4190.101
Discrete Mathematics

Chapter 6 Counting

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[Based on S. Srinivasa Rao’s Lecture Notes]

Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations

The Basics of Counting

Section 6.1

Section Summary

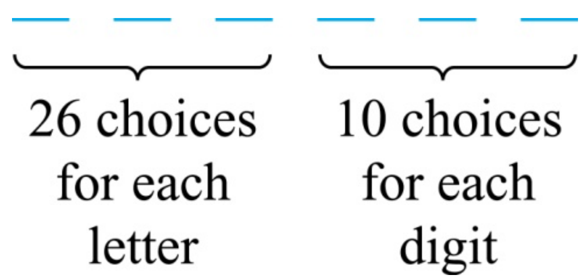
- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams

Basic Counting Principles: The Product Rule

- **The Product Rule**: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.
- **Example**: How many bit strings of length seven are there?
- **Solution**: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The Product Rule

- **Example**: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?
- **Solution**: By the product rule, there are 26 · 26 · 26 · 10 · 10 · 10 = 17,576,000 different possible license plates.



Counting Functions

- **Counting Functions**: How many functions are there from a set with m elements to a set with n elements?
- **Solution**: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

- **Counting one-to-one functions**: How many one-to-one functions are there from a set with m elements to one with n elements?
- **Solution**: Suppose the elements in the domain are a_1, a_2, … , a_m. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Telephone Numbering Plan

- **Example**: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.
  - Let *X* denote a digit from 0 through 9.
  - Let *N* denote a digit from 2 through 9.
  - Let *Y* denote a digit that is 0 or 1.
  - In the old plan (in use in the 1960s) the format was *NYX*-*NNX*-*XXXX*.
  - In the new plan, the format is *NXX*-*NXX*-*XXXX*.
- How many different telephone numbers are possible under the old plan and the new plan?
- **Solution**: Use the Product Rule.
  - There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format *NYX.*
  - There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format *NXX.*
  - There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format *NNX.*
  - There are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format *XXXX.*
- Number of old plan telephone numbers: $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$.
- Number of new plan telephone numbers: $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$.

Counting Subsets of a Finite Set

- **Counting subsets of a finite set**: Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$. (*In Section* 5.1, *mathematical induction was used to prove this same result*.)
- **Solution**: When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. When the i-th element is in the subset, the bit string has a 1 in the i-th position and a 0 otherwise.
- By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
  - The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, …, and an element in A_m.
- By the product rule, it follows that:

$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \times |A_2| \times \cdots \times |A_m|.$

DNA and Genomes

- A *gene* is a segment of a DNA molecule that encodes a particular protein and the entirety of genetic information of an organism is called its *genome*.
- DNA molecules consist of two strands of blocks known as nucleotides. Each nucleotide is composed of bases: adenine (A), cytosine (C), guanine (G), or thymine (T).
- The DNA of bacteria has between 10^5 and 10^7 links (one of the four bases). Mammals have between 10^8 and 10^{10} links. So, by the product rule there are at least $4^{10\text{^}5}$ different sequences of bases in the DNA of bacteria and $4^{10\text{^}8}$ different sequences of bases in the DNA of mammals.
- The human genome includes approximately 23,000 genes, each with 1,000 or more links.
- Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

Basic Counting Principles: The Sum Rule

- **The Sum Rule**: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.
- **Example**: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.
- **Solution**: By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick a representative.

The Sum Rule in Terms of Sets

- The sum rule can be phrased in terms of sets.
  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.
- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when $A_i \cap A_j = \emptyset$ for all i, j.

- The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Combining the Sum and Product Rule

- **Example**: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.
- **Solution**: Use the product rule.

  $$26 + 26 \cdot 10 = 286$$

Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.
- **Example**: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- **Solution**: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.
  - By the sum rule $P = P_6 + P_7 + P_8$.
  - To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

    $P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$

    $P_7 = 36^7 - 26^7 =$

    $78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$

    $P_8 = 36^8 - 26^8 =$

    $2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$
  - Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Internet Addresses

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid					hostid		
Class B	1	0	netid					hostid	
Class C	1	1	0	netid					hostid
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- **Class A Addresses**: used for the largest networks, a 0, followed by a 7-bit netid and a 24-bit hostid.
- **Class B Addresses**: used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid.
- **Class C Addresses**: used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid.
  - Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
  - 1111111 is not available as the netid of a Class A network.
  - Hostids consisting of all 0s and all 1s are not available in any network.

Counting Internet Addresses

- **Example**: How many different IPv4 addresses are available for computers on the internet?
- **Solution**: Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.
  - To find, x_A: $2^7 - 1 = 127$ netids. $2^{24} - 2 = 16{,}777{,}214$ hostids.
    $x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178$.
  - To find, x_B: $2^{14} = 16{,}384$ netids. $2^{16} - 2 = 16{,}534$ hostids.
    $x_B = 16{,}384 \cdot 16{,}534 = 1{,}073{,}709{,}056$.
  - To find, x_C: $2^{21} = 2{,}097{,}152$ netids. $2^8 - 2 = 254$ hostids.
    $x_C = 2{,}097{,}152 \cdot 254 = 532{,}676{,}608$.
  - Hence, the total number of available IPv4 addresses is

$$
\begin{aligned}
x &= x_A + x_B + x_C \\
&= 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608 \\
&= 3{,}737{,}091{,}842.
\end{aligned}
$$

Not Enough Today !!
The newer IPv6 protocol solves the problem of too few addresses.

