V-code: ???

Sinusoidal Steady State:
Impedance, Frequency Response, and Resonance

Lecture 16
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***Slide credits**: Prof. Anant Agarwal at MIT*

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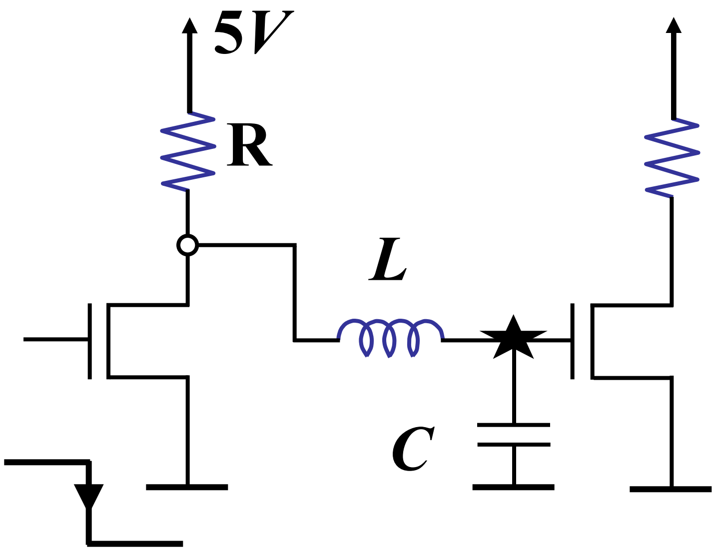
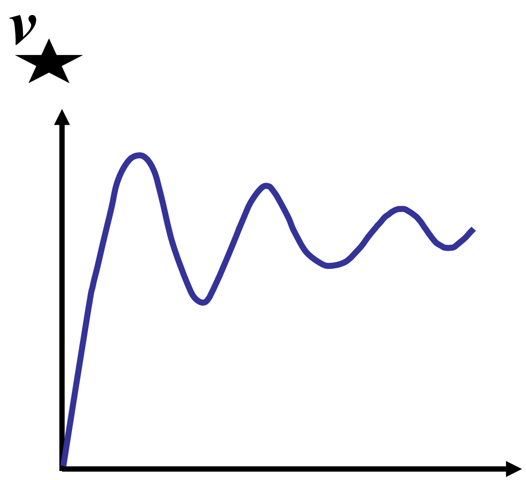
Review: Second-Order Transient

- We now understand the why of:





- Today, look at response of networks to sinusoidal drive.

Sinusoids important because signals can be represented as a sum of sinusoids. Response to sinusoids of various frequencies -- aka frequency response -- tells us a lot about the system

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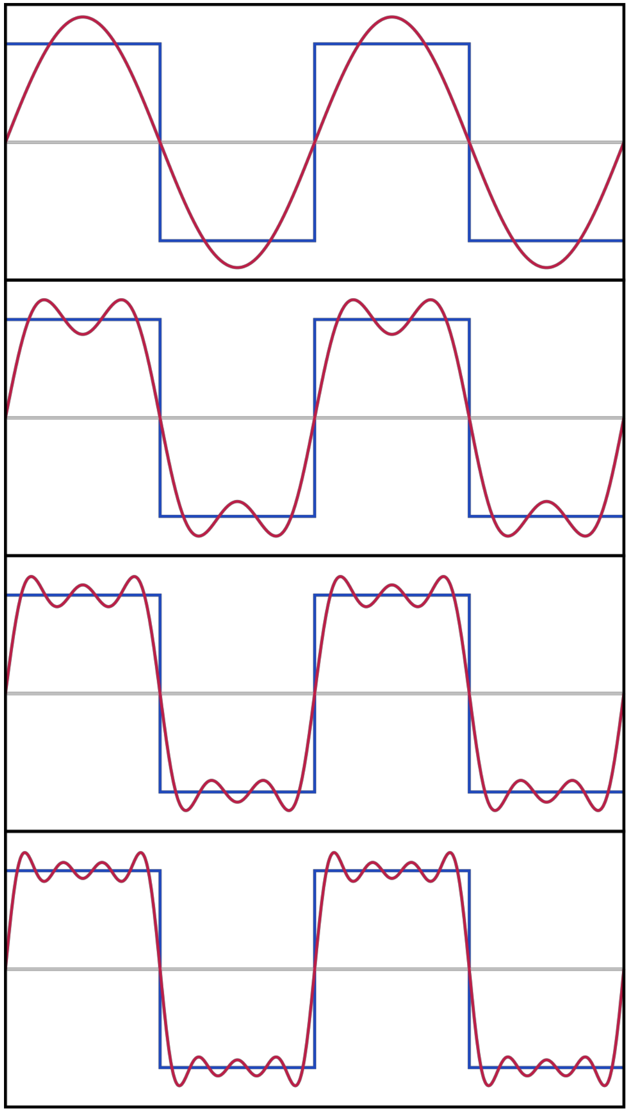
Introduction

- **Fourier synthesis of periodic function**
  - Any periodic signal is a sum of discrete sinusoidal components.
  - Example: Square wave
  - Demo: http://195.134.76.37/applets/AppletFourier/Appl_Fourier2.html



Source: https://upload.wikimedia.org/wikipedia/commons/2/2c/Fourier_Series.svg

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Outline

Textbook: 13.1, 13.2, 13.3, 13.4, 13.5, 14.2, 14.3

- **Sinusoidal Response of RC Network: Three Approaches**
- Impedance Model
- Frequency Response and Filters
- Frequency Response for Resonance Systems

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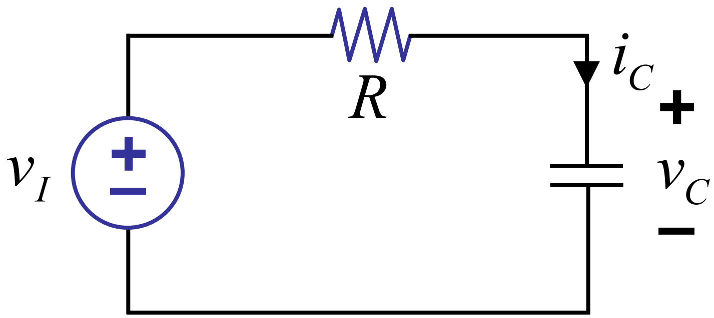
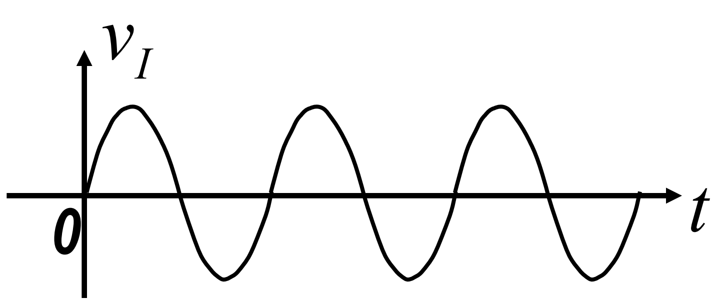
Sinusoidal Response of RC Network

Example:



$$v_I(t) = V_i \cos \omega t \quad \text{for } t \geq 0 \quad (V_i \text{ real})$$
$$= 0 \quad \text{for } t < 0$$

$$v_C(0) = 0 \quad \text{for } t = 0$$



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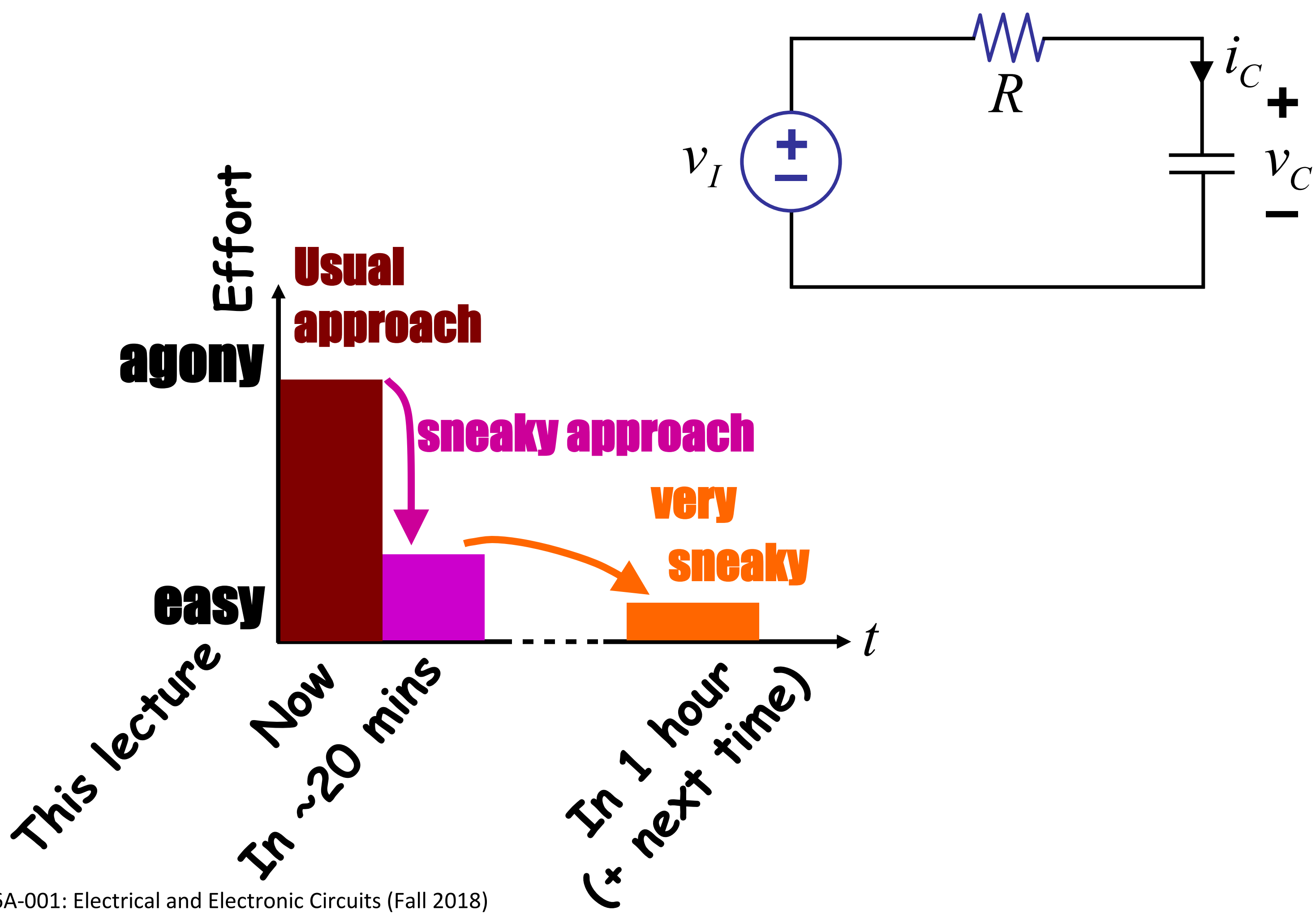
Sinusoidal Response of RC Network

- Three approaches to determine $v_C(t)$



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Approach 1: Usual Approach

Let's use the usual approach...

① Set up DE.

② Find v_p.

③ Find v_H.

④ $v_C = v_P + v_H$, solve for unknowns using initial conditions

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Approach 1: Usual Approach

1 Set up DE

$$RC\frac{dv_C}{dt} + v_C = v_I$$

$$= V_i \cos \omega t$$

That was easy!

