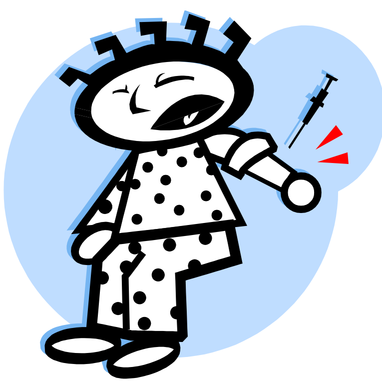
f {a,b,c,} is ? {y,z}

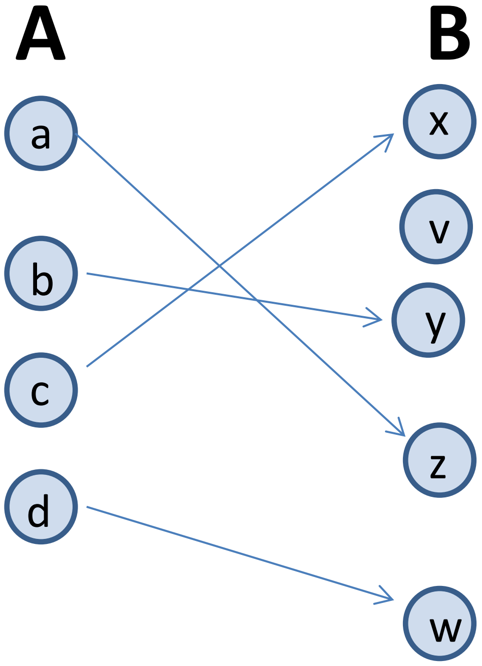
f {c,d} is ? {z}



Injections

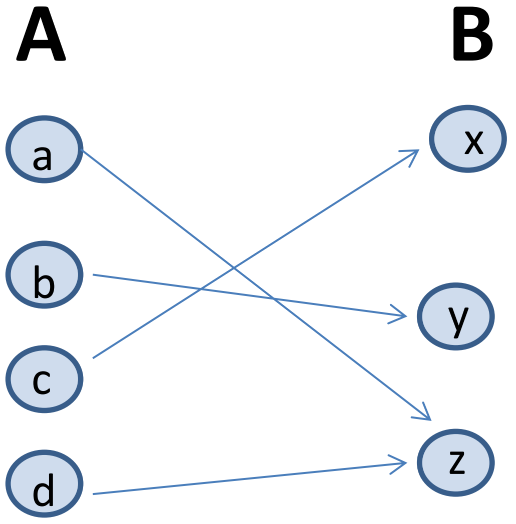
- **Definition**: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.





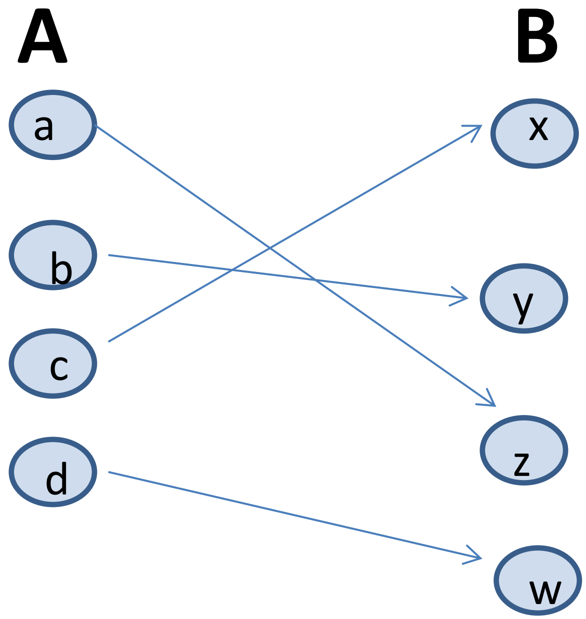
Surjections

- **Definition**: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a *surjection* if it is onto.



