Network Theorems

Lecture 3
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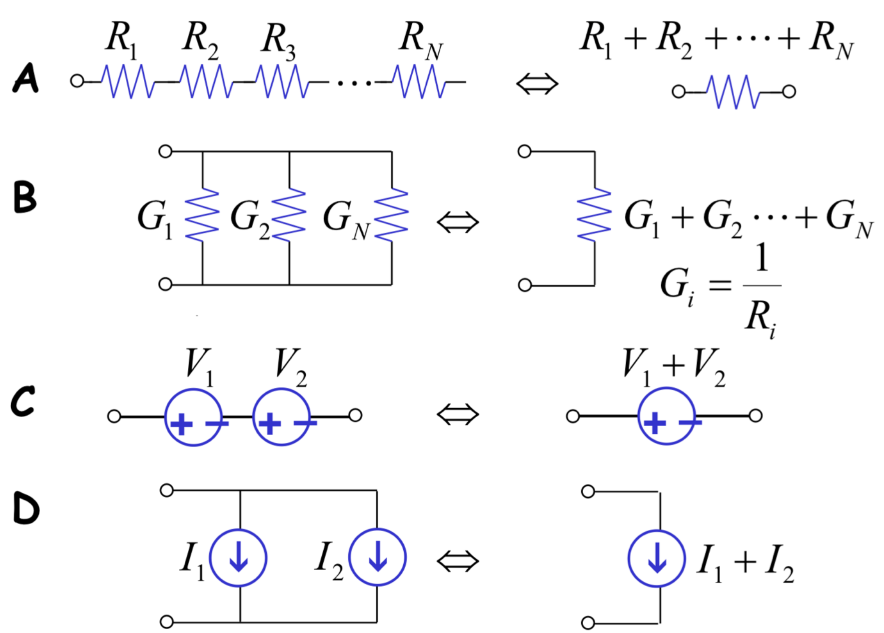
***Slide credits**: Prof. Anant Agarwal at MIT*

Review: Circuit Analysis Methods

- Circuit Analysis: Find all element v's and i's for a given circuit
- Method 1: Basic KVL, KCL method

$$\sum_{loop} V_i = 0 \qquad\qquad \sum_{node} I_i = 0$$

- Method 2: Apply element combination rules



Outline

Textbook: Ch. 3.1-3.6

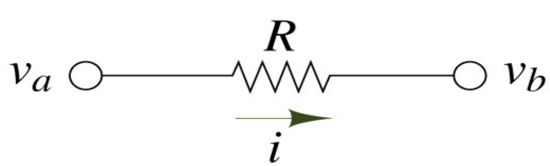
- **Node (Voltage) Analysis**
- Mesh (Loop) Analysis
- Linearity and Superposition Method
- Thevenin's Theorem and Norton's Theorem

Node Analysis (Method 3)

- **Node Analysis = Particular application of KVL, KCL method**
  - Node voltage: the potential difference between the given node and the reference (ground) node
  - Calculating branch current in Node Analysis

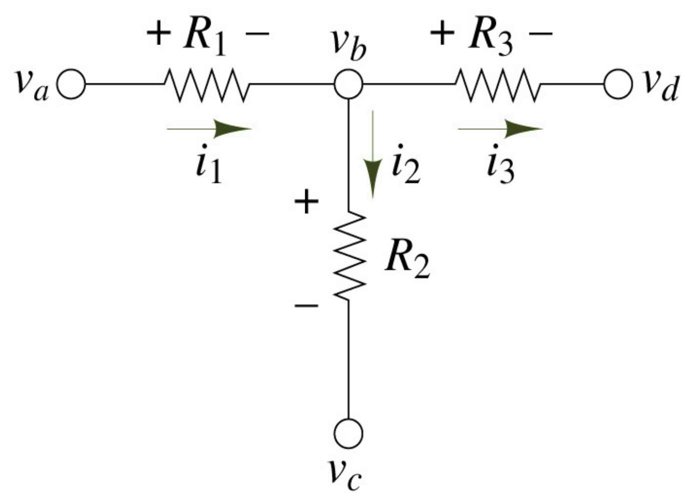
In the node voltage method, we assign the node voltages v_a and v_b; the branch current flowing from a to b is then expressed in terms of these node voltages.