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Approach 1: Usual Approach

② Find $\boxed{v_p}$ $$RC\frac{d\boxed{v_P}}{dt}+\boxed{v_P}=V_i\cos\omega t$$

First try: $\boxed{v_P}=A$ → nope

Second try: $\boxed{v_P}=A\cos\omega t$ → nope

Third try: $\boxed{v_P}=A\cos(\omega t+\phi)$ frequency

amplitude phase

$$-RCA\omega\sin(\omega t+\phi)+A\cos(\omega t+\phi)=V_i\cos\omega t$$

$$-RCA\omega\sin\omega t\cos\phi-RCA\omega\cos\omega t\sin\phi+$$
$$A\cos\omega t\cos\phi-A\sin\omega t\sin\phi\qquad=V_i\cos\omega t$$

: gasp !

works, but trig nightmare!

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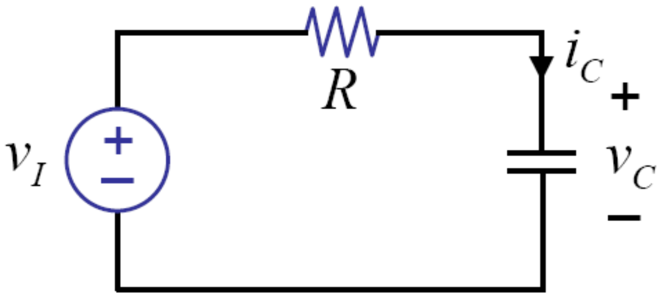
Approach 2: Sneaky Approach



$$\begin{aligned} v_I(t) &= V_i \cos\omega t && \text{for } t \ge 0 \quad (V_i \text{ real}) \\ &= 0 && \text{for } t < 0 \end{aligned}$$

$$v_C(0) = 0 \qquad \text{for } t = 0$$

$$v_I(t) = V_i \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Find the particular solution to (fake input) $V_i e^{j\omega t}$

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Approach 2: Sneaky Approach

Let's get sneaky!

Find particular solution to another input…

$$RC\frac{dv_{PS}}{dt} + v_{PS} = v_{IS}$$

$$= V_i e^{st}$$

(s: sneaky :-))

Try solution $v_{PS} = V_p e^{st}$

$$RC\frac{dV_p e^{st}}{dt} + V_p e^{st} = V_i e^{st}$$

$$sRCV_p \cancel{e^{st}} + V_p \cancel{e^{st}} = V_i \cancel{e^{st}}$$

$$(sRC + 1)V_p = V_i$$

$$\boxed{V_p = \frac{V_i}{1 + sRC}}$$

Nice property of exponentials

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Approach 2: Sneaky Approach

Let's get sneaky! (Cont'd)

Thus, $v_{PS} = \dfrac{V_i}{1+sRC} \cdot e^{st}$

easy!

is particular solution to $V_i e^{st}$

Similarly $\underbrace{\dfrac{V_i}{1+j\omega RC}}_{\boxed{V_p}} \cdot e^{j\omega t} \longrightarrow$ solution for $V_i e^{j\omega t}$

where we replace $s = j\omega$

$V_p \longrightarrow$ complex amplitude

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Approach 2: Sneaky Approach

② Fourth try to find $\boxed{v_P}$...
using the sneaky approach

Fact 1: Finding the response to

$$V_i e^{j\omega t}$$

was easy.

Fact 2:
$$v_I = V_i \cos \omega t$$
$$= \text{real}\left[V_i e^{j\omega t}\right] = \text{real}\left[v_{IS}\right]$$

from Euler relation,

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

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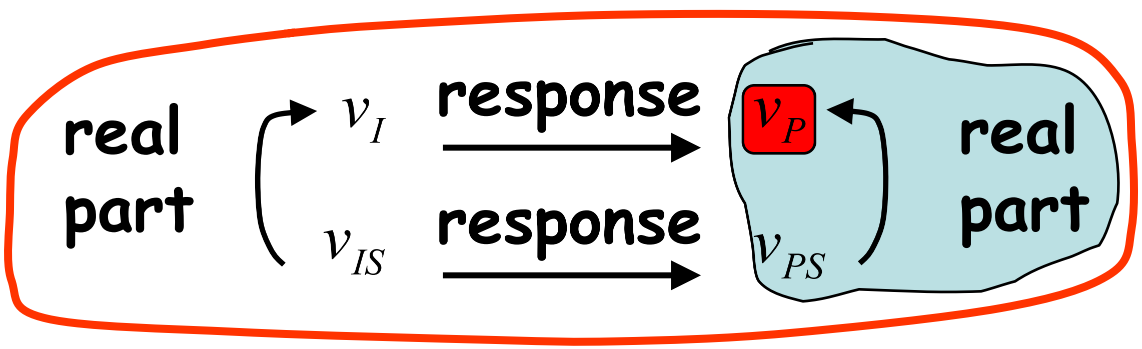
Approach 2: Sneaky Approach

② Fourth try to find $\boxed{v_P}$...

using the sneaky approach (cont'd)



an inverse superposition argument,
assuming system is real, linear.

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Approach 2: Sneaky Approach

② Fourth try to find v_P...

so,

$$\boxed{v_P} = Re[\underbrace{v_{PS}}_{\text{complex}}] = Re[V_p e^{j\omega t}]$$

$$= \mathrm{Re}\left[\frac{V_i}{1 + j\omega RC} \cdot e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\frac{V_i(1 - j\omega RC)}{1 + \omega^2 R^2 C^2} \cdot e^{j\omega t}\right]$$

$$= Re\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j\phi} e^{j\omega t}\right], \tan\phi = -\omega RC$$

$$= Re\left[\frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot e^{j(\omega t + \phi)}\right] \quad \boxed{v_P} = \frac{V_i}{\sqrt{1 + \omega^2 R^2 C^2}} \cdot \cos(\omega t + \phi)$$

Recall, $\boxed{v_P}$ is particular response to $V_i \cos\omega t$.

V-code: ???

Approach 2: Sneaky Approach

3 **Find** v_H

Recall, $v_H = Ae^{\frac{-t}{RC}}$

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Approach 2: Sneaky Approach

4 Find total solution

$$v_C = \boxed{v_P} + v_H$$

$$v_C = \frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} cos(\omega t + \phi) + Ae^{-\frac{t}{RC}}$$

$$where \quad \phi = tan^{-1}(-\omega RC)$$

Given $v_C(0) = 0$ **for** $t = 0$

so,

$$A = -\frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\phi)$$

Done! ***Phew!***

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Sinusoidal Steady State

We are usually interested only in the particular solution for sinusoids, i.e. after transients have died.

Notice when $t \rightarrow \infty, \quad v_C \rightarrow v_P$ **as** $e^{-\frac{t}{RC}} \rightarrow 0$

0

$$v_C = \frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t + \phi) + A e^{-\frac{t}{RC}}$$

$$where \quad \phi = tan^{-1}(-\omega RC)$$

$$A = -\frac{V_i}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\phi)$$

V_p $\quad$ $\angle V_p$

Described as

SSS: Sinusoidal Steady State

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Sinusoidal Steady State

All information about SSS is contained in V_p, the complex amplitude!

Recall $V_p = \dfrac{V_i}{1+j\omega RC}$

Steps ③ ④ were a waste of time!

$$\frac{V_p}{V_i} = \frac{1}{1+j\omega RC}$$

$$\frac{V_p}{V_i} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} e^{j\phi}$$

where

$$\phi = tan^{-1} - \omega RC$$

magnitude $\left|\dfrac{V_p}{V_i}\right| = \dfrac{1}{\sqrt{1+\omega^2 R^2 C^2}}$

phase ϕ : $\angle \dfrac{V_p}{V_i} = -\tan^{-1} \omega RC$

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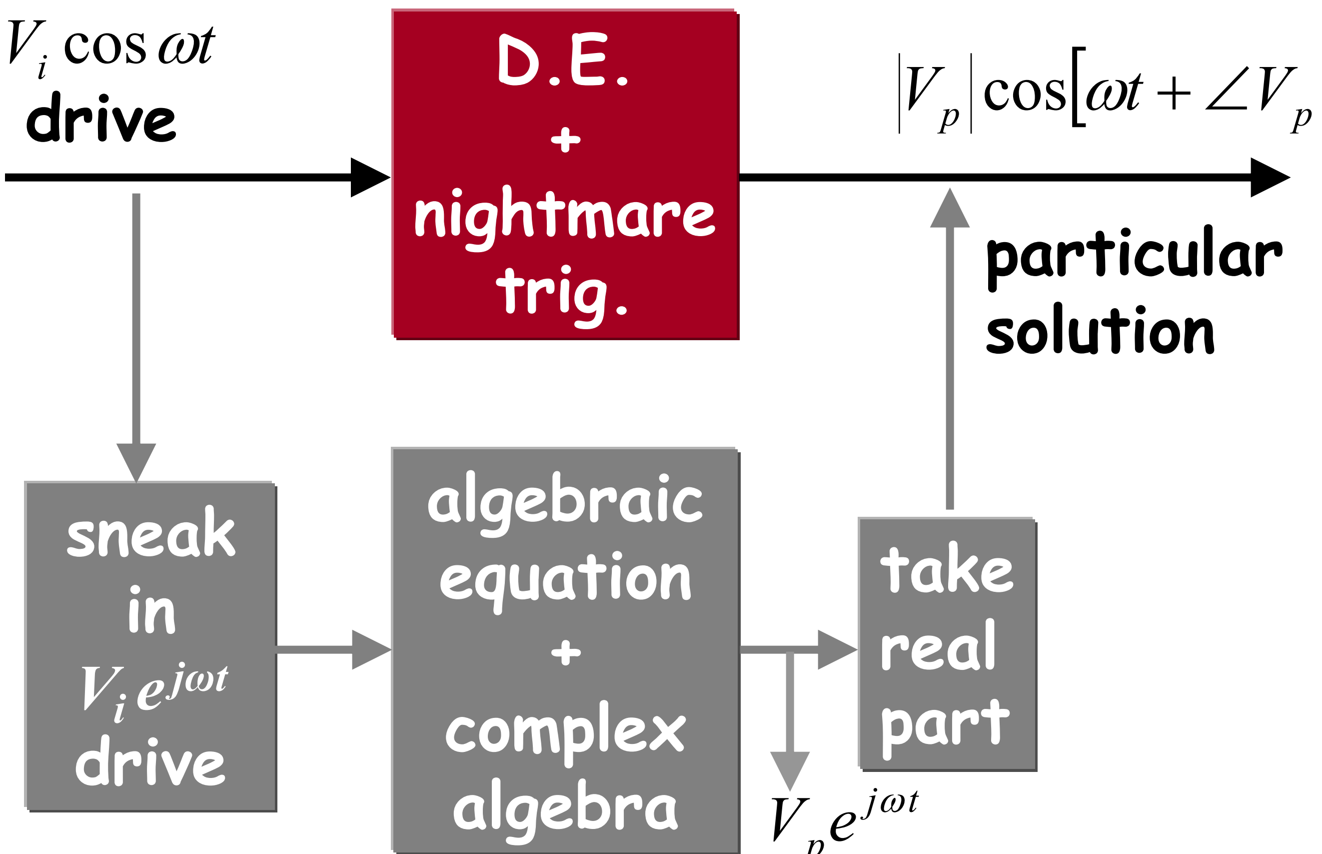
Sinusoidal Steady State

Visualizing the process of finding the particular solution v_P



the sneaky path!

