**4190.101**

# Discrete Mathematics

## Chapter 8 Advanced Counting Techniques

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[Based on S. Srinivasa Rao's Lecture Notes]

# Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

# Applications of Recurrence Relations

Section 8.1

# Section Summary

- Applications of Recurrence Relations
  - Fibonacci Numbers
  - The Tower of Hanoi
  - Counting Problems

# Recurrence Relations (recalling definitions from Chapter 2)

- **Definition:** A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses $a_n$ in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers $n$ with $n \geq n_0$, where $n_0$ is a non-negative integer.
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

# Rabbits and the Fiobonacci Numbers

- **Example**: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after $n$ months, assuming that rabbits never die.
- *This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century*.

# Rabbits and the Fiobonacci Numbers

- Modeling the Population Growth of Rabbits on an Island

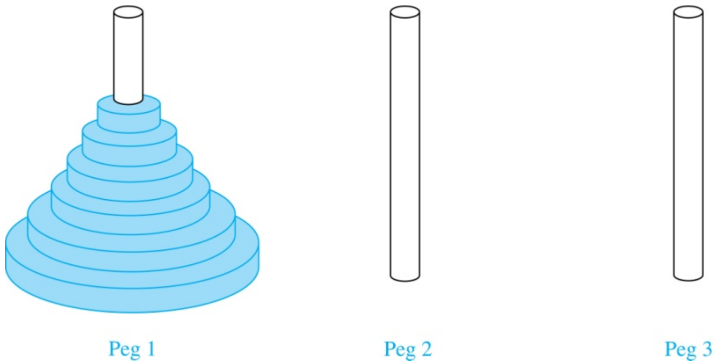
| Reproducing pairs (at least two months old) | Young pairs (less than two months old) | Month | Reproducing pairs | Young pairs | Total pairs |
|---|---|---|---|---|---|
| | | 1 | 0 | 1 | 1 |
| | | 2 | 0 | 1 | 1 |
| | | 3 | 1 | 1 | 2 |
| | | 4 | 1 | 2 | 3 |
| | | 5 | 2 | 3 | 5 |
| | | 6 | 3 | 5 | 8 |

# Rabbits and the Fiobonacci Numbers

- **Solution**: Let $f_n$ be the number of pairs of rabbits after $n$ months.
  - There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
  - We also have $f_2 = 1$ because the pair does not breed during the first month.
  - To find the number of pairs on the island after $n$ months, add the number on the island after the previous month, $f_{n-1}$, and the number of newborn pairs, which equals $f_{n-2}$, because each newborn pair comes from a pair at least two months old.
- Consequently the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.
- The number of pairs of rabbits on the island after $n$ months is given by the $n$-th Fibonacci number

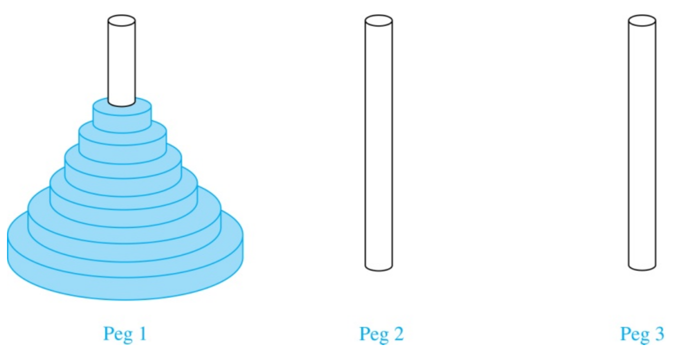
# The Tower of Hanoi

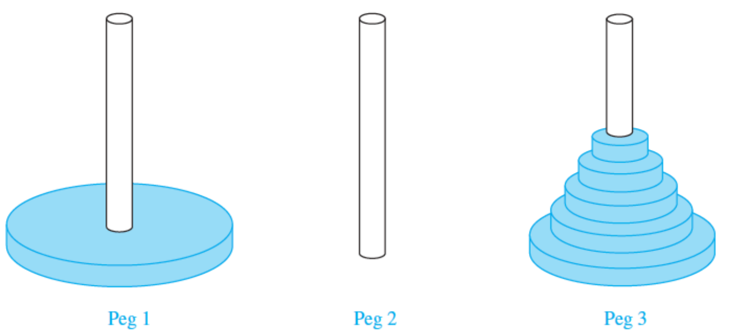
- The puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.
- Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.
- Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.