$$i = \frac{v_a - v_b}{R}$$

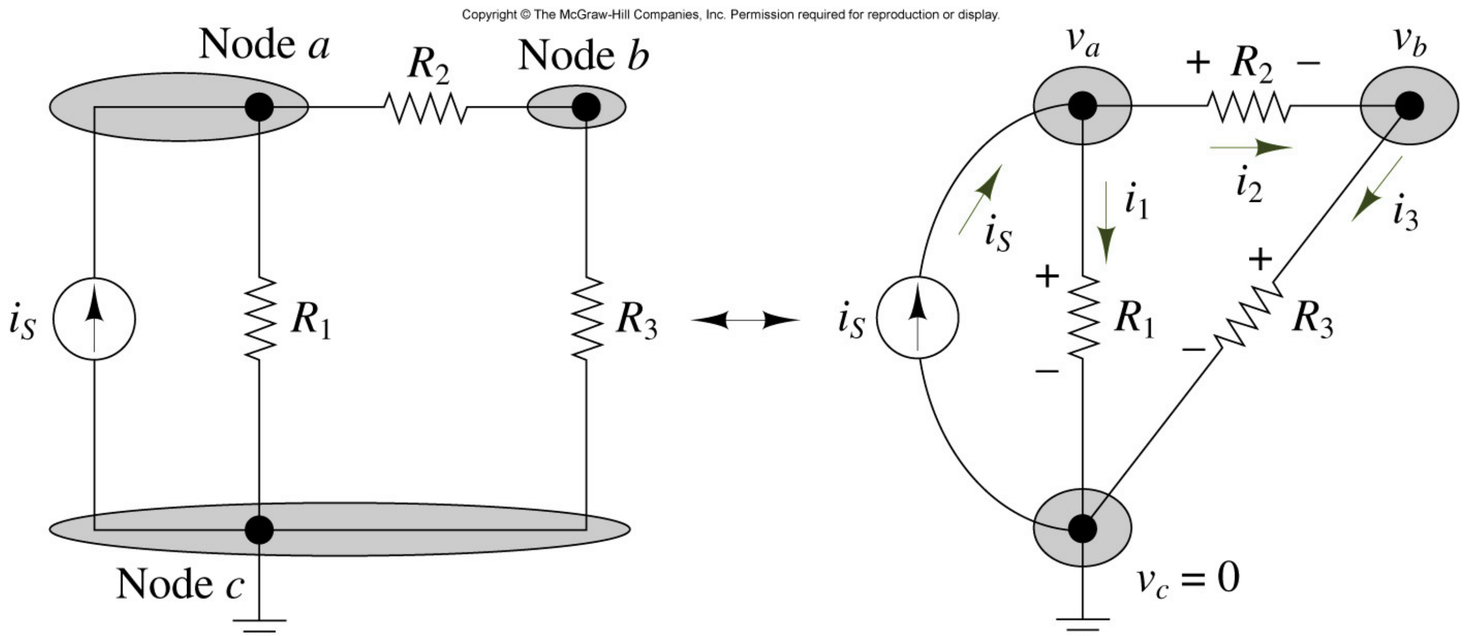


By KCL: $i_1 - i_2 - i_3 = 0$. In the node voltage method, we express KCL by

$$\frac{v_a - v_b}{R_1} - \frac{v_b - v_c}{R_2} - \frac{v_b - v_d}{R_3} = 0$$



Node Analysis: An Illustration

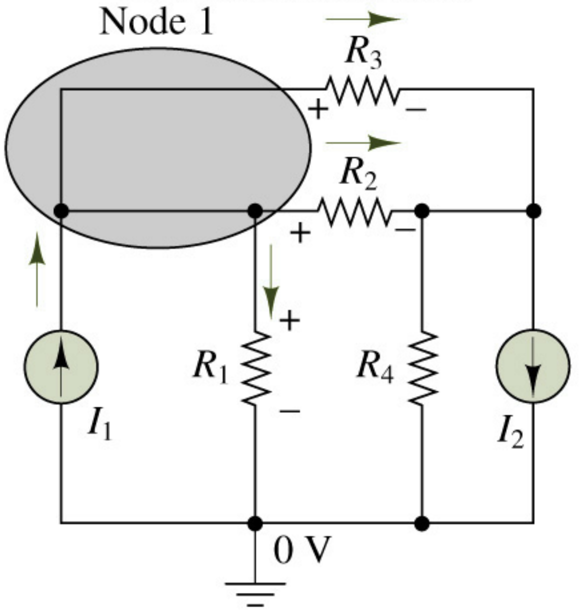


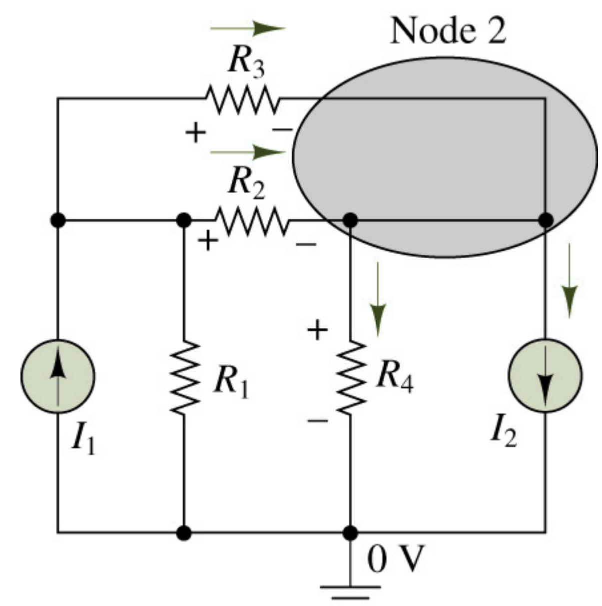
Node Analysis: Example

I1=10mA, I2=50mA, R1 = 1kΩ, R2 = 2kΩ

R3 = 10k, R4 = 2kΩ

Find all node voltages and branch currents



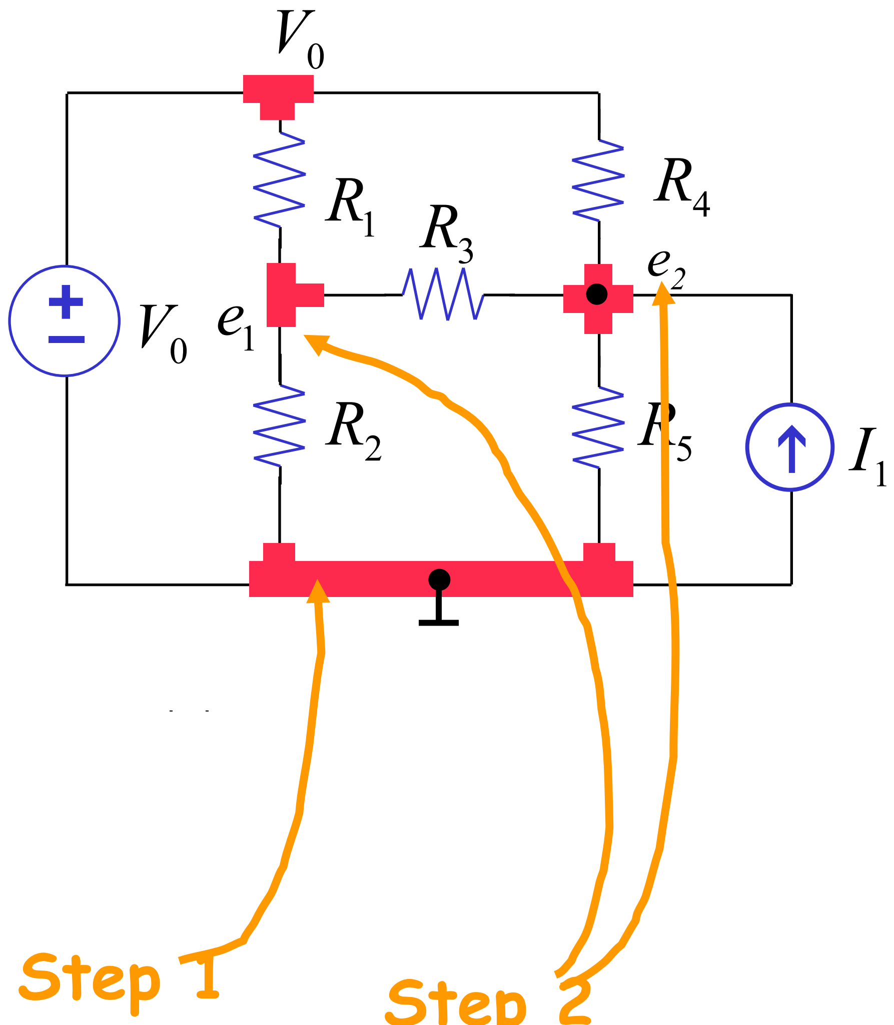


Node Analysis: Procedure

1. Select reference node (⊥ ground) from which voltages are measured.
2. Label voltages of remaining nodes with respect to ground. These are the primary unknowns.
3. Write KCL for all but the ground node, substituting device laws and KVL.
4. Solve for node voltages.
5. Back solve for branch voltages and currents (i.e., the secondary unknowns)

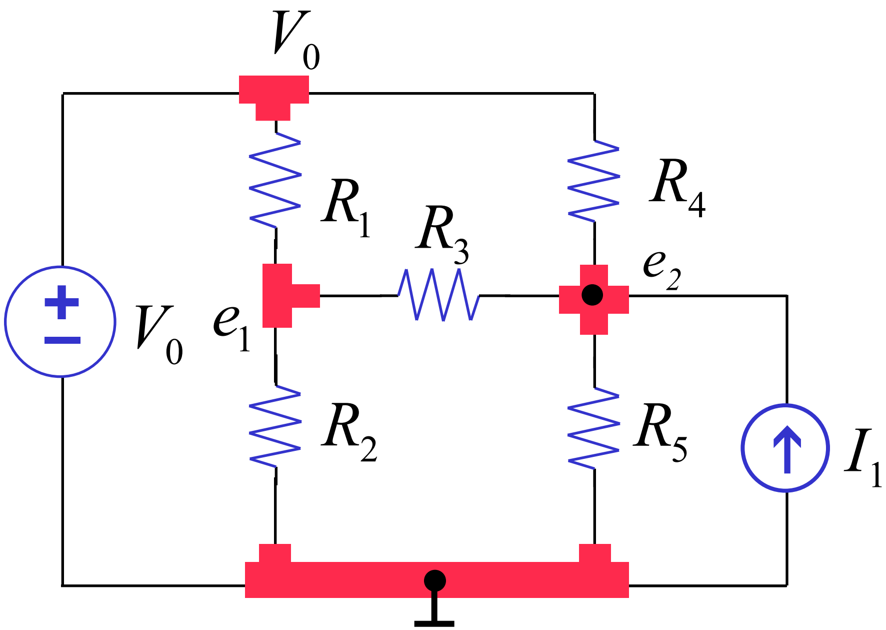
Node Analysis

- **Example with current source**



Node Analysis

- Example with current source



for convenience, write

$$G_i = \frac{1}{R_i}$$

KCL at e_1

$$(e_1 - V_0)G_1 + (e_1 - e_2)G_3 + (e_1)G_2 = 0$$

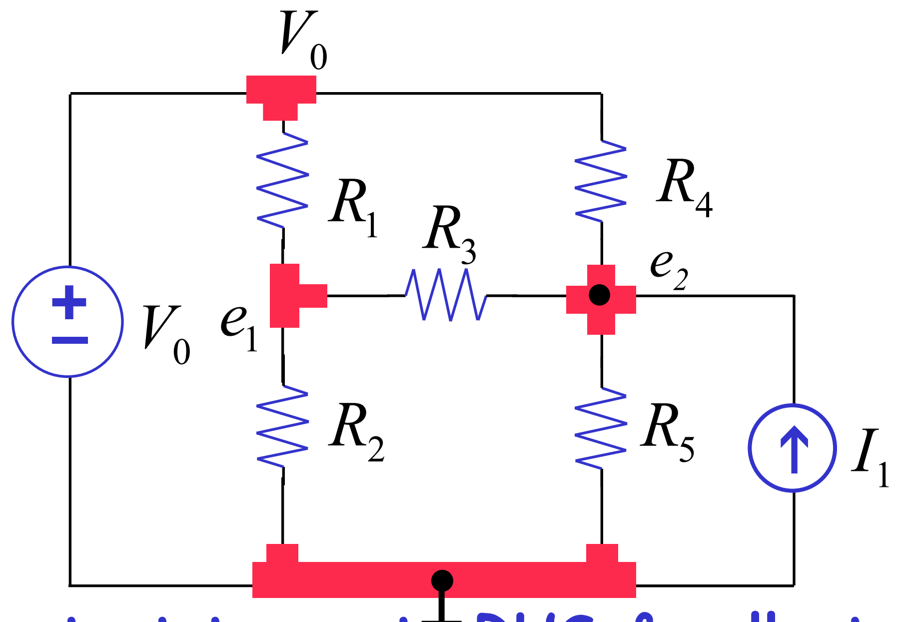
KCL at e_2

$$(e_2 - e_1)G_3 + (e_2 - V_0)G_4 + (e_2)G_5 - I_1 = 0$$

Step 3

Node Analysis

- Example with current source



for convenience, write

$$G_i = \frac{1}{R_i}$$

move constant terms to RHS & collect unknowns

$$e_1(G_1 + G_2 + G_3) + e_2(-G_3) = V_0(G_1)$$

$$e_1(-G_3) + e_2(G_3 + G_4 + G_5) = V_0(G_4) + I_1$$

2 equations, 2 unknowns $\longrightarrow$ Solve for e's

(compare units)

Step 4

Node Analysis

- Example (cont'd)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

In matrix form:

$$\begin{bmatrix} G_1+G_2+G_3 & -G_3 \\ -G_3 & G_3+G_4+G_5 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} G_1V_0 \\ G_4V_0+I_1 \end{bmatrix}$$

conductivity matrix

unknown node voltages

sources

Solve

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{\begin{bmatrix} G_3+G_4+G_5 & G_3 \\ G_3 & G_1+G_2+G_3 \end{bmatrix}\begin{bmatrix} G_1V_0 \\ G_4V_0+I_1 \end{bmatrix}}{(G_1+G_2+G_3)(G_3+G_4+G_5)-G_3^{\ 2}}$$

Notice: linear in V_0, I_1

no negatives

in denominator

$$e_1 = \frac{(G_3+G_4+G_5)(G_1V_0)+(G_3)(G_4V_0+I_1)}{G_1G_3+G_1G_4+G_1G_5+G_2G_3+G_2G_4+G_2G_5+G_3^{\ 2}+G_3G_4+G_3G}$$

$$e_2 = \frac{(G_3)(G_1V_0)+(G_1+G_2+G_3)(G_4V_0+I_1)}{G_1G_3+G_1G_4+G_1G_5+G_2G_3+G_2G_4+G_2G_5+G_3^{\ 2}+G_3G_4+G_3G_5}$$

(same denominator)

Node Analysis

- **Example (cont'd)**

Solve, given

$$\left.\begin{matrix} G_1 \\ G_5 \end{matrix}\right\} = \frac{1}{8.2K} \qquad \left.\begin{matrix} G_2 \\ G_4 \end{matrix}\right\} = \frac{1}{3.9K} \qquad G_3 = \frac{1}{1.5K}$$

$$I_1 = 0$$

$$e_2 = \frac{G_3 G_1 V_0 + (G_1 + G_2 + G_3)(G_4 V_0 + I_1)}{(G_1 + G_2 + G_3) + (G_3 + G_4 + G_5) - G_3^{\,2}}$$

$$G_1 + G_2 + G_3 = \frac{1}{8.2} + \frac{1}{3.9} + \frac{1}{1.5} = 1$$

$$G_3 + G_4 + G_5 = \frac{1}{1.5} + \frac{1}{3.9} + \frac{1}{8.2} = 1$$

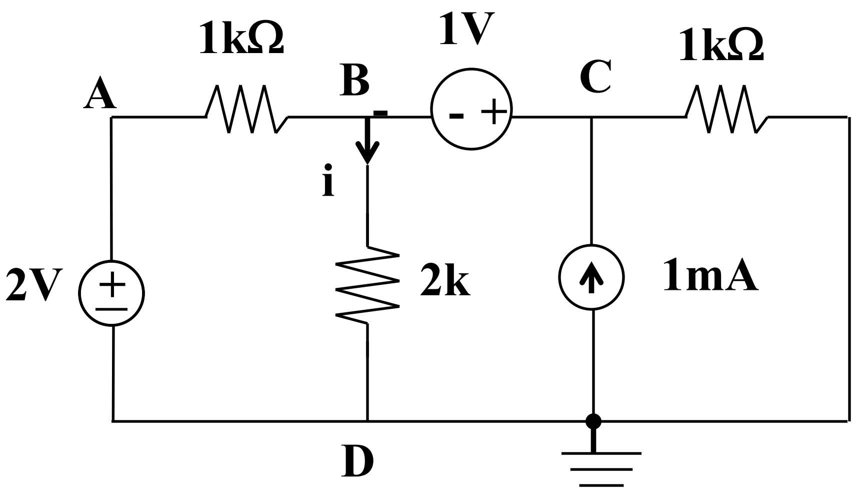
$$e_2 = \frac{\frac{1}{8.2} \times \frac{1}{1.5} + 1 \times \frac{1}{3.9}}{1 - \frac{1}{1.5^2}} V_0$$

$$e_2 = 0.6 V_0$$

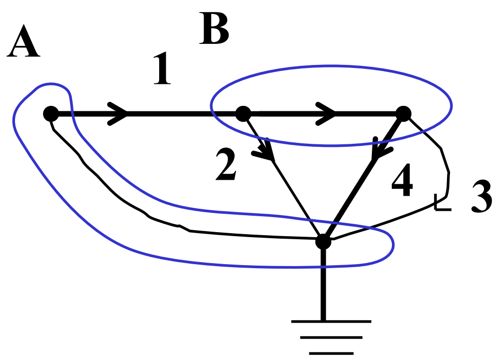
If $V_0 = 3V$, then $e_2 = 1.8 V_0$

Node Analysis

- **Example with Voltage Source**



(a)



