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PARTICULAR SOLUTION TO THE EULER-CAUCHY EQUATION WITH POLYNOMIAL NON-HOMOGENEITIES

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Abstract. The Euler-Cauchy differential equation is one of the first, and simplest, forms of a higher order non-constant coefficient ordinary differential equation that is encountered in an undergraduate differential equations course. For a non-homogeneous Euler-Cauchy equation, the particular solution is typically determined by either using the method of variation of parameters or transforming the equation to a constant-coefficient equation and applying the method of undetermined coefficients. This paper demonstrates the surprising form of the particular solution for the most general n^{th} order Euler-Cauchy equation when the non-homogeneity is a polynomial. In addition, a formula that can be used to compute the unknown coefficients in the form of the particular solution is presented.

1. **Introduction.** In most undergraduate differential equations classes, the Euler-Cauchy differential equation is one of the first higher order non-constant coefficient ordinary differential equations that students encounter (e.g, see the textbooks [2, 4]). Usually, one starts with the second order homogeneous Euler-Cauchy equation to simplify the calculations behind the motivation to find solutions of the form $y = t^\lambda$, where λ is a constant to be determined. Substituting $y = t^\lambda$ in the Euler-Cauchy equation yields the characteristic polynomial. Several forms of the general solution are then presented based on the roots of the characteristic polynomial. The particular solution to the non-homogeneous Euler-Cauchy equation is typically determined in one of two ways. In one approach, the method of variation of parameters is applied to the original equation. In the second approach, the Euler-Cauchy equation is first transformed into a constant-coefficient differential equation using the substitution $t = e^x$. Then, the particular solution is determined by either the method of undetermined coefficients or the method of variation of parameters, depending on the transformed non-homogeneity. Such calculations can become tedious and quite long at times, depending on the form of the particular solution, as well as the order of the Euler-Cauchy equation. The goal of this paper is to present a closed-form particular solution to the general n^{th} order non-homogeneous Euler-Cauchy equation for the case of a polynomial non-homogeneity. The formula presented for computing the unknown coefficients is a function of derivatives of the characteristic polynomial. Thus, a particular solution can be very quickly and efficiently computed for this special case of the right-hand side.

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Using an approach other than variation of parameters to determine the particular solution to nonhomogeneous Euler-Cauchy equations has been studied in only a few other contexts. In [5], D. De Leon demonstrates that the Euler-Cauchy equation may be solved using a method like undetermined coefficients for certain right-hand side functions, among them polynomials. The author does not provide a formula for such solution. Alternate approaches to determining the particular solution to the Euler-Cauchy equation have been discussed in other papers. In [1], M. Abualrub introduces a formula for solving the nonhomogeneous nth order Euler-Cauchy equation with a monomial of degree $m > n$ right-hand side. The method developed in Abualrub's paper is not applicable to equations with general polynomial right-hand sides.

In Section 2, we first describe the problem and present example problems demonstrating a traditional approach that may be taken in an undergraduate differential equations class. In Section 3, we present our solution to the problem and also provide a formula to compute the unknown coefficients. In Section 4, we demonstrate the use of the formula described in Section 3 by revisiting the examples considered in Section 2. Euler-Cauchy equations find applications in several areas of applied mathematics, such as electrostatics. In electrostatics, one wishes to determine the electric potential between two charged concentric spheres, and it turns out that the differential equation for the electric potential governing this application is a second-order Euler-Cauchy equation. In [3], Dev and Gleiser study the effects of pressure on the properties of objects that are assumed to be spherically symmetric and bound by gravity, such as stars. Under the assumptions that: (1) the interior pressure of the star is anisotropic (so the radial component of the pressure is different from the angular components), (2) the energy density is proportional to $1/r^2$, (3) the pressure anisotropy has the same form as the energy density, and (4) the star has no crust, the governing equation is an Euler-Cauchy equation. In addition, Schoenberger et al. [6] make various assumptions and approximations to transform the equations that govern the deceleration and oscillation of a blunt object moving along a planar trajectory into an Euler-Cauchy equation. Comparison of this simplified model with actual results shows that the analytic solution of this Euler-Cauchy equation can be useful for preliminarily assessing flight test data. When all parameters are designed so that the equation is dimensionless, the results provide a good approximation for the full-scale flight behavior of a decelerating blunt capsule.

