Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

It is worthwhile to revisit each of the mathematical induction proofs in Examples 1–14 to see how these steps are completed. It will be helpful to follow these guidelines in the solutions of the exercises that ask for proofs by mathematical induction. The guidelines that we presented can be adapted for each of the variants of mathematical induction that we introduce in the exercises and later in this chapter.

Exercises

1. There are infinitely many stations on a train route. Suppose that the train stops at the first station and suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.

2. Suppose that you know that a golfer plays the first hole of a golf course with an infinite number of holes and that if this golfer plays one hole, then the golfer goes on to play the next hole. Prove that this golfer plays every hole on the course.

Use mathematical induction in Exercises 3–17 to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

3. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

7. Prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

8. Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

9. **a)** Find a formula for the sum of the first n even positive integers.
b) Prove the formula that you conjectured in part (a).

10. **a)** Find a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.
b) Prove the formula you conjectured in part (a).

11. **a)** Find a formula for

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$$

by examining the values of this expression for small values of n.
b) Prove the formula you conjectured in part (a).

12. Prove that

$$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j=\frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$$

whenever n is a nonnegative integer.

13. Prove that $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2=(-1)^{n-1}n(n+1)/2$ whenever n is a positive integer.

14. Prove that for every positive integer n, $\sum_{k=1}^{n}k2^k=(n-1)2^{n+1}+2$.

15. Prove that for every positive integer n,

$$1\cdot 2+2\cdot 3+\cdots+n(n+1)=n(n+1)(n+2)/3.$$

16. Prove that for every positive integer n,

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)=n(n+1)(n+2)(n+3)/4.$$

17. Prove that $\sum_{j=1}^{n}j^4=n(n+1)(2n+1)(3n^2+3n-1)/30$ whenever n is a positive integer.

Use mathematical induction to prove the inequalities in Exercises 18–30.

18. Let $P(n)$ be the statement that $n!<n^n$, where n is an integer greater than 1.
a) What is the statement $P(2)$?
b) Show that $P(2)$ is true, completing the basis step of the proof.
c) What is the inductive hypothesis?
d) What do you need to prove in the inductive step?
e) Complete the inductive step.
f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

19. Let $P(n)$ be the statement that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n},$$

where n is an integer greater than 1.
a) What is the statement $P(2)$?
b) Show that $P(2)$ is true, completing the basis step of the proof.
c) What is the inductive hypothesis?
d) What do you need to prove in the inductive step?
e) Complete the inductive step.
f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

20. Prove that $3^n<n!$ if n is an integer greater than 6.

21. Prove that $2^n>n^2$ if n is an integer greater than 4.

22. For which nonnegative integers n is $n^2\le n!$? Prove your answer.

23. For which nonnegative integers n is $2n+3\le 2^n$? Prove your answer.

24. Prove that $1/(2n)\le[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]/(2\cdot 4\cdot\cdots\cdot 2n)$ whenever n is a positive integer.

***25.** Prove that if $h>-1$, then $1+nh\le(1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

***26.** Suppose that a and b are real numbers with $0<b<a$. Prove that if n is a positive integer, then $a^n-b^n\le na^{n-1}(a-b)$.

***27.** Prove that for every positive integer n,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>2(\sqrt{n+1}-1).$$

28. Prove that $n^2-7n+12$ is nonnegative whenever n is an integer with $n\ge 3$.

In Exercises 29 and 30, H_n denotes the nth harmonic number.

***29.** Prove that $H_{2^n}\le 1+n$ whenever n is a nonnegative integer.

***30.** Prove that

$$H_1+H_2+\cdots+H_n=(n+1)H_n-n.$$

Use mathematical induction in Exercises 31–37 to prove divisibility facts.

31. Prove that 2 divides n^2+n whenever n is a positive integer.

32. Prove that 3 divides n^3+2n whenever n is a positive integer.

33. Prove that 5 divides n^5-n whenever n is a nonnegative integer.

34. Prove that 6 divides n^3-n whenever n is a nonnegative integer.

***35.** Prove that n^2-1 is divisible by 8 whenever n is an odd positive integer.

***36.** Prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever n is a positive integer.

***37.** Prove that if n is a positive integer, then 133 divides $11^{n+1}+12^{2n-1}$.

Use mathematical induction in Exercises 38–46 to prove results about sets.

38. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j\subseteq B_j$ for $j=1,2,\ldots,n$, then

$$\bigcup_{j=1}^{n}A_j\subseteq\bigcup_{j=1}^{n}B_j.$$

39. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j.$$

40. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B \\ = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).$$

41. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B \\ = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

42. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \\ = (A_1 \cap A_2 \cap \cdots \cap A_n) - B.$$

43. Prove that if $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, then

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}.$$

44. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) \\ = (A_1 \cup A_2 \cup \cdots \cup A_n) - B.$$

45. Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

***46.** Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

In Exercises 47 and 48 we consider the problem of placing towers along a straight road, so that every building on the road receives cellular service. Assume that a building receives cellular service if it is within one mile of a tower.

47. Devise a greedy algorithm that uses the minimum number of towers possible to provide cell service to d buildings located at positions $x_1, x_2, \ldots, x_d$ from the start of the road. [*Hint:* At each step, go as far as possible along the road before adding a tower so as not to leave any buildings without coverage.]

***48.** Use mathematical induction to prove that the algorithm you devised in Exercise 47 produces an optimal solution, that is, that it uses the fewest towers possible to provide cellular service to all buildings.

Exercises 49–51 present incorrect proofs using mathematical induction. You will need to identify an error in reasoning in each exercise.

49. What is wrong with this "proof" that all horses are the same color?
Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color. Consider any $k+1$ horses; number these as horses $1, 2, 3, \ldots, k, k+1$. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Because the set of the first k horses and the set of the last k horses overlap, all $k+1$ must be the same color. This shows that $P(k+1)$ is true and finishes the proof by induction.

50. What is wrong with this "proof"?
"Theorem" For every positive integer n, $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$.

Basis Step: The formula is true for $n = 1$.

Inductive Step: Suppose that $\sum_{i=1}^{n} i = (n+\frac{1}{2})^2/2$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$. By the inductive hypothesis, $\sum_{i=1}^{n+1} i = (n+\frac{1}{2})^2/2 + n + 1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n+\frac{3}{2})^2/2 = [(n+1)+\frac{1}{2}]^2/2$, completing the inductive step.

51. What is wrong with this "proof"?
"Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x-1, y-1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

52. Suppose that m and n are positive integers with $m > n$ and f is a function from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Use mathematical induction on the variable n to show that f is not one-to-one.

***53.** Use mathematical induction to show that n people can divide a cake (where each person gets one or more separate pieces of the cake) so that the cake is divided fairly, that is, in the sense that each person thinks he or she got at least $(1/n)$th of the cake. [*Hint:* For the inductive step, take a fair division of the cake among the first k people, have each person divide their share into what this person thinks are $k+1$ equal portions, and then have the $(k+1)$st person select a portion from each of the k people. When showing this produces a fair division for $k+1$ people, suppose that person $k+1$ thinks that person i got p_i of the cake where $\sum_{i=1}^{k} p_i = 1$.]