Bijections

- **Definition**: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto

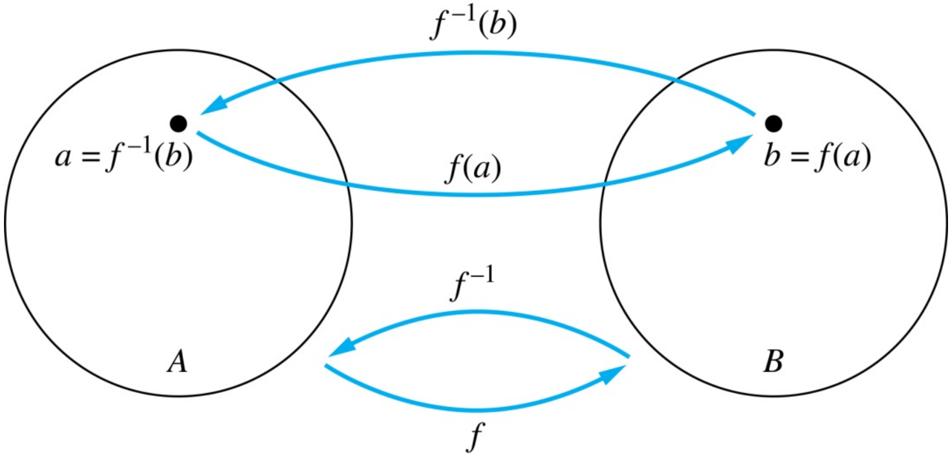
- **Example 1**: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?
- **Solution**: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.
- **Example 2**: Is the function $f(x) = x^2$ from the set of integers onto?
- **Solution**: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Inverse Functions

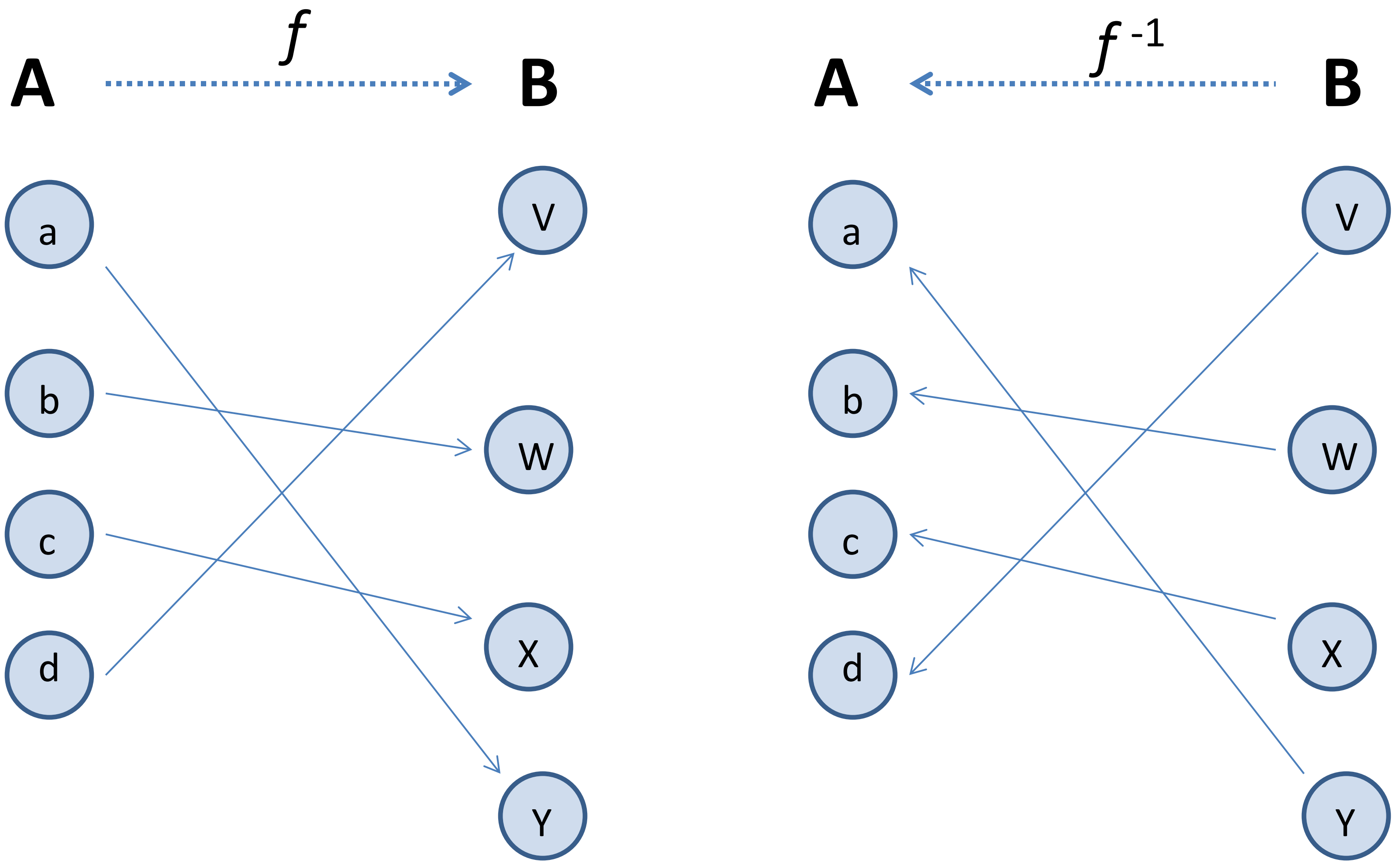
- **Definition**: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

- No inverse exists unless *f* is a bijection. Why?



