

Single Source Shortest Paths Problem

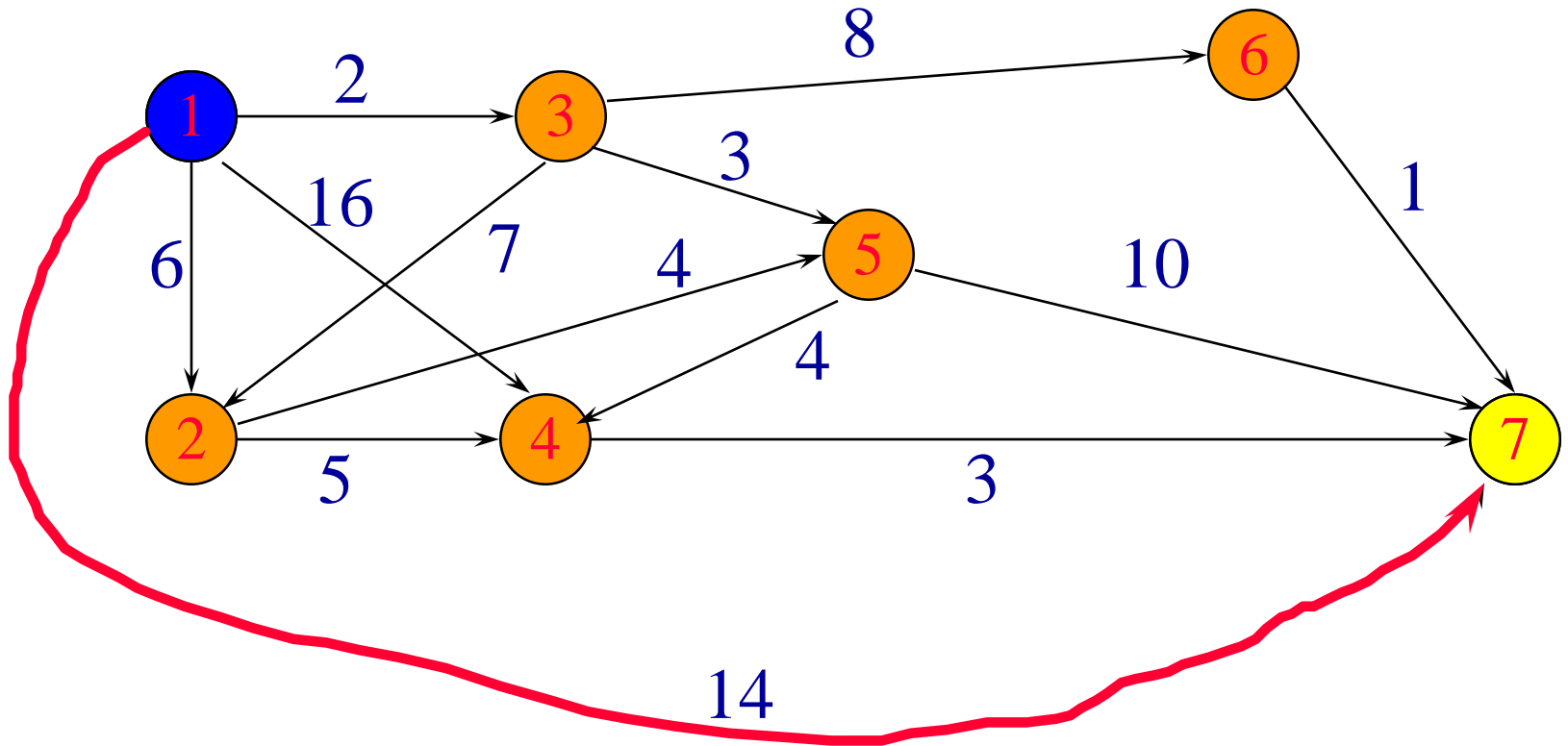
Data structures

Fall 2018

Shortest Path Problems

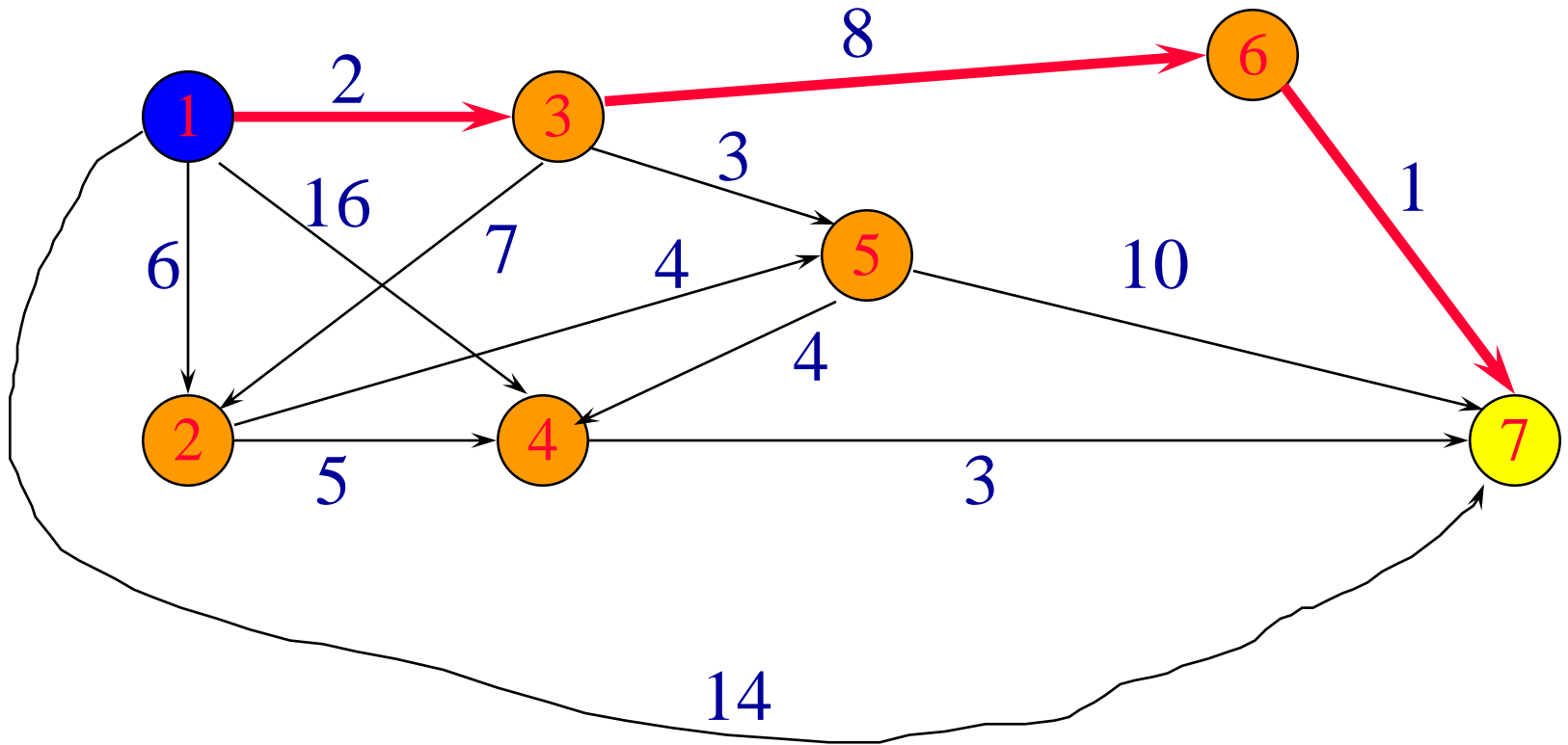
- Directed weighted graph.
- Path length is sum of weights of edges on path.
- The vertex at which the path begins is the **source** vertex.