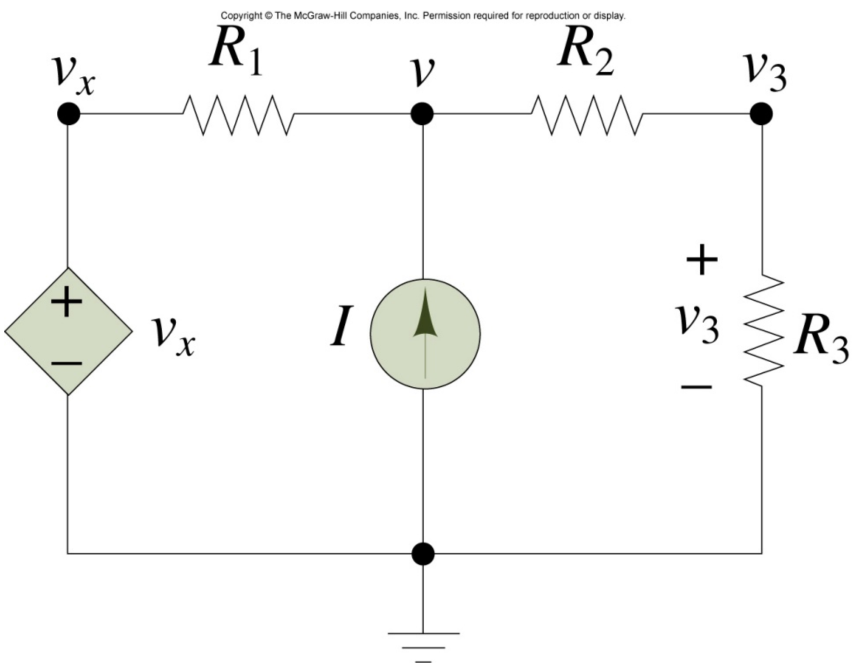
(b)

$$\frac{e_B - e_A}{1} + \frac{e_B}{2} + \frac{e_C}{1} = 1 \quad \Rightarrow \quad \frac{e_B - 2}{1} + \frac{e_B}{2} + \frac{e_B + 1}{1} = 1$$

$$e_B = 0.8, \quad i = 0.4(mA)$$

Node Analysis

- **Example with controlled sources**



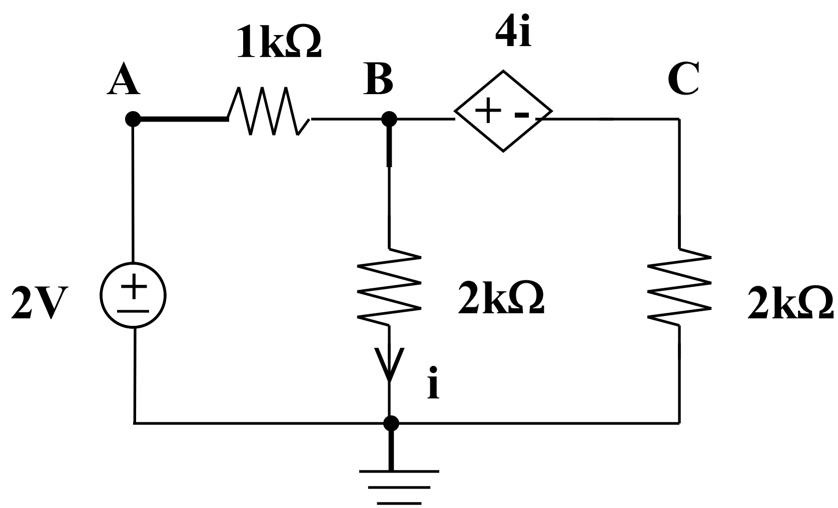
I = 0.5mA, Vx = 2 x v3

R1 = 5kΩ, R2 = 2kΩ, R3 = 4kΩ

Find v

Exercise

Find i using the node analysis technique



Outline

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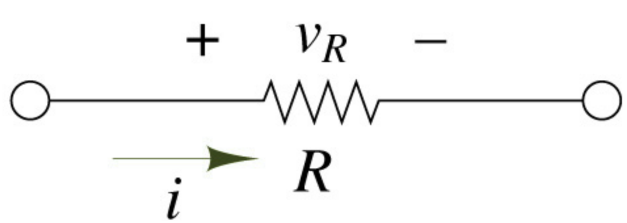
- Node (Voltage) Analysis
- **Mesh (Loop) Analysis**
- Linearity and Superposition Method
- Thevenin's Theorem and Norton's Theorem

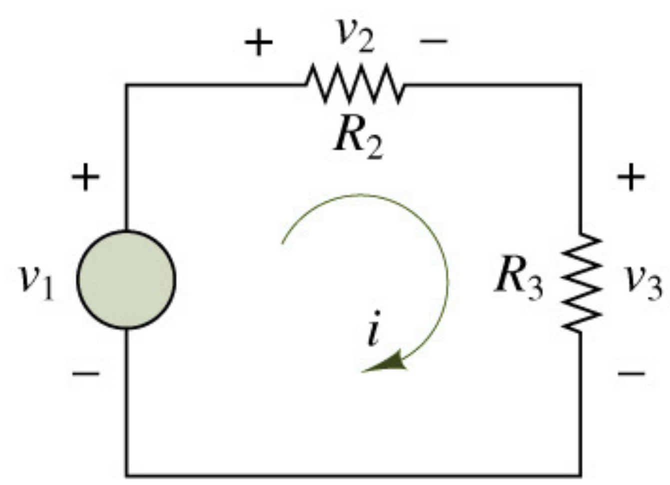
Mesh (Loop) Analysis

- **Simplified analysis method based on an astute choice of current variables that parallel Node Analysis**
- **Basic principle**

The current i, defined as flowing from left to right, establishes the polarity of the voltage across R.



Once the direction of current flow has been selected, KVL requires that $v_1 - v_2 - v_3 = 0$.



A mesh

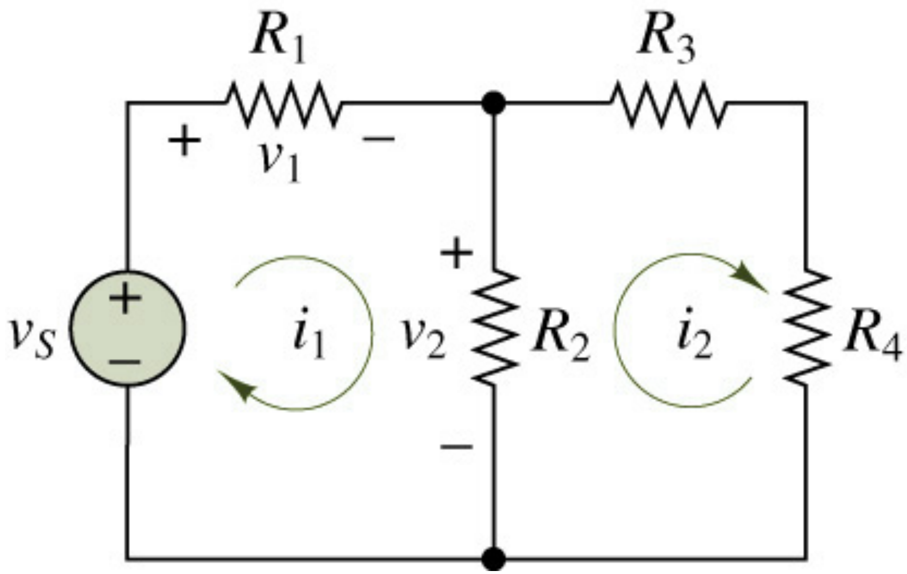
Mesh (Loop) Analysis

- **Two-mesh circuit**

Mesh 1: KVL requires that

$v_S - v_1 - v_2 = 0$, where $v_1 = i_1 R_1$,

$v_2 = (i_1 - i_2) R_1$.