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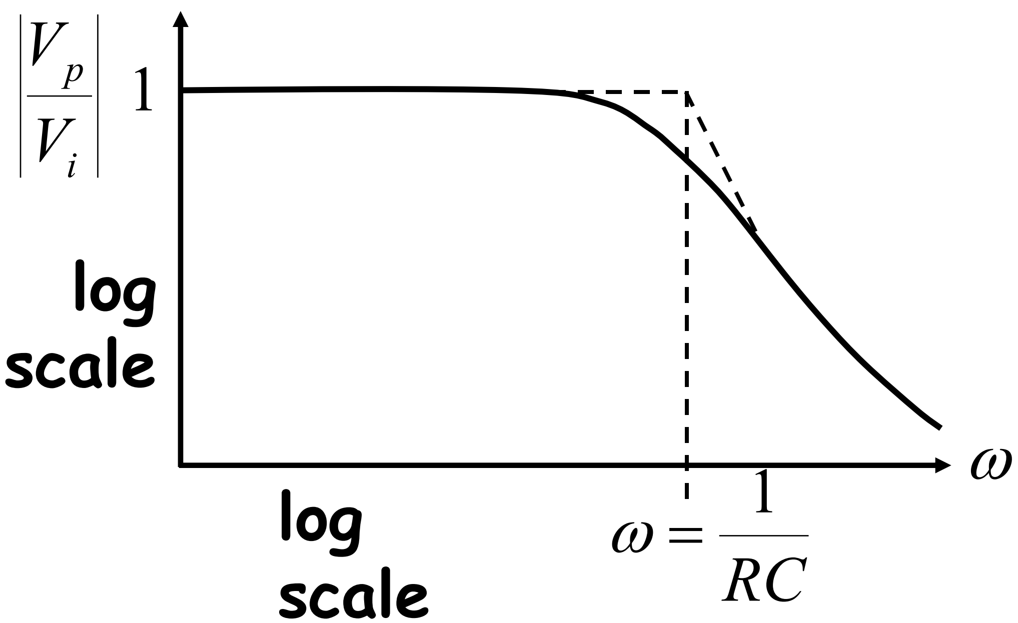
Sinusoidal Steady State

Magnitude Plot

transfer function

$$H(j\omega) = \frac{V_p}{V_i}$$

$$\left|\frac{V_p}{V_i}\right| = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$



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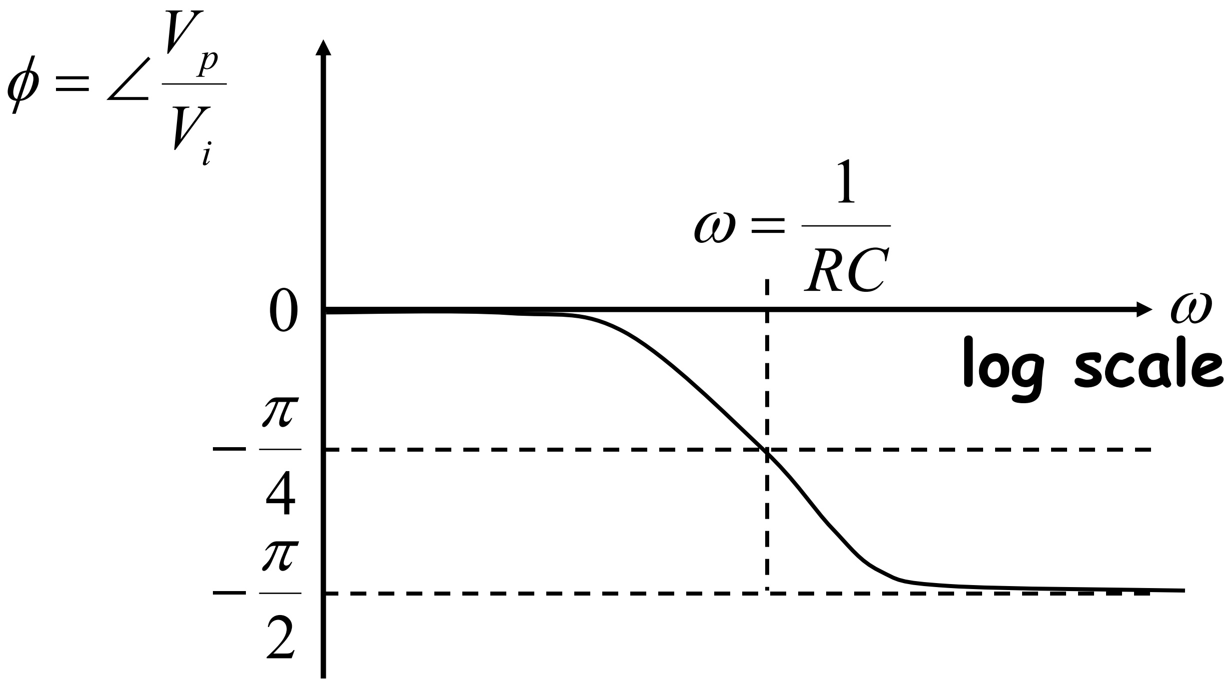
Sinusoidal Steady State

Phase Plot

$$\phi = tan^{-1} - \omega RC$$



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- **Impedance Model**
- Frequency Response and Filters
- Frequency Response for Resonance Systems

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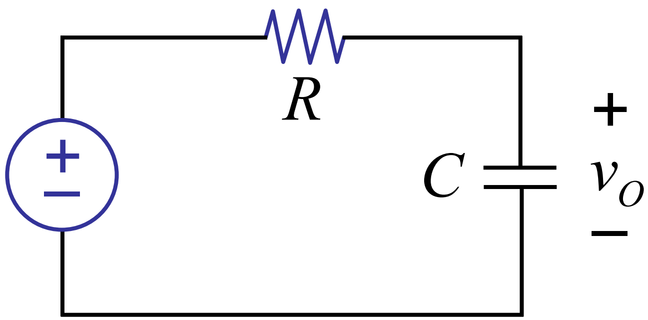
Review

- Sinusoidal Steady State (SSS)

$$v_I = V_i \cos \omega t$$

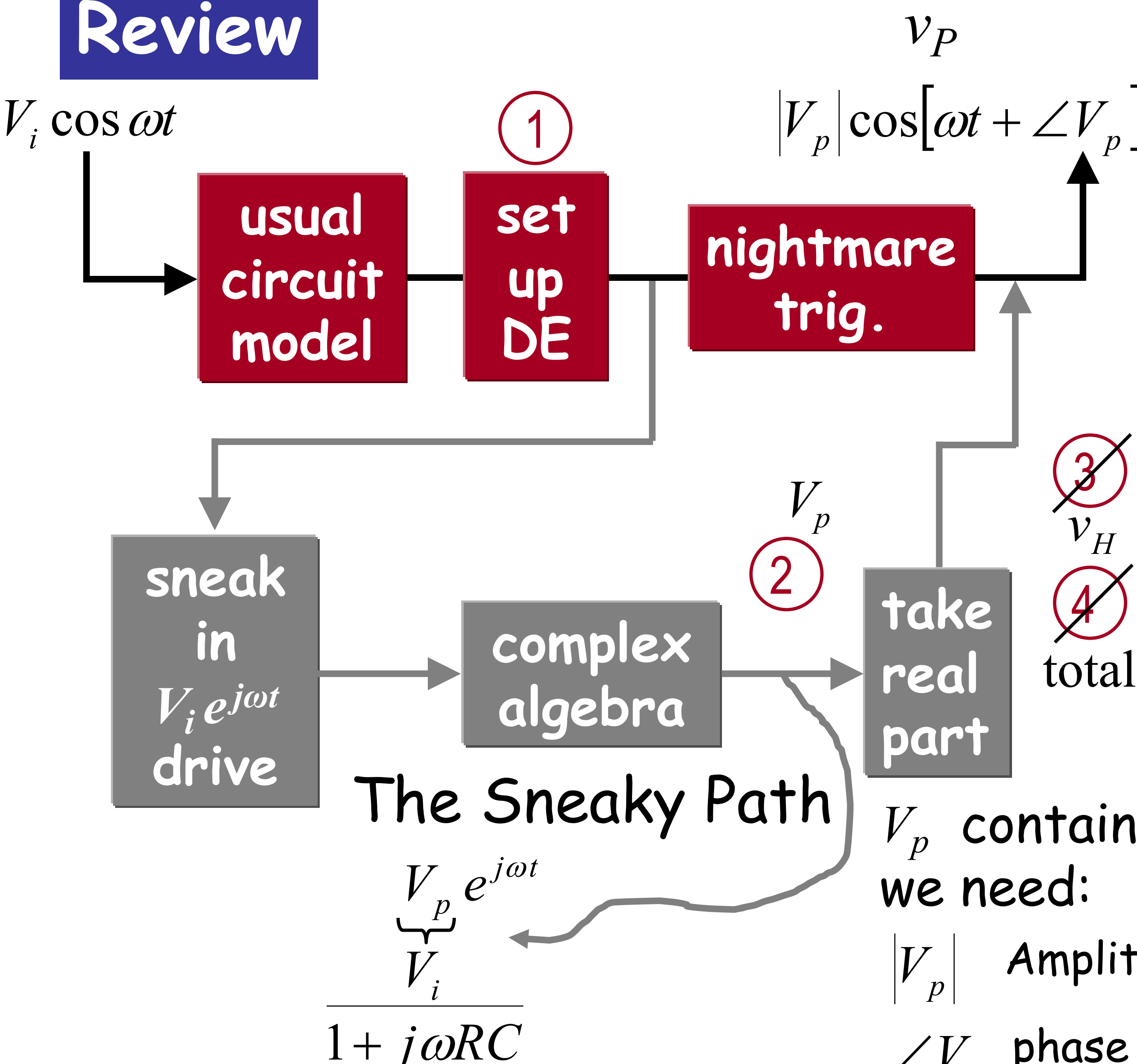


SSS

- Focus on steady state, only care about v_P as v_H dies away.
- Focus on sinusoids.

Review

V-code: ???



V_p contains all the info we need:

$|V_p|$ Amplitude of output cosine

$\angle V_p$ phase

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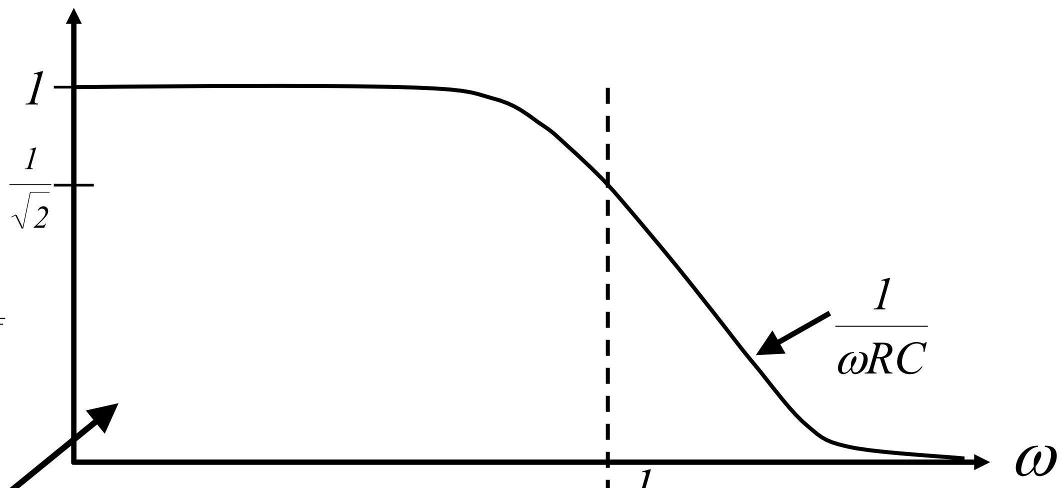
Review

$$v_O = |V_p| \cos(\omega t + \angle V_p)$$

$$\frac{V_p}{V_i} = \frac{1}{1 + j\omega RC} = H(j\omega)$$ transfer function

$$\left|\frac{V_p}{V_i}\right|$$

$$\frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$



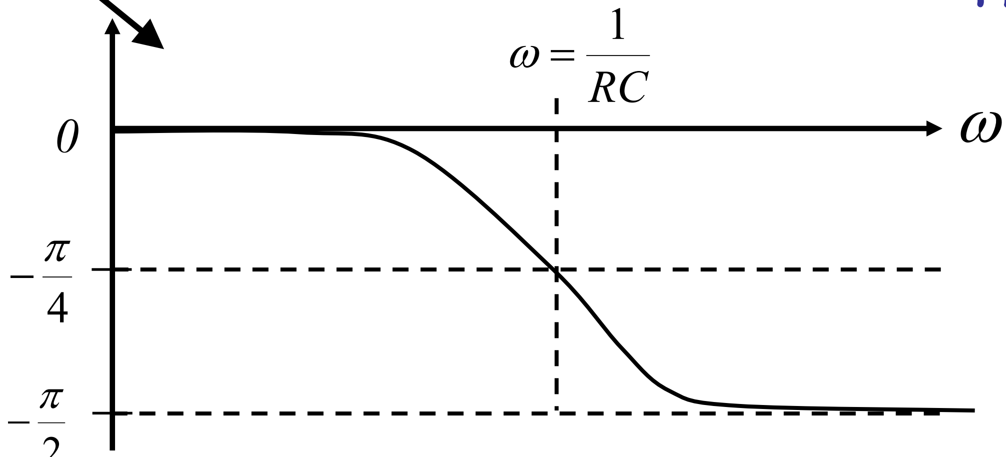
Bode plot

break frequency $\omega = \frac{1}{RC}$

The Frequency View

$$\angle \frac{V_p}{V_i}$$

$$tan^{-1}\left(\frac{-\omega RC}{1}\right)$$



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Approach 3: Impedance Model

Is there an even simpler way to get V_p ?

$$V_p = \frac{V_i}{1 + j\omega RC}$$

Divide numerator and denominator by $j\omega C$.

$$V_p = V_i \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R}$$

Hmmm… looks like a voltage divider relationship.