54. Use mathematical induction to show that given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

***55.** A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or two spaces horizontally (in either direction) and one space vertically (in either direction). Suppose that we have an infinite chessboard, made up of all squares (m, n) where m and n are nonnegative integers that denote the row number and the column number of the square, respectively. Use mathematical induction to show that a knight starting at $(0, 0)$ can visit every square using a finite sequence of moves. [*Hint:* Use induction on the variable $s = m + n$.]

56. Suppose that

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where a and b are real numbers. Show that

$$\mathbf{A}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every positive integer n.

57. (*Requires calculus*) Use mathematical induction to prove that the derivative of $f(x) = x^n$ equals nx^{n-1} whenever n is a positive integer. (For the inductive step, use the product rule for derivatives.)

58. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the property $\mathbf{AB} = \mathbf{BA}$. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

59. Suppose that m is a positive integer. Use mathematical induction to prove that if a and b are integers with $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$ whenever k is a nonnegative integer.

60. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ whenever $p_1, p_2, \ldots, p_n$ are propositions.

***61.** Show that

$$[(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_{n-1} \to p_n)] \\ \to [(p_1 \wedge p_2 \wedge \cdots \wedge p_{n-1}) \to p_n]$$

is a tautology whenever $p_1, p_2, \ldots, p_n$ are propositions, where $n \geq 2$.

***62.** Show that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

****63.** Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The **arithmetic mean** of these numbers is defined by

$$A = (a_1 + a_2 + \cdots + a_n)/n,$$

and the **geometric mean** of these numbers is defined by

$$G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$.

64. Use mathematical induction to prove Lemma 3 of Section 4.3, which states that if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where a_i is an integer for $i = 1, 2, 3, \ldots, n$, then $p \mid a_i$ for some integer i.

65. Show that if n is a positive integer, then

$$\sum_{\{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is over all nonempty subsets of the set of the n smallest positive integers.)

***66.** Use the well-ordering property to show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

Basis Step: $P(1)$ and $P(2)$ are true.

Inductive Step: For each positive integer k, if $P(k)$ and $P(k+1)$ are both true, then $P(k+2)$ is true.

67. Show that if $A_1, A_2, \ldots, A_n$ are sets where $n \geq 2$, and for all pairs of integers i and j with $1 \leq i < j \leq n$ either A_i is a subset of A_j or A_j is a subset of A_i, then there is an integer i, $1 \leq i \leq n$ such that A_i is a subset of A_j for all integers j with $1 \leq j \leq n$.

***68.** A guest at a party is a **celebrity** if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party, for if there were two, they would know each other. A particular party may have no celebrity. Your assignment is to find the celebrity, if one exists, at a party, by asking only one type of question—asking a guest whether they know a second guest. Everyone must answer your questions truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the party, then you can find the celebrity, if there is one, with $3(n-1)$ questions. [*Hint:* First ask a question to eliminate one person as a celebrity. Then use the inductive hypothesis to identify a potential celebrity. Finally, ask two more questions to determine whether that person is actually a celebrity.]

Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The **gossip problem** asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals. Exercises 69–71 deal with the gossip problem.

69. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

70. Use mathematical induction to prove that $G(n) \leq 2n - 4$ for $n \geq 4$. [*Hint:* In the inductive step, have a new person call a particular person at the start and at the end.]

****71.** Prove that $G(n) = 2n - 4$ for $n \geq 4$.

***72.** Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them. [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

***73.** Show that if $I_1, I_2, \ldots, I_n$ is a collection of open intervals on the real number line, $n \geq 2$, and every pair of these intervals has a nonempty intersection, that is, $I_i \cap I_j \neq \emptyset$ whenever $1 \leq i \leq n$ and $1 \leq j \leq n$, then the intersection of all these sets is nonempty, that is, $I_1 \cap I_2 \cap \cdots \cap I_n \neq \emptyset$. (Recall that an **open interval** is the set of real numbers x with $a < x < b$, where a and b are real numbers with $a < b$.)

Sometimes we cannot use mathematical induction to prove a result we believe to be true, but we can use mathematical induction to prove a stronger result. Because the inductive hypothesis of the stronger result provides more to work with, this process is called **inductive loading**. We use inductive loading in Exercise 74.

74. Suppose that we want to prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

b) Show that mathematical induction can be used to prove the stronger inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

for all integers greater than 1, which, together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

75. Let n be an even positive integer. Show that when n people stand in a yard at mutually distinct distances and each person throws a pie at their nearest neighbor, it is possible that everyone is hit by a pie.

76. Construct a tiling using right triominoes of the 4×4 checkerboard with the square in the upper left corner removed.

77. Construct a tiling using right triominoes of the 8×8 checkerboard with the square in the upper left corner removed.

78. Prove or disprove that all checkerboards of these shapes can be completely covered using right triominoes whenever n is a positive integer.

a) 3×2^n **b)** 6×2^n
c) $3^n \times 3^n$ **d)** $6^n \times 6^n$

***79.** Show that a three-dimensional $2^n \times 2^n \times 2^n$ checkerboard with one $1 \times 1 \times 1$ cube missing can be completely covered by $2 \times 2 \times 2$ cubes with one $1 \times 1 \times 1$ cube removed.

***80.** Show that an $n \times n$ checkerboard with one square removed can be completely covered using right triominoes if $n > 5$, n is odd, and $3 \nmid n$.

81. Show that a 5×5 checkerboard with a corner square removed can be tiled using right triominoes.

***82.** Find a 5×5 checkerboard with a square removed that cannot be tiled using right triominoes. Prove that such a tiling does not exist for this board.

83. Use the principle of mathematical induction to show that $P(n)$ is true for $n = b, b+1, b+2, \ldots,$ where b is an integer, if $P(b)$ is true and the conditional statement $P(k) \rightarrow P(k+1)$ is true for all integers k with $k \geq b$.

5.2 Strong Induction and Well-Ordering

Introduction

In Section 5.1 we introduced mathematical induction and we showed how to use it to prove a variety of theorems. In this section we will introduce another form of mathematical induction, called **strong induction**, which can often be used when we cannot easily prove a result using mathematical induction. The basis step of a proof by strong induction is the same as a proof of the same result using mathematical induction. That is, in a strong induction proof that $P(n)$ is true for all positive integers n, the basis step shows that $P(1)$ is true. However, the inductive steps in these two proof methods are different. In a proof by mathematical induction, the inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ is also true. In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers not exceeding k, then $P(k+1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$.