Inverse Functions



Questions

- **Example 1**: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

- **Solution**: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

- **Example 2**: Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

- **Solution**: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

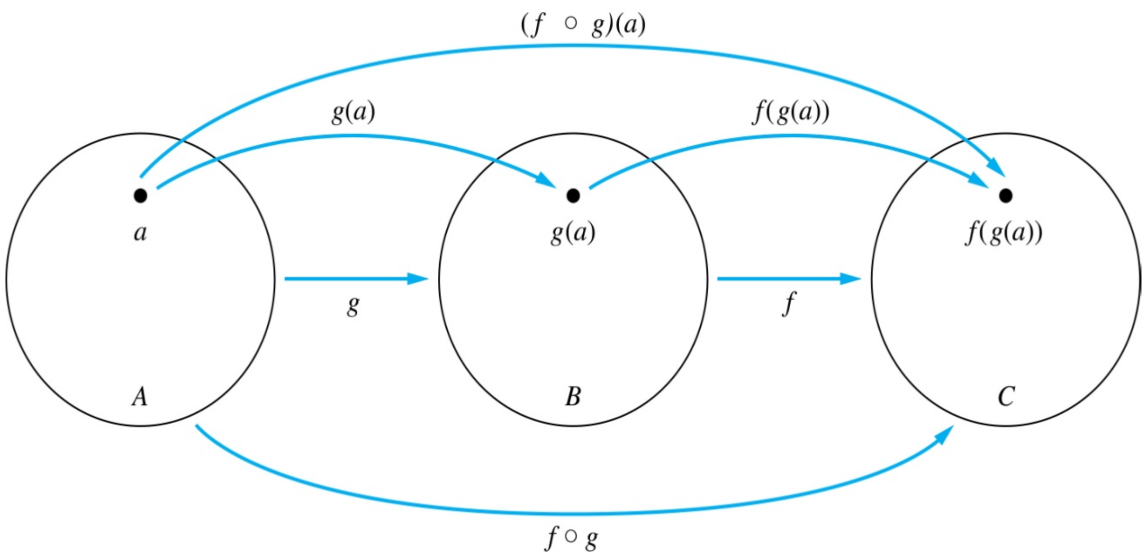
- **Example 3**: Let *f:* **R** $\rightarrow$ **R** be such that $f(x) = x^2$. Is *f* invertible, and if so, what is its inverse?

- **Solution**: The function *f* is not invertible because it is not one-to-one .

