V-code: ???

Transients in Second-Order Circuits

Lecture 15
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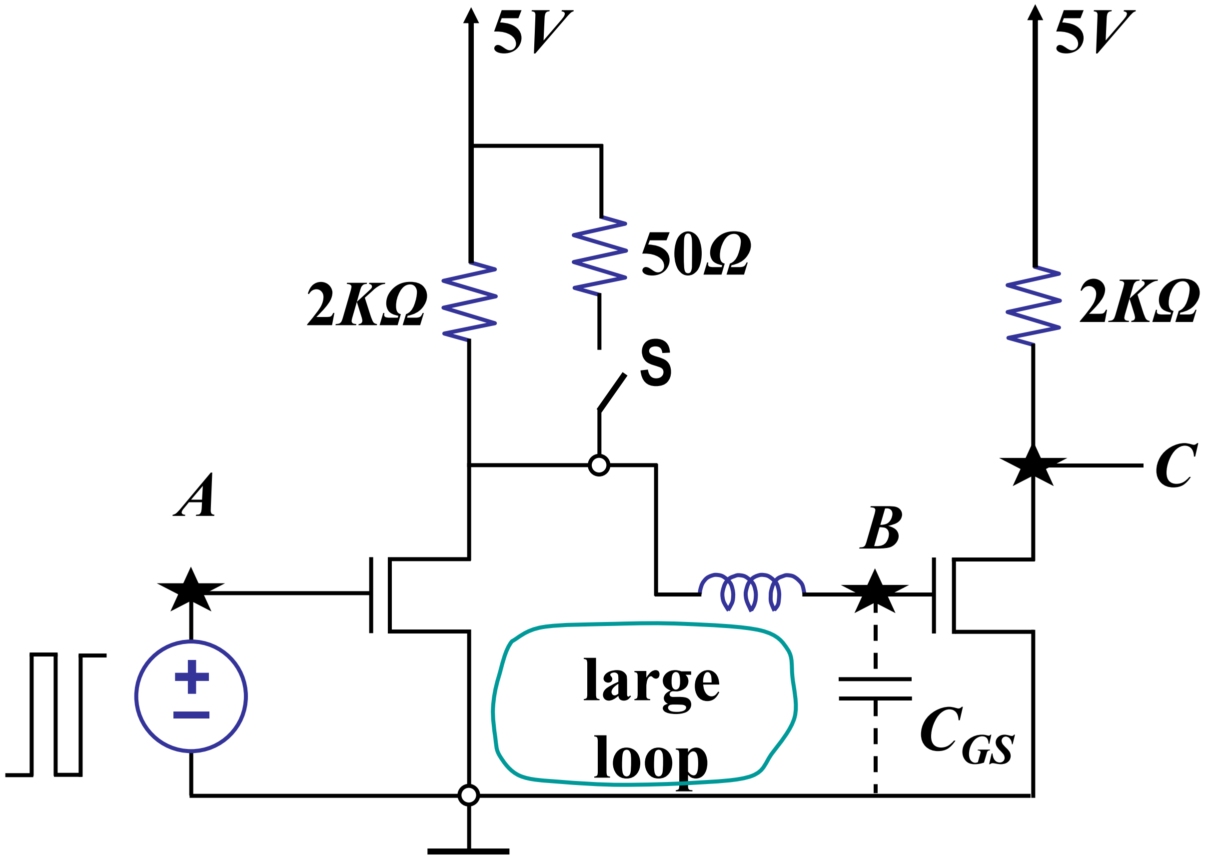
***Slide credits**: Prof. Anant Agarwal at MIT*

V-code: ???

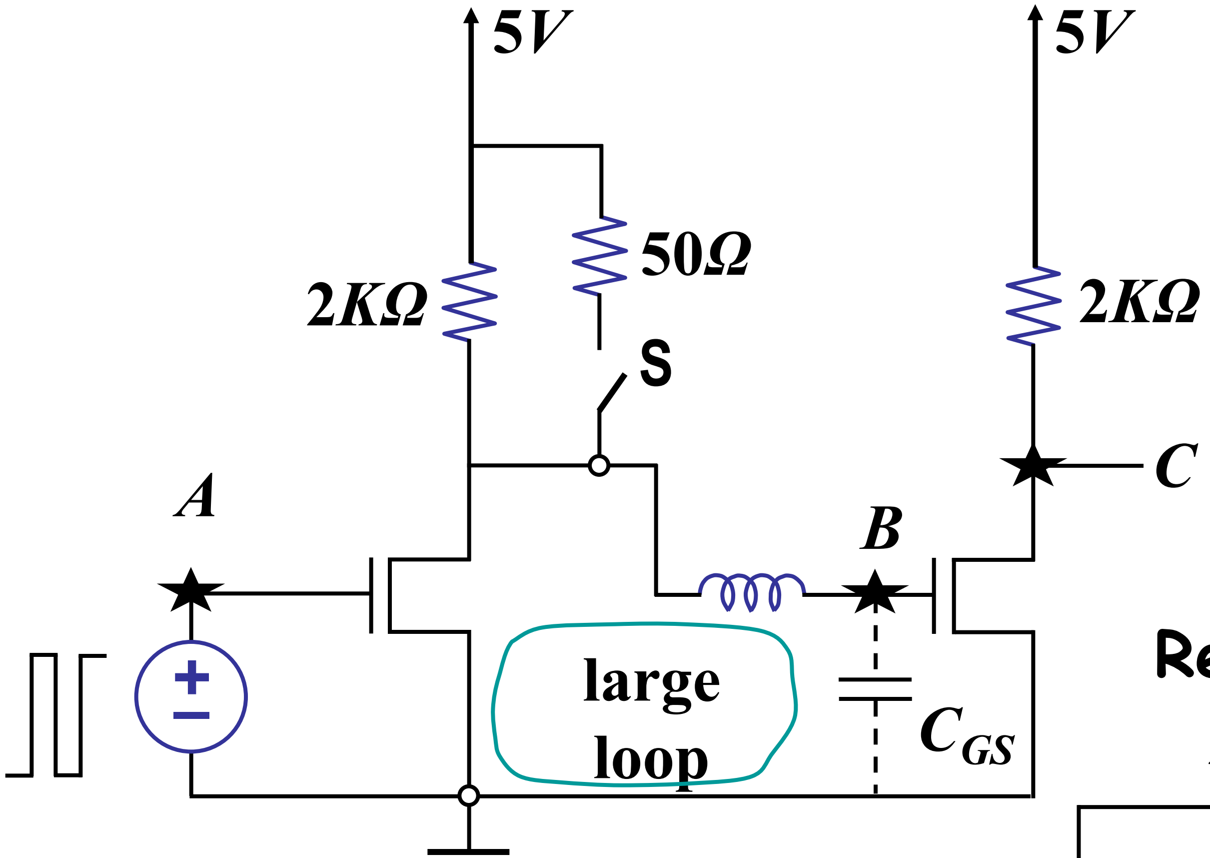
Second-Order Systems



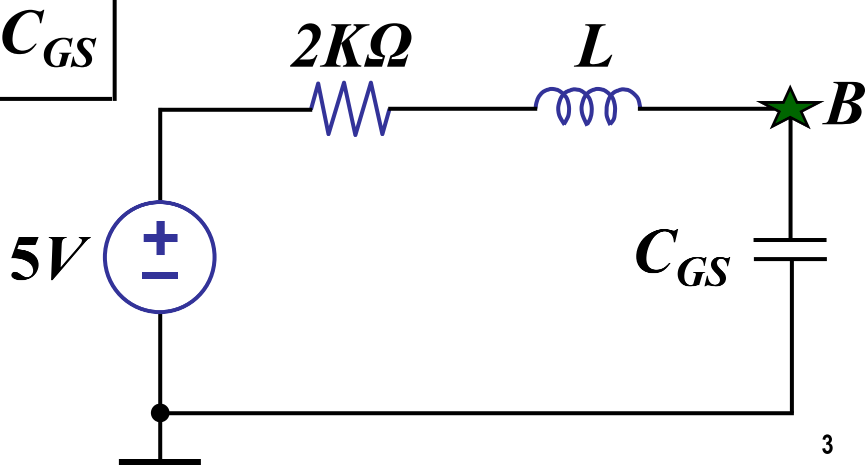
Our old friend, the inverter, driving another.
The parasitic inductance of the wire and the gate-to-source capacitance of the MOSFET are shown

V-code: ???