- The vertex at which the path ends is the **destination** vertex.

Example



A path from 1 to 7.
Path length is 14.

Example



Another path from 1 to 7.
Path length is 11.

Shortest Path Problems

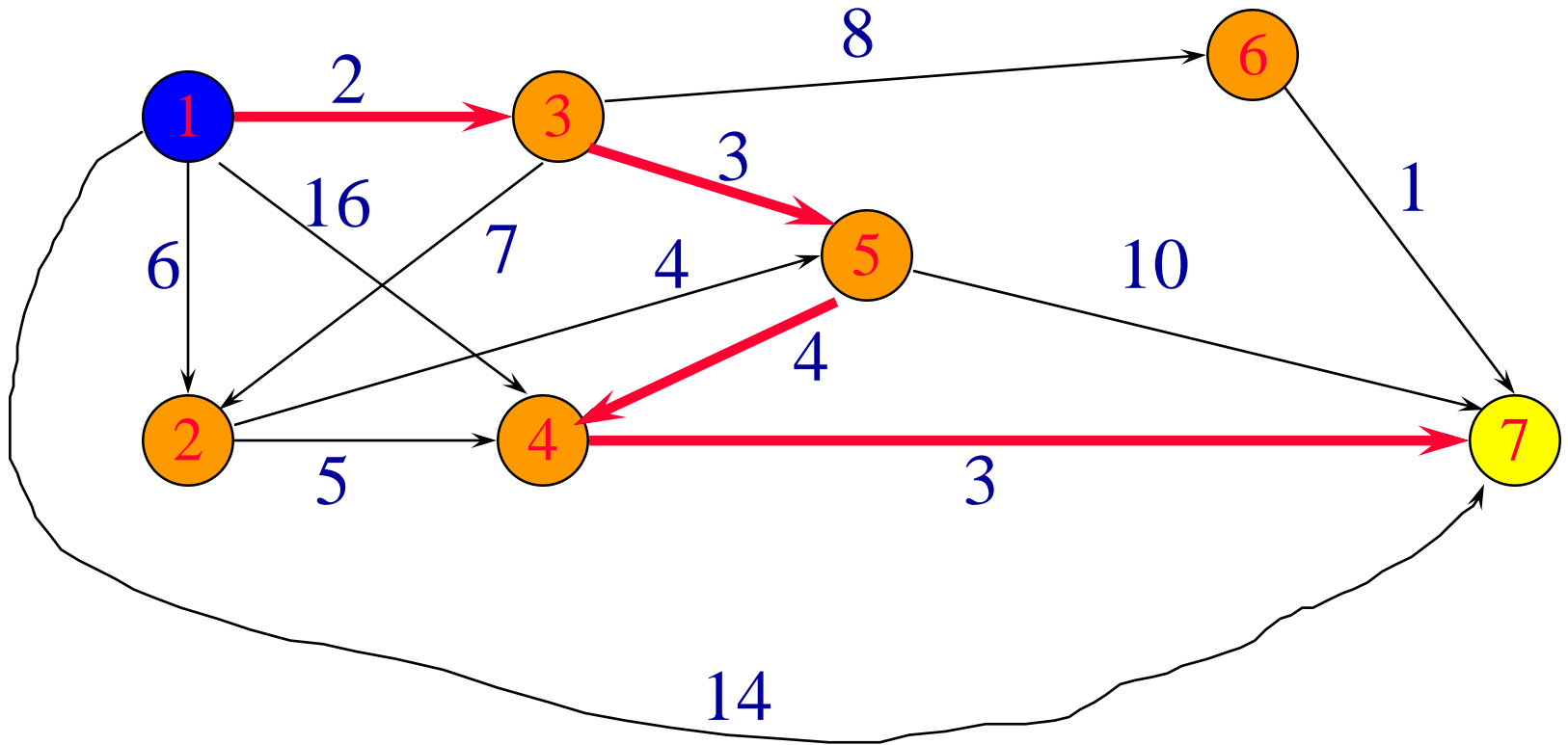
- Single source single destination.
- Single source all destinations.
- All pairs (every vertex is a source and destination).