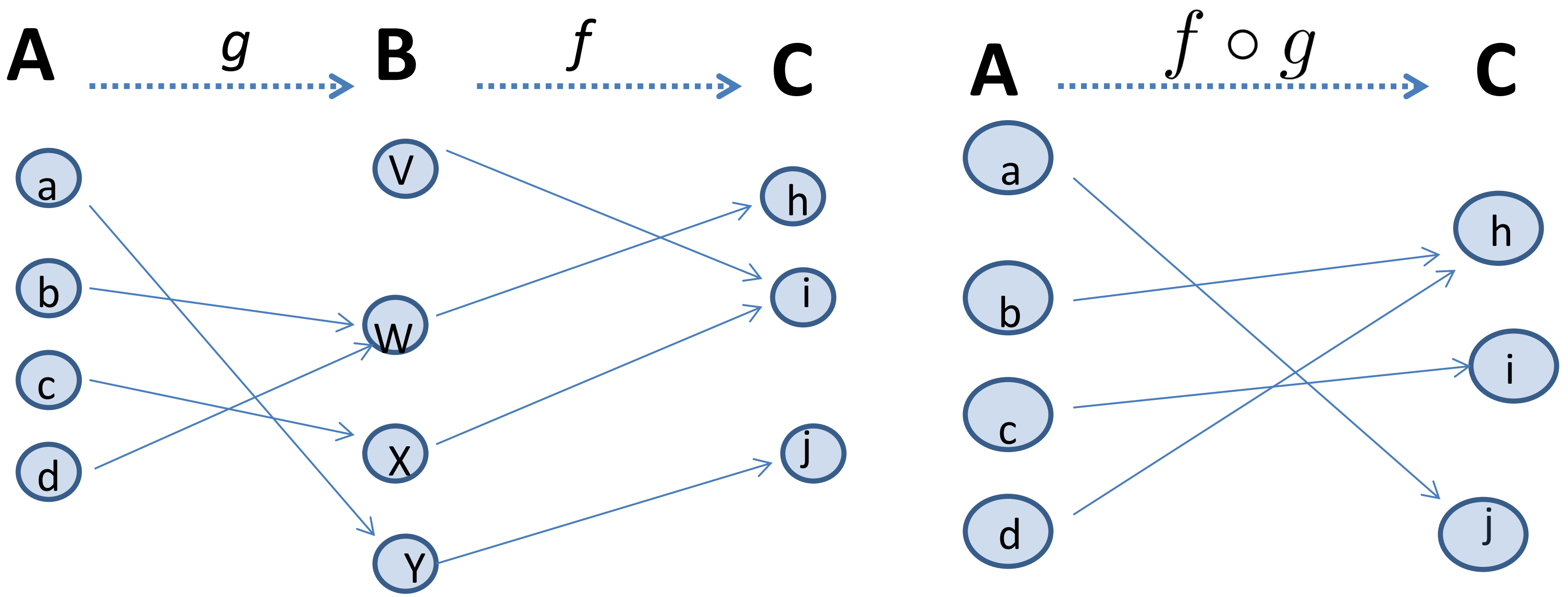
Composition

- **Definition**: Let f: $B \to C$, g: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) \;=\; f(g(x))$$



Composition



Composition

- **Example 1**: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

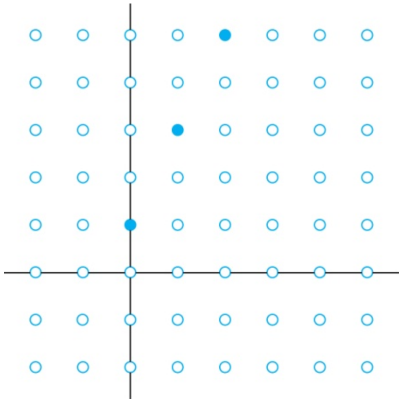
- **Example 2**: Let g be a function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be a function from the set $\{a,b,c\}$ to set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.
- What is the composition of f and g, and what is the composition of g and f.
- **Solution:** The composition $f \circ g$ is defined by
  - $f \circ g\ (a) = f(g(a)) = f(b) = 2.$
  - $f \circ g\ (b) = f(g(b)) = f(c) = 1.$
  - $f \circ g\ (c) = f(g(c)) = f(a) = 3.$
  - Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

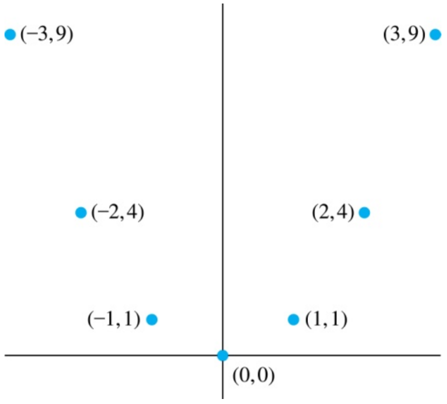
- **Example 3**: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.
- What is the composition of f and g, and also the composition of g and f ?
- **Solution:**
  - $f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$
  - $g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from **Z** to **Z**