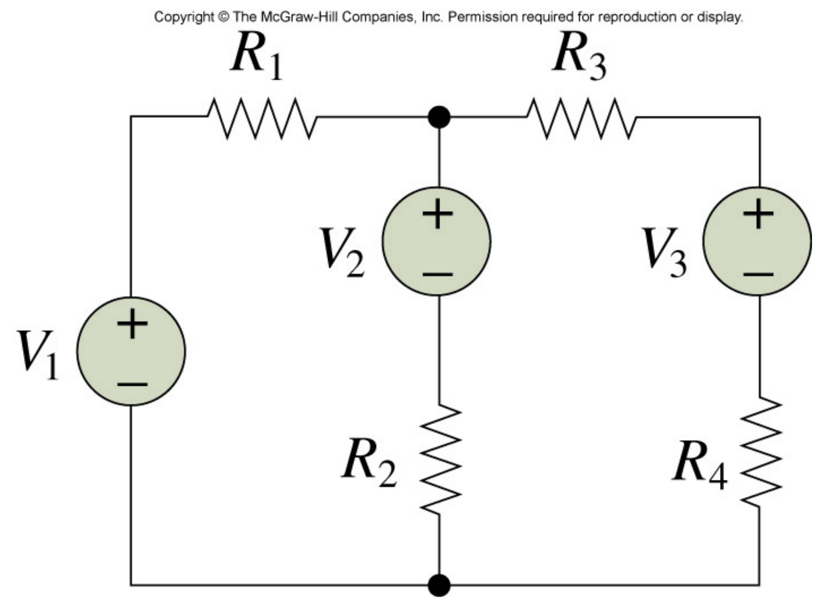
Mesh (Loop) Analysis

- Example

V1 = 10V, V2 = 9V, V3 = 1V,

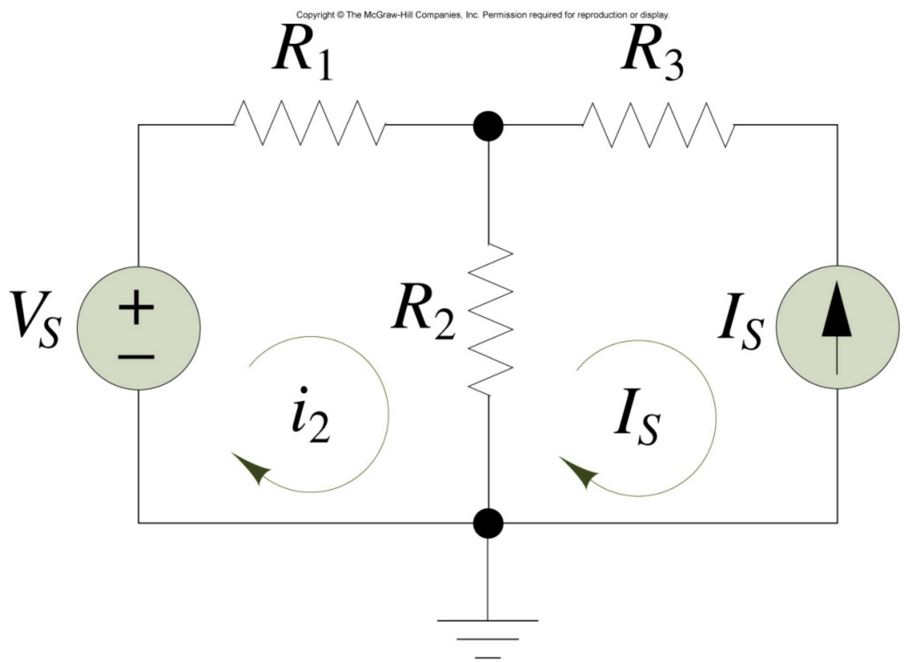
R1 = 5kΩ, R2 = 10kΩ,
R3 = 5kΩ, R4 = 5kΩ

Find i



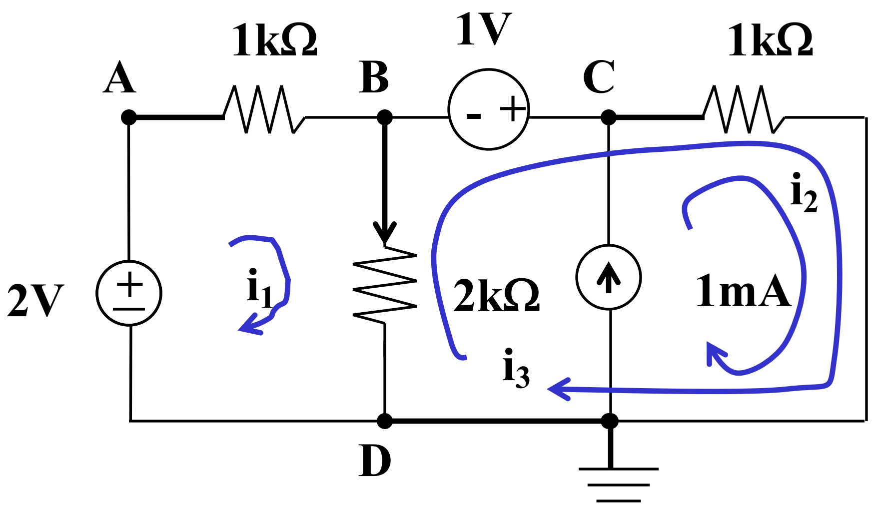
Mesh (Loop) Analysis

- **Example with current sources**



Mesh (Loop) Analysis

- (Another) Example
  - **Every element** must be traversed by **at least one loop current**.

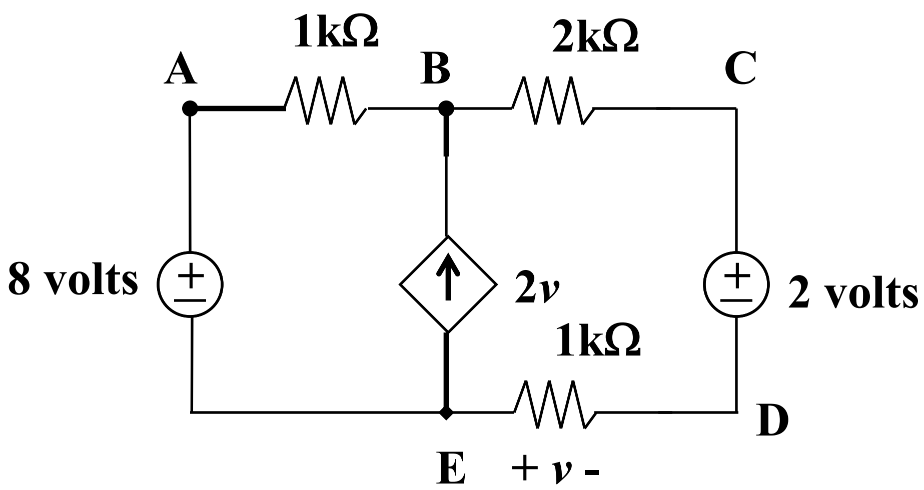


$$\{A,B,D,A\} \quad i_1 + 2(i_1 - i_3) = 2$$

$$\{B,C,D,B\} \quad (i_3 + i_2) + 2(i_3 - i_1) = 1, \quad i_2 = 1$$

Exercise

Find *v* using the loop analysis technique.



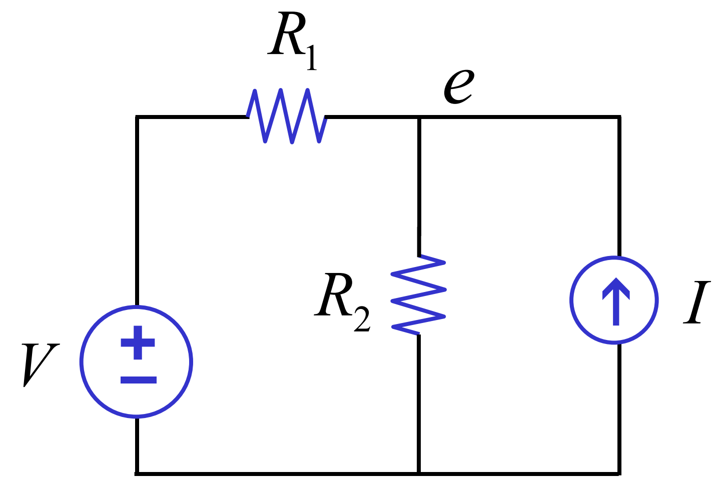
Outline

Textbook: Ch. 3.1-3.6

- Node (Voltage) Analysis
- Mesh (Loop) Analysis
- **Linearity and Superposition Method**
- Thevenin's Theorem and Norton's Theorem

Linearity

Consider



Write node equations –

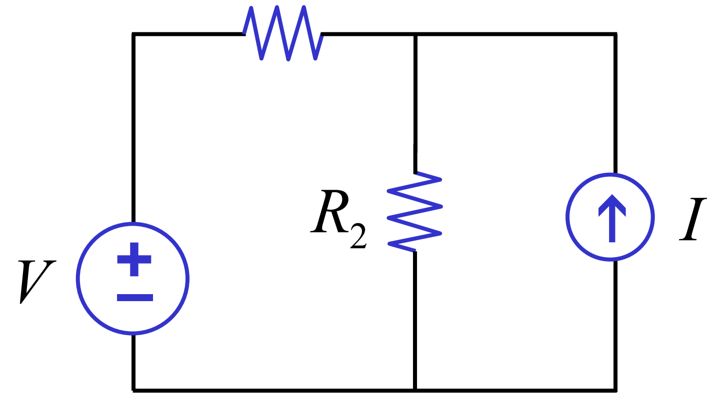
$$\frac{e-V}{R_1}+\frac{e}{R_2}-I=0$$

Notice:
linear in e, V, I

No eV, VI terms

Linearity

Consider



Write node equations --

$$\frac{e-V}{R_1}+\frac{e}{R_2}-I=0$$ **linear in** e, V, I

Rearrange --

$$\underbrace{\left[\frac{1}{R_1}+\frac{1}{R_2}\right]}_{\text{conductance matrix}}\underbrace{e}_{\text{node voltages}}=\underbrace{\frac{V}{R_1}+I}_{\text{linear sum of sources}}$$

$$G \quad e = S$$

Linearity

Write node equations --

$$\frac{e-V}{R_1}+\frac{e}{R_2}-I=0$$ linear in e, V, I

Rearrange --

$$\left[\frac{1}{R_1} + \frac{1}{R_2}\right]e = \frac{V}{R_1} + I$$

conductance matrix

node voltages

linear sum of sources

$$G \qquad e = S$$

or $$e=\frac{R_2}{R_1+R_2}V+\frac{R_1R_2}{R_1+R_2}I$$

$$\boxed{e=a_1V_1+a_2V_2+\ldots+b_1I_1+b_2I_2+\ldots}$$

Linear!

Linearity

Linearity $\Rightarrow$ Homogeneity
Superposition

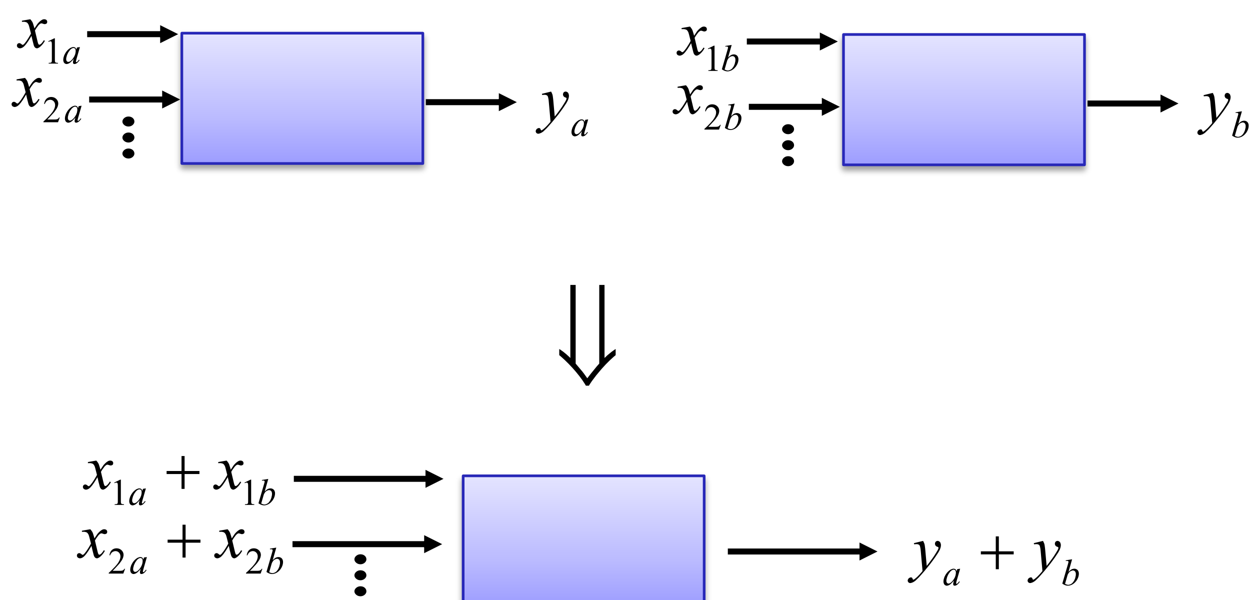
Linearity $\Rightarrow$ Homogeneity, Superposition