Single Source Single Destination

Possible greedy algorithm:

- Leave source vertex using cheapest/shortest edge.
- Leave new vertex using cheapest edge subject to the constraint that a new vertex is reached.
- Continue until destination is reached.

Greedy Shortest 1 To 7 Path



Path length is 12.

Not shortest path. Algorithm doesn't work!

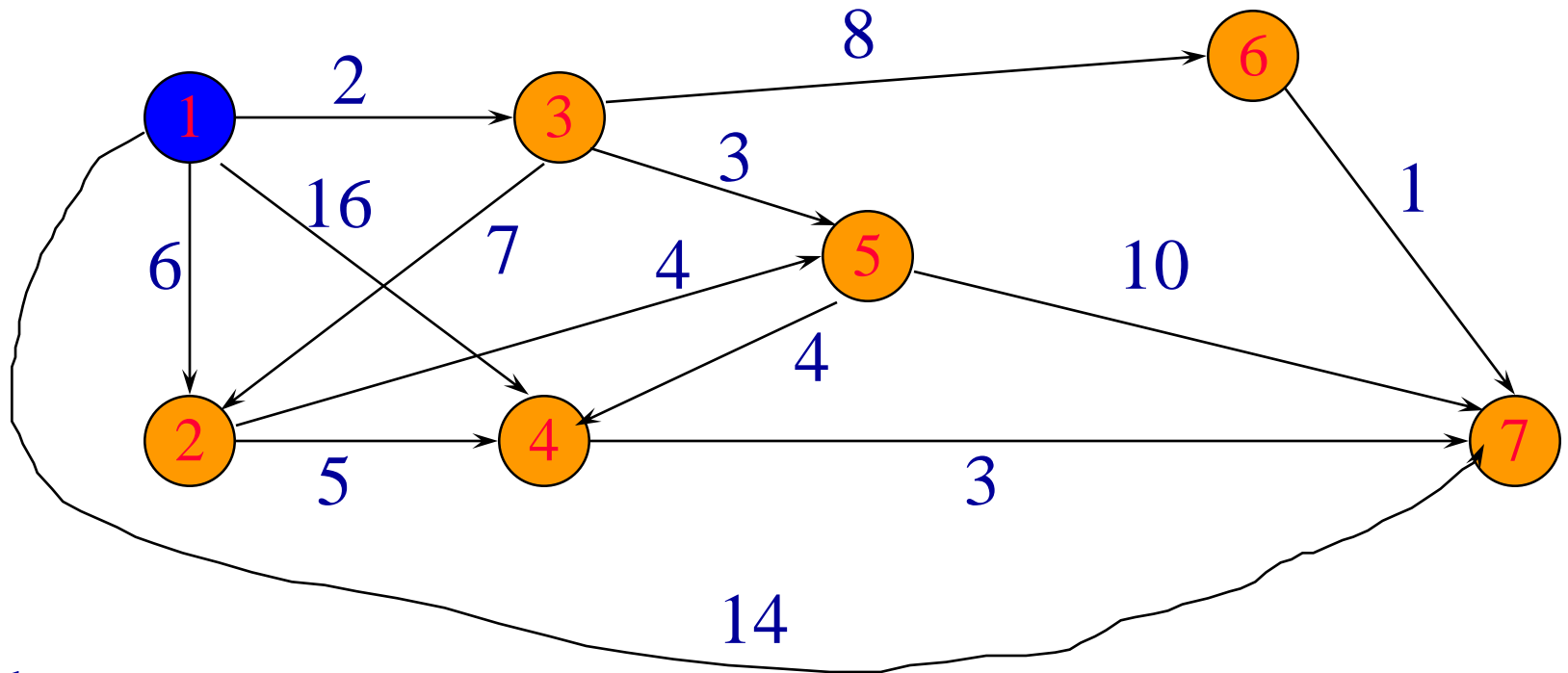
Single Source All Destinations

Need to generate up to n (n is number of vertices) paths (including path from source to itself).

Greedy method:

- Construct these up to n paths in order of increasing length.
- Assume edge costs (lengths) are ≥ 0 .
- So, no path has length < 0 .
- First shortest path is from the source vertex to itself. The length of this path is 0 .

Greedy Single Source All Destinations



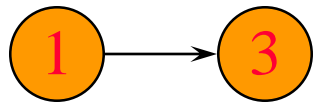
Path	Length		
1	0	1 → 2	6
1 → 3	2	1 → 3 → 5 → 4	9
1 → 3 → 5	5	1 → 3 → 6	10
		1 → 3 → 6 → 7	11