Graph of $f(x) = x^2$ from **Z** to **Z**

Some Important Functions

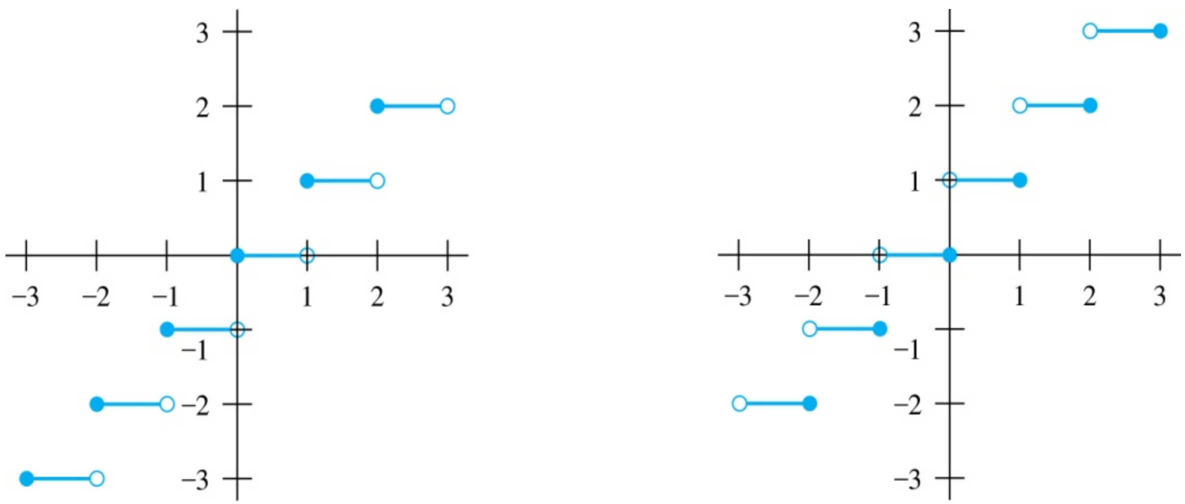
- The *floor* function, denoted $f(x) = \lfloor x \rfloor$
  is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted $f(x) = \lceil x \rceil$
  is the smallest integer greater than or equal to *x*
- **Example***:*

$$\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$$

$$\lceil -1.5 \rceil = -1 \quad \lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions



(a) $y = [x]$ (b) $y = [x]$

Graph of (a) Floor and (b) Ceiling functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

- **Example**: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

- **Solution**: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.
- *Case 1:* $\varepsilon < ½$
  - $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
  - $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
  - Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.
- *Case 2:* $\varepsilon \geq ½$
  - $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
  - $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
  - Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Factorial Function

- **Definition**: f: **N → Z**$^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a non-negative integer.
  - $f(n) = 1 \cdot 2 \cdots (n-1) \cdot n$, $f(0) = 0! = 1$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

- **Examples:**
  - $f(1) = 1! = 1$
  - $f(2) = 2! = 1 \cdot 2 = 2$
  - $f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$
  - $f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000$

Sequences and Summations

Section 2.4

Section Summary

- Sequences
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
- Special Integer Sequences (*optional*)

Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

- **Definition**: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set S.
- The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of $f(n)$ where f is a function from {0,1,2,…..} to S. We call a_n a *term* of the sequence.

Sequences

- **Example**: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

- **Definition**: A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

- **Examples**:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

- **Definition**: A *arithmetic progression* is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

  where the *initial term a* and the *common difference d* are real numbers.
- **Examples**:
  1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

  2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

  3. Let $a = 1$ and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

- **Definition**: A *string* is a finite sequence of characters from a finite set (an alphabet).
- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

- **Definition**: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations

- **Example 1**: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

  [Here $a_0 = 2$ is the initial condition.]

- **Solution**: We see from the recurrence relation that
  - $a_1 = a_0 + 3 = 2 + 3 = 5$
  - $a_2 = 5 + 3 = 8$
  - $a_3 = 8 + 3 = 11$

Questions about Recurrence Relations

- **Example 2**: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} – a_{n\text{-}2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?
  [Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

- **Solution**: We see from the recurrence relation that
  - $a_2 = a_1 - a_0 = 5 – 3 = 2$
  - $a_3 = a_2 – a_1 = 2 – 5 = –3$

Fibonacci Sequence

- **Example 2**: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by*:*
  - Initial Conditions: $f_0 = 0$, $f_1 = 1$
  - Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

- **Example**: Find f_2, f_3, f_4, f_5 and f_6.

- **Answer***:*
  - $f_2 = f_1 + f_0 = 1 + 0 = 1,$
  - $f_3 = f_2 + f_1 = 1 + 1 = 2,$
  - $f_4 = f_3 + f_2 = 2 + 1 = 3,$
  - $f_5 = f_4 + f_3 = 3 + 2 = 5,$
  - $f_6 = f_5 + f_4 = 5 + 3 = 8.$

Solving Recurrence Relations

- Finding a formula for the *n*-th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Iterative Solution Example

- **Method 1**: Working upward, forward substitution. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots.$ and suppose that $a_1 = 2$.
  - $a_2 = 2 + 3$
  - $a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$
  - $a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$

    .

    .