# The Tower of Hanoi

- Let $H_n$ denote the number of moves needed to solve the Tower of Hanoi problem with $n$ disks. Set up a recurrence relation for the sequence.
- **Solution**: Begin with $n$ disks on peg 1.
  - We can transfer the top $n - 1$ disks to peg 3 using $H_{n-1}$ moves.
  - Then, we use one move to transfer the largest disk to peg 2.
  - We can transfer the $n - 1$ disks on peg 3 to peg 2 using $H_{n-1}$ additional moves.
  - Therefore, $H_n = 2H_{n-1} + 1$.



# The Tower of Hanoi

- We can use an iterative approach to solve this recurrence relation by repeatedly expressing $H_n$ in terms of the previous terms of the sequence.

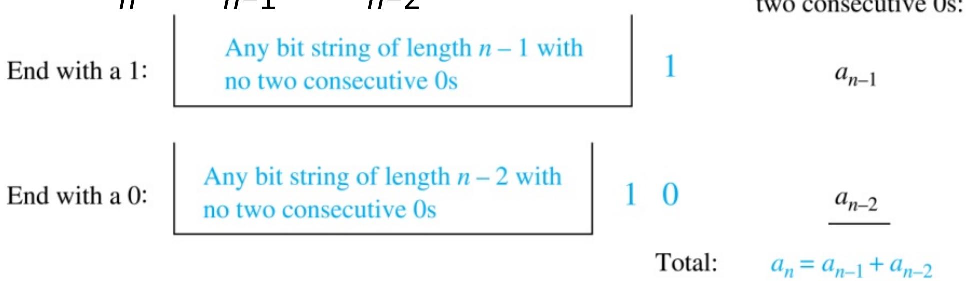
$$H_n = 2H_{n-1} + 1$$
$$= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1$$
$$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1$$
$$\vdots$$
$$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$$
$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 \quad \text{because } H_1 = 1$$
$$= 2^n - 1 \quad \text{using the formula for the sum of the terms of a geometric series}$$

  - There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
  - Using this formula for the 64 gold disks of the myth, $2^{64} - 1$ = 18,446, 744,073, 709,551,615 days are needed to solve the puzzle (more than 500 billion years).

# Counting Bit Strings

- **Example 3**: Find a recurrence relation and give initial conditions for the number of bit strings of length $n$ without two consecutive 0s. How many such bit strings are there of length five?
- **Solution**: Let $a_n$ denote the number of bit strings of length $n$ without two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$ the number of bit strings of length $n$ that do not have two consecutive 0s is the number of bit strings ending with a 0 plus the number of such bit strings ending with a 1.
- Now assume that $n \geq 3$.
  - The bit strings of length $n$ ending with 1 without two consecutive 0s are the bit strings of length $n-1$ with no two consecutive 0s with a 1 at the end. Hence, there are $a_{n-1}$ such bit strings.
  - The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length $n-2$ with no two consecutive 0s with 10 at the end. Hence, there are $a_{n-2}$ such bit strings.
- We conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

Number of bit strings of length $n$ with no two consecutive 0s:



# Counting Bit Strings

- The initial conditions are:
  - $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
  - $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.
- To obtain $a_5$, we use the recurrence relation three times to find that:
  - $a_3 = a_2 + a_1 = 3 + 2 = 5$
  - $a_4 = a_3 + a_2 = 5+ 3 = 8$
  - $a_5 = a_4 + a_3 = 8+ 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

# Counting Bit Strings

- **Example**: Find a recurrence relation for $C_n$ , the number of ways to parenthesize the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, to specify the order of multiplication.
  $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$ , $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$, $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$, $x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$, …
- **Solution**: Note that however parentheses are inserted in $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, one "$\cdot$" operator remains outside all parentheses. This final operator appears between two of the $n + 1$ numbers, say $x_k$ and $x_{k+1}$. Since there are $C_k$ ways to insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_k$ and $C_{n-k-1}$ ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

$$\begin{aligned} C_n &= C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0 \\ &= \sum_{k=0}^{n-1} C_k C_{n-k-1} \end{aligned}$$

- The initial conditions are $C_0 = 1$ and $C_1 = 1$.

The sequence $\{C_n\}$ is the sequence of **Catalan Numbers**. This recurrence relation can be solved using the method of generating functions; see Exercise 41 in Section 8.4.