Second-Order Systems



Relevant circuit:

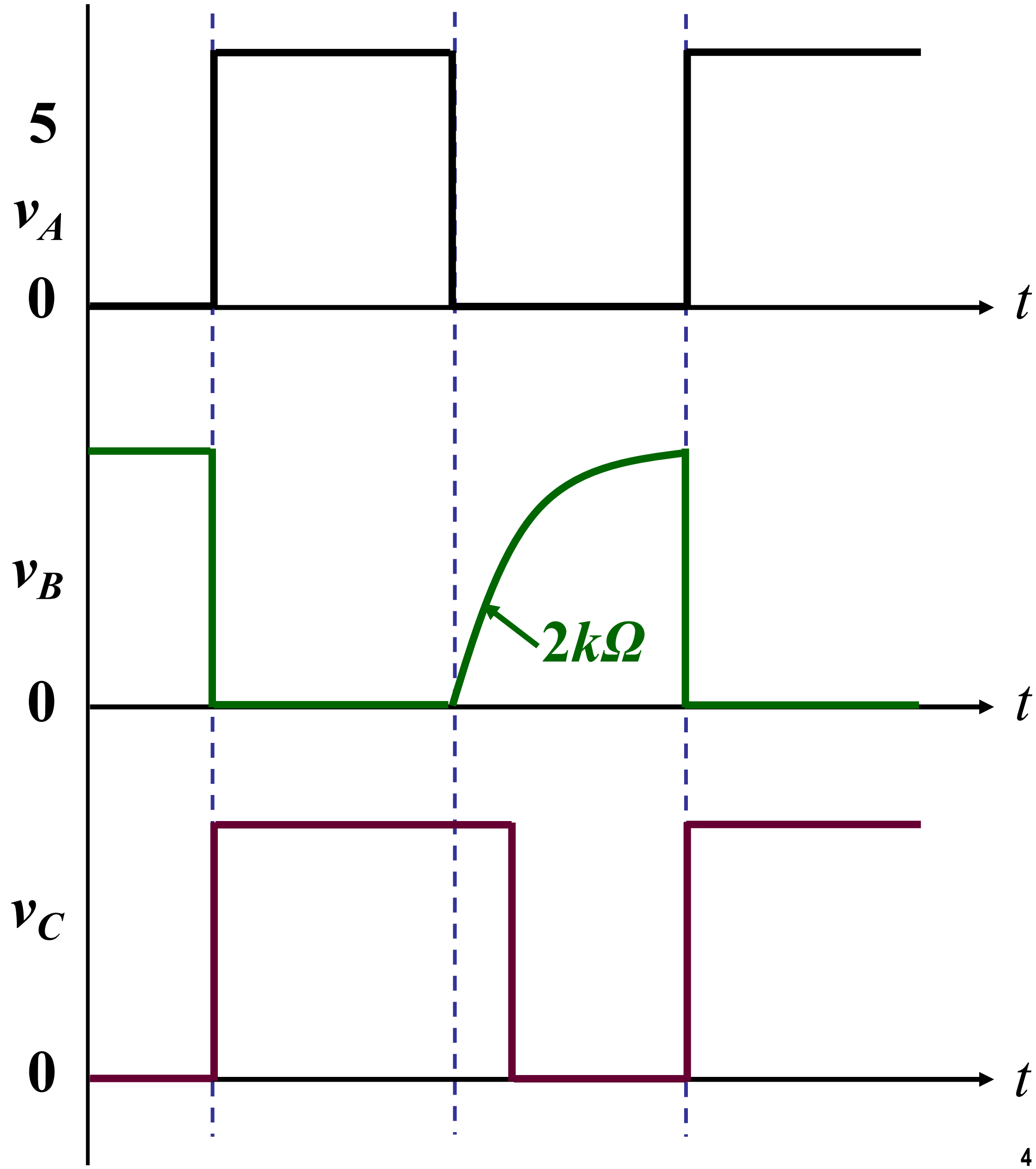


Second-Order Systems

V-code: ???

Observed Output

$2k\Omega$



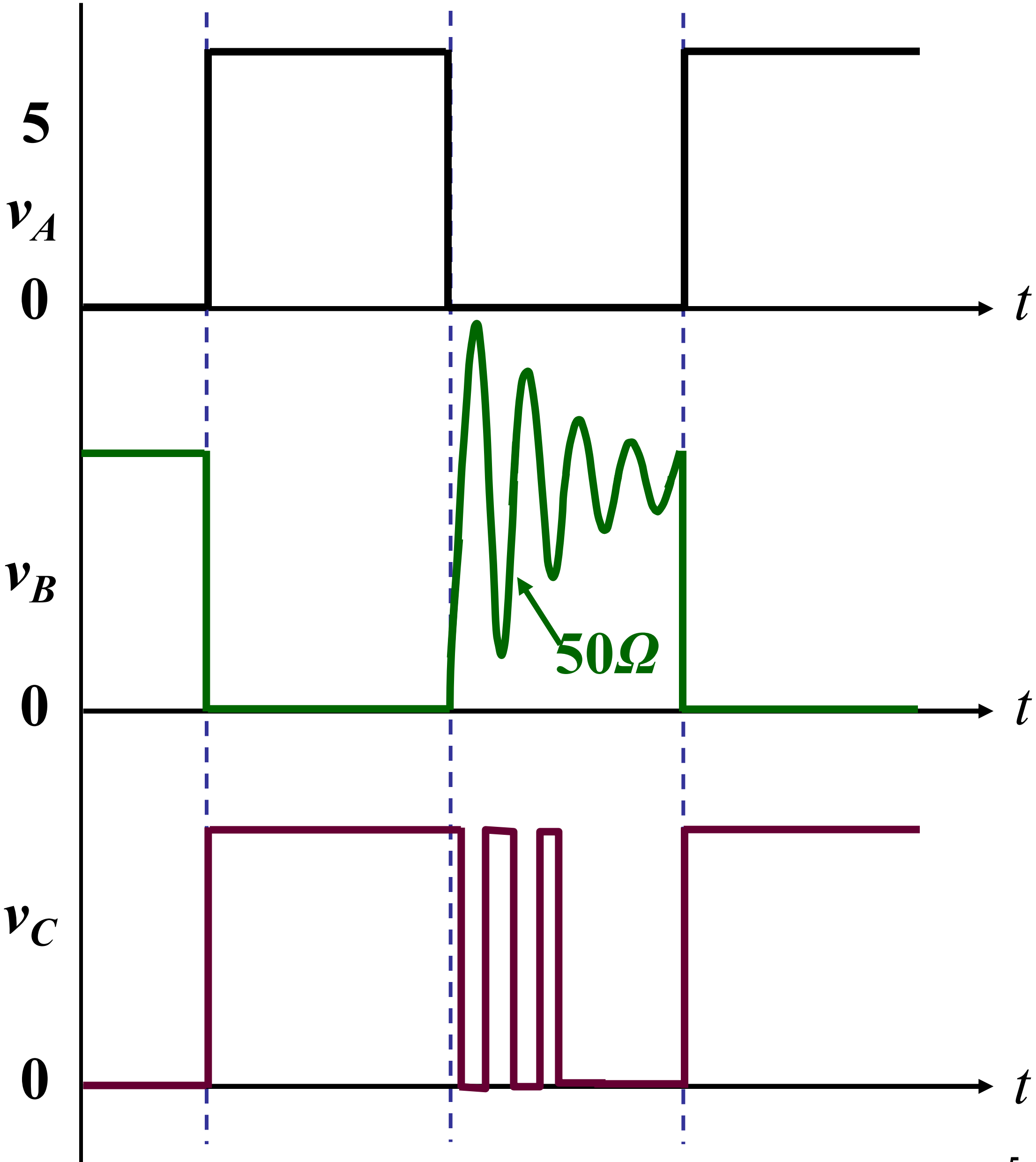
Now, let's try to speed up our inverter by closing the switch S to lower the effective resistance

Second-Order Systems

V-code: ???

Observed Output

~50Ω



V-code: ???