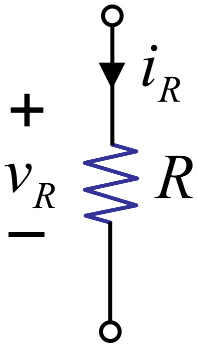
$$V_p = V_i \frac{Z_C}{Z_C + R}$$

Let's explore further…

Approach 3: Impedance Model

V-code: ???

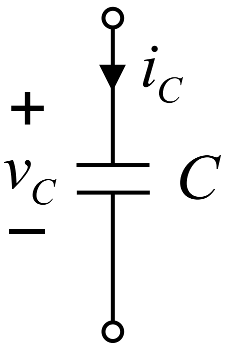
Consider:



$$i_R = I_r e^{j\omega t} \qquad v_R = R i_R$$

$$v_R = V_r e^{j\omega t} \qquad V_r \cancel{e^{j\omega t}} = R I_r \cancel{e^{j\omega t}}$$

Resistor $\boxed{V_r = R I_r}$

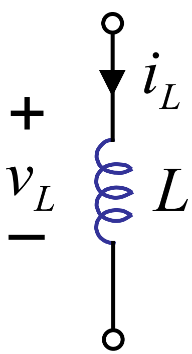


$$i_C = I_C e^{j\omega t} \qquad i_C = C\frac{dv_C}{dt}$$

$$v_C = V_C e^{j\omega t} \qquad I_C \cancel{e^{j\omega t}} = C V_C j\omega \cancel{e^{j\omega t}}$$

Capacitor $\boxed{V_C = \left(\frac{1}{j\omega C}\right) I_C}$

Z_C



$$i_L = I_l e^{j\omega t} \qquad v_L = L\frac{di_L}{dt}$$

$$v_L = V_l e^{j\omega t} \qquad V_l \cancel{e^{j\omega t}} = L I_l j\omega \cancel{e^{j\omega t}}$$

Z_L

Inductor $\boxed{V_l = (j\omega L)\, I_l}$

Approach 3: Impedance Model

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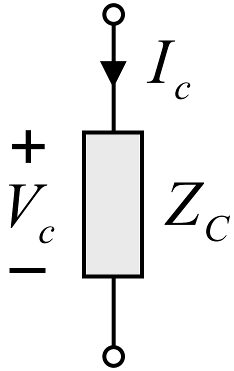
In other words,

capacitor

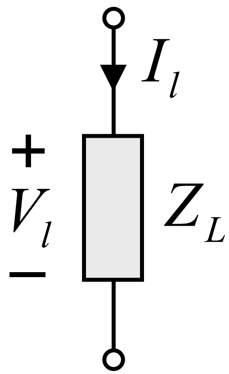


$$V_c = Z_C \, I_c$$

$$Z_C = \frac{1}{j\omega C}$$

impedance

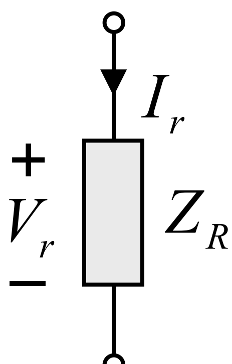
inductor



$$V_l = Z_l I_l$$

$$Z_l = j\omega L$$

resistor



$$V_r = Z_r I_r$$

$$Z_r = R$$

For a drive of the form $V_c e^{j\omega t}$, complex amplitude V_c is related to the complex amplitude I_c algebraically, by a generalization of Ohm's Law.

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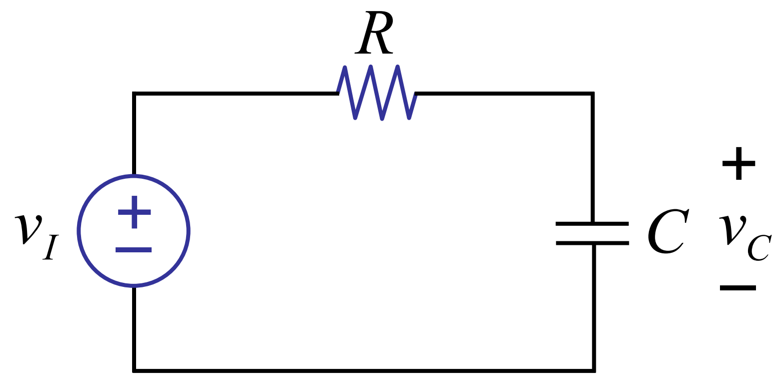
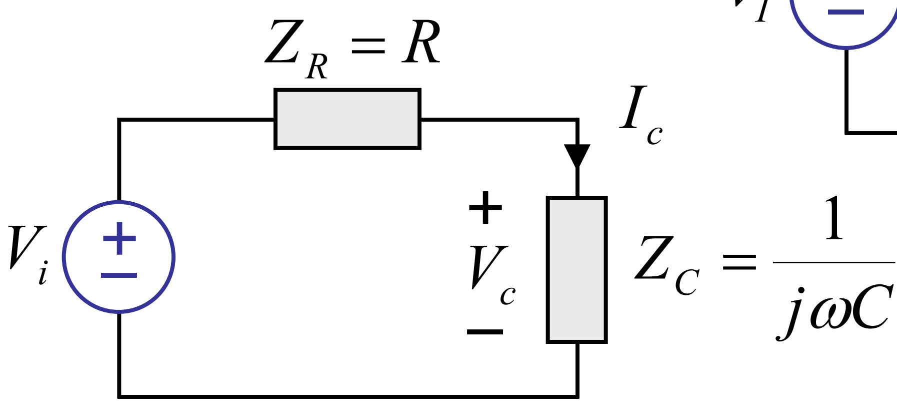
Approach 3: Impedance Model

Back to RC example…



Impedance model:



$$V_c = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} V_i = \frac{Z_C}{Z_C + Z_R} V_i$$

$$\boxed{V_c = \frac{1}{1 + j\omega RC} V_i}$$

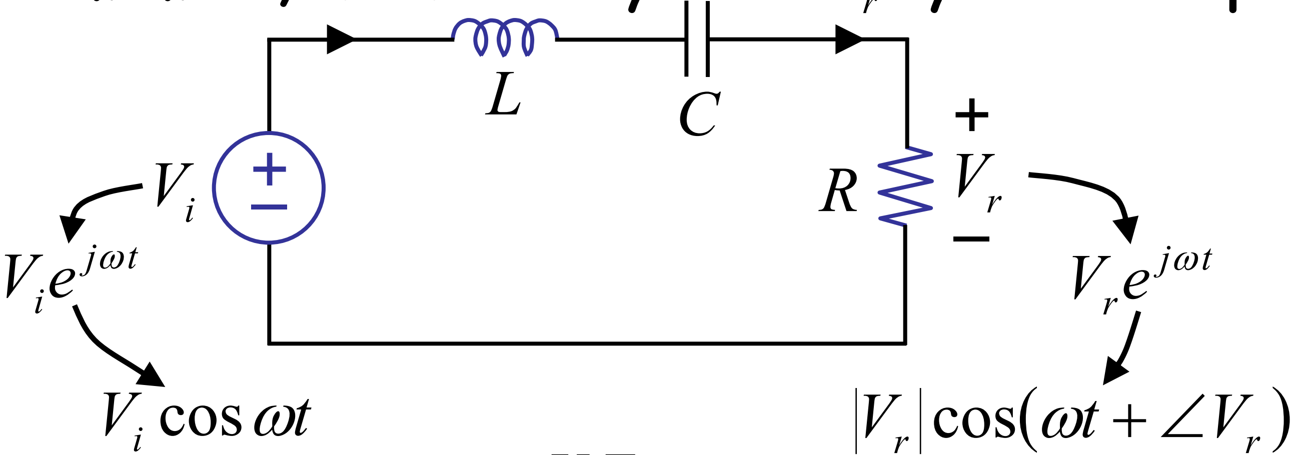
Done! All our old friends apply! KVL, KCL, superposition…

V-code: ???

Approach 3: Impedance Model

Another example, recall series RLC:

Remember, we want only the steady-state response to sinusoid



$$V_r = \frac{V_i Z_R}{Z_L + Z_C + Z_R}$$

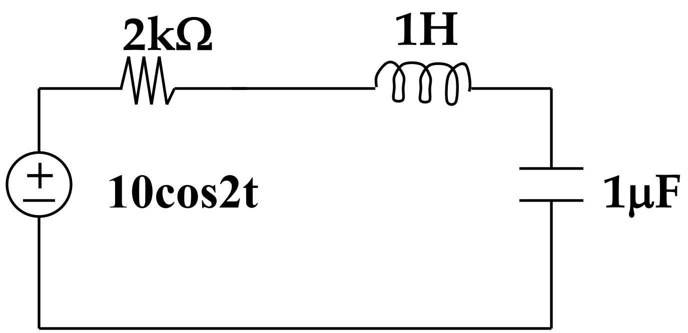
$$V_r = \frac{V_i R}{j\omega L + \dfrac{1}{j\omega C} + R} \Rightarrow V_r = \frac{V_i j\omega CR}{-\omega^2 LC + 1 + j\omega CR}$$

We will study this and other functions in more detail in the next lecture.

V-code: ???