# Solving Linear Recurrence Relations

Section 8.2

# Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

# Linear Homogeneous Recurrence Relations

- **Definition:** A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k}$, where $c_1, c_2, \ldots\ldots, c_k$ are real numbers, and $c_k \neq 0$

• It is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of $n$.
• It is *homogeneous* because no terms occur that are not multiples of the $a_j$s. Each coefficient is a constant.
• The *degree* is $k$ because $a_n$ is expressed in terms of the previous $k$ terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the $k$ initial conditions $a_0 = C_1$, $a_0 = C_1$ ,… , $a_{k-1} = C_{k-1}$.

# Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ Linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ Linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ Not linear
- $H_n = 2H_{n-1} + 1$ Not homogeneous
- $B_n = nB_{n-1}$ Coefficients are not constants

# Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where $r$ is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_k r^{n-k}$.
- Algebraic manipulation yields the *characteristic equation*: $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0$.
- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if $r$ is a solution to the characteristic equation.
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

# Solving Linear Homogeneous Recurrence Relations of Degree Two

- **Theorem 1**: Let $c_1$ and $c_2$ be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0,1,2,\ldots$ , where $\alpha_1$ and $\alpha_2$ are constants.

# Using Theorem 1

- **Example**: What is the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?
- **Solution**: The characteristic equation is $r^2 - r - 2 = 0$. Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants $\alpha_1$ and $\alpha_2$.
- To find the constants $\alpha_1$ and $\alpha_2$, note that $a_0 = 2 = \alpha_1 + \alpha_2$ and $a_1 = 7 = \alpha_1 2 + \alpha_2(-1)$.
- Solving these equations, we find $\alpha_1 = 3$ and $\alpha_2 = -1$.
- Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3\cdot2^n - (-1)^n$.

# An Explicit Formula for the Fibonacci Numbers

- We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.
- **Solution**: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

*continued* →

# Fibonacci Numbers

- Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

  for some constants $\alpha_1$ and $\alpha_2$.
- Using the initial conditions $f_0$ = 0 and $f_1$ = 1, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

- Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.
- Hence, $f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$

# The Solution When There is a Repeated Root

- **Theorem 2**: Let $c_1$ and $c_2$ be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root $r_0$. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0,1,2,\ldots$ , where $\alpha_1$ and $\alpha_2$ are constants.

# Using Theorem 2

- **Example**: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?
- **Solution**: The characteristic equation is $r^2 - 6r + 9 = 0$.
- The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

  where $\alpha_1$ and $\alpha_2$ are constants.
- To find the constants $\alpha_1$ and $\alpha_2$, note that

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

- Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.
- Hence, $a_n = 3^n + n3^n$.

# Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

- This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.
- **Theorem 3**: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has $k$ distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

# The General Case with Repeated Roots Allowed

- **Theorem 4**: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

  $$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

  has $t$ distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k}$$

  if and only if

  $$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

  for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

# Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

- **Definition:** A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

  $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots.. + c_k\, a_{n-k} + F(n)$ ,

  where $c_1, c_2, \ldots.,c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on $n$.
- The recurrence relation

  $$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots.. + c_k\, a_{n-k} ,$$

  is called the associated homogeneous recurrence relation.

# Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

- The following are linear nonhomogeneous recurrence relations with constant coefficients:
  $a_n = a_{n-1} + 2^n$ ,
  $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,
  $a_n = 3a_{n-1} + n3^n$ ,
  $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$
  where the following are the associated linear homogeneous recurrence relations, respectively:
  $a_n = a_{n-1}$ ,
  $a_n = a_{n-1} + a_{n-2}$,
  $a_n = 3a_{n-1}$ ,
  $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

# Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

- **Theorem 5**: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

  then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

# Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

- **Example**: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
- **Solution**: The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where $\alpha$ is a constant.
- Because $F(n) = 2n$ is a polynomial in $n$ of degree one, to find a particular solution we might try a linear function in $n$, say $p_n = cn + d$, where $c$ and $d$ are constants. Suppose that $p_n = cn + d$ is such a solution.
- Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n- 1) + d)+ 2n.$
- Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = - 1$ and $d = -3/2$.
- Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

*continued* →

# Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

- **Example**: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
- **Solution**: The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.
- By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.
- Let's find the solution with $a_1 = 3$. Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$.
- Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.