Outline

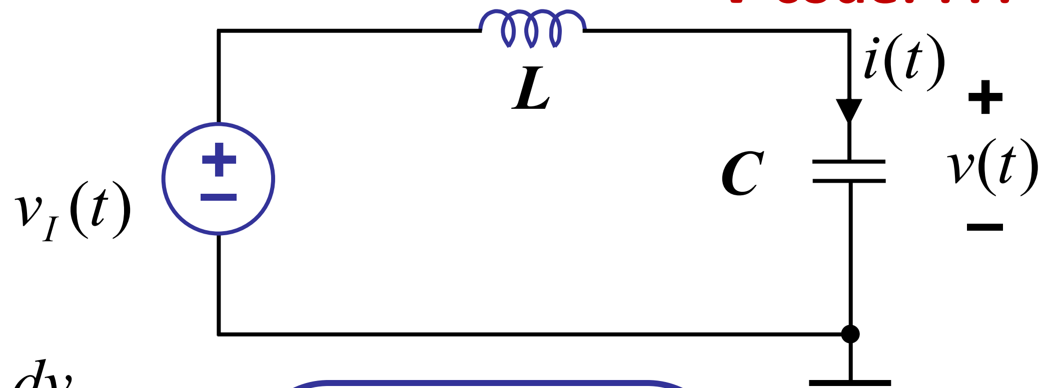
Textbook: 12.1, 12.2, 12.3, 12.5, 12.7

- **LC Circuits**
- RLC Circuits
- Idealized Analysis

V-code: ???

Series LC Circuits

First, let's analyze the LC network



Node method: $i(t) = C\dfrac{dv}{dt}$

$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v)\,dt = C\frac{dv}{dt}$$

$$\frac{1}{L}(v_I - v) = C\frac{d^2v}{dt^2}$$

$$\boxed{\underbrace{LC}_{\text{time}^2}\frac{d^2v}{dt^2} + v = v_I}$$

Recall

$$v_I - v = L\frac{di}{dt}$$

$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v)\,dt = i$$

v, i **state variables**

V-code: ???

Series LC Circuits

Solving

Recall, the method of homogeneous and particular solutions:

① Find the particular solution.

② Find the homogeneous solution.
↓
4 steps

③ The total solution is the sum of the particular and homogeneous.
Use initial conditions to solve for the remaining constants.

$$v = v_P(t) + v_H(t)$$

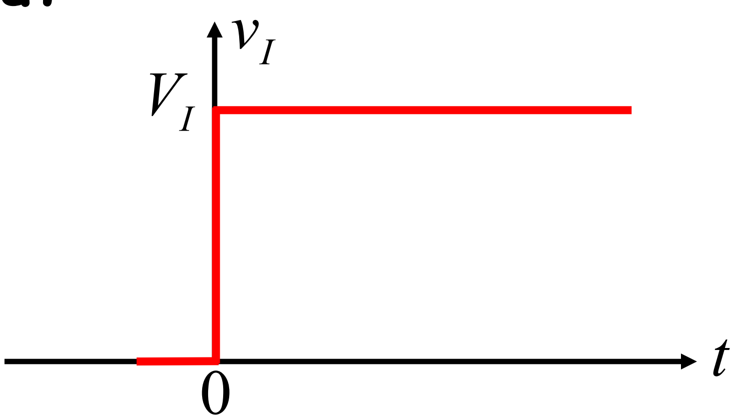
V-code: ???

Series LC Circuits

Let's solve

$$LC\frac{d^2v}{dt^2}+v=v_I$$

For input



And for initial conditions

$v(0)=0 \qquad i(0)=0 \qquad$ [ZSR]

V-code: ???

Series LC Circuits

① Particular solution

$$LC\frac{d^2v_P}{dt^2}+v_P=V_0$$

$v_P=V_0$ **is a solution.**

V-code: ???

Series LC Circuits

2 Homogeneous solution

Solution to

$$LC\frac{d^2v_H}{dt^2}+v_H=0$$

Recall, v_H : solution to homogeneous equation (drive set to zero)

Four-step method:

A Assume solution of the form*

$$v_H = Ae^{st} \qquad , \quad A,\ s\ =\ ?$$

*Differential equations are commonly solved by guessing solutions

so,

$$LC\cancel{A}s^2\cancel{e^{st}}+\cancel{Ae^{st}}=0$$

B

$$s^2=-\frac{1}{LC} \quad \text{characteristic equation}$$

$$s=\pm j\sqrt{\frac{1}{LC}} \qquad j=\sqrt{-1}$$

V-code: ???

Series LC Circuits

2 Homogeneous solution

Solution to

$$LC\frac{d^2v_H}{dt^2}+v_H=0$$

Four-step method: (cont'd)

C Roots $s=\pm j\omega_o$

$$\omega_o=\sqrt{\frac{1}{LC}}$$

General solution,

D $v_H=A_1e^{j\omega_o t}+A_2e^{-j\omega_o t}$

V-code: ???

Series LC Circuits

3 Total solution

$$v(t) = v_P(t) + v_H(t)$$

$$v(t) = V_0 + A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$$

Find unknowns from initial conditions.

$$v(0) = 0$$

$$0 = V_0 + A_1 + A_2$$

$$i(0) = 0$$

$$i(t) = C\frac{dv}{dt}$$

$$i(t) = CA_1 j\omega_o e^{j\omega_o t} - CA_2 j\omega_o e^{-j\omega_o t}$$

so, $0 = CA_1 j\omega_o - CA_2 j\omega_o$

or, $A_1 = A_2$

$$-V_0 = 2A$$

$$A_1 = -\frac{V_0}{2}$$

so, $v(t) = V_0 - \frac{V_0}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right)$

V-code: ???

Series LC Circuits

3 Total solution

Remember Euler relation

$$e^{jx} = \cos x + j \sin x$$

(verify using Taylor's expansion)