- **Homogeneity**

$$\Downarrow$$

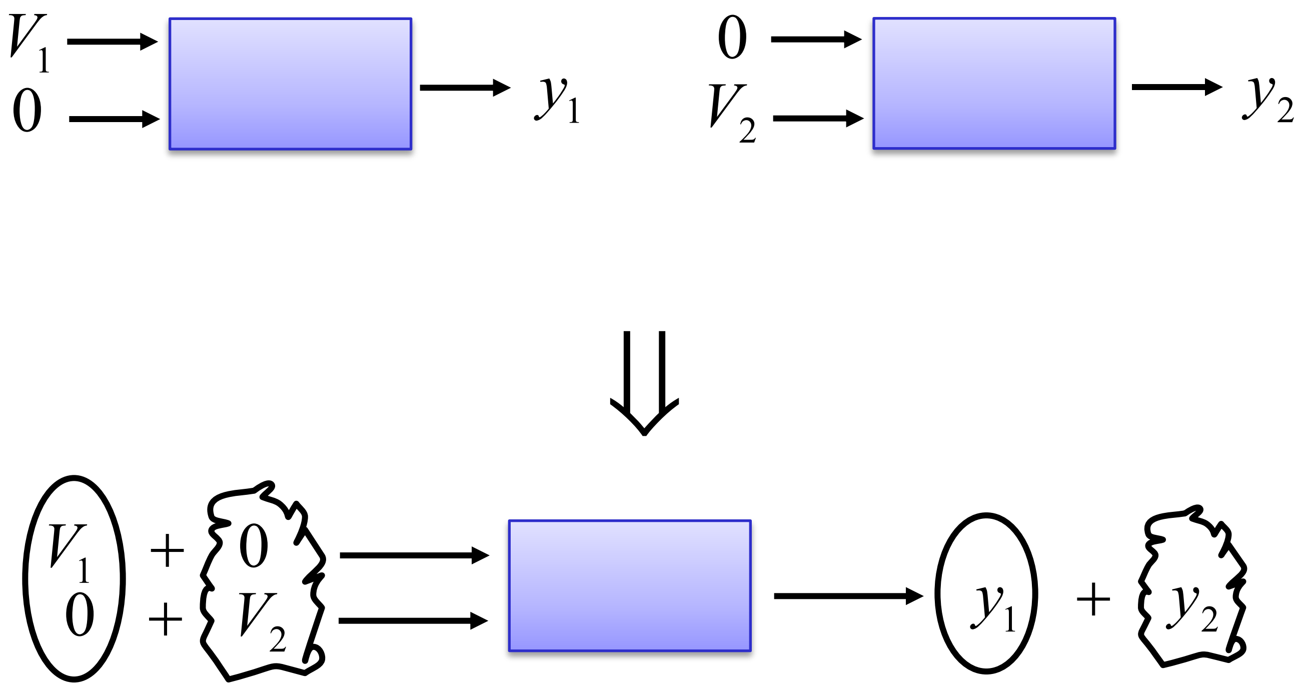
Linearity $\Rightarrow$ Homogeneity, Superposition

- **Superposition**



Linearity $\Rightarrow$ Homogeneity, Superposition

- **Specific superposition example**



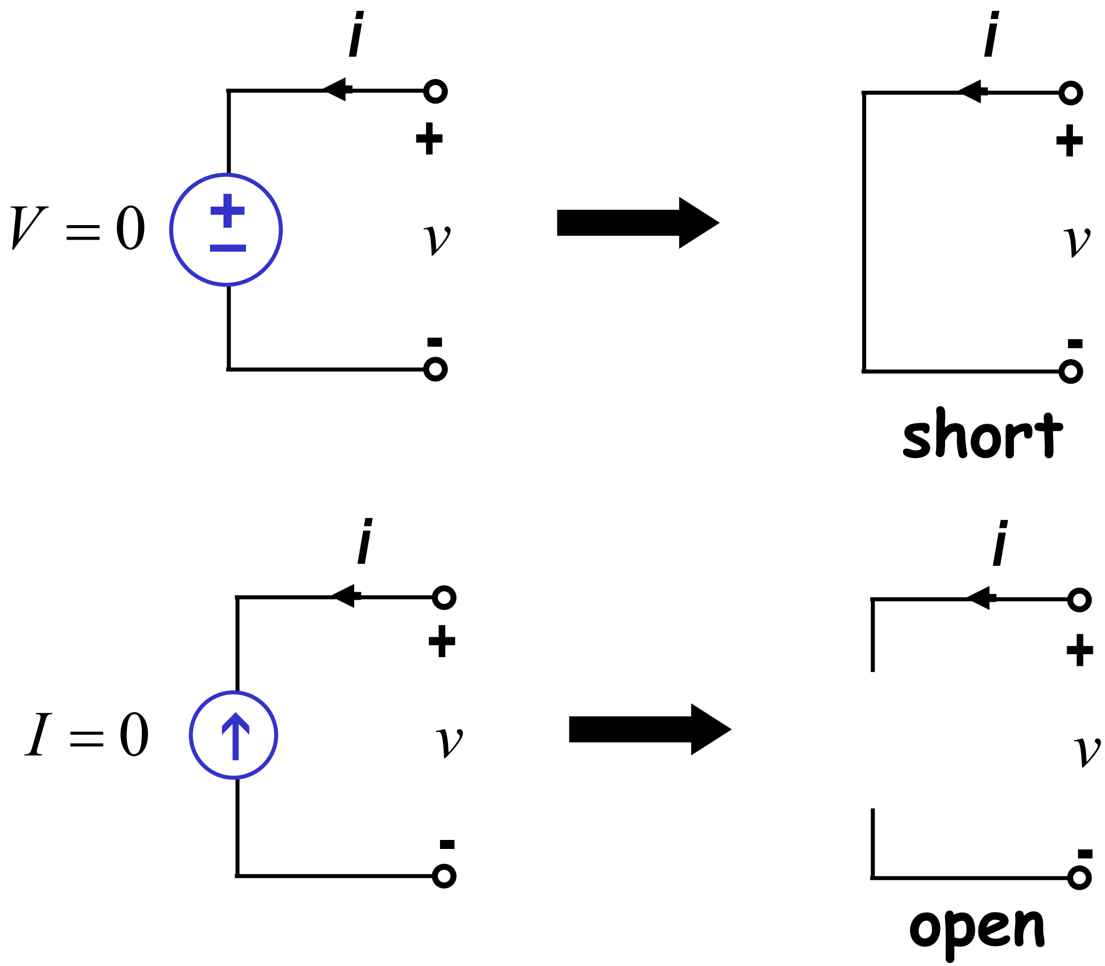
Superposition Method (Method 4)

The output of a circuit is determined by summing the responses to each source acting alone.