The validity of both mathematical induction and strong induction follow from the well-ordering property in Appendix 1. In fact, mathematical induction, strong induction, and well-ordering are all equivalent principles (as shown in Exercises 41, 42, and 43). That is, the validity of each can be proved from either of the other two. This means that a proof using one of these two principles can be rewritten as a proof using either of the other two principles. Just as it is sometimes the case that it is much easier to see how to prove a result using strong induction rather than mathematical induction, it is sometimes easier to use well-ordering than one of the showed that the principle of mathematical induction follows from the well-ordering property. The other parts of this equivalence are left as Exercises 31, 42, and 43.

THE WELL-ORDERING PROPERTY Every nonempty set of nonnegative integers has a least element.

The well-ordering property can often be used directly in proofs.

EXAMPLE 5 Use the well-ordering property to prove the division algorithm. Recall that the division algorithm states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.

Extra Examples

Solution: Let S be the set of nonnegative integers of the form $a - dq$, where q is an integer. This set is nonempty because $-dq$ can be made as large as desired (taking q to be a negative integer with large absolute value). By the well-ordering property, S has a least element $r = a - dq_0$.

The integer r is nonnegative. It is also the case that $r < d$. If it were not, then there would be a smaller nonnegative element in S, namely, $a - d(q_0 + 1)$. To see this, suppose that $r \geq d$. Because $a = dq_0 + r$, it follows that $a - d(q_0 + 1) = (a - dq_0) - d = r - d \geq 0$. Consequently, there are integers q and r with $0 \leq r < d$. The proof that q and r are unique is left as Exercise 37. ◀

EXAMPLE 6 In a round-robin tournament every player plays every other player exactly once and each match has a winner and a loser. We say that the players $p_1, p_2, \ldots, p_m$ form a *cycle* if p_1 beats p_2, p_2 beats $p_3, \ldots, p_{m-1}$ beats p_m, and p_m beats p_1. Use the well-ordering principle to show that if there is a cycle of length m ($m \geq 3$) among the players in a round-robin tournament, there must be a cycle of three of these players.

Solution: We assume that there is no cycle of three players. Because there is at least one cycle in the round-robin tournament, the set of all positive integers n for which there is a cycle of length n is nonempty. By the well-ordering property, this set of positive integers has a least element k, which by assumption must be greater than three. Consequently, there exists a cycle of players $p_1, p_2, p_3, \ldots, p_k$ and no shorter cycle exists.

Because there is no cycle of three players, we know that $k > 3$. Consider the first three elements of this cycle, p_1, p_2, and p_3. There are two possible outcomes of the match between p_1 and p_3. If p_3 beats p_1, it follows that p_1, p_2, p_3 is a cycle of length three, contradicting our assumption that there is no cycle of three players. Consequently, it must be the case that p_1 beats p_3. This means that we can omit p_2 from the cycle $p_1, p_2, p_3, \ldots, p_k$ to obtain the cycle $p_1, p_3, p_4, \ldots, p_k$ of length $k - 1$, contradicting the assumption that the smallest cycle has length k. We conclude that there must be a cycle of length three. ◀

Exercises

1. Use strong induction to show that if you can run one mile or two miles, and if you can always run two more miles once you have run a specified number of miles, then you can run any number of miles.

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.

3. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 8$.
   a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of the proof.
   b) What is the inductive hypothesis of the proof?
   c) What do you need to prove in the inductive step?
   d) Complete the inductive step for $k \geq 10$.
   e) Explain why these steps show that this statement is true whenever $n \geq 8$.

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The

parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

5. a) Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

6. a) Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

7. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

8. Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

***9.** Use strong induction to prove that $\sqrt{2}$ is irrational. [*Hint:* Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.]

10. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

11. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j + 2$, or $4j + 3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer j.

12. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

***13.** A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n - 1$ moves are required to assemble a puzzle with n pieces.

14. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n - 1)/2$.

15. Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is square. [*Hint:* Use strong induction to show that this strategy works. For the first move, the first player chomps all cookies except those in the left and top edges. On subsequent moves, after the second player has chomped cookies on either the top or left edge, the first player chomps cookies in the same relative positions in the left or top edge, respectively.]

***16.** Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is two squares wide, that is, a $2 \times n$ board. [*Hint:* Use strong induction. The first move of the first player should be to chomp the cookie in the bottom row at the far right.]

17. Use strong induction to show that if a simple polygon with at least four sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon.

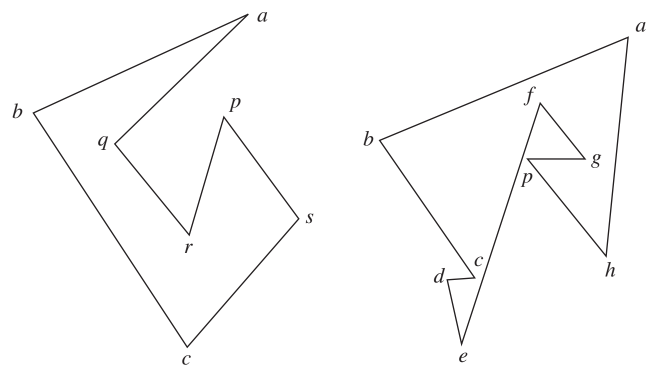
***18.** Use strong induction to show that when a simple polygon P with consecutive vertices $v_1, v_2, \ldots, v_n$ is triangulated into $n - 2$ triangles, the $n - 2$ triangles can be numbered $1, 2, \ldots, n - 2$ so that v_i is a vertex of triangle i for $i = 1, 2, \ldots, n - 2$.

***19.** **Pick's theorem** says that the area of a simple polygon P in the plane with vertices that are all lattice points (that is, points with integer coordinates) equals $I(P) + B(P)/2 - 1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of P and on the boundary of P, respectively. Use strong induction on the number of vertices of P to prove Pick's theorem. [*Hint:* For the basis step, first prove the theorem for rectangles, then for right triangles, and finally for all triangles by noting that the area of a triangle is the area of a larger rectangle containing it with the areas of at most three triangles subtracted. For the inductive step, take advantage of Lemma 1.]

20. ** Suppose that P is a simple polygon with vertices $v_1, v_2, \ldots, v_n$ listed so that consecutive vertices are connected by an edge, and v_1 and v_n are connected by an edge. A vertex v_i is called an **ear** if the line segment connecting the two vertices adjacent to v_i is an interior diagonal of the simple polygon. Two ears v_i and v_j are called **nonoverlapping** if the interiors of the triangles with vertices v_i and its two adjacent vertices and v_j and its two adjacent vertices do not intersect. Prove that every simple polygon with at least four vertices has at least two nonoverlapping ears.

21. In the proof of Lemma 1 we mentioned that many incorrect methods for finding a vertex p such that the line segment bp is an interior diagonal of P have been published. This exercise presents some of the incorrect ways p has been chosen in these proofs. Show, by considering one of the polygons drawn here, that for each of these choices of p, the line segment bp is not necessarily an interior diagonal of P.

- **a)** p is the vertex of P such that the angle $\angle abp$ is smallest.
- **b)** p is the vertex of P with the least x-coordinate (other than b).
- **c)** p is the vertex of P that is closest to b.