$$\frac{e^{jx} + e^{-jx}}{2} = \cos x$$

so,

$$v(t) = V_0 - V_0 \cos \omega_o t$$

$$i(t) = CV_0 \omega_o \sin \omega_o t$$

where

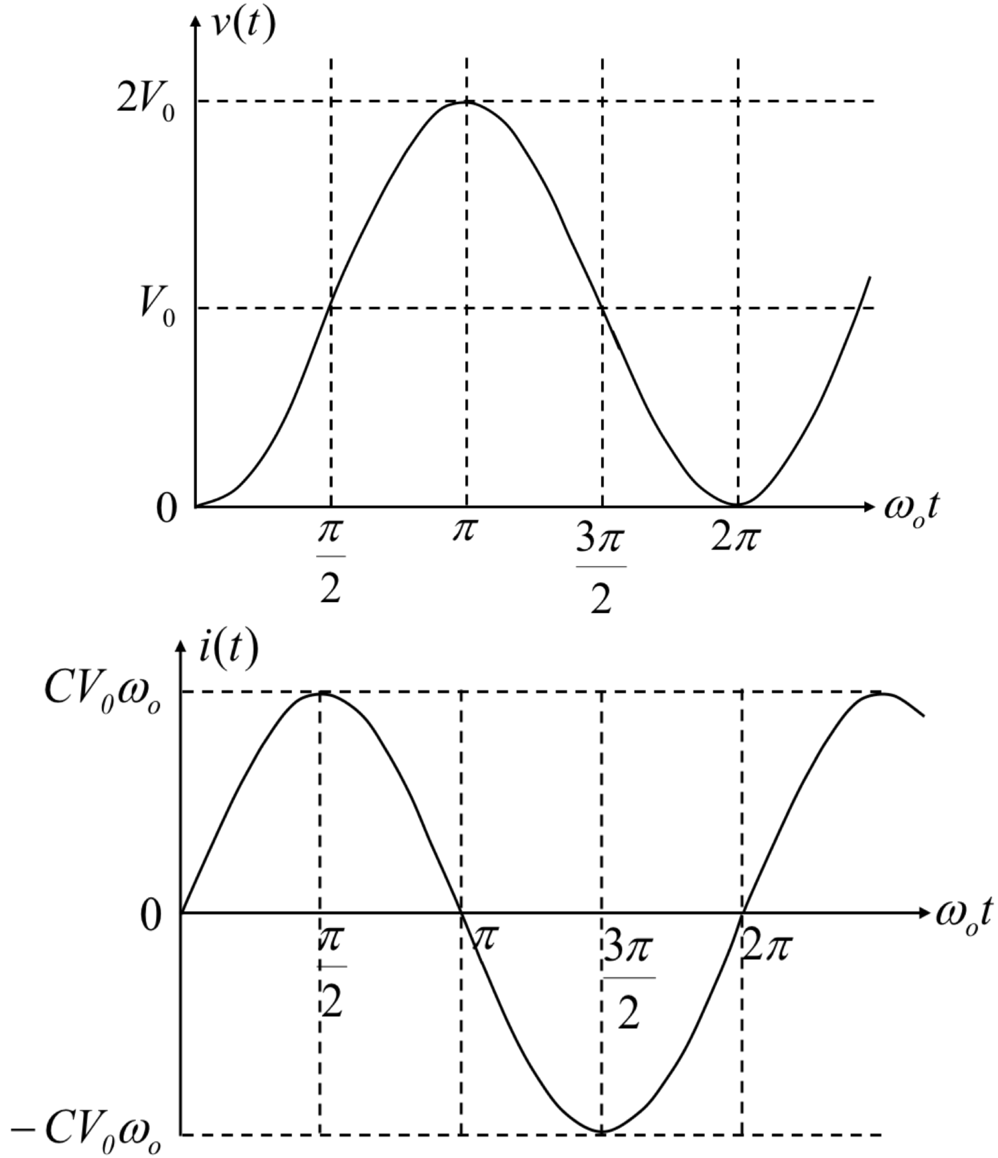
$$\omega_o = \frac{1}{\sqrt{LC}}$$

The output looks sinusoidal

V-code: ???

Series LC Circuits

③ Total solution:
Plotting it



V-code: ???

Series LC Circuits
Summary of Method

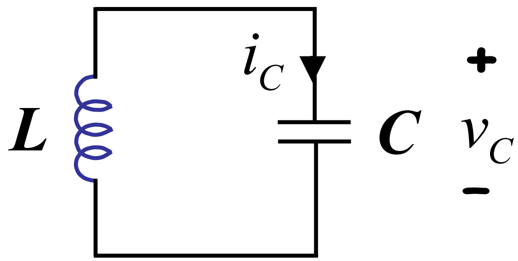
1. Write DE for circuit by applying node method.
2. Find particular solution v_P by guessing and trial & error.
3. Find homogeneous solution v_H
   - A. Assume solution of the form Ae^{st}.
   - B. Obtain characteristic equation.
   - C. Solve characteristic equation for roots s_i.
   - D. Form v_H by summing $A_i e^{s_i t}$ terms.
4. Total solution is $v_P + v_H$, solve for remaining constants using initial conditions.

V-code: ???

Undriven LC Circuits

Example

What if we have:



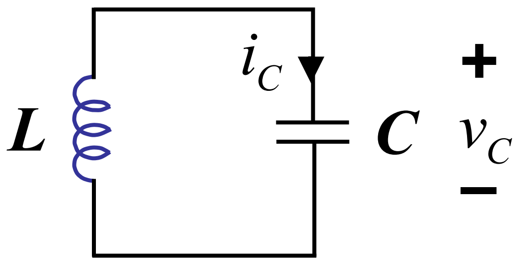
$$v_C(0) = V$$
$$i_C(0) = 0$$

We can obtain the answer directly from the homogeneous solution (V_0 = 0).

V-code: ???

Undriven LC Circuits

Example



$$v_C(0) = V$$

$$i_C(0) = 0$$

We can obtain the answer directly from the homogeneous solution (V_0 = 0).

$$v_C(t) = A_1 e^{j\omega_o t} + A_2 e^{-j\omega_o t}$$

$v_C(0) = V$ $\qquad V = A_1 + A_2$

$i_C(0) = 0$ $\qquad 0 = CA_1 j\omega_o - CA_2 j\omega_o$

or $A_1 = A_2 = \frac{V}{2}$

or $v_C = \frac{V}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right)$

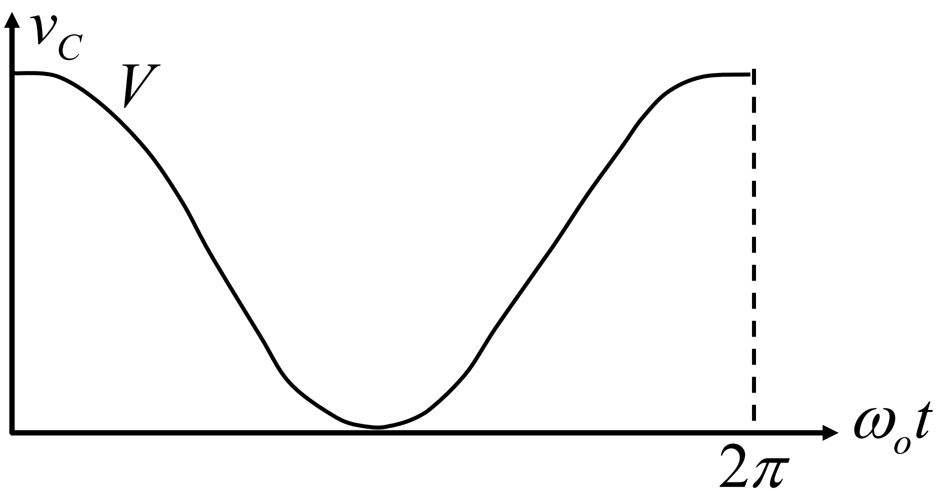
$$\boxed{v_C = V\cos\omega_o t}$$

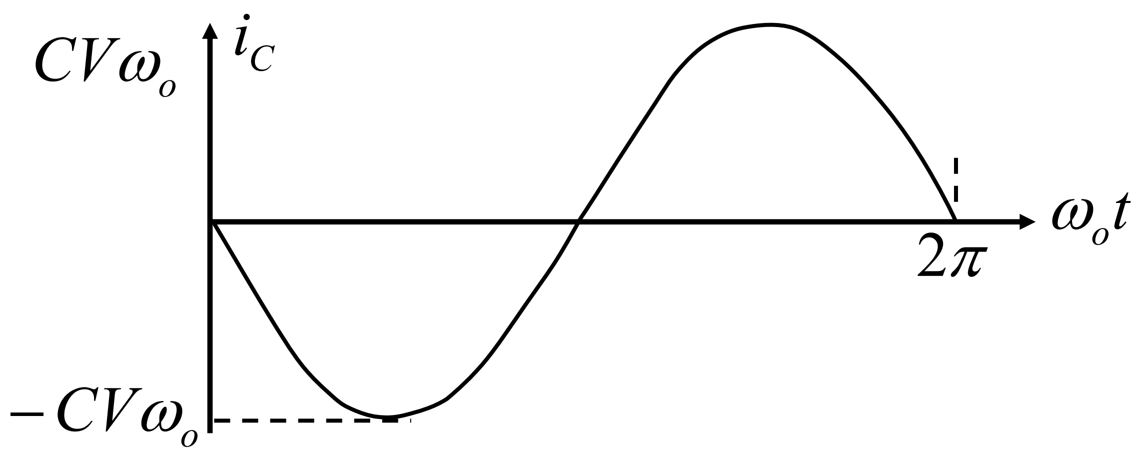
$$\boxed{i_C = -CV\,\omega_o \sin\omega_o t}$$

V-code: ???