    .
  - $a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$

Iterative Solution Example

- **Method 2**: Working downward, backward substitution. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.
  - $a_n = a_{n-1} + 3$

$$
\begin{aligned}
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Financial Application

- **Example**: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?
- Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Continued on next slide →

Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

- **Solution**: Forward Substitution
  - $P_1 = (1.11)P_0$
  - $P_2 = (1.11)P_1 = (1.11)^2 P_0$
  - $P_3 = (1.11)P_2 = (1.11)^3 P_0$

    :
  - $P_n = (1.11)P_{n-1} = (1.11)^n \mathrm{P}_0 = (1.11)^n\, 10{,}000$
  - $P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

Special Integer Sequences (*opt*)

- Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.
- Some questions to ask?
  - Are there repeated terms of the same value?
  - Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
  - Can you obtain a term by combining the previous terms in some way?
  - Are they cycles among the terms?
  - Do the terms match those of a well known sequence?

Questions on Special Integer Sequences (*opt*)

- **Example 1**: Find formulae for the sequences with the following first five terms: 1, ½, ¼, 1/8, 1/16
- **Solution:** Note that the denominators are powers of 2. The sequence with $a_n = (1/2)^n$ is a possible match. This is a geometric progression with $a = 1$ and $r = ½$.
- **Example 2**: Consider 1,3,5,7,9
- **Solution:** Note that each term is obtained by adding 2 to the previous term. A possible formula is $a_n = 2n + 1$. This is an arithmetic progression with $a = 1$ and $d = 2$.
- **Example 3**: 1, -1, 1, -1,1
- **Solution:** The terms alternate between 1 and -1. A possible sequence is $a_n = (-1)^n$. This is a geometric progression with $a = 1$ and $r = -1$.

Useful Sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Guessing Sequences (*optional*)

- **Example**: Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

- **Solution**: Note the ratio of each term to the previous approximates 3. So now compare with the sequence 3^n. We notice that the nth term is 2 less than the corresponding power of 3. So a good conjecture is that $a_n = 3^n - 2$.

Integer Sequences (*optional*)

- Integer sequences appear in a wide range of contexts. Later we will see the sequence of prime numbers (Chapter 4), the number of ways to order *n* discrete objects (Chapter 6), the number of moves needed to solve the Tower of Hanoi puzzle with *n* disks (Chapter 8), and the number of rabbits on an island after *n* months (Chapter 8).
- Integer sequences are useful in many fields such as biology, engineering, chemistry and physics.
- On-Line Encyclopedia of Integer Sequences (OESIS) contains over 200,000 sequences. Began by Neil Stone in the 1960s (printed form). Now found at http://oeis.org/Spuzzle.html

Integer Sequences (*optional*)

- Here are three interesting sequences to try from the OESIS site. To solve each puzzle, find a rule that determines the terms of the sequence.
- Guess the rules for forming for the following sequences:
  - 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, ...
    - Hint: Think of adding and multiplying by numbers to generate this sequence.
  - 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, ...
    - Hint: Think of the English names for the numbers representing the position in the sequence and the Roman Numerals for the same number.
  - 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, ...
    - Hint: Think of the English names for numbers, and whether or not they have the letter 'e.'

- The answers and many more can be found at

http://oeis.org/Spuzzle.html

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$
  from the sequence $\{a_n\}$
- The notation:

$$\sum_{j=m}^{n} a_j \quad \textstyle\sum_{j=m}^{n} a_j \quad \textstyle\sum_{m \leq j \leq n} a_j$$

  represents

$$a_m + a_{m+1} + \cdots + a_n$$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations

- More generally for a set *S*:

$$\sum_{j \in S} a_j$$

- **Examples**: $r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation (*optional*)

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$

- The notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{m \leq j \leq n} a_j$$

represents $a_m \times a_{m+1} \times \cdots \times a_n$

Geometric Series

- Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

- **Proof**: Let

$$S_n = \sum_{j=0}^{n} ar^j$$

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

Continued on next slide →

Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$ From previous slide.

$$= \sum_{k=1}^{n+1} ar^{k}$$ Shifting the index of summation with *k* = *j* + 1.

$$= \left(\sum_{k=0}^{n} ar^{k}\right) + (ar^{n+1} - a)$$ Removing *k* = *n* + 1 term and adding *k* = 0 term.

$$= S_n + (ar^{n+1} - a)$$ Substituting *S* for summation formula

$$\therefore \quad rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$ if r $\neq$1

$$S_n = \sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} a = (n+1)a$$ if r = 1

Some Useful Summation Formulae

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)