2. **The Problem.** Consider the n^{th} order non-homogeneous Euler-Cauchy equation with a polynomial (of degree m) right-hand side:

$$\sum_{i=0}^{n} a_i t^i y^{(i)} = \sum_{j=0}^{m} A_j t^j. \tag{1}$$

We will assume that $t > 0$ for the remainder of this paper. We wish to compute a particular solution to (1). It is well-known (and is usually the approach taken by most undergraduate differential equations textbooks) that this equation can be transformed into a constant-coefficient differential equation by setting $t = e^x$, and then the standard techniques for finding the particular solution can be applied to the transformed constant-coefficient differential equation. However, it should be noted that even after the form of the particular solution is determined by this procedure, it still remains to compute the unknown coefficients associated with the particular solution. This is often done by computing the derivatives of all (required) orders of the particular solution and then plugging them back into the transformed differential equation to solve for the unknown coefficients. This process can be tedious, as well as time-consuming. In a previous paper by one of the authors [5], a form of the particular solution for monomial non-homogeneities is presented, which allows the particular solution to be computed using an undetermined coefficients type of method. The current paper presents a novel proof of this more succinct form of the particular solution to (1), which results in a simple new formula for computing the unknown coefficients of the various terms in the particular solution. Let us first take a look at some examples of computing particular solutions to Euler-Cauchy equations of various orders by using the standard approach of transforming them into constant-coefficient differential equations.

1. Compute a particular solution to the Euler-Cauchy equation:
   $t^2y'' - 3ty' + 4y = 6t^2$.
   Solution:
   Setting $t = e^x$, we have,

$$\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) - 3\frac{dy}{dx} + 4y = 6e^{2x} \Longrightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 6e^{2x}.$$

   The particular solution of the above constant-coefficient differential equation is of the form $y_p = Ax^2e^{2x}$ because $x = 2$ is a root of the characteristic polynomial of multiplicity 2. In order to determine A, we take derivatives of y_p and plug them back into the constant-coefficient differential equation obtained above, yielding

$$\left(2Ae^{2x} + 8Axe^{2x} + 4Ax^2e^{2x}\right) - 4\left(2Axe^{2x} + 2Ax^2e^{2x}\right) + 4Ax^2e^{2x} = 6e^{2x} \Longrightarrow A = 3.$$

   Transforming back to t, we obtain $y_p = 3t^2(\ln t)^2$.

2. Compute a particular solution to the Euler-Cauchy equation:
   $t^3y''' - 3t^2y'' + 6ty' - 6y = 2t + 7t^2$.
   Solution:
   Setting $t = e^x$, we have,

$$\left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right) - 3\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + 6\frac{dy}{dx} - 6y = 2e^x + 7e^{2x}$$
$$\Longrightarrow \frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 2e^x + 7e^{2x}.$$

   The particular solution of the above constant-coefficient differential equation is of the form $y_p = Axe^x + Bxe^{2x}$ because $x = 1$ and $x = 2$ are both roots of the characteristic polynomial of multiplicity 1. Note that we are applying the Superposition Principle here. In order to determine A and B, we take derivatives of y_p and plug them back into the constant-coefficient differential equation obtained above, yielding

$$\left(3Ae^x + Axe^x + 12Be^{2x} + 8Bxe^{2x}\right) - 6\left(2Ae^x + Axe^x + 4Be^{2x} + 4Bxe^{2x}\right)$$
$$+ 11\left(Ae^x + Axe^x + Be^{2x} + 2Bxe^{2x}\right) - 6\left(Axe^x + Bxe^{2x}\right) = 2e^x + 7e^{2x}$$
$$\Longrightarrow A = 1, B = -7.$$

Transforming back to t, we obtain $y_p = t\ln t - 7t^2 \ln t$.

In the next section, we prove that the particular solution for such polynomial nonhomogenieties has, in fact, a very special form in which the unknown coefficients are related to the characteristic polynomial. In Section 4, we revisit the above examples and solve them using the proposed approach.