Undriven LC Circuits

Example





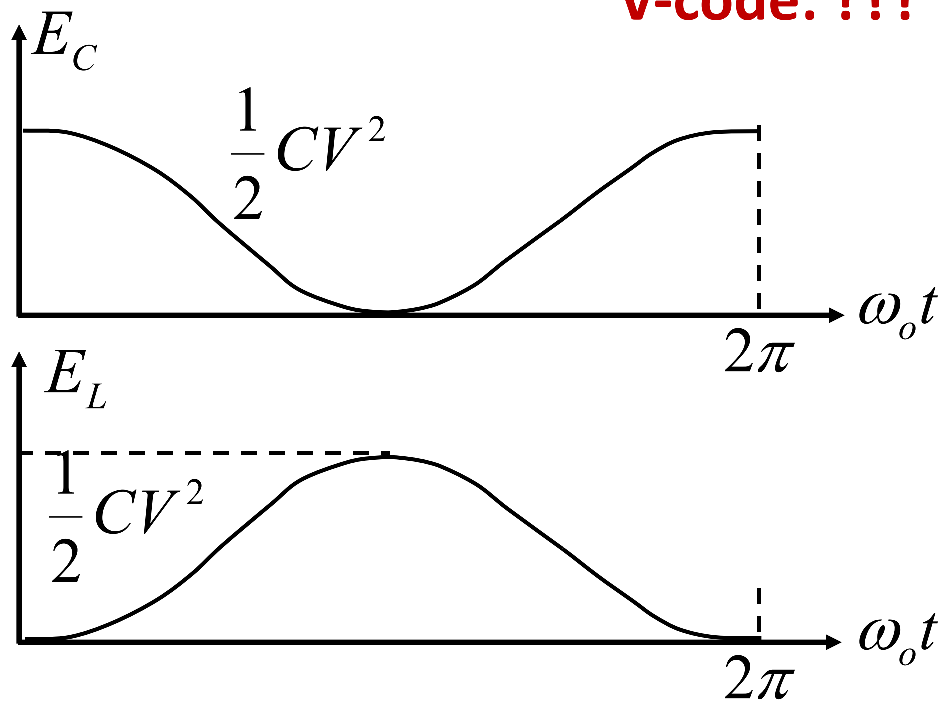
V-code: ???

Undriven LC Circuits

Energy

$$C: \frac{1}{2} C {v_C}^2$$

$$L: \frac{1}{2} L {i_C}^2$$



Notice $$\frac{1}{2} C {v_C}^2 + \frac{1}{2} L {i_C}^2 = \frac{1}{2} CV^2$$

Total energy in the system is a constant, but it sloshes back and forth between the Capacitor and the inductor

V-code: ???

Undriven LC Circuits

- **Example 12.1: Oscillation frequency of i_L and v_C? Peak values?**

$$C = 1\mu F,\ L = 100\mu H,\ (i_L = 0.5A,\ v_C = 10V\ at\ some\ t)$$

V-code: ???

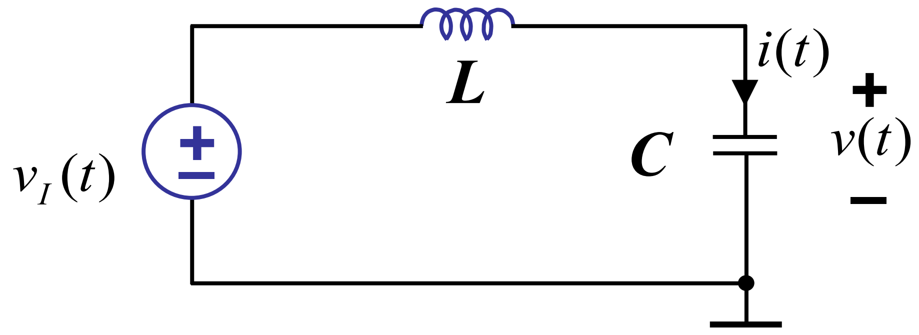
Outline

Textbook: 12.1, 12.2, 12.3, 12.5, 12.7

- LC Circuits
- **RLC Circuits**
- Idealized Analysis

V-code: ???

Review: LC Circuit

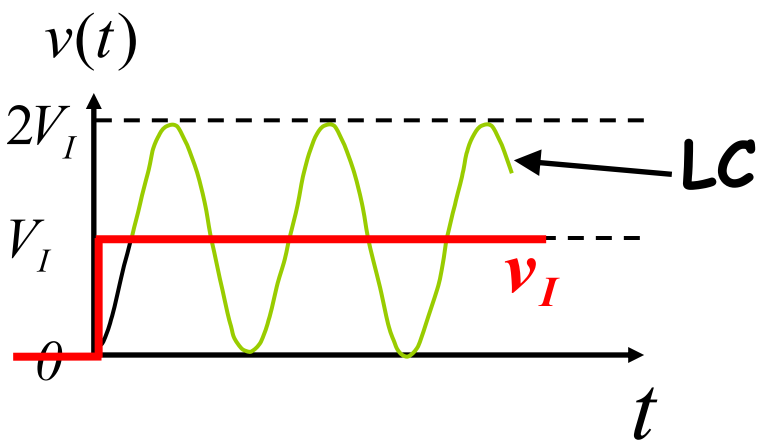


Total solution

$$v(t) = V_I - V_I \cos \omega_o t$$

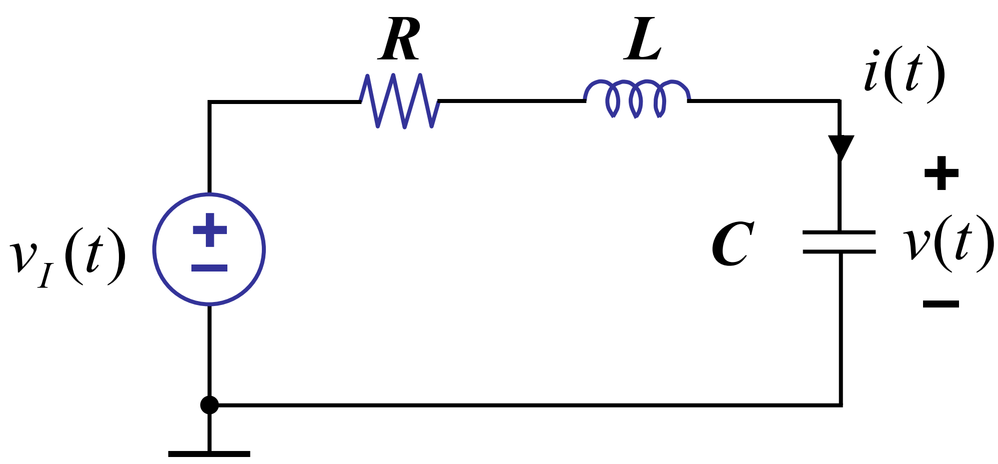
where

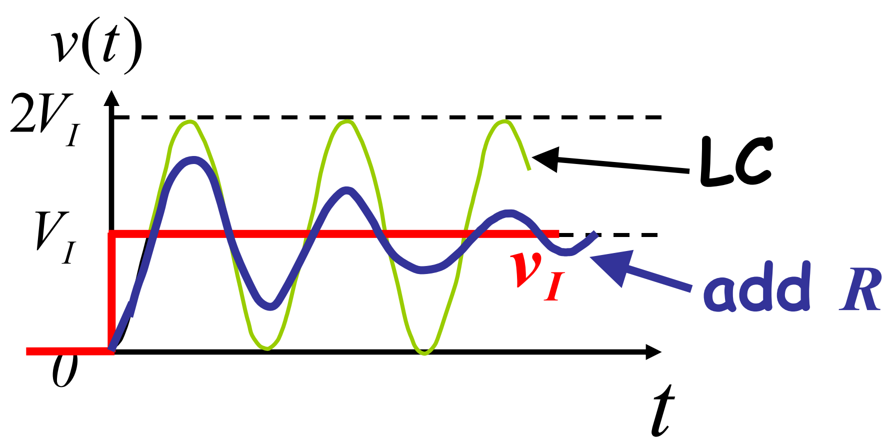
$$\omega_o = \frac{1}{\sqrt{LC}}$$



V-code: ???

Series RLC Circuits



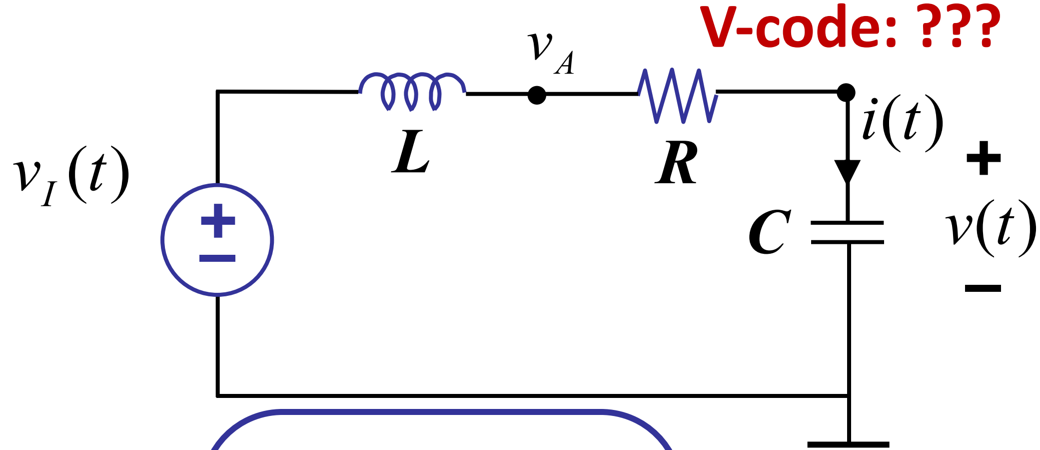


Damped sinusoids with R!