Greedy Single Source All Destinations

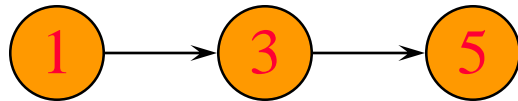
Path Length



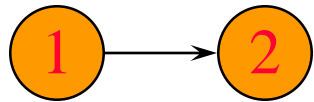
0



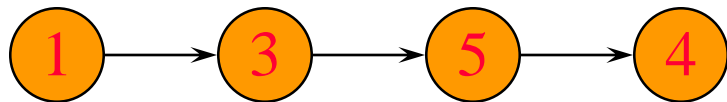
2



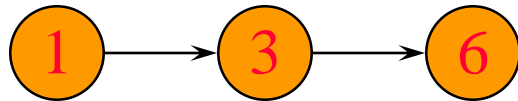
5



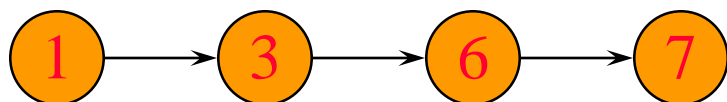
6



9



10



11

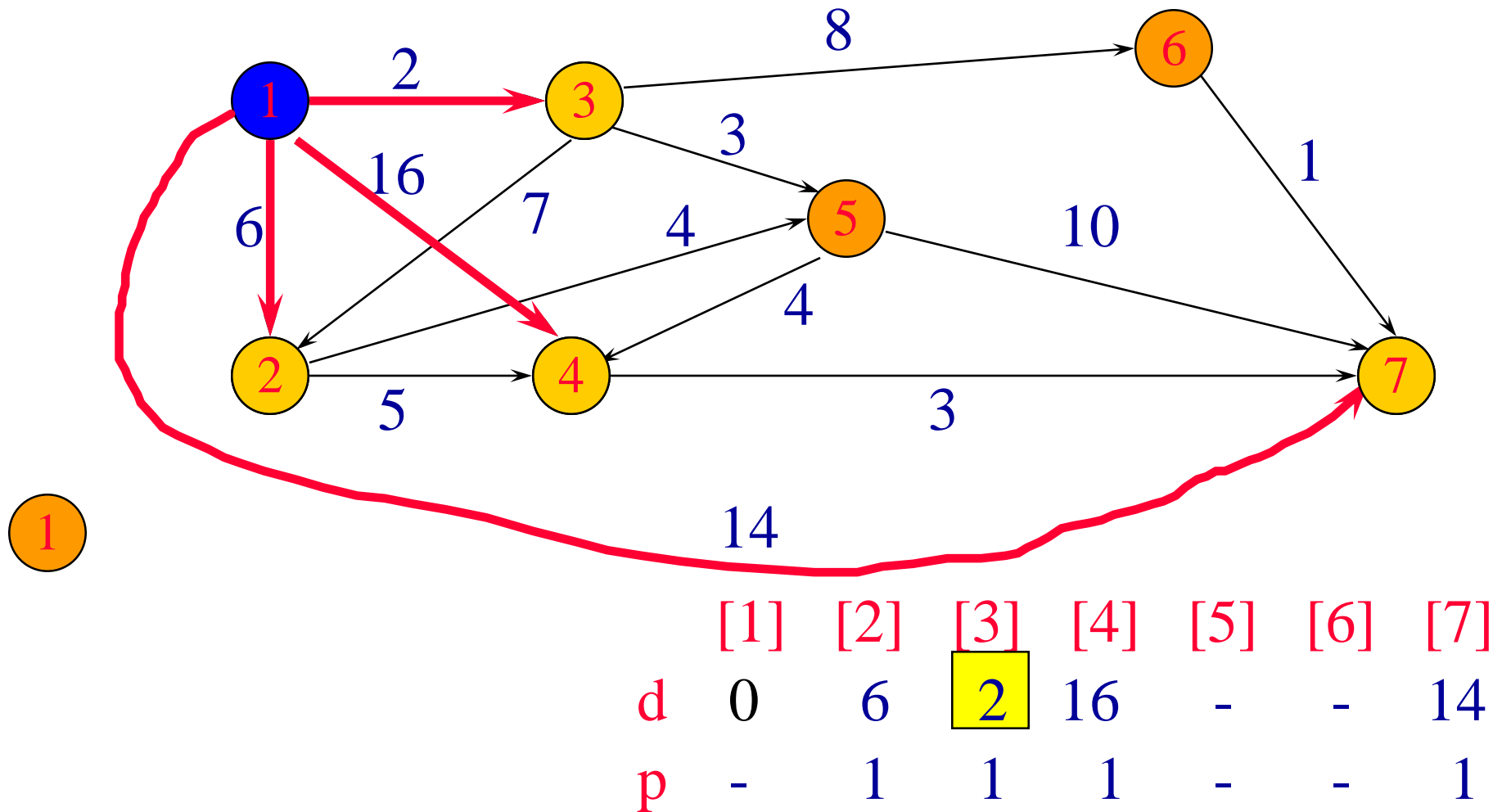
- Each path (other than first) is a one edge extension of a previous path.

- Next shortest path is the shortest one edge extension of an already generated shortest path.