independent sources only

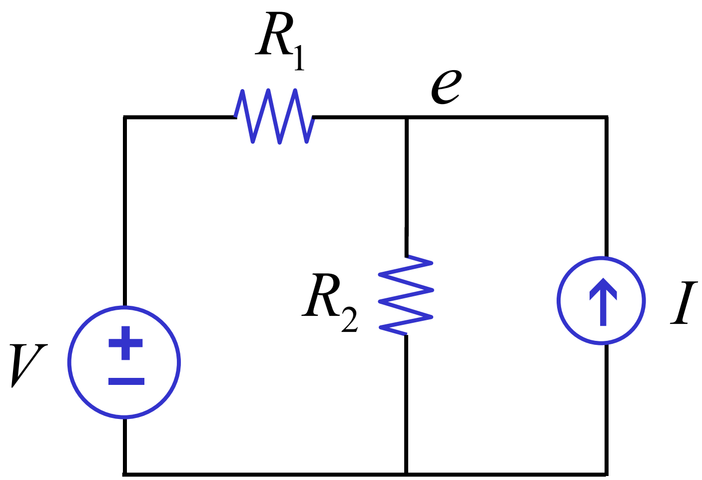
Superposition Method

- **Zeroing voltage and current sources**



Superposition Method

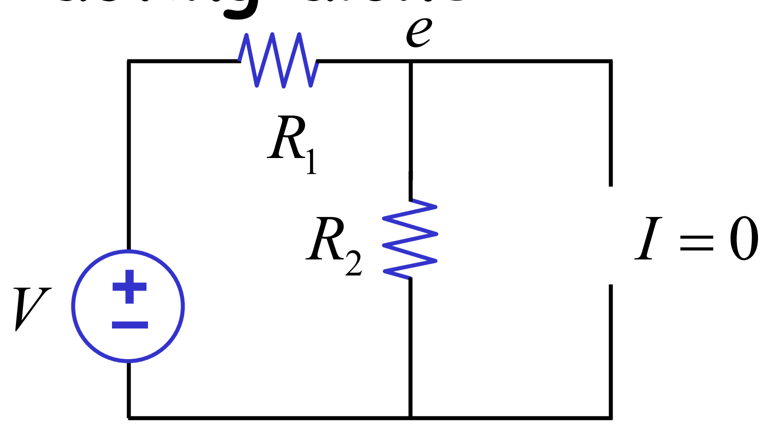
- **Back to the example**



Superposition Method

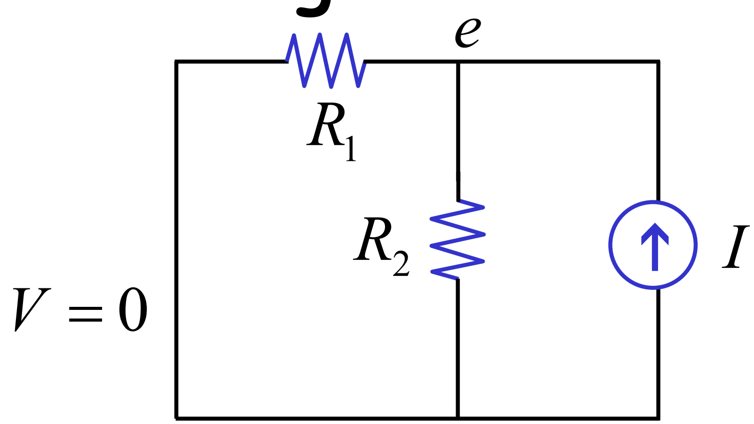
- Back to the example

V acting alone



$$e_V = \frac{R_2}{R_1 + R_2} V$$

I acting alone

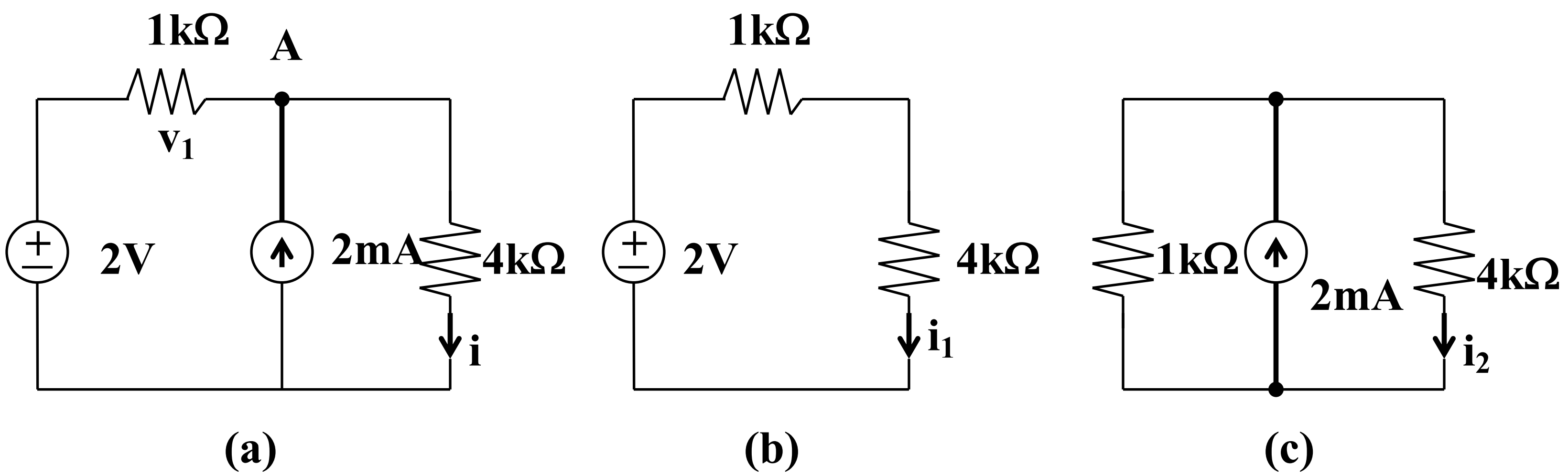


$$e_I = \frac{R_1 R_2}{R_1 + R_2} I$$

sum → superposition

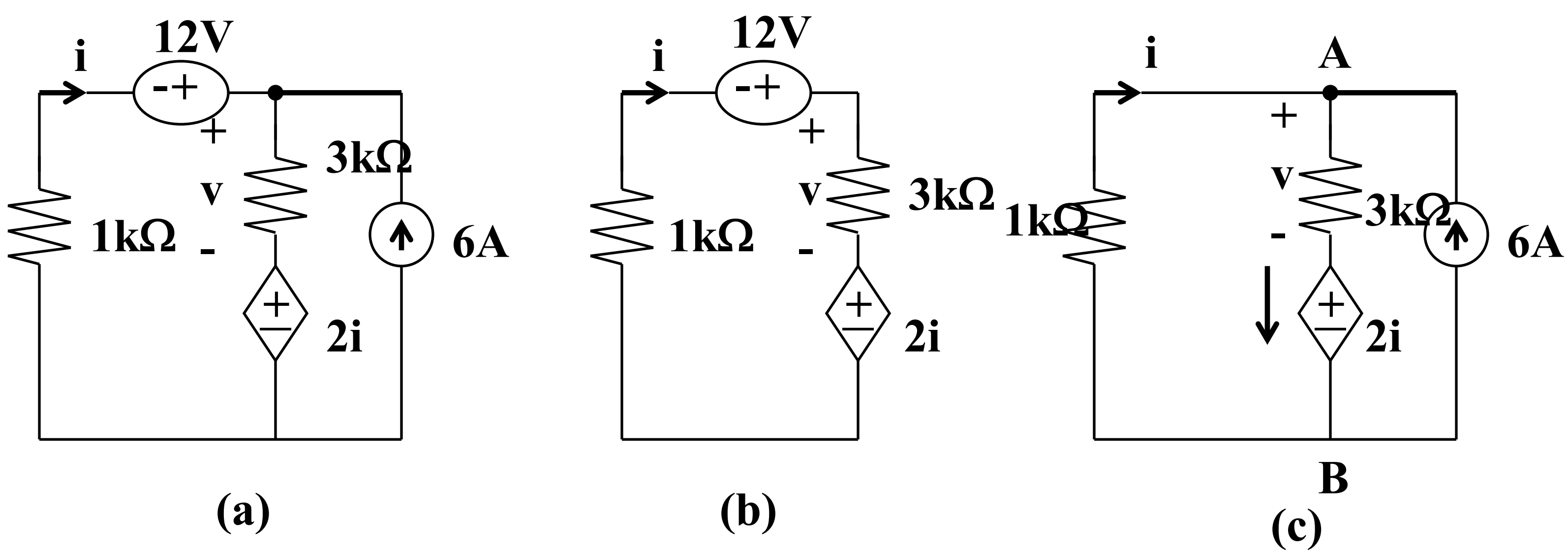
$$e = e_V + e_I = \frac{R_2}{R_1 + R_2} V + \frac{R_1 R_2}{R_1 + R_2} I$$

Exercise



Superposition Method

- **Example with controlled sources**



$$v = v_{(b)} + v_{(c)}$$

$$(b): 12 = 3i + 2i + i \quad \rightarrow i = 2, v = 6$$

$$(c): i + 3(i+6) + 2i = 0 \quad \rightarrow i = -3, v = 9$$

Outline

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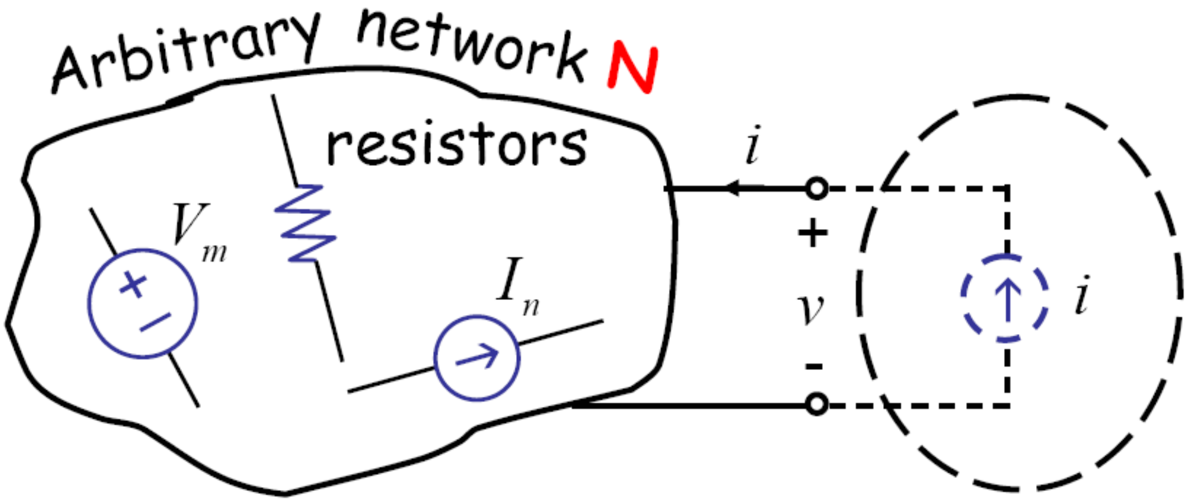
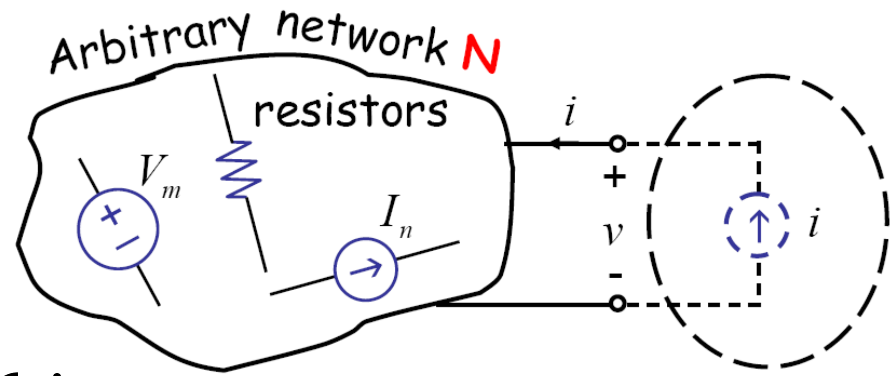
- Node (Voltage) Analysis
- Mesh (Loop) Analysis
- Linearity and Superposition Method
- **Thevenin's Theorem and Norton's Theorem**

Thevenin's Theorem

- **Derivation of Thevenin's network**
  - Thevenin's network abstracts the behavior of a linear network at a given pair of terminals as a voltage source in series with a resistor
  - Consider



Thevenin's Theorem



- Derivation of Thevenin's network

By superposition

$$v = \sum_m \alpha_m V_m + \sum_n \beta_n I_n + Ri$$

α_m: no units

β_n, R: resistance units

R: also independent of external excitement & behaves like a resistor

By setting

$\forall_n I_n = 0,\ i = 0$

$\forall_m V_m = 0,\ i = 0$

All

$\forall_n I_n = 0,$
$\forall_m V_m = 0$

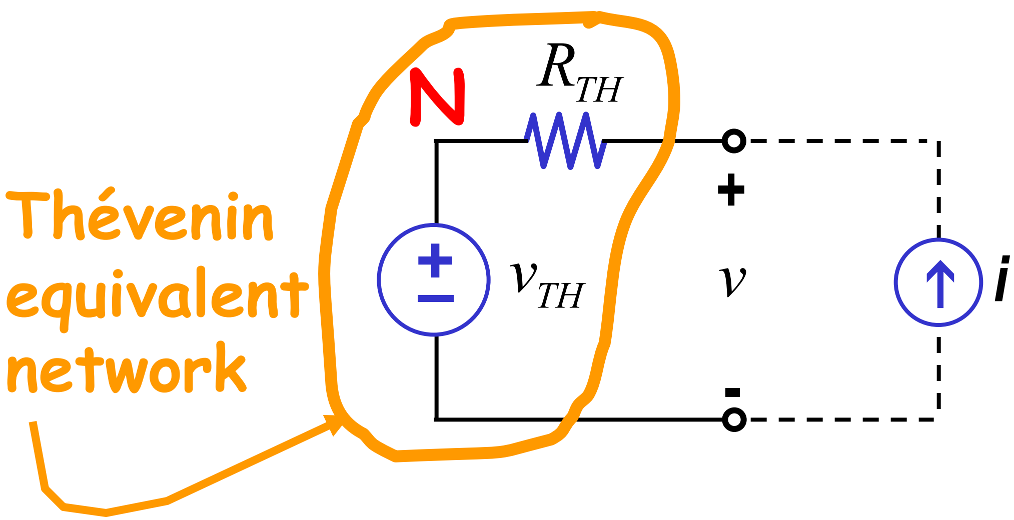
independent of external excitation and behaves like a voltage "v_{TH}"