V-code: ???

Series RLC Circuits

Let's analyze the RLC network



Node method:

v_A :
$$\frac{1}{L}\int_{-\infty}^{t}(v_I - v_A)\,dt = \frac{v_A - v}{R}$$

v :
$$\frac{v_A - v}{R} = C\frac{dv}{dt}$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

Recall element rules

L:

$$v_L = L\frac{di}{dt}$$

$$\frac{1}{L}\int_{-\infty}^{t} v_L\,dt = i$$

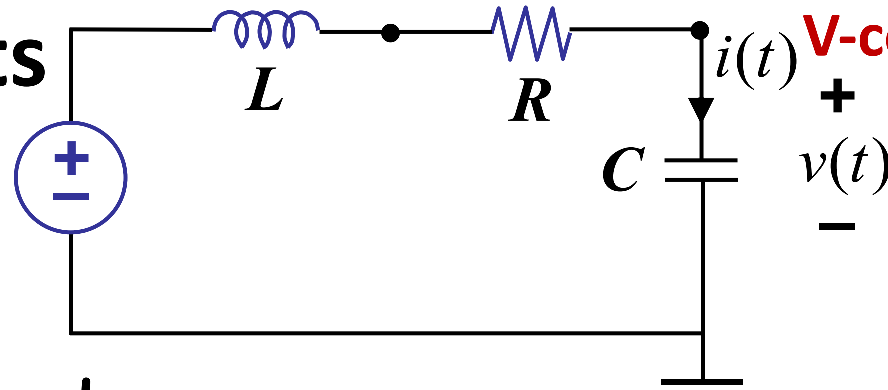
C:

$$i_C = C\frac{dv_C}{dt}$$

v, i state variables

Series RLC Circuits

V-code: ???



Let's analyze the RLC network

Node method:

$$v_A: \quad \frac{1}{L}\int_{-\infty}^{t}(v_I - v_A)\,dt = \frac{v_A - v}{R}$$

$$v: \quad \frac{v_A - v}{R} = C\frac{dv}{dt}$$

$$\frac{1}{L}(v_I - v_A) = C\frac{d^2v}{dt^2}$$

$$\frac{1}{LC}(v_I - v_A) = \frac{d^2v}{dt^2}$$

$$v_A = RC\frac{dv}{dt} + v$$

$$\frac{1}{LC}\left(v_I - RC\frac{dv}{dt} - v\right) = \frac{d^2v}{dt^2}$$

$$\boxed{\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I}$$

V-code: ???

Series RLC Circuits

Solving

Recall, the method of homogeneous and particular solutions:

1. Find the particular solution.
2. Find the homogeneous solution.
   ↓
   4 steps
3. The total solution is the sum of the particular and homogeneous.
   Use initial conditions to solve for the remaining constants.

$$v = v_P(t) + v_H(t)$$

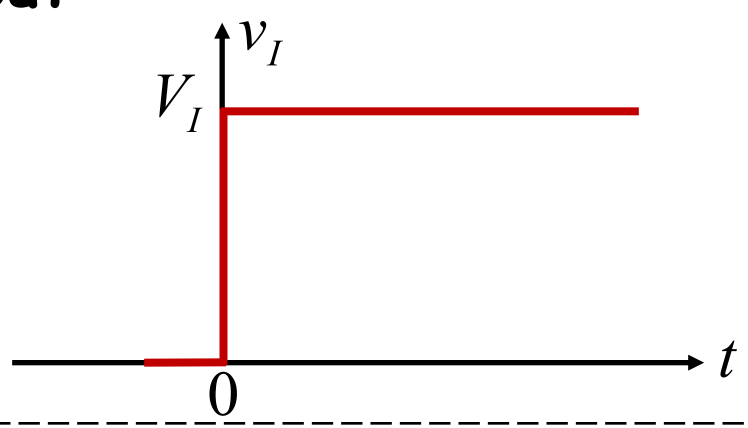
V-code: ???

Series RLC Circuits

Let's solve

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

For input



And for initial conditions

$v(0) = 0$ $\quad i(0) = 0$ $\quad$ [ZSR]

V-code: ???

Series RLC Circuits

1 Particular solution

$$\frac{d^2 v_P}{dt^2} + \frac{R}{L}\frac{dv_P}{dt} + \frac{1}{LC} v_P = \frac{1}{LC} V_I$$

$v_P = V_I$ is a solution.

V-code: ???

Series RLC Circuits

2 Homogeneous solution

Solution to $$\frac{d^2 v_H}{dt^2}+\frac{R}{L}\frac{dv_H}{dt}+\frac{1}{LC}v_H=0$$

Recall, v_H : solution to homogeneous equation (drive set to zero)

Four-step method:

A Assume solution of the form

$$v_H = Ae^{st} \quad , \; A, \; s = ?$$

B Form the characteristic equation f(s)

C Find the roots of the characteristic equation s_1, s_2

D General solution $v_H = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

V-code: ???

Series RLC Circuits

2 Homogeneous solution

Solution to $$\frac{d^2v_H}{dt^2}+\frac{R}{L}\frac{dv_H}{dt}+\frac{1}{LC}v_H=0$$

A Assume solution of the form

$$v_H = Ae^{st} \quad , \quad A,\ s\ =\ ?$$

so,
$$\cancel{A}s^2\cancel{e^{st}}+\frac{R}{L}\cancel{A}s\cancel{e^{st}}+\frac{1}{LC}\cancel{A}\cancel{e^{st}}=0$$

B
$$s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

characteristic equation

$$s^2+2\alpha s+\omega_o^2=0$$

$$\omega_o=\sqrt{\frac{1}{LC}}$$

$$\alpha=\frac{R}{2L}$$

V-code: ???

Series RLC Circuits

2 Homogeneous solution

Solution to $$\frac{d^2v_H}{dt^2}+\frac{R}{L}\frac{dv_H}{dt}+\frac{1}{LC}v_H=0$$

(Cont'd)

C Roots $$s_1=-\alpha+\sqrt{\alpha^2-\omega^2_o}$$
$$s_2=-\alpha-\sqrt{\alpha^2-\omega^2_o}$$

D General solution

$$v_H=A_1e^{\left(-\alpha+\sqrt{\alpha^2-\omega^2_o}\right)t}+A_2e^{\left(-\alpha-\sqrt{\alpha^2-\omega^2_o}\right)t}$$

V-code: ???

Series RLC Circuits

$$v(t) = v_P(t) + v_H(t)$$

③ Total solution

$$v(t) = V_I + A_1 e^{\left(-\alpha + \sqrt{\alpha^2 - \omega^2_o}\right)t} + A_2 e^{\left(-\alpha - \sqrt{\alpha^2 - \omega^2_o}\right)t}$$

Find unknowns from initial conditions.

$v(0) = 0$: $0 = V_I + A_1 + A_2$

$i(0) = 0$:

$$i(t) = C\frac{dv}{dt}$$

$$= CA_1\left(-\alpha + \sqrt{\alpha^2 - \omega^2_o}\right)e^{\left(-\alpha + \sqrt{\alpha^2 - \omega^2_o}\right)t} +$$

$$CA_2\left(-\alpha - \sqrt{\alpha^2 - \omega^2_o}\right)e^{\left(-\alpha - \sqrt{\alpha^2 - \omega^2_o}\right)t}$$

so, $0 = A_1\left(-\alpha + \sqrt{\alpha^2 - \omega^2_o}\right) + A_2\left(-\alpha - \sqrt{\alpha^2 - \omega^2_o}\right)$

Mathematically: solve for unknowns, done.

SeriesRLC Circuits

V-code: ???