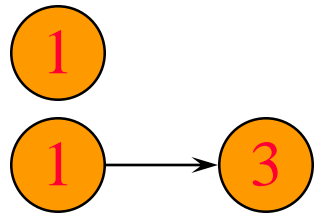
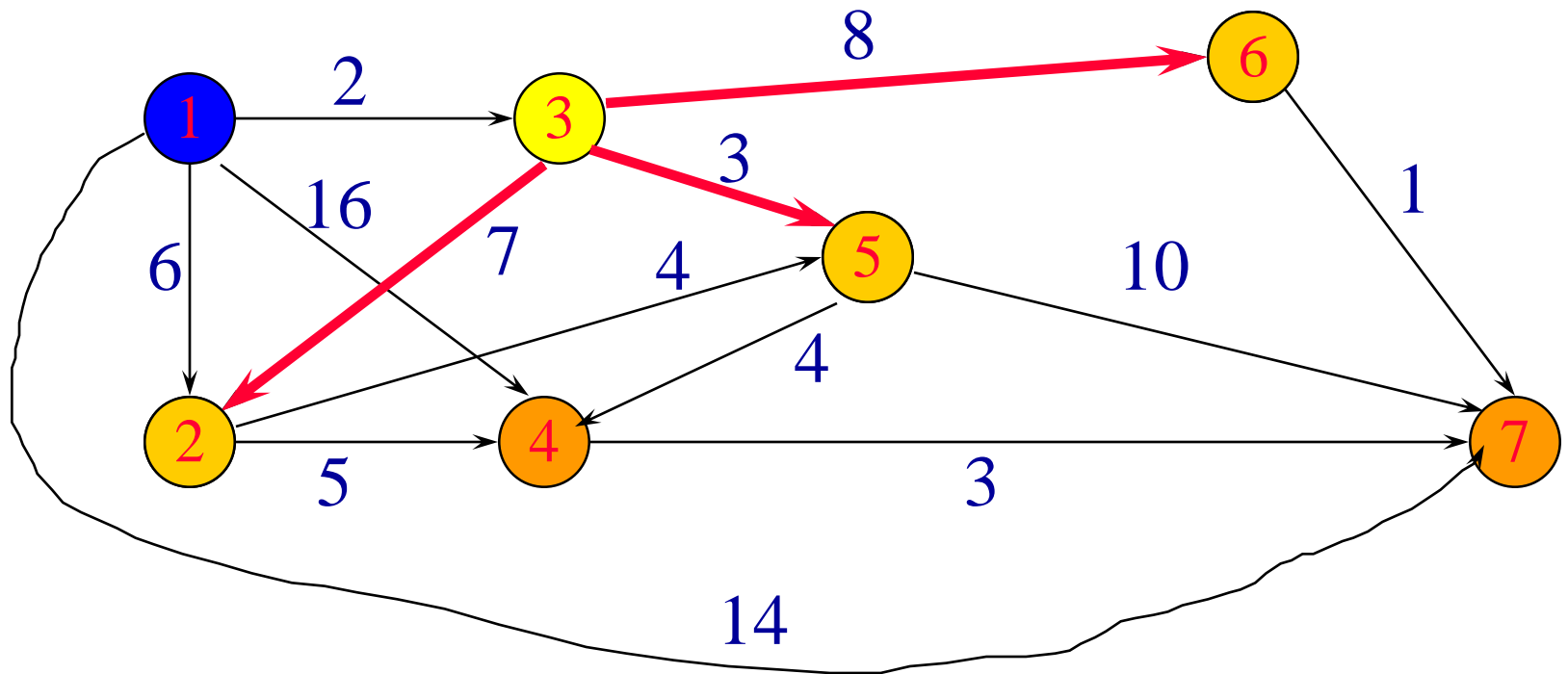
Greedy Single Source All Destinations

- Let $d(i)$ (`distanceFromSource(i)`) be the length of a shortest one edge extension of an already generated shortest path, the one edge extension ends at vertex i .
- The next shortest path is to an as yet unreached vertex for which the $d()$ value is least.
- Let $p(i)$ (`predecessor(i)`) be the vertex just before vertex i on the shortest one edge extension to i .

Greedy Single Source All Destinations

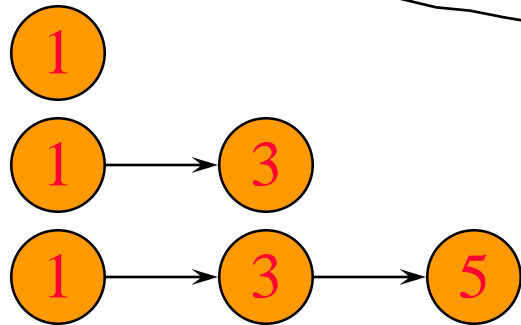
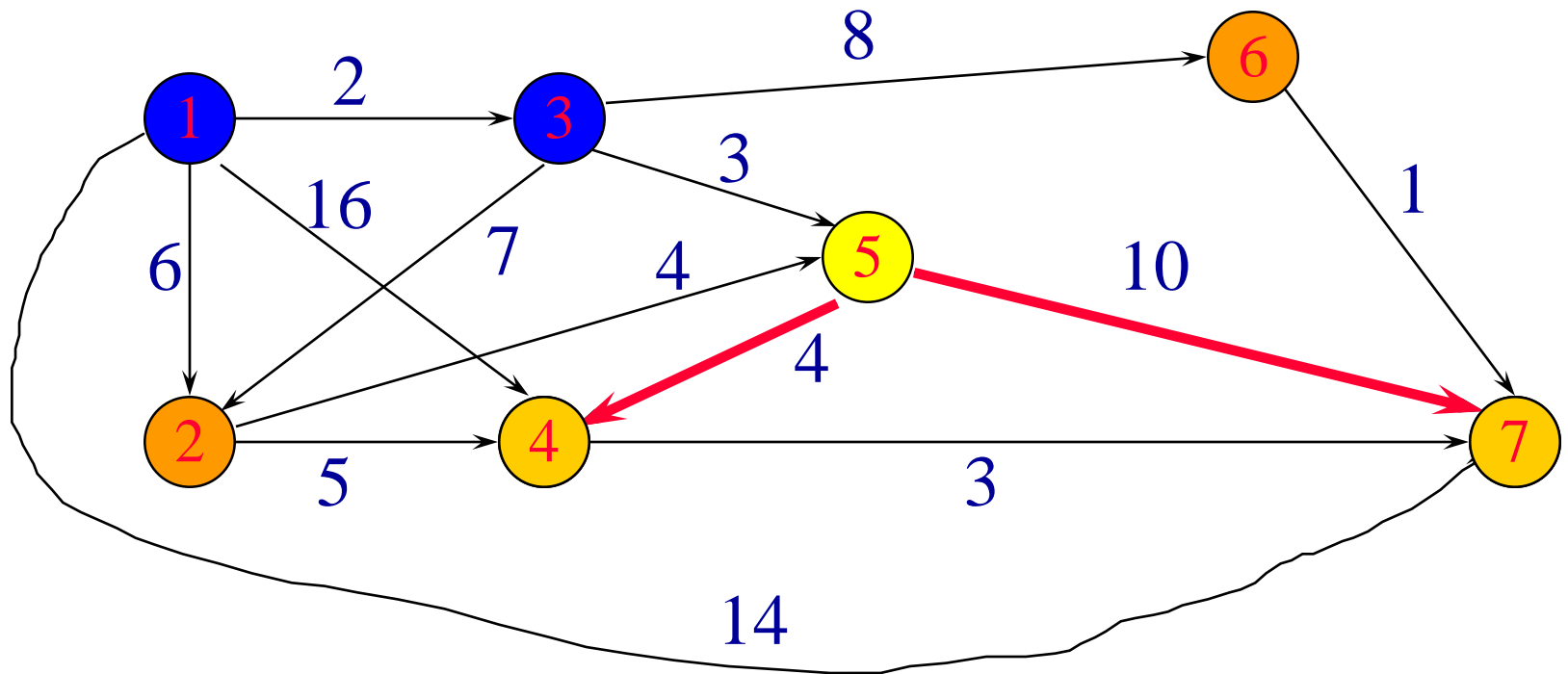