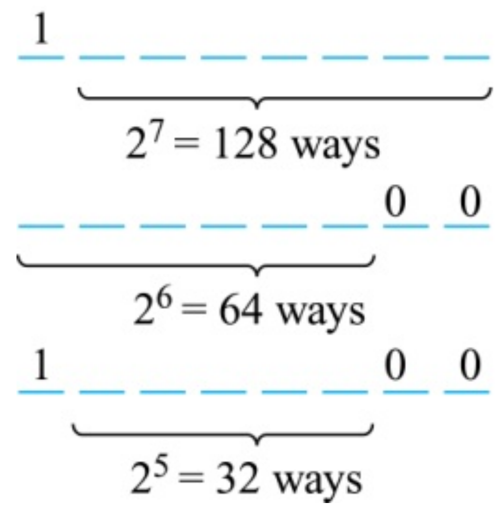
Basic Counting Principles: Subtraction Rule

- **Subtraction Rule**: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

- **Example**: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?



- **Solution**: Use the subtraction rule.
  - Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
  - Number of bit strings of length eight that start with bits 00: $2^6 = 64$
  - Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$
- Hence, the number is $128 + 64 - 32 = 160$.

Basic Counting Principles: Division Rule

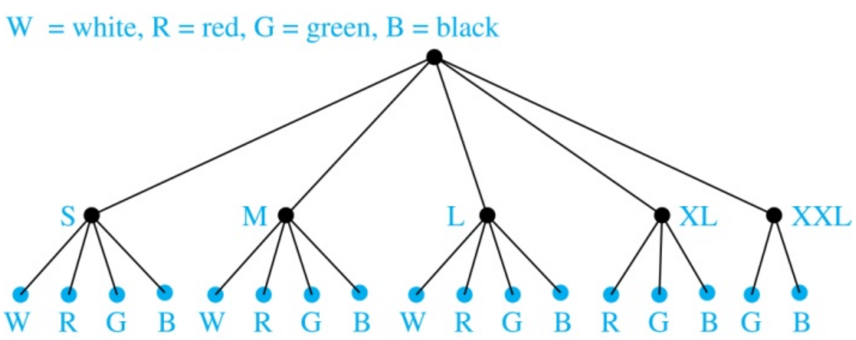
- **Division Rule**: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.
- Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.
- In terms of functions: If f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

Basic Counting Principles: Division Rule

- **Example**: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?
- **Solution**: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are 4! = 24 ways to order the four people. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.
- Therefore, by the division rule, there are 24/4 = 6 different seating arrangements.

Tree Diagrams

- **Tree Diagrams**: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of stores that the campus book store needs to stock to have one of each size and color available?
- **Solution**: Draw the tree diagram.



- The store must stock 17 T-shirts.

The Pigeonhole Principle

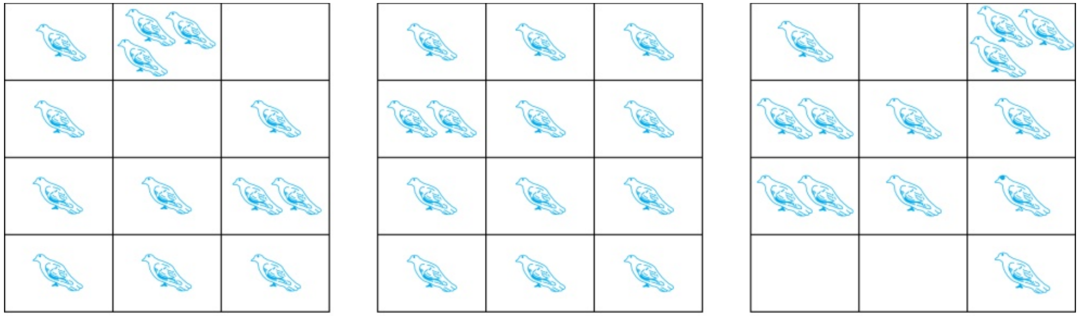
Section 6.2

Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a) (b) (c)

- **Pigeonhole Principle**: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.
- **Proof**: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects. ◀

The Pigeonhole Principle

- **Corollary 1**: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.
- **Proof**: Use the pigeonhole principle.
  – Create a box for each element y in the codomain of f.
  – Put in the box for y all of the elements x from the domain such that $f(x) = y$.
  – Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.
- Hence, f can't be one-to-one. ◀

The Pigeonhole Principle

- **Example**: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
- **Example** (*optional*): Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.
- **Solution**: Let n be a positive integer. Consider the $n + 1$ integers 1, 11, 111, …., 11…1 (where the last has $n + 1$ 1s). There are n possible remainders when an integer is divided by n. By the pigeonhole principle, when each of the $n + 1$ integers is divided by n, at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion. ◀

The Generalized Pigeonhole Principle

- **The Generalized Pigeonhole Principle**: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
- **Proof**: We use a proof by contradiction. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < N/k + 1$ has been used. This is a contradiction because there are a total of N objects. ◀

- **Example**: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

The Generalized Pigeonhole Principle

- **Example**: a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

  b) How many must be selected to guarantee that at least three hearts are selected?
- **Solution**: a) We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards. At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.

  b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)