Thevenin's Theorem

- **Derivation of Thevenin's network**

$$v = v_{TH} + R_{TH} i$$

  - As far as the external world is concerned (for the purpose of I-V relation), "Arbitrary network N" is indistinguishable from:



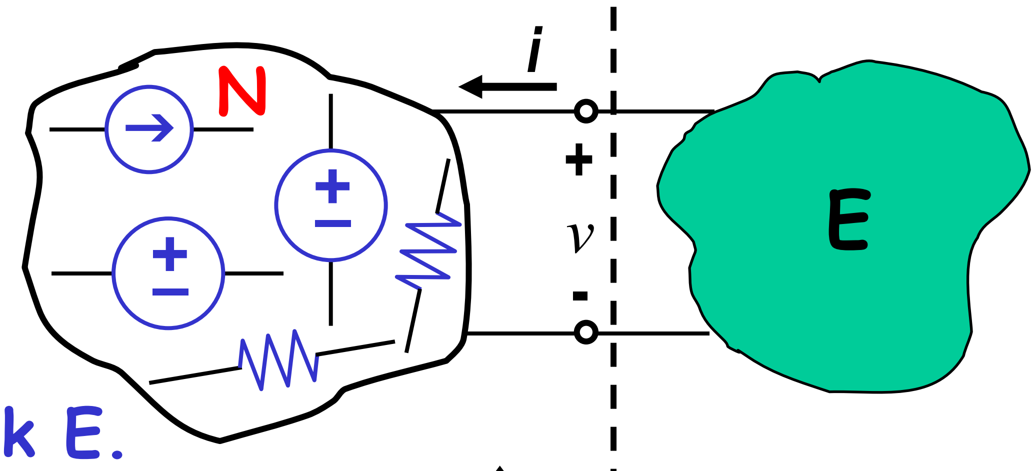
v_{TH} → **open circuit voltage at terminal pair (a.k.a. port)**

R_{TH} → **resistance of network seen from port (V_m's, I_n's set to 0)**

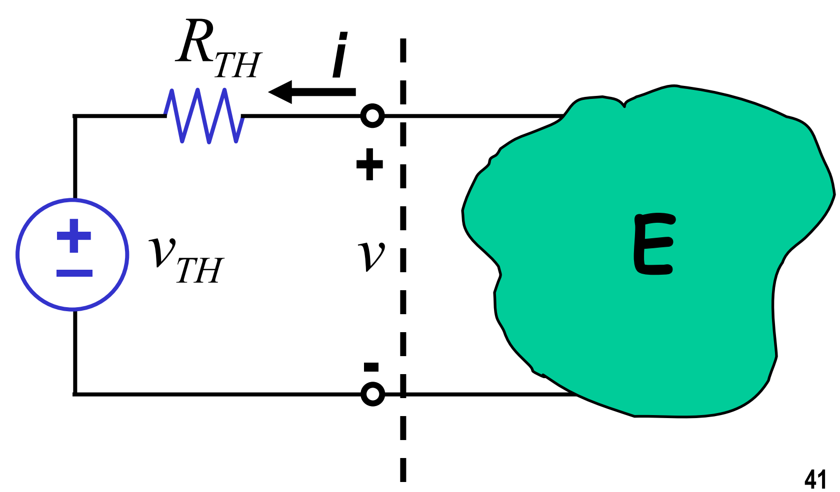
Thevenin's Theorem

- The Thévenin Method (Method 5-1)

Replace network N with its Thévenin equivalent, then solve external network E.

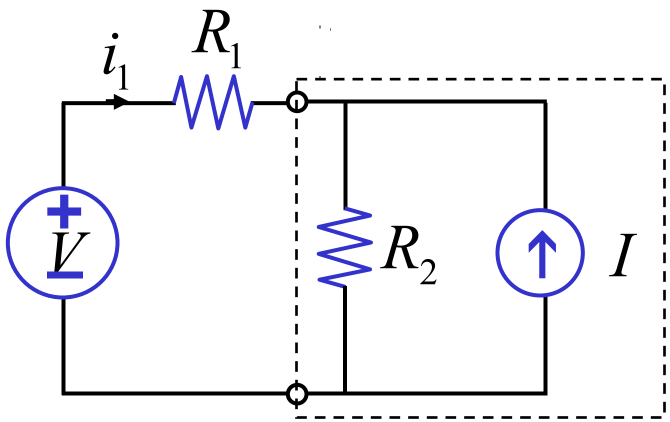


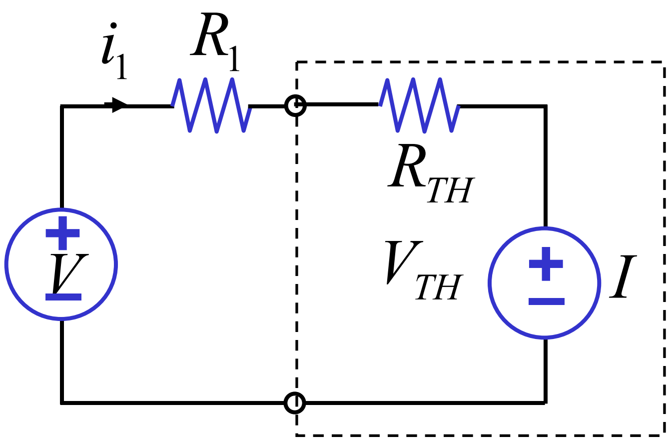
Thévenin equivalent



Thevenin's Theorem

- **Example**





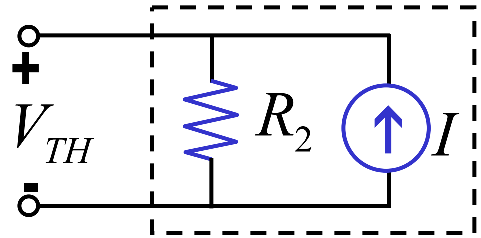
$$i_1 = \frac{V - V_{TH}}{R_1 + R_{TH}}$$

Thevenin's Theorem

- **Example (Cont'd)**

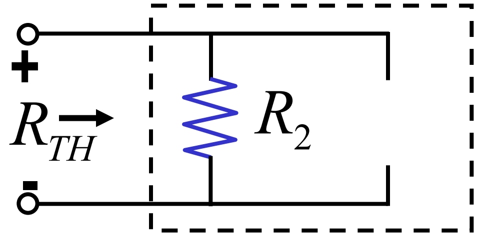
V_{TH} :

$$V_{TH} = IR_2$$



R_{TH} :

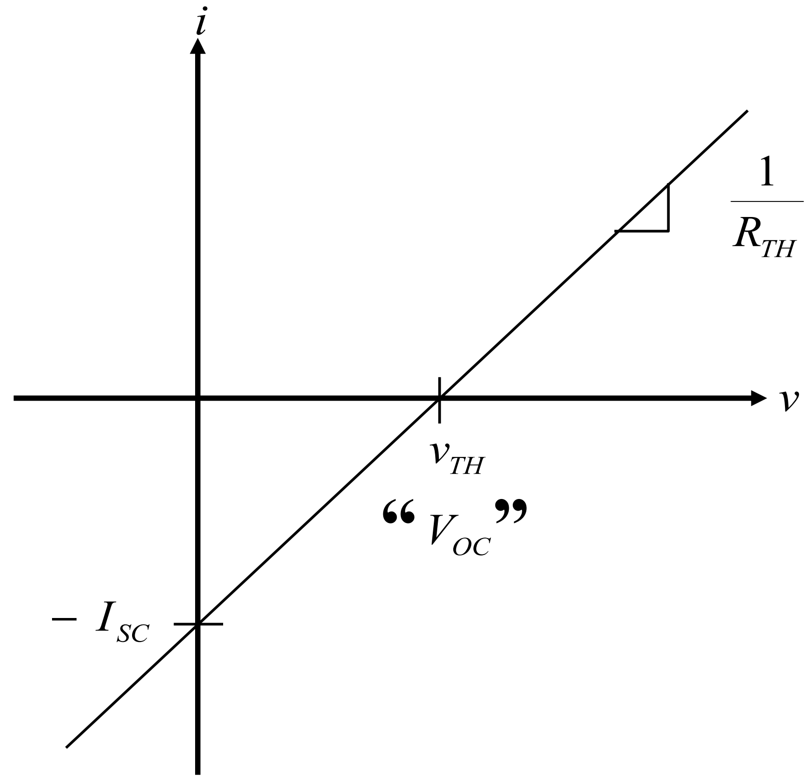
$$R_{TH} = R_2$$



Thevenin's Theorem

- **Graphical interpretation**

$$v = v_{TH} + R_{TH} i$$



Open circuit
$(i \equiv 0)$

$$v = v_{TH}$$

$$V_{OC}$$

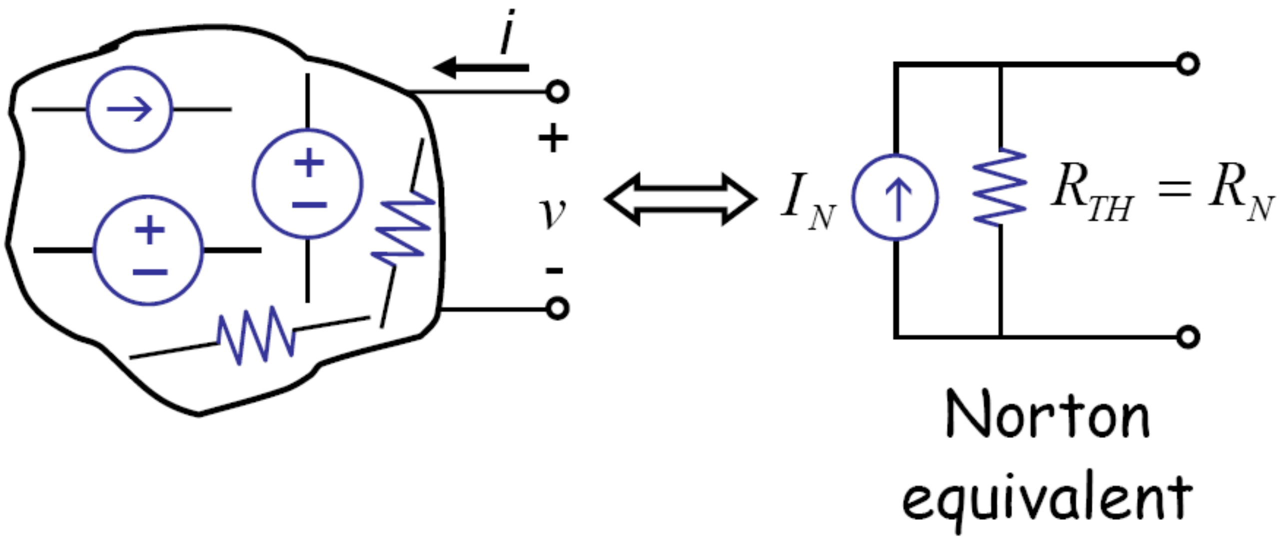
Short circuit
$(v \equiv 0)$

$$i = \frac{-\ v_{TH}}{R_{TH}}$$

$$-\ I_{SC}$$

Norton's Theorem (Method 5-2)