Greedy Single Source All Destinations



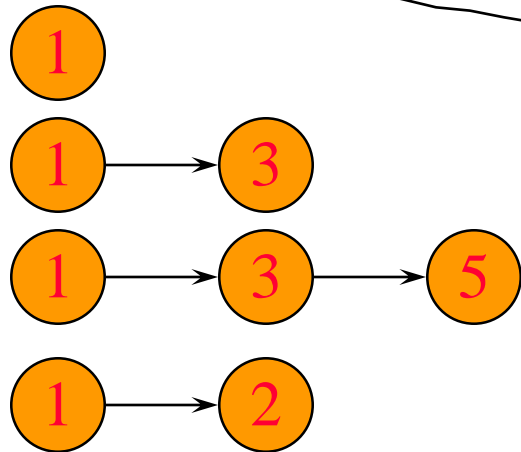
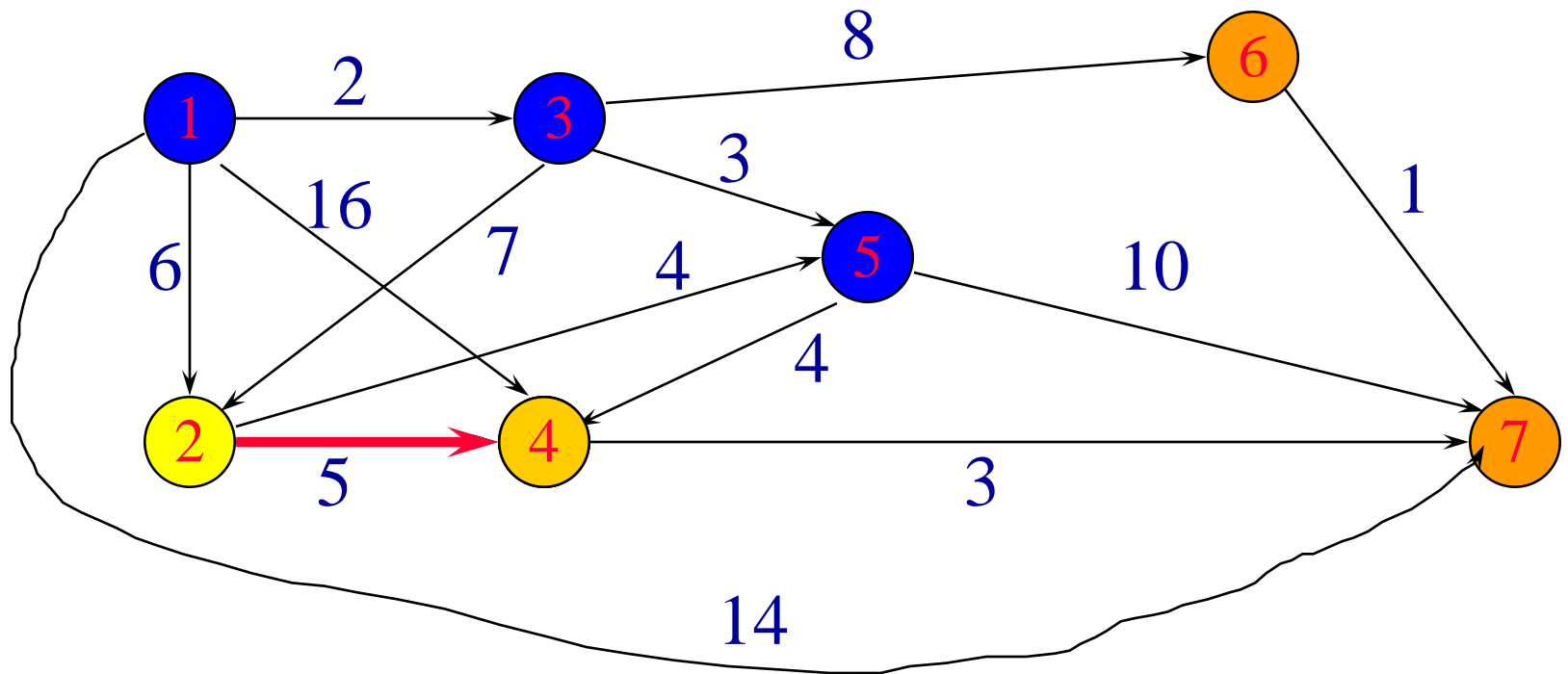
	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	16	5	10	14
p	-	1	1	1	3	3	1