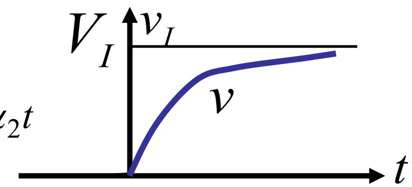
Let's stare at this a while longer…

$$v(t) = V_I + A_1 e^{-\alpha t} e^{\left(\sqrt{\alpha^2 - \omega^2{}_o}\right)t} + A_2 e^{-\alpha t} e^{\left(-\sqrt{\alpha^2 - \omega^2{}_o}\right)t}$$

3 cases:

$\alpha > \omega_o$ Overdamped

$$v(t) = V_I + A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$$



$\alpha < \omega_o$ Underdamped

$$v(t) = V_I + A_1 e^{-\alpha t} e^{\left(j\sqrt{\omega^2{}_o - \alpha^2}\right)t} + A_2 e^{-\alpha t} e^{\left(-j\sqrt{\omega_o{}^2 - \alpha^2}\right)t}$$

$$= V_I + A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$= V_I + K_1 e^{-\alpha t} \cos\omega_d t + K_2 e^{-\alpha t} \sin\omega_d t$$

$$\omega_d = \sqrt{\omega^2{}_o - \alpha^2}$$

$$e^{j\omega_d t} = \cos\omega_d t + j\sin\omega_d t$$

$\alpha = \omega_o$ Critically damped

Later…

Series RLC Circuits

V-code: ???

Let's stare at underdamped a bit longer…

$\alpha < \omega_o$ Underdamped contd…

$$v(t) = V_I + K_1 e^{-\alpha t}\cos\omega_d t + K_2 e^{-\alpha t}\sin\omega_d t$$

$$v(0) = 0: \quad K_1 = -V_I$$

$$i(0) = 0: \; i(t) = C\frac{dv}{dt}$$

$$= -CK_1\alpha e^{-\alpha t}\cos\omega_d t - CK_2\omega_d e^{-\alpha t}\sin\omega_d t$$

$$- CK_1\alpha e^{-\alpha t}\sin\omega_d t + CK_2\omega_d e^{-\alpha t}\cos\omega_d t$$

$$0 = -K_1\alpha + K_2\omega_d$$

$$K_2 = -\frac{V_1\alpha}{\omega_d}$$

$$v(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - V_I\frac{\alpha}{\omega_d}e^{-\alpha t}\sin\omega_d t$$

Note:
For $R = 0 \Longrightarrow \alpha = 0$

$$v(t) = V_I - V_I\cos\omega_o t$$

Same as LC as expected

Series RLC Circuits

V-code: ???

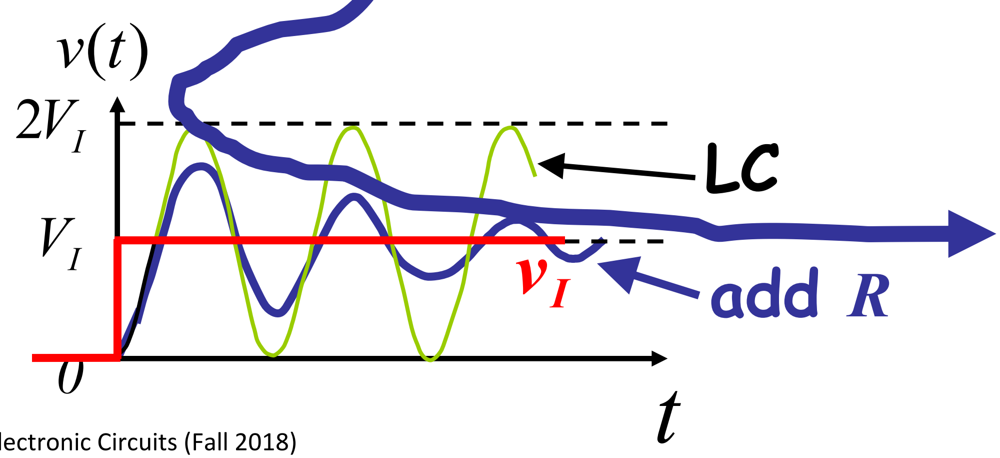
Let's stare at underdamped a bit longer…

$\alpha < \omega_o$ **Underdamped contd…**

$$v(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t}\sin\omega_d t$$

Remember, scaled sum of sines (of the same frequency) are also sines! -- Appendix B.7

$$v(t) = V_I - V_I \frac{\omega_o}{\omega_d} e^{-\alpha t}\cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$



V-code: ???

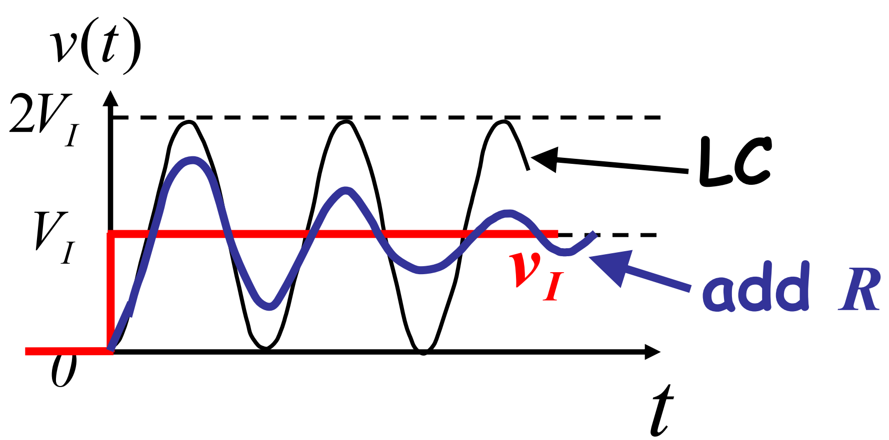
Series RLC Circuits

$\alpha < \omega_o$ Underdamped contd…

$$v(t) = V_I - V_I e^{-\alpha t} \cos \omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t} \sin \omega_d t$$

Remember, scaled sum of sines (of the same frequency) are also sines! -- Appendix B.7

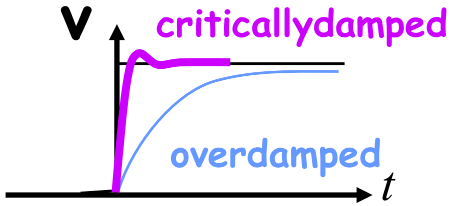
$$v(t) = V_I - V_I \frac{\omega_o}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1} \frac{\alpha}{\omega_d} \right)$$



V-code: ???