Exercises 22 and 23 present examples that show inductive loading can be used to prove results in computational geometry.

22. * Let $P(n)$ be the statement that when nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two vertices of the polygon are not endpoints of any of these diagonals.

- **a)** Show that when we attempt to prove $P(n)$ for all integers n with $n \geq 3$ using strong induction, the inductive step does not go through.
- **b)** Show that we can prove that $P(n)$ is true for all integers n with $n \geq 3$ by proving by strong induction the stronger assertion $Q(n)$, for $n \geq 4$, where $Q(n)$ states that whenever nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two *nonadjacent* vertices are not endpoints of any of these diagonals.

23. Let $E(n)$ be the statement that in a triangulation of a simple polygon with n sides, at least one of the triangles in the triangulation has two sides bordering the exterior of the polygon.

- **a)** Explain where a proof using strong induction that $E(n)$ is true for all integers $n \geq 4$ runs into difficulties.
- **b)** Show that we can prove that $E(n)$ is true for all integers $n \geq 4$ by proving by strong induction the stronger statement $T(n)$ for all integers $n \geq 4$, which states that in every triangulation of a simple polygon, at least two of the triangles in the triangulation have two sides bordering the exterior of the polygon.

24. * A stable assignment, defined in the preamble to Exercise 60 in Section 3.1, is called **optimal for suitors** if no stable assignment exists in which a suitor is paired with a suitee whom this suitor prefers to the person to whom this suitor is paired in this stable assignment. Use strong induction to show that the deferred acceptance algorithm produces a stable assignment that is optimal for suitors.

25. Suppose that $P(n)$ is a propositional function. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

- **a)** $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+2)$ is true.
- **b)** $P(1)$ and $P(2)$ are true; for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.
- **c)** $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.
- **d)** $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

26. Suppose that $P(n)$ is a propositional function. Determine for which nonnegative integers n the statement $P(n)$ must be true if

- **a)** $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ is true.
- **b)** $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+3)$ is true.
- **c)** $P(0)$ and $P(1)$ are true; for all nonnegative integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.
- **d)** $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ and $P(n+3)$ are true.

27. Show that if the statement $P(n)$ is true for infinitely many positive integers n and $P(n+1) \to P(n)$ is true for all positive integers n, then $P(n)$ is true for all positive integers n.

28. Let b be a fixed integer and j a fixed positive integer. Show that if $P(b), P(b+1), \ldots, P(b+j)$ are true and $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for every integer $k \geq b+j$, then $P(n)$ is true for all integers n with $n \geq b$.

29. What is wrong with this "proof" by strong induction?

"Theorem" For every nonnegative integer n, $5n = 0$.

Basis Step: $5 \cdot 0 = 0$.

Inductive Step: Suppose that $5j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. Write $k+1 = i+j$, where i and j are natural numbers less than $k+1$. By the inductive hypothesis, $5(k+1) = 5(i+j) = 5i + 5j = 0 + 0 = 0$.

***30.** Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0 = 1$ is true by the definition of a^0.

Inductive Step: Assume that $a^j = 1$ for all nonnegative integers j with $j \le k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

***31.** Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

32. Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \le k$ using just three-cent and four-cent stamps. We can then form postage of $k + 1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

33. Show that we can prove that $P(n, k)$ is true for all pairs of positive integers n and k if we show

a) $P(1, 1)$ is true and $P(n, k) \to [P(n + 1, k) \wedge P(n, k + 1)]$ is true for all positive integers n and k.

b) $P(1, k)$ is true for all positive integers k, and $P(n, k) \to P(n + 1, k)$ is true for all positive integers n and k.

c) $P(n, 1)$ is true for all positive integers n, and $P(n, k) \to P(n, k + 1)$ is true for all positive integers n and k.

34. Prove that $\sum_{j=1}^{n} j(j + 1)(j + 2) \cdots (j + k - 1) = n(n + 1)(n + 2) \cdots (n + k)/(k + 1)$ for all positive integers k and n. [*Hint*: Use a technique from Exercise 33.]

***35.** Show that if $a_1, a_2, \ldots, a_n$ are n distinct real numbers, exactly $n - 1$ multiplications are used to compute the product of these n numbers no matter how parentheses are inserted into their product. [*Hint:* Use strong induction and consider the last multiplication.]

***36.** The well-ordering property can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let S be the set of positive integers of the form $as + bt$, where s and t are integers.

a) Show that S is nonempty.

b) Use the well-ordering property to show that S has a smallest element c.

c) Show that if d is a common divisor of a and b, then d is a divisor of c.

d) Show that $c \mid a$ and $c \mid b$. [*Hint:* First, assume that $c \nmid a$. Then $a = qc + r$, where $0 < r < c$. Show that $r \in S$, contradicting the choice of c.]

e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing that this greatest common divisor is unique.

37. Let a be an integer and d be a positive integer. Show that the integers q and r with $a = dq + r$ and $0 \le r < d$, which were shown to exist in Example 5, are unique.

38. Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

****39.** Can you use the well-ordering property to prove the statement: "Every positive integer can be described using no more than fifteen English words"? Assume the words come from a particular dictionary of English. [*Hint:* Suppose that there are positive integers that cannot be described using no more than fifteen English words. By well ordering, *the smallest positive integer that cannot be described using no more than fifteen English words* would then exist.]

40. Use the well-ordering principle to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$. [*Hint:* Use the Archimedean property, given in Appendix 1, to find a positive integer A with $A > 1/(y - x)$. Then show that there is a rational number r with denominator A between x and y by looking at the numbers $\lfloor x \rfloor + j/A$, where j is a positive integer.]

***41.** Show that the well-ordering property can be proved when the principle of mathematical induction is taken as an axiom.

***42.** Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

***43.** Show that we can prove the well-ordering property when we take strong induction as an axiom instead of taking the well-ordering property as an axiom.

5.3 Recursive Definitions and Structural Induction

Introduction

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called **recursion**. For instance, the picture shown in Figure 1 is produced recursively. First, an original picture is given. Then a process of successively superimposing centered smaller pictures on top of the previous pictures is carried out.

EXAMPLE 13 Suppose that $a_{m,n}$ is defined recursively for $(m, n) \in \mathbf{N} \times \mathbf{N}$ by $a_{0,0} = 0$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + n & \text{if } n > 0. \end{cases}$$

Show that $a_{m,n} = m + n(n + 1)/2$ for all $(m, n) \in \mathbf{N} \times \mathbf{N}$, that is, for all pairs of nonnegative integers.

Solution: We can prove that $a_{m,n} = m + n(n + 1)/2$ using a generalized version of mathematical induction. The basis step requires that we show that this formula is valid when $(m, n) = (0, 0)$. The induction step requires that we show that if the formula holds for all pairs smaller than (m, n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$, then it also holds for (m, n).

BASIS STEP: Let $(m, n) = (0, 0)$. Then by the basis case of the recursive definition of $a_{m,n}$ we have $a_{0,0} = 0$. Furthermore, when $m = n = 0$, $m + n(n + 1)/2 = 0 + (0 \cdot 1)/2 = 0$. This completes the basis step.