Approach 3: Impedance Model

- **Exercise 1**



find v_C

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Notation for Voltages and Currents

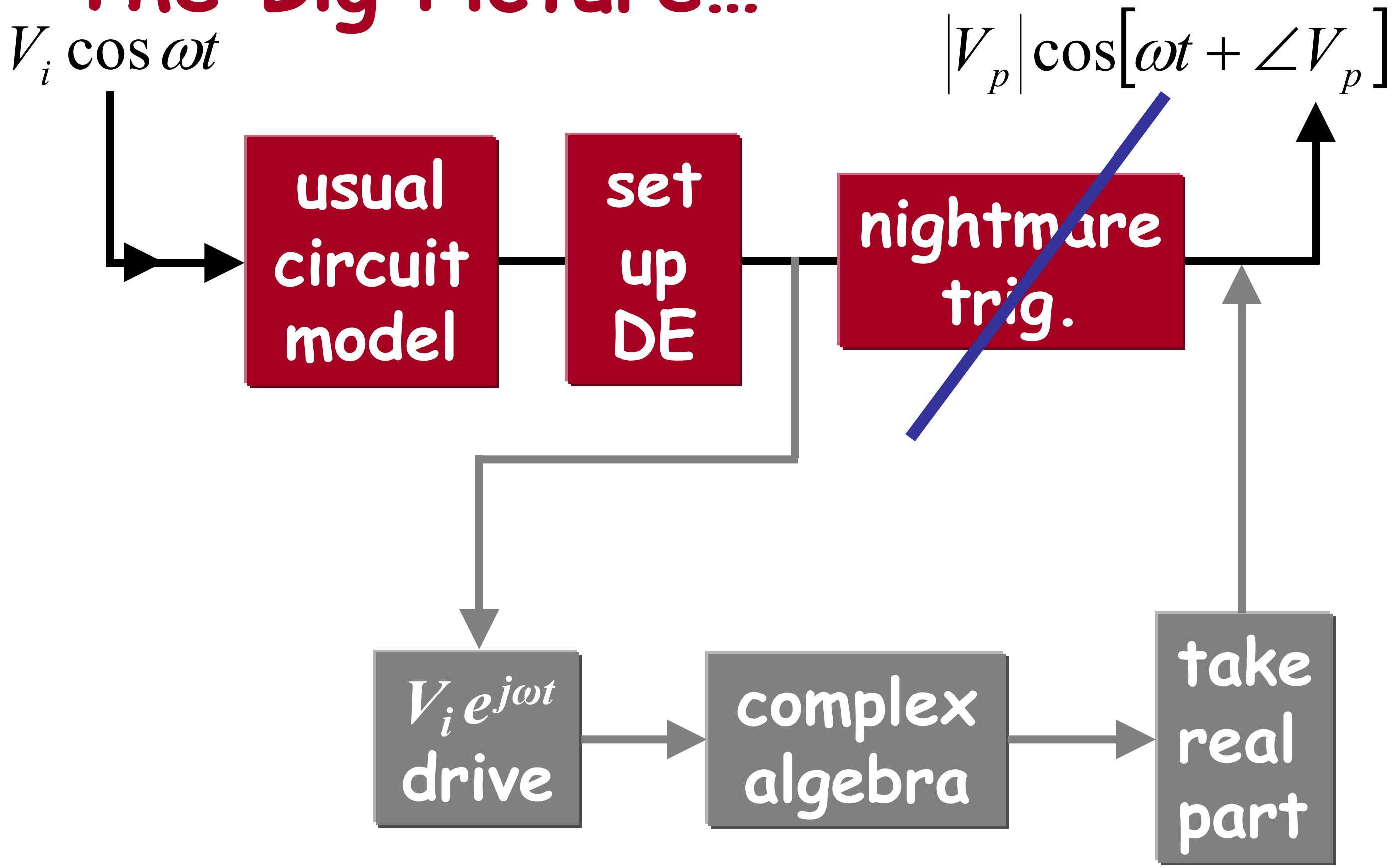
- **We abide by the international standard!**

- DC or operating-point variables: uppercase symbols with uppercase subscripts (for example, V_A)
- Total instantaneous variables: lowercase symbols with uppercase subscripts (for example, v_A)
- Incremental instantaneous variables: lowercase symbols with lowercase subscripts (for example, v_a)
- Complex amplitudes or complex amplitudes of incremental components, and real amplitudes of sinusoidal input sources: uppercase symbols with lowercase subscripts (for example, V_a)

Approach 3: Impedance Model

V-code: ???

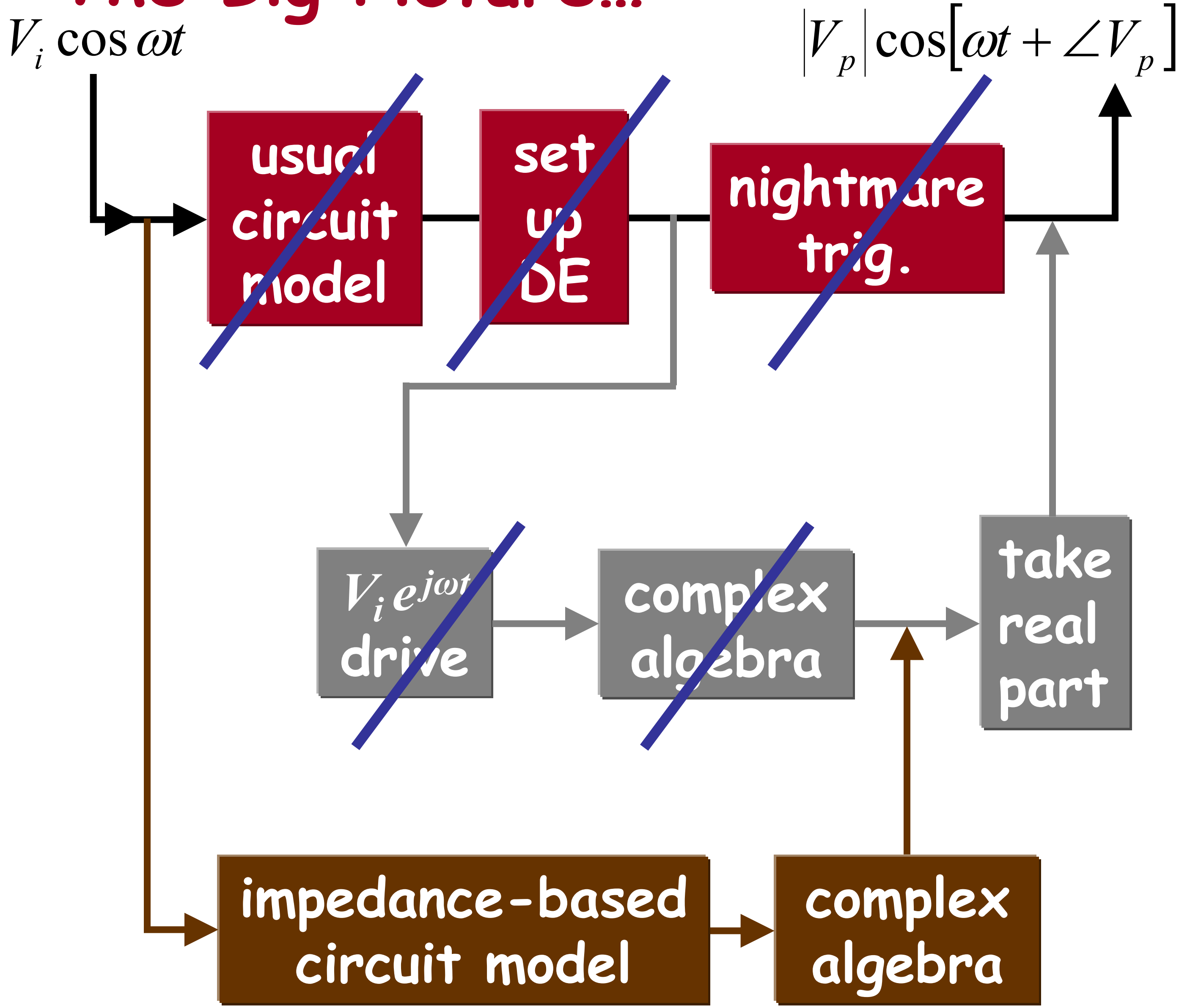
The Big Picture…



Approach 3: Impedance Model

V-code: ???

The Big Picture…



No D.E.s, no trig!

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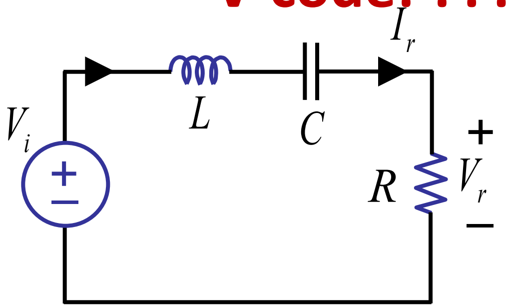
Approach 3: Impedance Model



Back to

$$\frac{V_r}{V_i} = \frac{j\omega RC}{1 + j\omega RC - \omega^2 LC}$$

Let's study this transfer function

$$\frac{V_r}{V_i} = \frac{j\omega RC}{1 + j\omega RC - \omega^2 LC}$$

$$= \frac{j\omega RC}{\left(1-\omega^2 LC\right) + j\omega RC} \cdot \frac{\left(1-\omega^2 LC\right) - j\omega RC}{\left(1-\omega^2 LC\right) - j\omega RC}$$

$$\left|\frac{V_r}{V_i}\right| = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}}$$

Observe

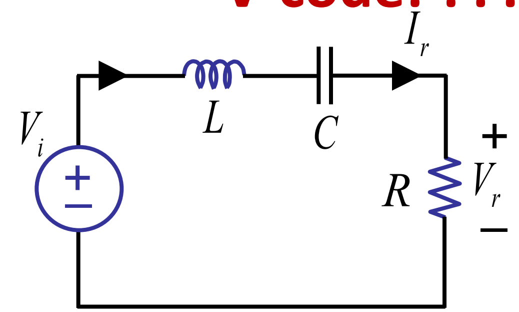
$$\text{Low } \omega: \quad \approx \omega RC$$

$$\text{High } \omega: \quad \approx \frac{R}{\omega L}$$

$$\omega\sqrt{LC} = 1: \quad \approx 1$$

V-code: ???

Approach 3: Impedance Model

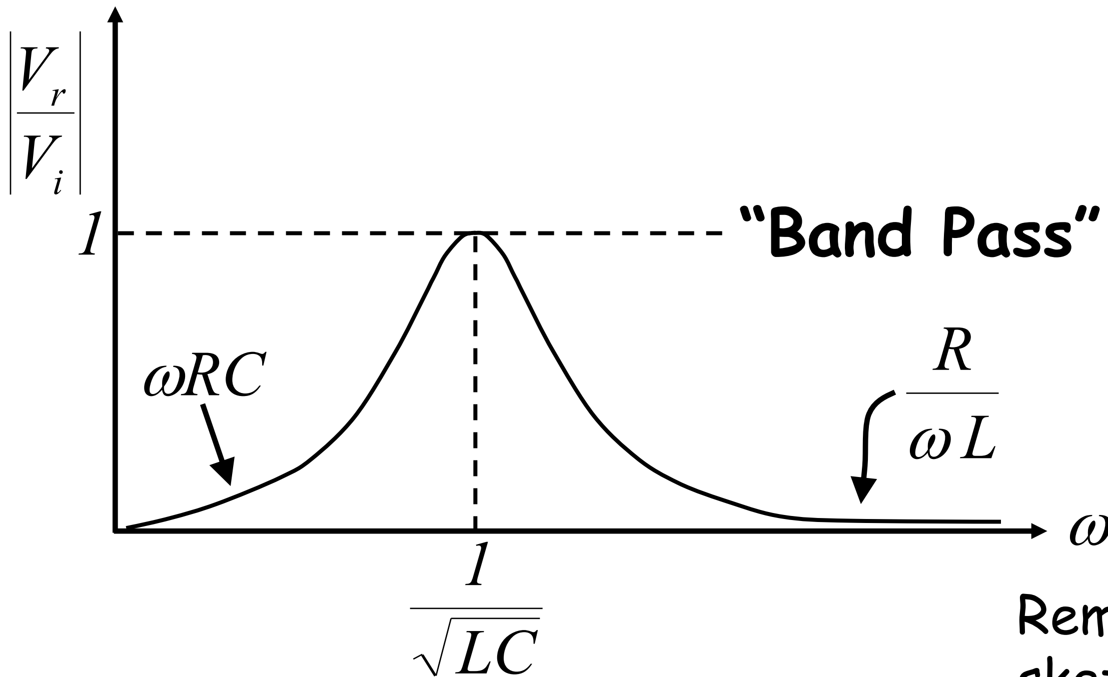


$$\left|\frac{V_r}{V_i}\right| = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2+\left(\omega RC\right)^2}}$$

$$\text{Low } \omega: \quad \approx \omega RC$$

$$\text{High } \omega: \quad \approx \frac{R}{\omega L}$$

$$\omega\sqrt{LC} = 1: \quad \approx 1$$



Remember this trick to sketch the form of transfer functions quickly.

V-code: ???