Series RLC Circuits

$\alpha = \omega_o$ Critically damped



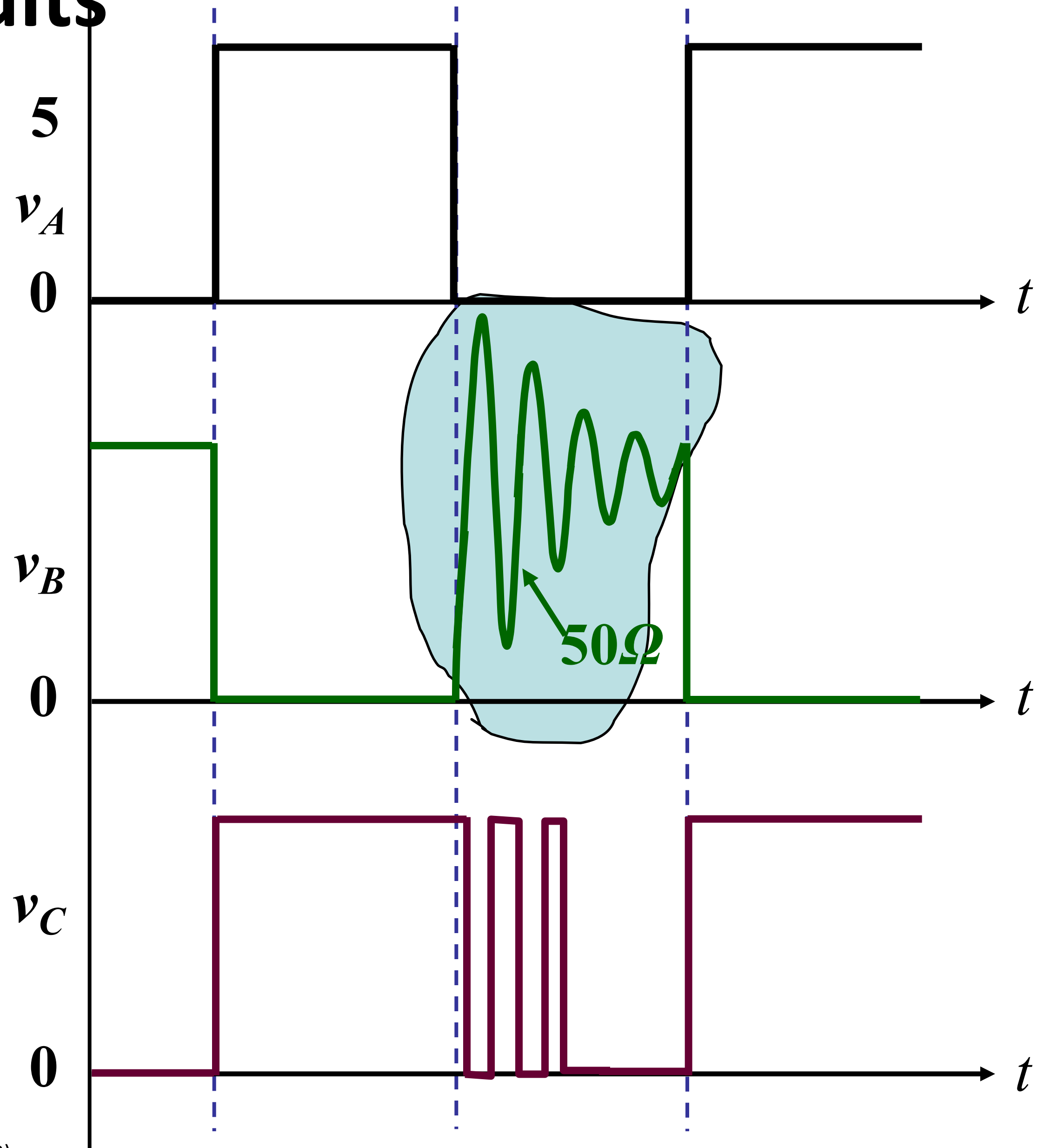
Section 12.2.3

V-code: ???

Series RLC Circuits

Remember this?

Closed the loop…



V-code: ???

Outline

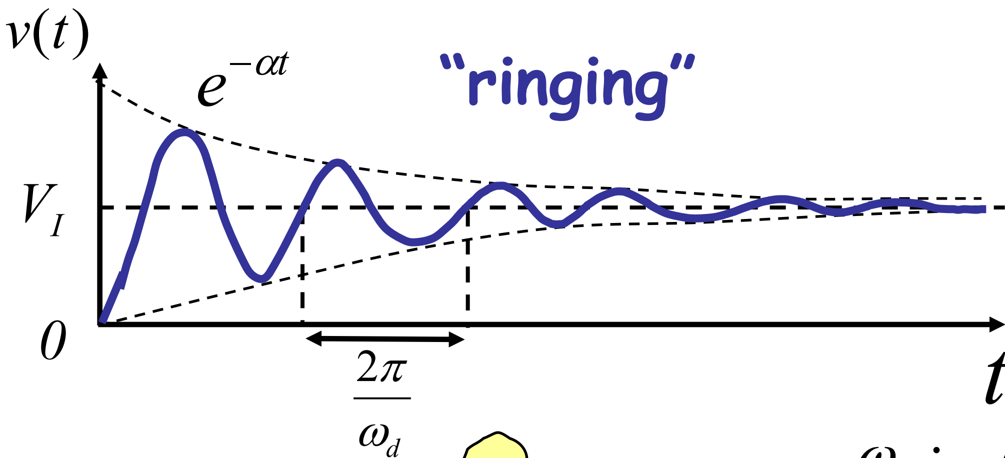
Textbook: 12.1, 12.2, 12.3, 12.5, 12.7

- LC Circuits
- RLC Circuits
- **Idealized Analysis**

V-code: ???

Intuitive Analysis

Underdamped $$v(t) = V_I - V_I \frac{\omega_o}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$



Characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

ω_d : **Oscillation frequency**

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

α : **Governs rate of decay**

V_I : **Final value**

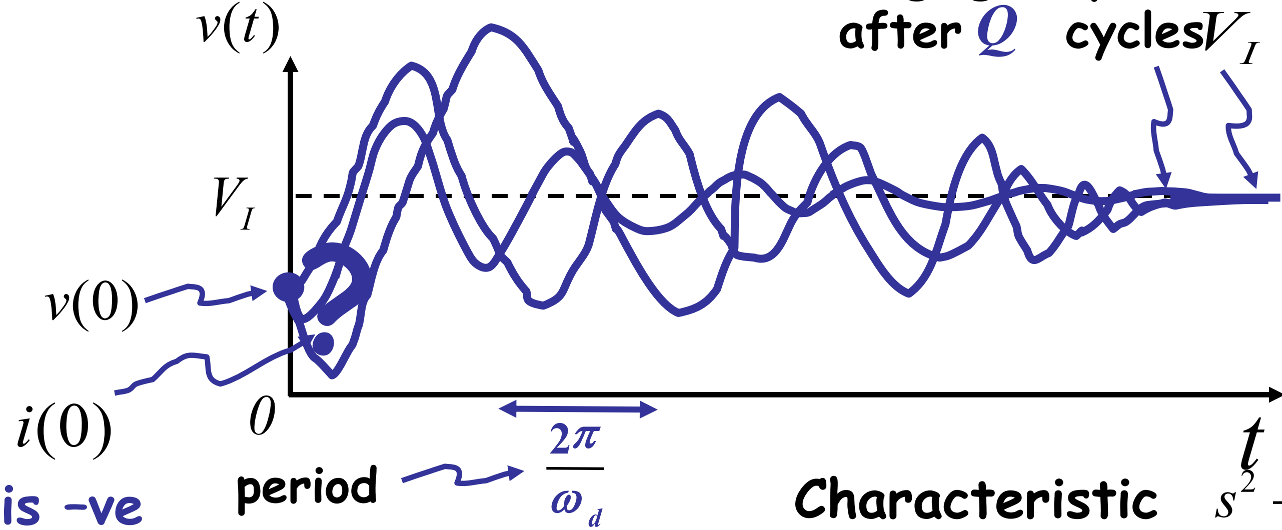
$v(0)$: **Initial value**

$Q = \frac{\omega_o}{2\alpha}$: **Quality factor (approximately the number of cycles of ringing)**

V-code: ???

Intuitive Analysis

Ringing stops after Q cycles



is -ve so v(t) must drop

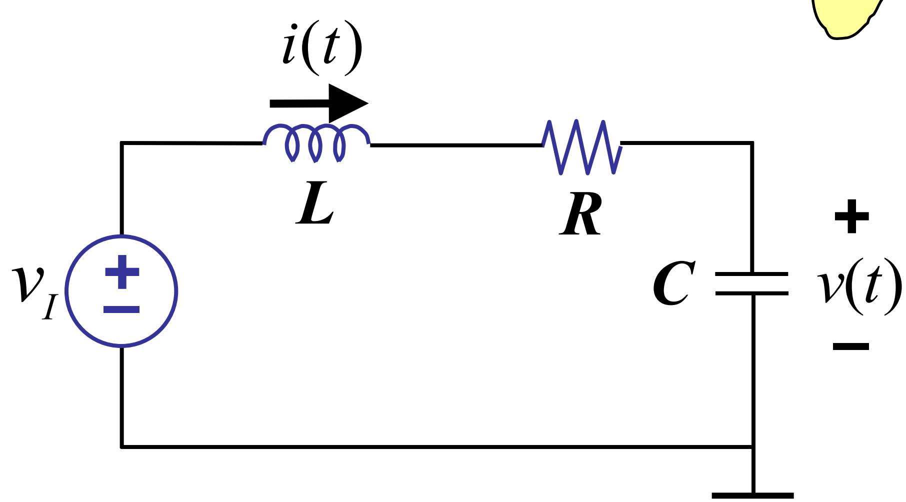
Characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

$$Q = \frac{\omega_o}{2\alpha}$$



given

$i(0)$ -ve

$v(0)$ +ve