INDUCTIVE STEP: Suppose that $a_{m',n'} = m' + n'(n' + 1)/2$ whenever (m', n') is less than (m, n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$. By the recursive definition, if $n = 0$, then $a_{m,n} = a_{m-1,n} + 1$. Because $(m - 1, n)$ is smaller than (m, n), the inductive hypothesis tells us that $a_{m-1,n} = m - 1 + n(n + 1)/2$, so that $a_{m,n} = m - 1 + n(n + 1)/2 + 1 = m + n(n + 1)/2$, giving us the desired equality. Now suppose that $n > 0$, so $a_{m,n} = a_{m,n-1} + n$. Because $(m, n - 1)$ is smaller than (m, n), the inductive hypothesis tells us that $a_{m,n-1} = m + (n - 1)n/2$, so $a_{m,n} = m + (n - 1)n/2 + n = m + (n^2 - n + 2n)/2 = m + n(n + 1)/2$. This finishes the inductive step. ◀

As mentioned, we will justify this proof technique in Section 9.6.

Exercises

1. Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 0, 1, 2, \ldots$
   a) $f(n + 1) = f(n) + 2$.
   b) $f(n + 1) = 3f(n)$.
   c) $f(n + 1) = 2^{f(n)}$.
   d) $f(n + 1) = f(n)^2 + f(n) + 1$.

2. Find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if $f(n)$ is defined recursively by $f(0) = 3$ and for $n = 0, 1, 2, \ldots$
   a) $f(n + 1) = -2f(n)$.
   b) $f(n + 1) = 3f(n) + 7$.
   c) $f(n + 1) = f(n)^2 - 2f(n) - 2$.
   d) $f(n + 1) = 3^{f(n)/3}$.

3. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = -1$, $f(1) = 2$, and for $n = 1, 2, \ldots$
   a) $f(n + 1) = f(n) + 3f(n - 1)$.
   b) $f(n + 1) = f(n)^2 f(n - 1)$.
   c) $f(n + 1) = 3f(n)^2 - 4f(n - 1)^2$.
   d) $f(n + 1) = f(n - 1)/f(n)$.

4. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and for $n = 1, 2, \ldots$
   a) $f(n + 1) = f(n) - f(n - 1)$.
   b) $f(n + 1) = f(n)f(n - 1)$.
   c) $f(n + 1) = f(n)^2 + f(n - 1)^3$.
   d) $f(n + 1) = f(n)/f(n - 1)$.

5. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.
   a) $f(0) = 0$, $f(n) = 2f(n - 2)$ for $n \geq 1$
   b) $f(0) = 1$, $f(n) = f(n - 1) - 1$ for $n \geq 1$
   c) $f(0) = 2$, $f(1) = 3$, $f(n) = f(n - 1) - 1$ for $n \geq 2$
   d) $f(0) = 1$, $f(1) = 2$, $f(n) = 2f(n - 2)$ for $n \geq 2$
   e) $f(0) = 1$, $f(n) = 3f(n - 1)$ if n is odd and $n \geq 1$ and $f(n) = 9f(n - 2)$ if n is even and $n \geq 2$

6. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.
   a) $f(0) = 1$, $f(n) = -f(n - 1)$ for $n \geq 1$
   b) $f(0) = 1$, $f(1) = 0$, $f(2) = 2$, $f(n) = 2f(n - 3)$ for $n \geq 3$
   c) $f(0) = 0$, $f(1) = 1$, $f(n) = 2f(n + 1)$ for $n \geq 2$
   d) $f(0) = 0$, $f(1) = 1$, $f(n) = 2f(n - 1)$ for $n \geq 1$
   e) $f(0) = 2$, $f(n) = f(n - 1)$ if n is odd and $n \geq 1$ and $f(n) = 2f(n - 2)$ if $n \geq 2$

7. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 6n$. **b)** $a_n = 2n + 1$.
c) $a_n = 10^n$. **d)** $a_n = 5$.

8. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n - 2$. **b)** $a_n = 1 + (-1)^n$.
c) $a_n = n(n + 1)$. **d)** $a_n = n^2$.

9. Let F be the function such that $F(n)$ is the sum of the first n positive integers. Give a recursive definition of $F(n)$.

10. Give a recursive definition of $S_m(n)$, the sum of the integer m and the nonnegative integer n.

11. Give a recursive definition of $P_m(n)$, the product of the integer m and the nonnegative integer n.

In Exercises 12–19 f_n is the nth Fibonacci number.

12. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

13. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ when n is a positive integer.

***14.** Show that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

***15.** Show that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$ when n is a positive integer.

***16.** Show that $f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ when n is a positive integer.

17. Determine the number of divisions used by the Euclidean algorithm to find the greatest common divisor of the Fibonacci numbers f_n and f_{n+1}, where n is a nonnegative integer. Verify your answer using mathematical induction.

18. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Show that

$$\mathbf{A}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

when n is a positive integer.

19. By taking determinants of both sides of the equation in Exercise 18, prove the identity given in Exercise 14. (Recall that the determinant of the matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$.)

***20.** Give a recursive definition of the functions max and min so that $\max(a_1, a_2, \ldots, a_n)$ and $\min(a_1, a_2, \ldots, a_n)$ are the maximum and minimum of the n numbers $a_1, a_2, \ldots, a_n$, respectively.

***21.** Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ be real numbers. Use the recursive definitions that you gave in Exercise 20 to prove these.

a) $\max(-a_1, -a_2, \ldots, -a_n) = -\min(a_1, a_2, \ldots, a_n)$

b) $\max(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \leq \max(a_1, a_2, \ldots, a_n) + \max(b_1, b_2, \ldots, b_n)$

c) $\min(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \geq \min(a_1, a_2, \ldots, a_n) + \min(b_1, b_2, \ldots, b_n)$

22. Show that the set S defined by $1 \in S$ and $s + t \in S$ whenever $s \in S$ and $t \in S$ is the set of positive integers.

23. Give a recursive definition of the set of positive integers that are multiples of 5.

24. Give a recursive definition of

a) the set of odd positive integers.
b) the set of positive integer powers of 3.
c) the set of polynomials with integer coefficients.

25. Give a recursive definition of

a) the set of even integers.
b) the set of positive integers congruent to 2 modulo 3.
c) the set of positive integers not divisible by 5.

26. Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

a) List the elements of S produced by the first five applications of the recursive definition.
b) Use strong induction on the number of applications of the recursive step of the definition to show that $5 \mid a + b$ when $(a, b) \in S$.
c) Use structural induction to show that $5 \mid a + b$ when $(a, b) \in S$.

27. Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a, b + 1) \in S$, $(a + 1, b + 1) \in S$, and $(a + 2, b + 1) \in S$.

a) List the elements of S produced by the first four applications of the recursive definition.
b) Use strong induction on the number of applications of the recursive step of the definition to show that $a \leq 2b$ whenever $(a, b) \in S$.
c) Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.