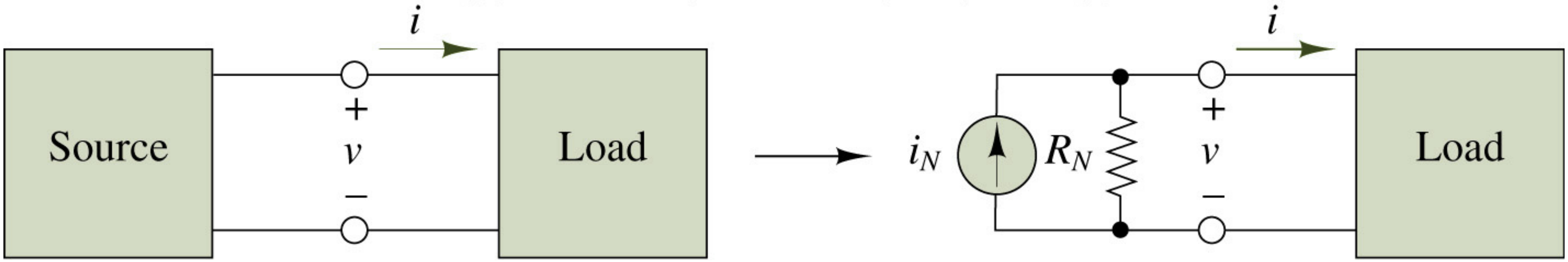
- **Norton equivalent circuit**



$$I_N = \frac{V_{TH}}{R_{TH}}$$

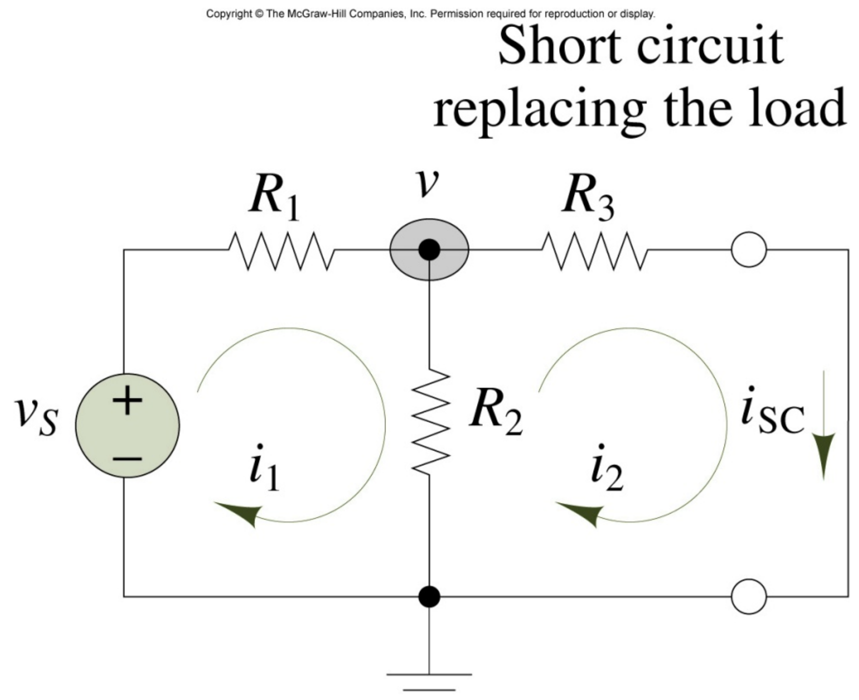
Norton's Theorem

- **Illustration of Norton's Theorem**



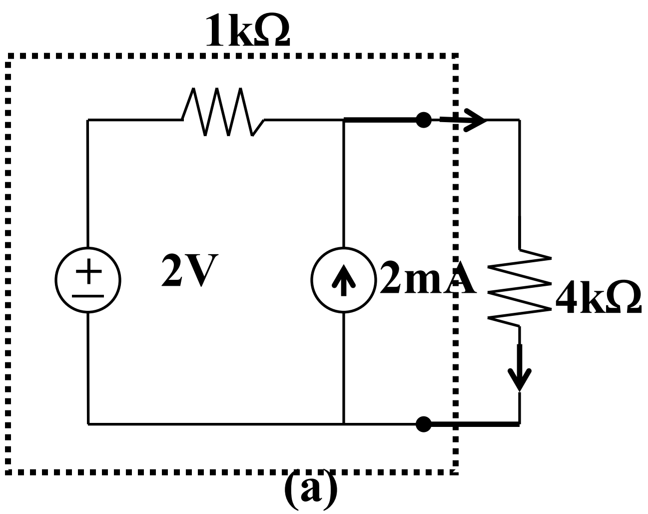
Norton's Theorem

- **Computation of Norton current**

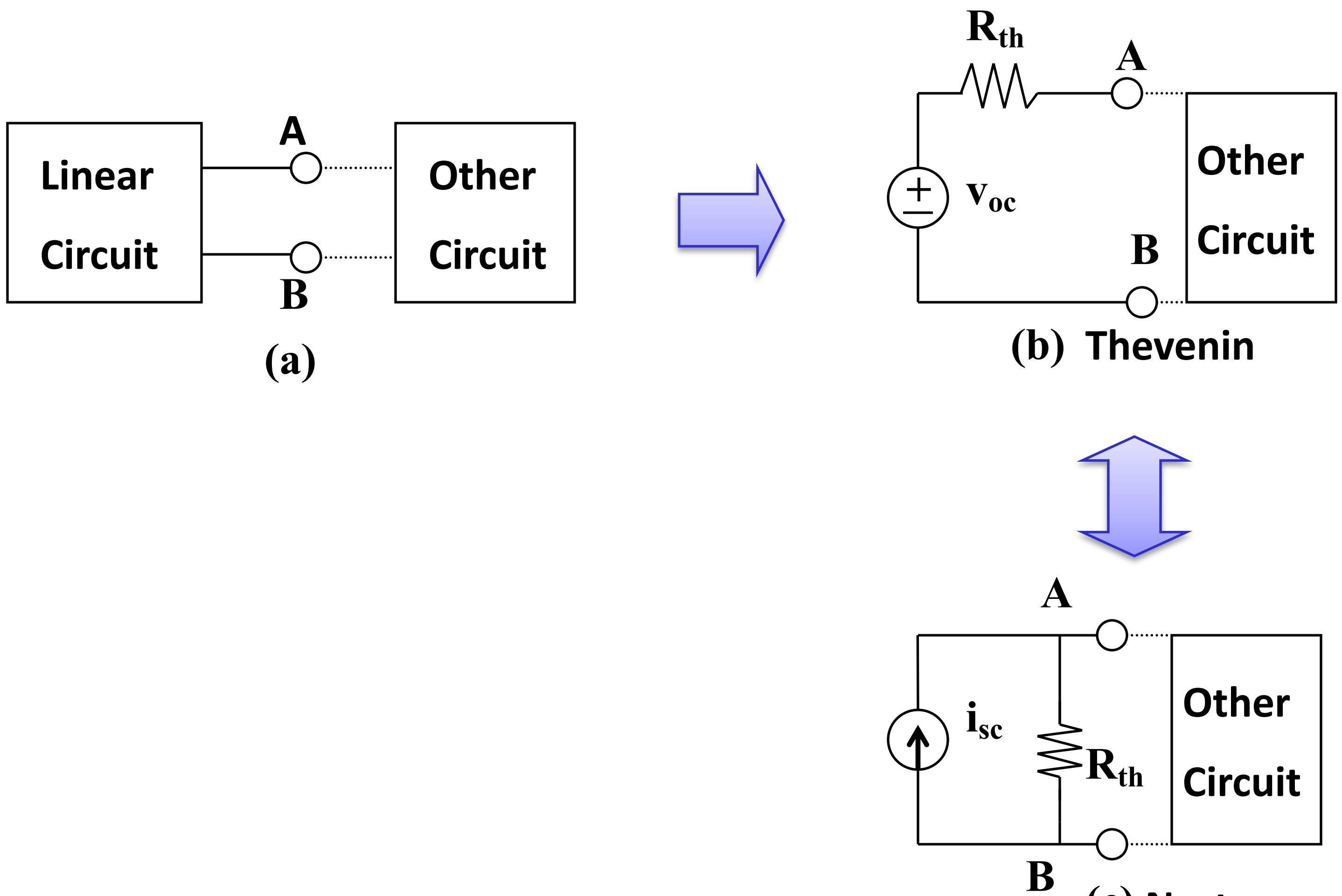


Norton's Theorem

- **Example**



Source Transformation



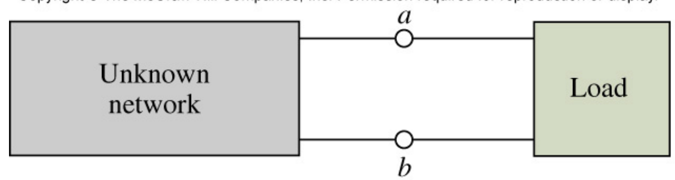
(a)

(b) Thevenin

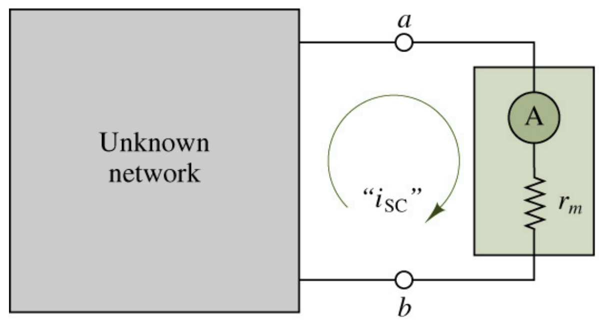
(c) Norton

Source Transformation

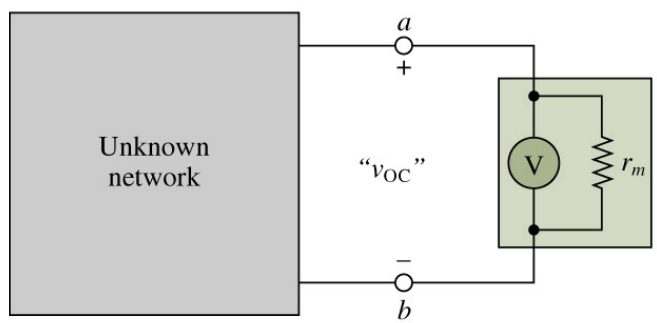
- **Measurement of open-circuit voltage and short-circuit current**



An unknown network connected to a load



Network connected for measurement of short-circuit current



Network connected for measurement of open-circuit voltage