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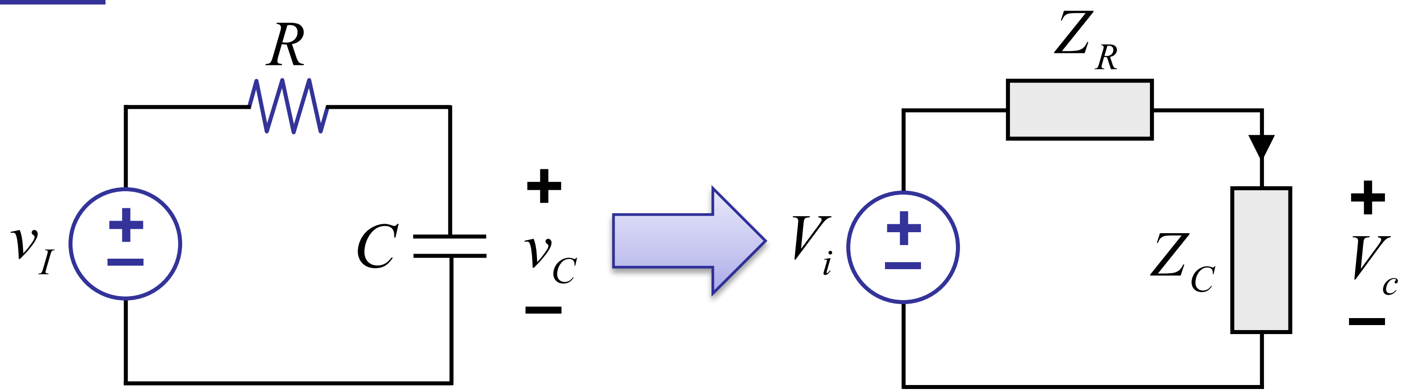
Review

V-code: ???



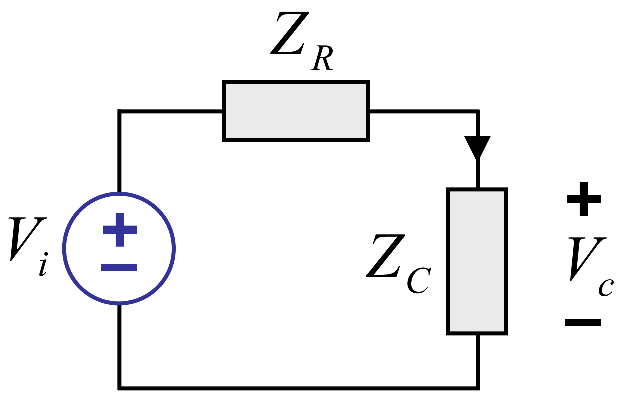
$$V_c = \frac{Z_C}{Z_C + Z_R} \cdot V_i$$

$$\frac{V_c}{V_i} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC}$$

V-code: ???

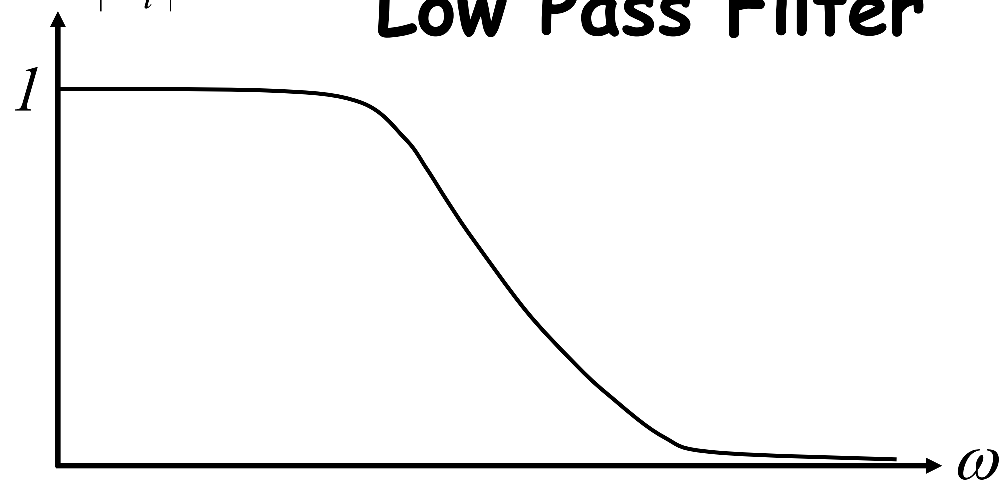
Review

A Filter



$$|H(\omega)| = \left|\frac{V_c}{V_i}\right|$$

"Low Pass Filter"



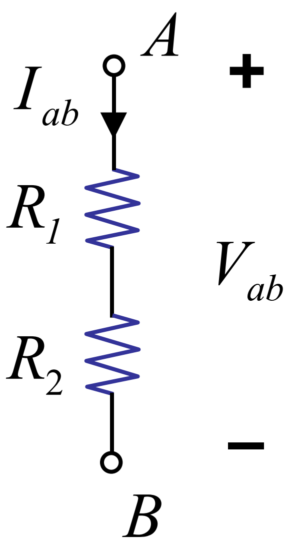
$$V_c = \frac{Z_C}{Z_C + Z_R} \cdot V_i = \frac{1}{1 + j\omega RC}$$

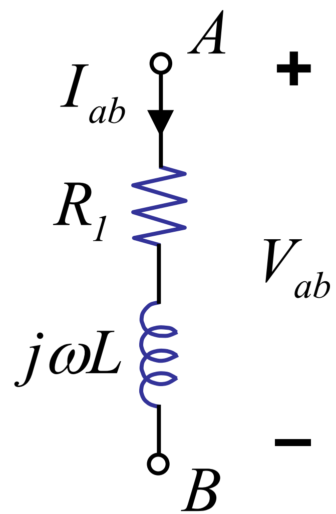
Review

V-code: ???

Quick Review of Impedances-

Just as





$$R_{AB} = \frac{V_{ab}}{I_{ab}} = R_1 + R_2$$

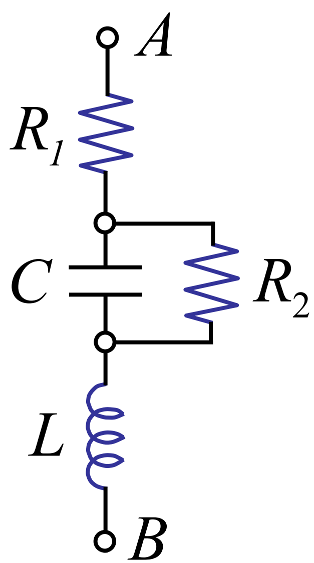
$$Z_{AB} = \frac{V_{ab}}{I_{ab}} = R_1 + j\omega L$$

Review

V-code: ???

Quick Review of Impedances

Similarly



$$Z_{AB} = R_1 + Z_C \,||\, R_2 + Z_L$$

$$= R_1 + \frac{Z_C R_2}{Z_C + R_2} + Z_L$$

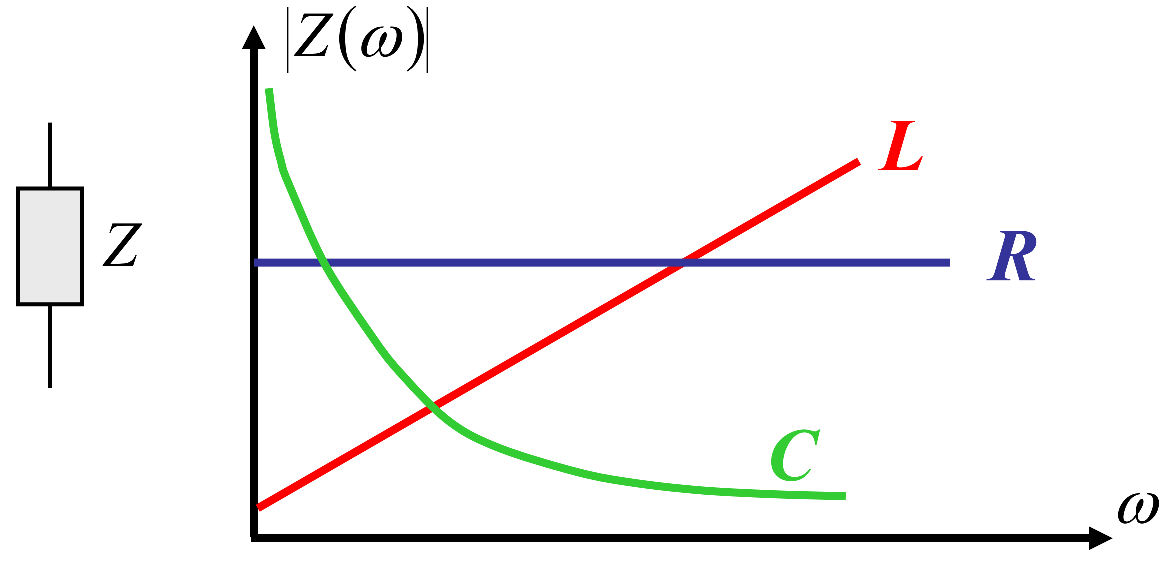
$$= R_1 + \frac{R_2}{1 + j\omega C R_2} + j\omega L$$

V-code: ???

Filters

We can build other filters by combining impedances.

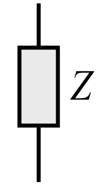
Filter = only passing signals within a specific range of frequency while rejecting ones outside the range.

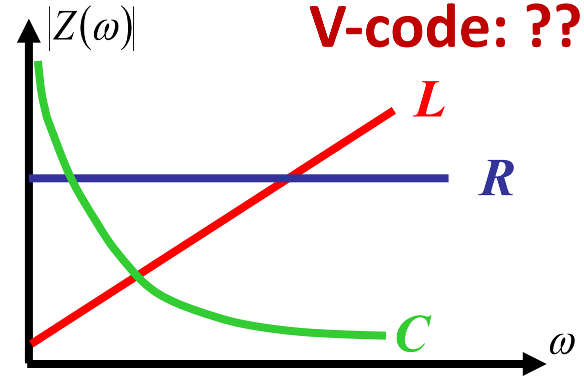


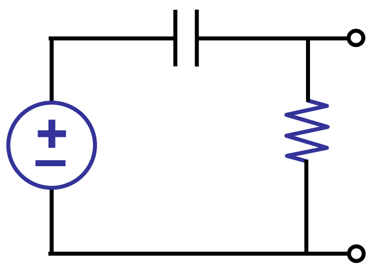
V-code: ???

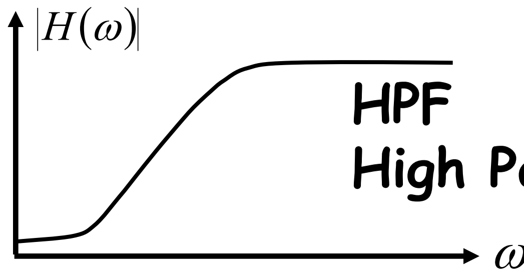
Filters

We can build other filters by combining impedances.