28. Give a recursive definition of each of these sets of ordered pairs of positive integers. [*Hint:* Plot the points in the set in the plane and look for lines containing points in the set.]

a) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{and } a + b \text{ is odd}\}$
b) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{and } a \mid b\}$
c) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{and } 3 \mid a + b\}$

29. Give a recursive definition of each of these sets of ordered pairs of positive integers. Use structural induction to prove that the recursive definition you found is correct. [*Hint:* To find a recursive definition, plot the points in the set in the plane and look for patterns.]

a) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{and } a + b \text{ is even}\}$
b) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, \text{and } a \text{ or } b \text{ is odd}\}$
c) $S = \{(a, b) \mid a \in \mathbf{Z}^+, b \in \mathbf{Z}^+, a + b \text{ is odd, and } 3 \mid b\}$

30. Prove that in a bit string, the string 01 occurs at most one more time than the string 10.

31. Define well-formed formulae of sets, variables representing sets, and operators from $\{^-, \cup, \cap, -\}$.

32. **a)** Give a recursive definition of the function $ones(s)$, which counts the number of ones in a bit string s.
b) Use structural induction to prove that $ones(st) = ones(s) + ones(t)$.

33. **a)** Give a recursive definition of the function $m(s)$, which equals the smallest digit in a nonempty string of decimal digits.
b) Use structural induction to prove that $m(st) = \min(m(s), m(t))$.

The **reversal** of a string is the string consisting of the symbols of the string in reverse order. The reversal of the string w is denoted by w^R.

34. Find the reversal of the following bit strings.
a) 0101 **b)** 1 1011 **c)** 1000 1001 0111

35. Give a recursive definition of the reversal of a string. [*Hint:* First define the reversal of the empty string. Then write a string w of length $n + 1$ as xy, where x is a string of length n, and express the reversal of w in terms of x^R and y.]

***36.** Use structural induction to prove that $(w_1w_2)^R = w_2^R w_1^R$.

37. Give a recursive definition of w^i, where w is a string and i is a nonnegative integer. (Here w^i represents the concatenation of i copies of the string w.)

***38.** Give a recursive definition of the set of bit strings that are palindromes.

39. When does a string belong to the set A of bit strings defined recursively by

$$\lambda \in A$$
$$0x1 \in A \text{ if } x \in A,$$

where λ is the empty string?

***40.** Recursively define the set of bit strings that have more zeros than ones.

41. Use Exercise 37 and mathematical induction to show that $l(w^i) = i \cdot l(w)$, where w is a string and i is a nonnegative integer.

***42.** Show that $(w^R)^i = (w^i)^R$ whenever w is a string and i is a nonnegative integer; that is, show that the ith power of the reversal of a string is the reversal of the ith power of the string.

43. Use structural induction to show that $n(T) \geq 2h(T) + 1$, where T is a full binary tree, $n(T)$ equals the number of vertices of T, and $h(T)$ is the height of T.

The set of leaves and the set of internal vertices of a full binary tree can be defined recursively.

Basis step: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices.

Recursive step: The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and of T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices of T_2.

44. Use structural induction to show that $l(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal vertices of T.

45. Use generalized induction as was done in Example 13 to show that if $a_{m,n}$ is defined recursively by $a_{0,0} = 0$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + 1 & \text{if } n > 0, \end{cases}$$

then $a_{m,n} = m + n$ for all $(m, n) \in \mathbf{N} \times \mathbf{N}$.

46. Use generalized induction as was done in Example 13 to show that if $a_{m,n}$ is defined recursively by $a_{1,1} = 5$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 2 & \text{if } n = 1 \text{ and } m > 1 \\ a_{m,n-1} + 2 & \text{if } n > 1, \end{cases}$$

then $a_{m,n} = 2(m + n) + 1$ for all $(m, n) \in \mathbf{Z}^+ \times \mathbf{Z}^+$.

***47.** A **partition** of a positive integer n is a way to write n as a sum of positive integers where the order of terms in the sum does not matter. For instance, $7 = 3 + 2 + 1 + 1$ is a partition of 7. Let P_m equal the number of different partitions of m, and let $P_{m,n}$ be the number of different ways to express m as the sum of positive integers not exceeding n.
a) Show that $P_{m,m} = P_m$.
b) Show that the following recursive definition for $P_{m,n}$ is correct:

$$P_{m,n} = \begin{cases} 1 & \text{if } m = 1 \\ 1 & \text{if } n = 1 \\ P_{m,m} & \text{if } m < n \\ 1 + P_{m,m-1} & \text{if } m = n > 1 \\ P_{m,n-1} + P_{m-n,n} & \text{if } m > n > 1. \end{cases}$$

c) Find the number of partitions of 5 and of 6 using this recursive definition.

Consider an inductive definition of a version of **Ackermann's function**. This function was named after Wilhelm Ackermann, a German mathematician who was a student of the great mathematician David Hilbert. Ackermann's function plays an important role in the theory of recursive functions and in the study of the complexity of certain algorithms involving set unions. (There are several different variants of this function. All are called Ackermann's function and have similar properties even though their values do not always agree.)

$$A(m, n) = \begin{cases} 2n & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}$$

Exercises 48–55 involve this version of Ackermann's function.

48. Find these values of Ackermann's function.
a) $A(1, 0)$ **b)** $A(0, 1)$
c) $A(1, 1)$ **d)** $A(2, 2)$

49. Show that $A(m, 2) = 4$ whenever $m \geq 1$.

50. Show that $A(1, n) = 2^n$ whenever $n \geq 1$.

51. Find these values of Ackermann's function.
a) $A(2, 3)$ ***b)** $A(3, 3)$

***52.** Find $A(3, 4)$.

**53. Prove that $A(m, n+1) > A(m, n)$ whenever m and n are nonnegative integers.

*54. Prove that $A(m+1, n) \geq A(m, n)$ whenever m and n are nonnegative integers.

55. Prove that $A(i, j) \geq j$ whenever i and j are nonnegative integers.

56. Use mathematical induction to prove that a function F defined by specifying $F(0)$ and a rule for obtaining $F(n+1)$ from $F(n)$ is well defined.

57. Use strong induction to prove that a function F defined by specifying $F(0)$ and a rule for obtaining $F(n+1)$ from the values $F(k)$ for $k = 0, 1, 2, \ldots, n$ is well defined.

58. Show that each of these proposed recursive definitions of a function on the set of positive integers does not produce a well-defined function.

a) $F(n) = 1 + F(\lfloor n/2 \rfloor)$ for $n \geq 1$ and $F(1) = 1$.
b) $F(n) = 1 + F(n-3)$ for $n \geq 2$, $F(1) = 2$, and $F(2) = 3$.
c) $F(n) = 1 + F(n/2)$ for $n \geq 2$, $F(1) = 1$, and $F(2) = 2$.
d) $F(n) = 1 + F(n/2)$ if n is even and $n \geq 2$, $F(n) = 1 - F(n-1)$ if n is odd, and $F(1) = 1$.
e) $F(n) = 1 + F(n/2)$ if n is even and $n \geq 2$, $F(n) = F(3n-1)$ if n is odd and $n \geq 3$, and $F(1) = 1$.