Greedy Single Source All Destinations



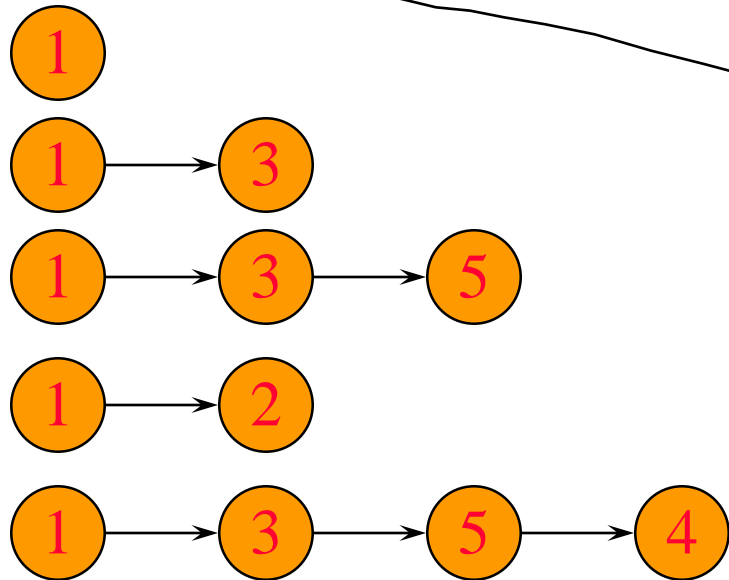
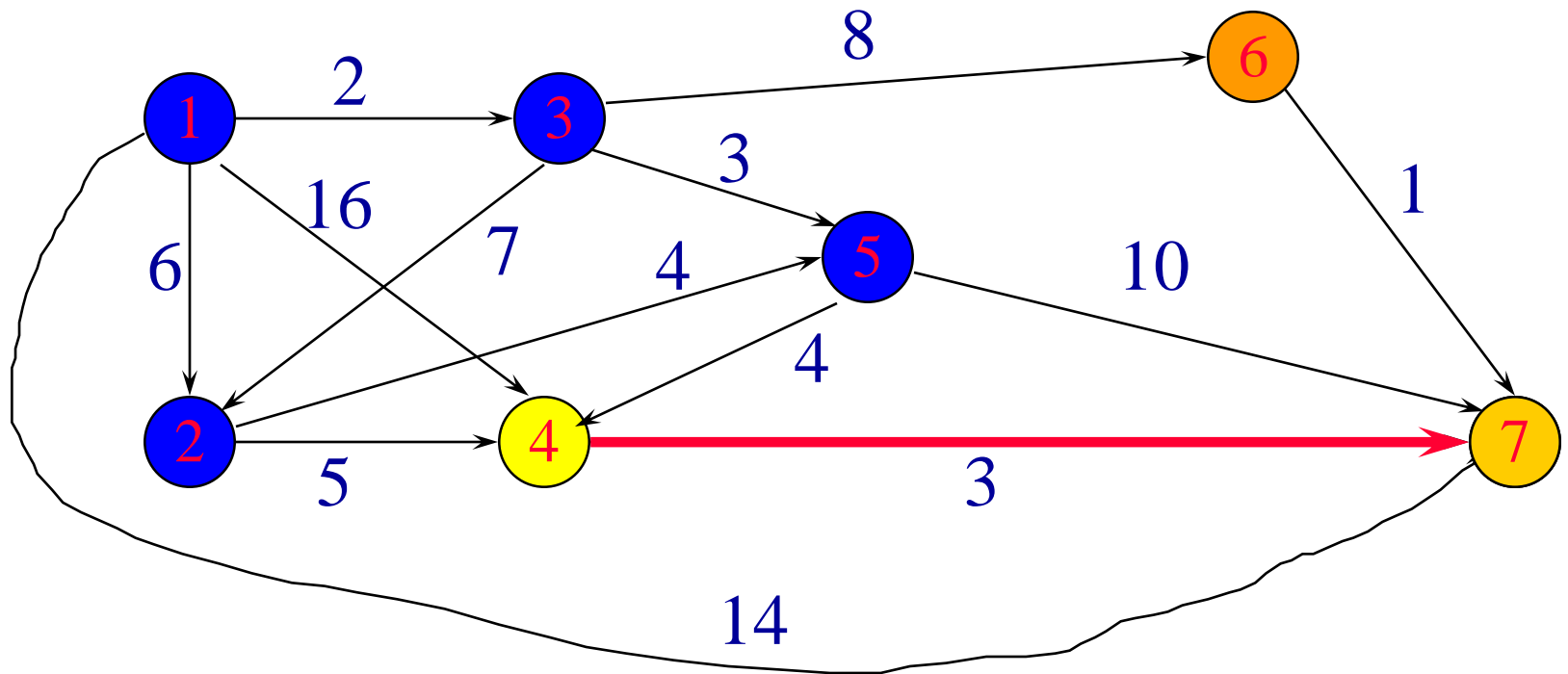
	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	9	5	10	14
p	-	1	1	5	3	3	1