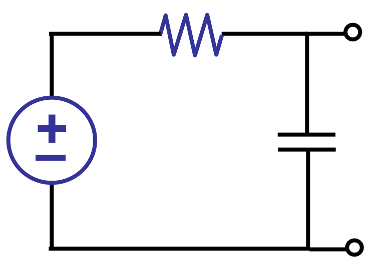


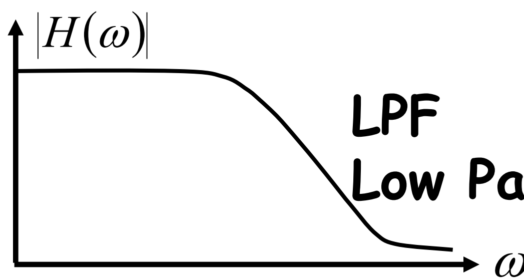




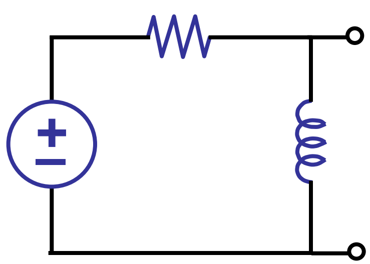


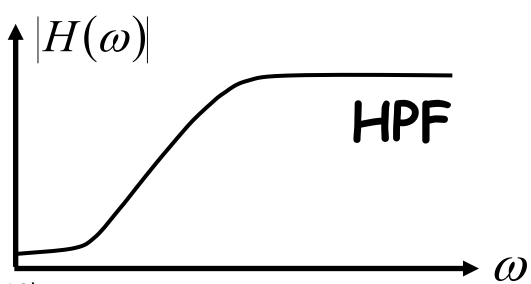
HPF
High Pass Filter





LPF
Low Pass Filter



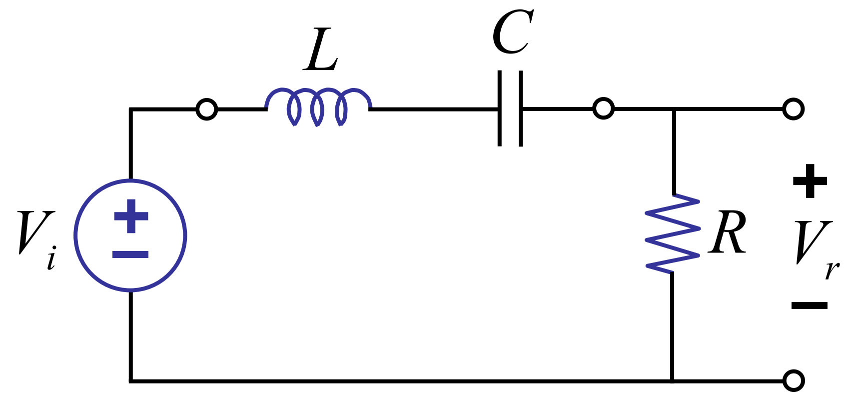


HPF

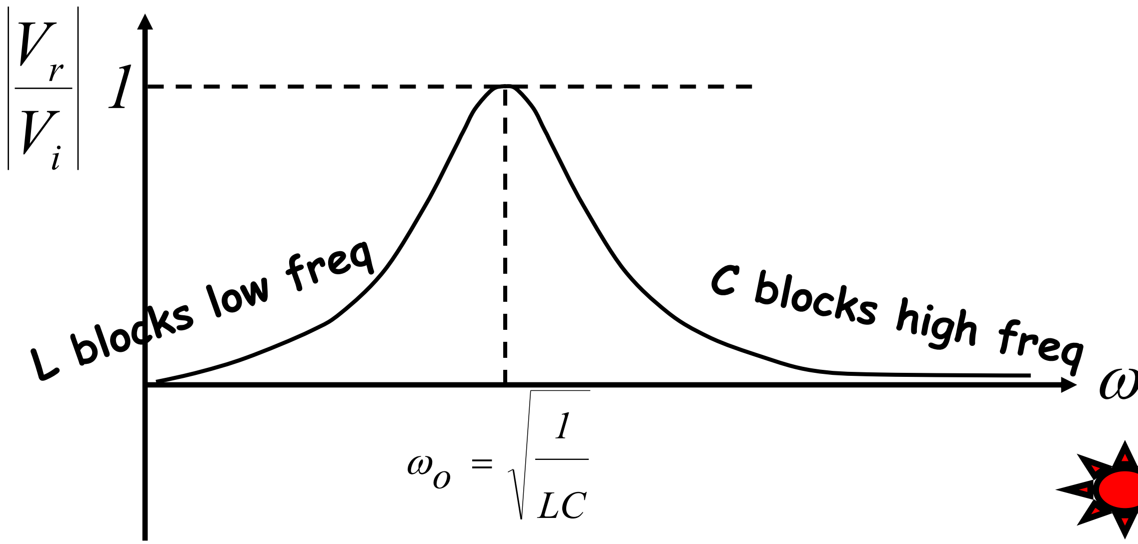
V-code: ???

Filters

Check out:



Intuitively:



$$\frac{V_r}{V_i} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R} = \frac{j\omega RC}{1 - \omega^2 LC + j\omega RC}$$

$$\left|\frac{V_r}{V_i}\right| = \frac{\omega RC}{\sqrt{\left(1 - \omega^2 LC\right)^2 + (\omega RC)^2}}$$

At resonance,

$$\omega = \omega_o$$

and

$$Z_L + Z_C = 0,$$

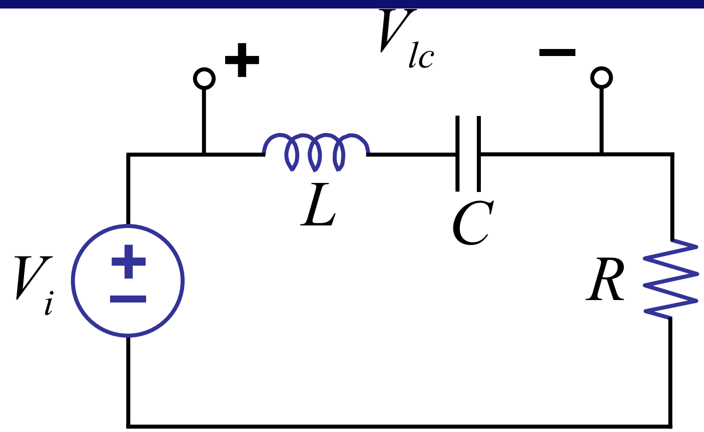
so V_i sees only R!

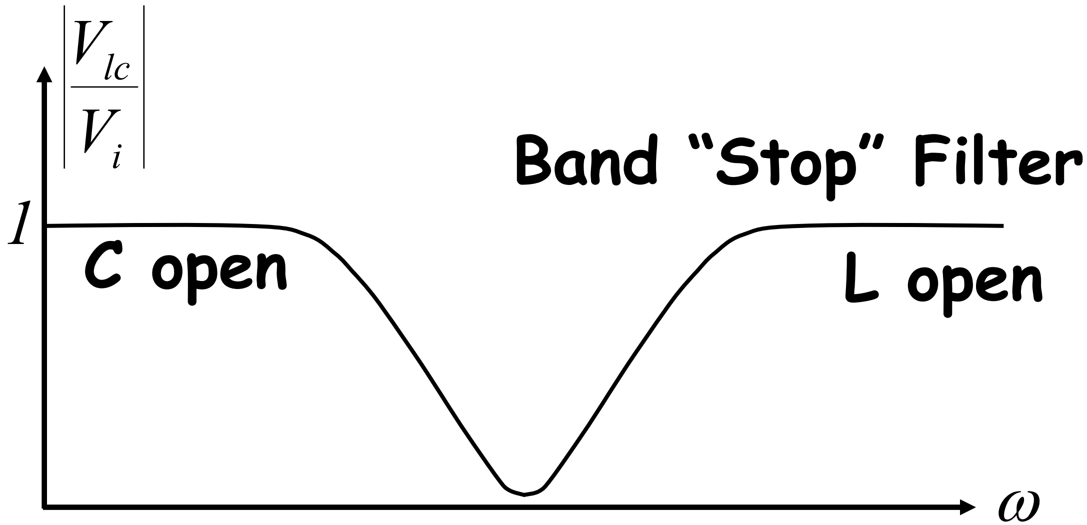
→ *Maximum Response*

V-code: ???

Filters

What about:

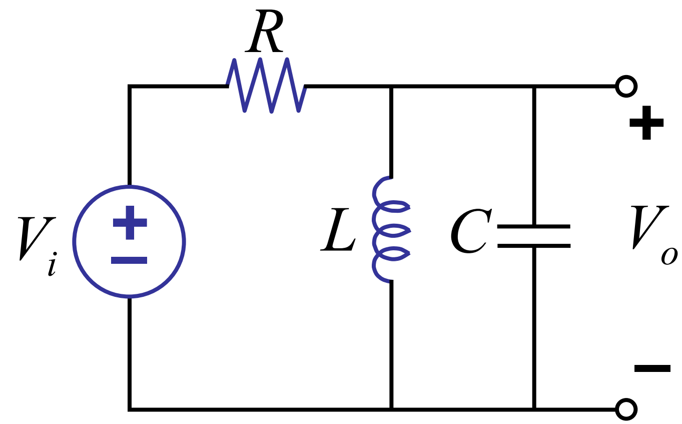


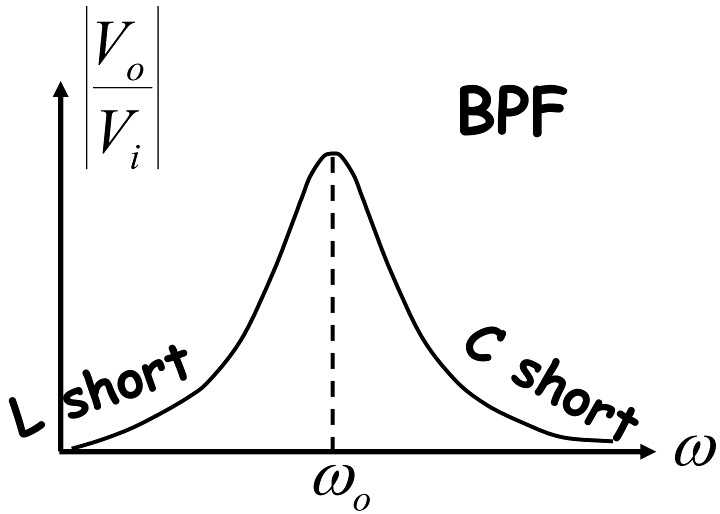


V-code: ???

Filters

Another example:





Application: see AM radio coming up shortly

V-code: ???

Outline

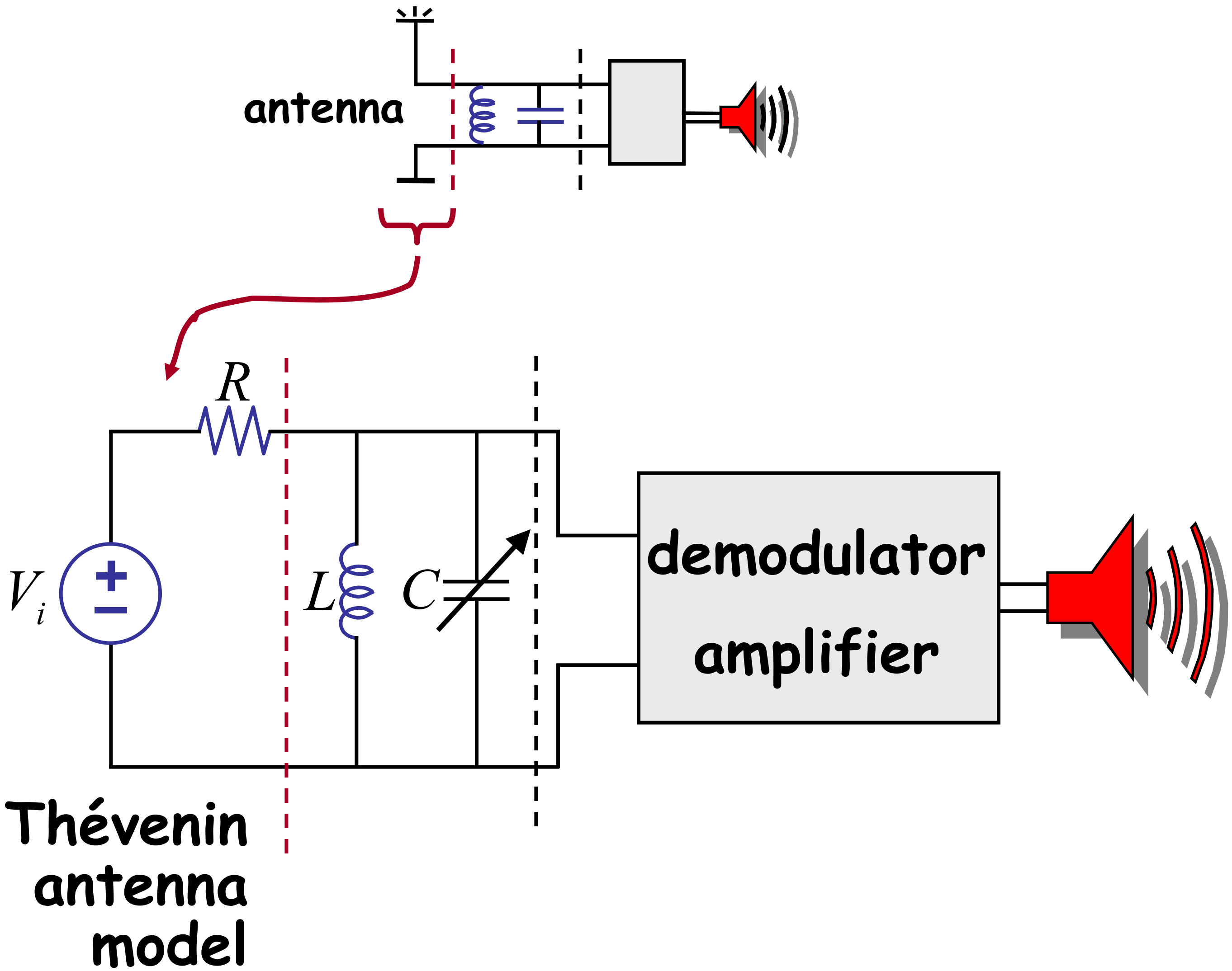
Textbook: 13.1, 13.2, 13.3, 13.4, 13.5, 14.2, 14.3