59. Show that each of these proposed recursive definitions of a function on the set of positive integers does not produce a well-defined function.

a) $F(n) = 1 + F(\lfloor (n+1)/2 \rfloor)$ for $n \geq 1$ and $F(1) = 1$.
b) $F(n) = 1 + F(n-2)$ for $n \geq 2$ and $F(1) = 0$.
c) $F(n) = 1 + F(n/3)$ for $n \geq 3$, $F(1) = 1$, $F(2) = 2$, and $F(3) = 3$.
d) $F(n) = 1 + F(n/2)$ if n is even and $n \geq 2$, $F(n) = 1 + F(n-2)$ if n is odd, and $F(1) = 1$.
e) $F(n) = 1 + F(F(n-1))$ if $n \geq 2$ and $F(1) = 2$.

Exercises 60–62 deal with iterations of the logarithm function. Let $\log n$ denote the logarithm of n to the base 2, as usual. The function $\log^{(k)} n$ is defined recursively by

$$\log^{(k)} n = \begin{cases} n & \text{if } k = 0 \\ \log(\log^{(k-1)} n) & \text{if } \log^{(k-1)} n \text{ is defined and positive} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The **iterated logarithm** is the function $\log^* n$ whose value at n is the smallest nonnegative integer k such that $\log^{(k)} n \leq 1$.

60. Find these values.

a) $\log^{(2)} 16$
b) $\log^{(3)} 256$
c) $\log^{(3)} 2^{65536}$
d) $\log^{(4)} 2^{2^{65536}}$

61. Find the value of $\log^* n$ for these values of n.

a) 2
b) 4
c) 8
d) 16
e) 256
f) 65536
g) 2^{2048}

62. Find the largest integer n such that $\log^* n = 5$. Determine the number of decimal digits in this number.

Exercises 63–65 deal with values of iterated functions. Suppose that $f(n)$ is a function from the set of real numbers, or positive real numbers, or some other set of real numbers, to the set of real numbers such that $f(n)$ is monotonically increasing [that is, $f(n) < f(m)$ when $n < m$) and $f(n) < n$ for all n in the domain of f.] The function $f^{(k)}(n)$ is defined recursively by

$$f^{(k)}(n) = \begin{cases} n & \text{if } k = 0 \\ f(f^{(k-1)}(n)) & \text{if } k > 0. \end{cases}$$

Furthermore, let c be a positive real number. The **iterated function** f_c^* is the number of iterations of f required to reduce its argument to c or less, so $f_c^*(n)$ is the smallest nonnegative integer k such that $f^k(n) \leq c$.

63. Let $f(n) = n - a$, where a is a positive integer. Find a formula for $f^{(k)}(n)$. What is the value of $f_0^*(n)$ when n is a positive integer?

64. Let $f(n) = n/2$. Find a formula for $f^{(k)}(n)$. What is the value of $f_1^*(n)$ when n is a positive integer?

65. Let $f(n) = \sqrt{n}$. Find a formula for $f^{(k)}(n)$. What is the value of $f_2^*(n)$ when n is a positive integer?

5.4 Recursive Algorithms

Introduction

Here's a famous humorous quote: "To understand recursion, you must first understand recursion."

Sometimes we can reduce the solution to a problem with a particular set of input values to the solution of the same problem with smaller input values. For instance, the problem of finding the greatest common divisor of two positive integers a and b, where $b > a$, can be reduced to finding the greatest common divisor of a pair of smaller integers, namely, b **mod** a and a, because $\gcd(b \textbf{ mod } a, a) = \gcd(a, b)$. When such a reduction can be done, the solution to the original problem can be found with a sequence of reductions, until the problem has been reduced to some initial case for which the solution is known. For instance, for finding the greatest common divisor, the reduction continues until the smaller of the two numbers is zero, because $\gcd(a, 0) = a$ when $a > 0$.

We will see that algorithms that successively reduce a problem to the same problem with smaller input are used to solve a wide variety of problems.

by Lemma 1, each of these mergers can be carried out using at most $2^{m-k} + 2^{m-k} - 1 = 2^{m-k+1} - 1$ comparisons. Hence, going from level k to $k-1$ can be accomplished using at most $2^{k-1}(2^{m-k+1} - 1)$ comparisons.

Summing all these estimates shows that the number of comparisons required for the merge sort is at most

$$\sum_{k=1}^{m} 2^{k-1}(2^{m-k+1} - 1) = \sum_{k=1}^{m} 2^m - \sum_{k=1}^{m} 2^{k-1} = m2^m - (2^m - 1) = n \log n - n + 1,$$

because $m = \log n$ and $n = 2^m$. (We evaluated $\sum_{k=1}^{m} 2^m$ by noting that it is the sum of m identical terms, each equal to 2^m. We evaluated $\sum_{k=1}^{m} 2^{k-1}$ using the formula for the sum of the terms of a geometric progression from Theorem 1 of Section 2.4.)

Theorem 1 summarizes what we have discovered about the worst-case complexity of the merge sort algorithm.

THEOREM 1 The number of comparisons needed to merge sort a list with n elements is $O(n \log n)$.

In Chapter 11 we will show that the fastest comparison-based sorting algorithm have $O(n \log n)$ time complexity. (A comparison-based sorting algorithm has the comparison of two elements as its basic operation.) Theorem 1 tells us that the merge sort achieves this best possible big-O estimate for the complexity of a sorting algorithm. We describe another efficient algorithm, the quick sort, in the preamble to Exercise 50.

Exercises

1. Trace Algorithm 1 when it is given $n = 5$ as input. That is, show all steps used by Algorithm 1 to find 5!, as is done in Example 1 to find 4!.

2. Trace Algorithm 1 when it is given $n = 6$ as input. That is, show all steps used by Algorithm 1 to find 6!, as is done in Example 1 to find 4!.

3. Trace Algorithm 3 when it finds $\gcd(8, 13)$. That is, show all the steps used by Algorithm 3 to find $\gcd(8, 13)$.

4. Trace Algorithm 3 when it finds $\gcd(12, 17)$. That is, show all the steps used by Algorithm 3 to find $\gcd(12, 17)$.

5. Trace Algorithm 4 when it is given $m = 5$, $n = 11$, and $b = 3$ as input. That is, show all the steps Algorithm 4 uses to find $3^{11} \mathbf{mod}\ 5$.

6. Trace Algorithm 4 when it is given $m = 7$, $n = 10$, and $b = 2$ as input. That is, show all the steps Algorithm 4 uses to find $2^{10} \mathbf{mod}\ 7$.

7. Give a recursive algorithm for computing nx whenever n is a positive integer and x is an integer, using just addition.