Greedy Single Source All Destinations



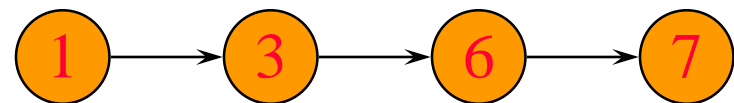
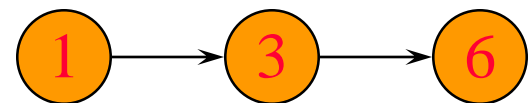
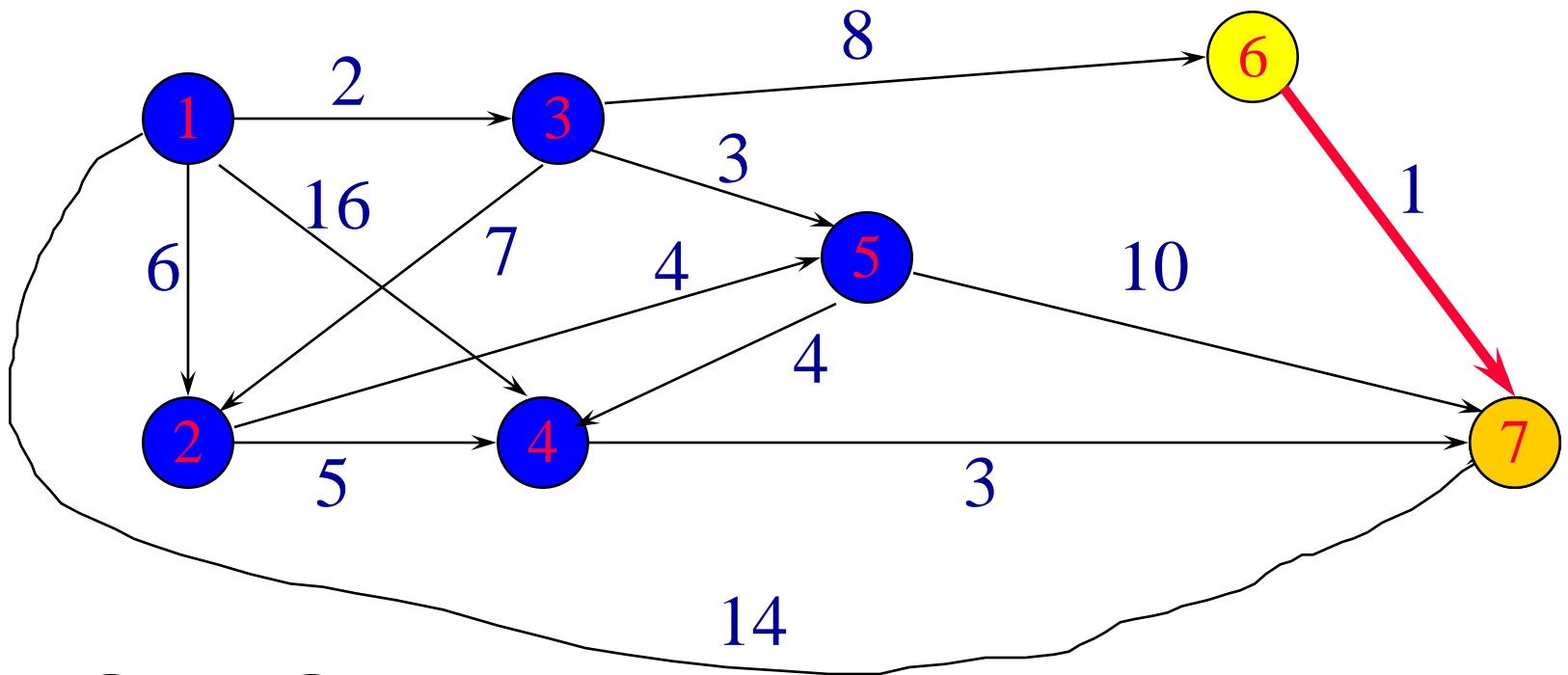
	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	9	5	10	14
p	-	1	1	5	3	3	1

Greedy Single Source All Destinations



	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	9	5	10	12
p	-	1	1	5	3	3	4

Greedy Single Source All Destinations



	[1]	[2]	[3]	[4]	[5]	[6]	[7]
d	0	6	2	9	5	10	11
p	-	1	1	5	3	3	6

Greedy Single Source All Destinations

Path	Length	[1]	[2]	[3]	[4]	[5]	[6]	[7]
1	0	0	-	-	-	-	-	-
1 → 3	2	6	1	-	-	-	-	-
1 → 3 → 5	5	2	1	5	3	-	-	-
1 → 2	6	9	-	-	-	-	-	-
1 → 3 → 5 → 4	9	5	3	3	6	-	-	-
1 → 3 → 6	10	-	-	-	-	-	-	-
1 → 3 → 6 → 7	11	-	-	-	-	-	-	-

Single Source Single Destination

Terminate single source all destinations greedy algorithm as soon as shortest path to desired vertex has been generated.

Data Structures For Dijkstra's Algorithm

- The greedy single source all destinations algorithm is known as Dijkstra's algorithm.
- Implement $d()$ and $p()$ as 1D arrays.
- Keep a linear list L of reachable vertices to which shortest path is yet to be generated.
- Select and remove vertex v in L that has smallest $d()$ value.
- Update $d()$ and $p()$ values of vertices adjacent to v .

Complexity



- $O(n)$ to select next destination vertex.
- updating $d()$ and $p()$ values:
 - $O(\text{out-degree})$ when adjacency lists are used.
 - $O(n)$ when adjacency matrix is used.
- Selection and update done once for each vertex to which a shortest path is found.
- Total time is $O(n^2 + e) = O(n^2)$.

Complexity



- When a min heap of $d()$ values is used in place of the linear list L of reachable vertices, total time is $O((n+e) \log n)$, because $O(n)$ remove min operations and $O(e)$ change key ($d()$ value) operations are done.
- When e is $O(n^2)$, using a min heap is worse than using a linear list.
- When a Fibonacci heap is used, the total time is $O(n \log n + e)$.