- Sinusoidal Response of RC Network: Three Approaches
- Impedance Model
- Frequency Response and Filters
- **Frequency Response for Resonance Systems**

V-code: ???

Filters

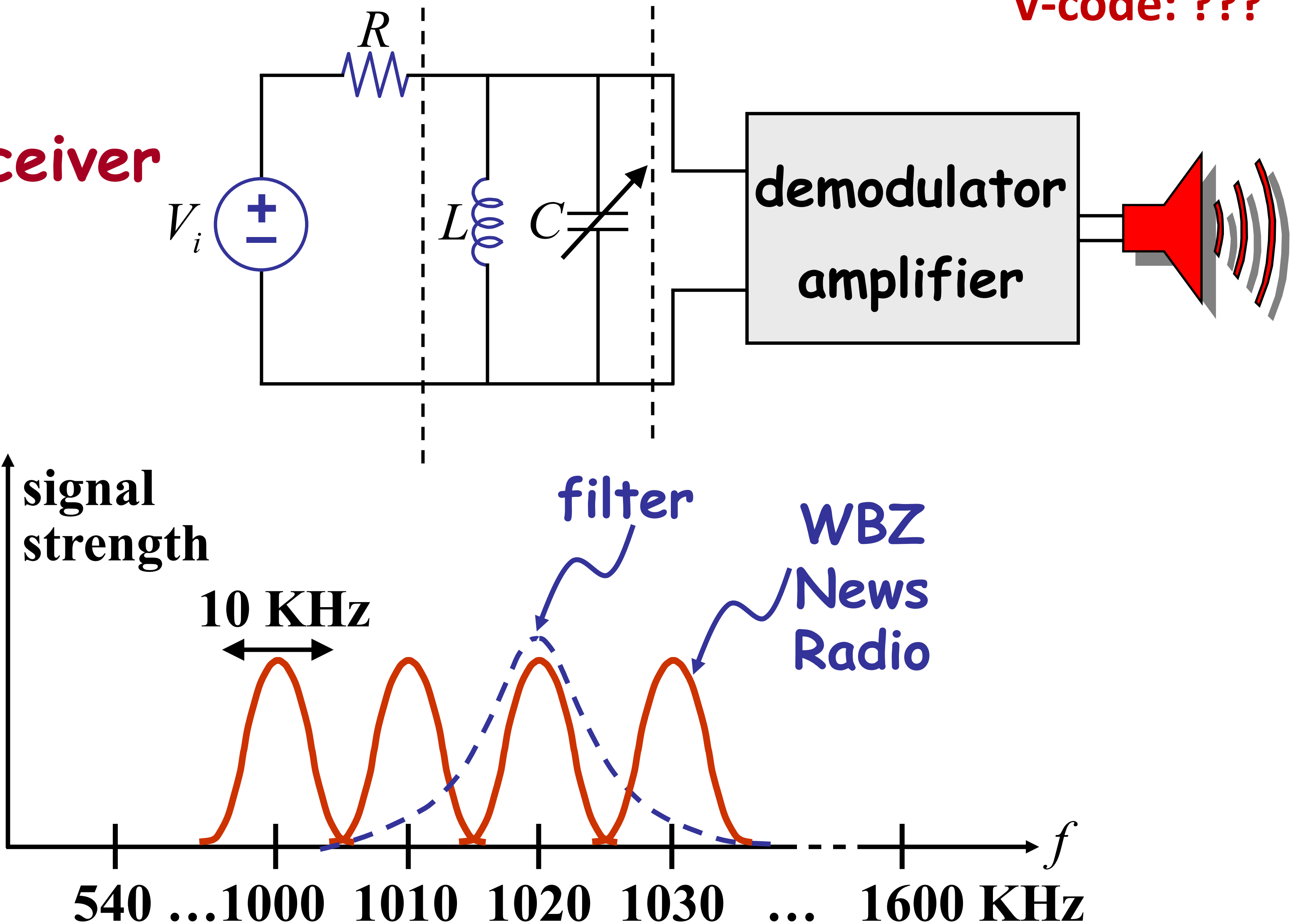
AM Radio Receiver



V-code: ???

Filters

AM Receiver

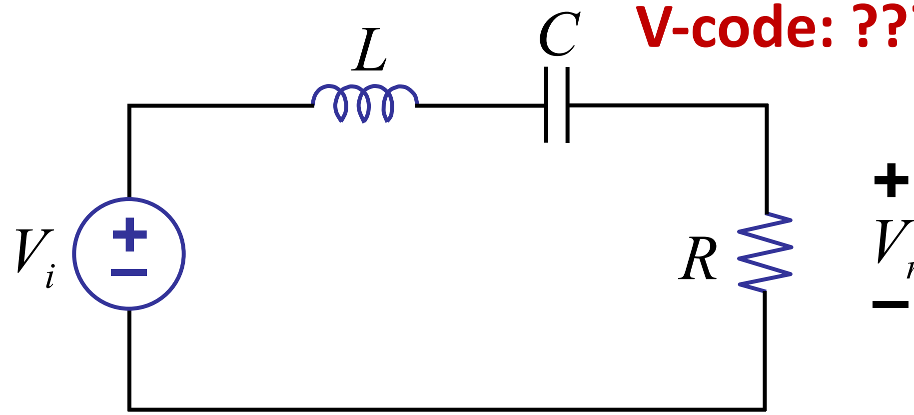


"Selectivity" important —
relates to a parameter "Q" for the filter. Next…

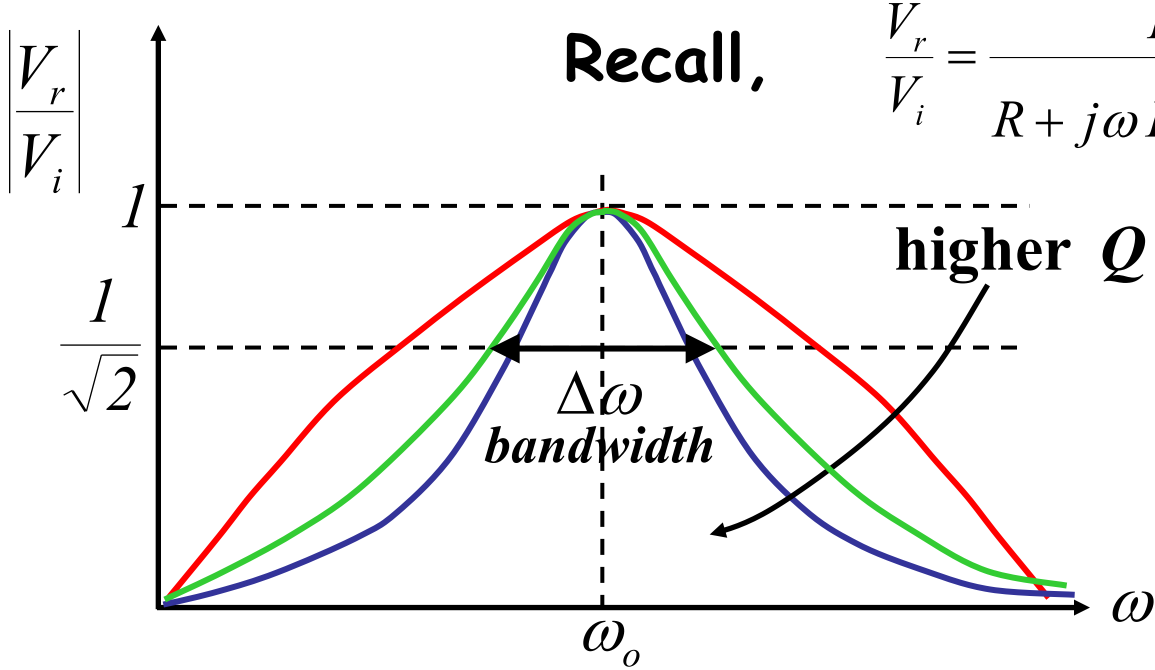
V-code: ???

Filters

Selectivity:
Look at series RLC in more detail



Recall, $$\frac{V_r}{V_i} = \frac{R}{R + j\omega L + \dfrac{1}{j\omega C}}$$



Define $Q = \dfrac{\omega_o}{\Delta\omega}$ ***quality factor***

high Q $\Rightarrow$ **more selective**

V-code: ???

Quality Factor Q

$$Q = \frac{\omega_o}{\Delta\omega}$$

ω_o:

$$\frac{V_r}{V_i} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{1 + j\underbrace{\left(\omega\frac{L}{R} - \frac{1}{\omega CR}\right)}_{\text{at } \omega_o = 0}}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$\Delta\omega$?

V-code: ???

Quality Factor Q

$$Q = \frac{\omega_o}{\Delta\omega}$$

$\Delta\omega$:

Note that abs magnitude is $\frac{1}{\sqrt{2}}$

when $$\frac{V_r}{V_i} = \frac{1}{1 + j\left(\omega\frac{L}{R} - \frac{1}{\omega CR}\right)} = \frac{1}{1 \pm j1}$$

i.e. when $$\frac{\omega L}{R} - \frac{1}{\omega CR} = \pm 1$$

$$\omega^2 \mp \frac{\omega R}{L} - \frac{1}{LC} = 0$$

Looking at the roots of both equations,

$$\omega_1 = \frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} + \frac{4}{LC}} \qquad \omega_2 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}$$

$$\Delta\omega = \omega_1 - \omega_2 = \frac{R}{L}$$

V-code: ???

Quality Factor Q

$$Q = \frac{\omega_o}{\Delta\omega}$$

$$Q = \frac{\omega_o}{\frac{R}{L}} = \frac{\omega_o L}{R} \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

The lower the R (for series R), the sharper the peak

V-code: ???

Quality Factor Q

Another way of looking at Q:

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per cycle}}$$

$$= 2\pi \frac{\frac{1}{2} L |I_r|^2}{\frac{1}{2} |I_r|^2 R \frac{2\pi}{\omega_0}}$$

$$Q = \frac{\omega_o L}{R}$$

V-code: ???

Exercise

Find the resonance frequency and the equivalent impedance at resonance.