8. Give a recursive algorithm for finding the sum of the first n positive integers.

9. Give a recursive algorithm for finding the sum of the first n odd positive integers.

10. Give a recursive algorithm for finding the maximum of a finite set of integers, making use of the fact that the maximum of n integers is the larger of the last integer in the list and the maximum of the first $n - 1$ integers in the list.

11. Give a recursive algorithm for finding the minimum of a finite set of integers, making use of the fact that the minimum of n integers is the smaller of the last integer in the list and the minimum of the first $n - 1$ integers in the list.

12. Devise a recursive algorithm for finding $x^n \mathbf{mod}\ m$ whenever n, x, and m are positive integers based on the fact that $x^n \mathbf{mod}\ m = (x^{n-1} \mathbf{mod}\ m \cdot x \mathbf{mod}\ m) \mathbf{mod}\ m$.

13. Give a recursive algorithm for finding $n! \mathbf{mod}\ m$ whenever n and m are positive integers.

14. Give a recursive algorithm for finding a **mode** of a list of integers. (A **mode** is an element in the list that occurs at least as often as every other element.)

15. Devise a recursive algorithm for computing the greatest common divisor of two nonnegative integers a and b with $a < b$ using the fact that $\gcd(a, b) = \gcd(a, b - a)$.

16. Prove that the recursive algorithm for finding the sum of the first n positive integers you found in Exercise 8 is correct.

17. Describe a recursive algorithm for multiplying two nonnegative integers x and y based on the fact that $xy = 2(x \cdot (y/2))$ when y is even and $xy = 2(x \cdot \lfloor y/2 \rfloor) + x$ when y is odd, together with the initial condition $xy = 0$ when $y = 0$.

18. Prove that Algorithm 1 for computing $n!$ when n is a nonnegative integer is correct.

19. Prove that Algorithm 3 for computing $\gcd(a, b)$ when a and b are positive integers with $a < b$ is correct.

20. Prove that the algorithm you devised in Exercise 17 is correct.

21. Prove that the recursive algorithm that you found in Exercise 7 is correct.

22. Prove that the recursive algorithm that you found in Exercise 10 is correct.

23. Devise a recursive algorithm for computing n^2 where n is a nonnegative integer, using the fact that $(n+1)^2 = n^2 + 2n + 1$. Then prove that this algorithm is correct.

24. Devise a recursive algorithm to find a^{2^n}, where a is a real number and n is a positive integer. [*Hint:* Use the equality $a^{2^{n+1}} = (a^{2^n})^2$.]

25. How does the number of multiplications used by the algorithm in Exercise 24 compare to the number of multiplications used by Algorithm 2 to evaluate a^{2^n}?

***26.** Use the algorithm in Exercise 24 to devise an algorithm for evaluating a^n when n is a nonnegative integer. [*Hint:* Use the binary expansion of n.]

***27.** How does the number of multiplications used by the algorithm in Exercise 26 compare to the number of multiplications used by Algorithm 2 to evaluate a^n?

28. How many additions are used by the recursive and iterative algorithms given in Algorithms 7 and 8, respectively, to find the Fibonacci number f_7?

29. Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1, a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$, for $n = 2, 3, 4, \ldots$.

30. Devise an iterative algorithm to find the nth term of the sequence defined in Exercise 29.

31. Is the recursive or the iterative algorithm for finding the sequence in Exercise 29 more efficient?

32. Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n = 3, 4, 5, \ldots$.

33. Devise an iterative algorithm to find the nth term of the sequence defined in Exercise 32.

34. Is the recursive or the iterative algorithm for finding the sequence in Exercise 32 more efficient?

35. Give iterative and recursive algorithms for finding the nth term of the sequence defined by $a_0 = 1, a_1 = 3, a_2 = 5$, and $a_n = a_{n-1} \cdot a_{n-2}^2 \cdot a_{n-3}^3$. Which is more efficient?

36. Give a recursive algorithm to find the number of partitions of a positive integer based on the recursive definition given in Exercise 47 in Section 5.3.

37. Give a recursive algorithm for finding the reversal of a bit string. (See the definition of the reversal of a bit string in the preamble of Exercise 34 in Section 5.3.)

38. Give a recursive algorithm for finding the string w^i, the concatenation of i copies of w, when w is a bit string.

39. Prove that the recursive algorithm for finding the reversal of a bit string that you gave in Exercise 37 is correct.

40. Prove that the recursive algorithm for finding the concatenation of i copies of a bit string that you gave in Exercise 38 is correct.

***41.** Give a recursive algorithm for tiling a $2^n \times 2^n$ checkerboard with one square missing using right triominoes.

42. Give a recursive algorithm for triangulating a simple polygon with n sides, using Lemma 1 in Section 5.2.

43. Give a recursive algorithm for computing values of the Ackermann function. [*Hint:* See the preamble to Exercise 48 in Section 5.3.]

44. Use a merge sort to sort 4, 3, 2, 5, 1, 8, 7, 6 into increasing order. Show all the steps used by the algorithm.

45. Use a merge sort to sort *b, d, a, f, g, h, z, p, o, k* into alphabetic order. Show all the steps used by the algorithm.

46. How many comparisons are required to merge these pairs of lists using Algorithm 10?

a) 1, 3, 5, 7, 9; 2, 4, 6, 8, 10
b) 1, 2, 3, 4, 5; 6, 7, 8, 9, 10
c) 1, 5, 6, 7, 8; 2, 3, 4, 9, 10

47. Show that for all positive integers m and n there are sorted lists with m elements and n elements, respectively, such that Algorithm 10 uses $m + n - 1$ comparisons to merge them into one sorted list.

***48.** What is the least number of comparisons needed to merge any two lists in increasing order into one list in increasing order when the number of elements in the two lists are

a) 1, 4? **b)** 2, 4? **c)** 3, 4? **d)** 4, 4?

***49.** Prove that the merge sort algorithm is correct.

The **quick sort** is an efficient algorithm. To sort $a_1, a_2, \ldots, a_n$, this algorithm begins by taking the first element a_1 and forming two sublists, the first containing those elements that are less than a_1, in the order they arise, and the second containing those elements greater than a_1, in the order they arise. Then a_1 is put at the end of the first sublist. This procedure is repeated recursively for each sublist, until all sublists contain one item. The ordered list of n items is obtained by combining the sublists of one item in the order they occur.

50. Sort 3, 5, 7, 8, 1, 9, 2, 4, 6 using the quick sort.

51. Let $a_1, a_2, \ldots, a_n$ be a list of n distinct real numbers. How many comparisons are needed to form two sublists from this list, the first containing elements less than a_1 and the second containing elements greater than a_1?

52. Describe the quick sort algorithm using pseudocode.

53. What is the largest number of comparisons needed to order a list of four elements using the quick sort algorithm?

54. What is the least number of comparisons needed to order a list of four elements using the quick sort algorithm?

55. Determine the worst-case complexity of the quick sort algorithm in terms of the number of comparisons used.