3. **The Solution.** In order to prove the main results of this paper, we need to first establish some basic results on permutations. As such, we begin with the following lemmas.

Lemma 3.1. *Let* $P(n,r) = \dfrac{n!}{(n-r)!}$ *denote the usual permutation. Then*

$$P(m, k+1) = (m-k)P(m,k). \tag{2}$$

Proof.

$$\begin{aligned}(m-k)P(m,k) &= (m-k)\frac{m!}{(m-k)!} \\ &= \frac{m!}{(m-k-1)!} \\ &= P(m,k+1).\end{aligned}$$

□

Lemma 3.2. *Let* $P(n,r) = \dfrac{n!}{(n-r)!}$ *denote the usual permutation. Then*

$$P^{(j)}(m,k+1) = jP^{(j-1)}(m,k) + (m-k)P^{(j)}(m,k); \quad j = 1, \cdots, k, \tag{3}$$

where $P^{(j)}(m,k)$ *denotes the* j^{th} *derivative of* $P(\lambda,k)$ *at* $\lambda = m$.

Proof. We shall prove this by induction. For the base case, $j = 1$, using (2) gives

$$P'(m,k+1) = P(m,k) + (m-k)P'(m,k).$$

Hence, the result is true for the base case. Let us assume that the result holds for $j = n-1$; that is, suppose

$$P^{(n-1)}(m,k+1) = (n-1)P^{(n-2)}(m,k) + (m-k)P^{(n-1)}(m,k).$$

We shall now prove that it holds for $j = n$.

$$\begin{aligned}P^{(n)}(m,k+1) &= \frac{d}{d\lambda}\left[P^{(n-1)}(\lambda,k+1)\right]|_{\lambda=m} \\ &= \frac{d}{d\lambda}\left[(n-1)P^{(n-2)}(\lambda,k) + (\lambda-k)P^{(n-1)}(\lambda,k)\right]|_{\lambda=m} \\ &= nP^{(n-1)}(m,k) + (m-k)P^{(n)}(m,k).\end{aligned}$$

Hence, by mathematical induction, the result follows. □

Lemma 3.3. *Let* $P(n,r) = \dfrac{n!}{(n-r)!}$ *denote the usual permutation. Then*

$$P^{(k+1)}(m,k+1) = (k+1)P^{(k)}(m,k). \tag{4}$$

Proof. Notice that $P(\lambda,i)$ is an i^{th} degree monic polynomial in λ, and so $P^{(i)}(m,i) = i!$. Thus, the result follows trivially. $\square$

Theorem 3.4. *Define* $y_p = \tilde{A}t^m(\ln t)^r$. *Then*

$$y_p^{(i)} = \sum_{j=0}^{i} \binom{r}{j} P^{(j)}(m,i)\tilde{A}t^{m-i}(\ln t)^{r-j}, \tag{5}$$

where $\binom{r}{k} = 0$ *for* $k > r$.

Proof. We shall prove this by induction on i. For the base case, $i = 0$, we have

$$y_p = \binom{r}{0} P(m,0)\tilde{A}t^m(\ln t)^r = \tilde{A}t^m(\ln t)^r.$$

Hence, the result is true for the base case. Let us assume that the result holds for $i = k$; that is, suppose

$$y_p^{(k)} = \sum_{j=0}^{k} \binom{r}{j} P^{(j)}(m,k)\tilde{A}t^{m-k}(\ln t)^{r-j}.$$

We shall now prove that it holds for $i = k+1$.

$$\begin{aligned}
&y_p^{(k+1)}\\
&= [y_p^{(k)}]'\\
&= \left[\sum_{j=0}^{k} \binom{r}{j} P^{(j)}(m,k)\tilde{A}t^{m-k}(\ln t)^{r-j}\right]'\\
&= \sum_{j=0}^{k} \binom{r}{j} P^{(j)}(m,k)\tilde{A}\left[(m-k)t^{m-k-1}(\ln t)^{r-j} + (r-j)t^{m-k-1}(\ln t)^{r-j-1}\right]\\
&= \sum_{j=0}^{k} \binom{r}{j}(m-k)P^{(j)}(m,k)\tilde{A}t^{m-k-1}(\ln t)^{r-j} +\\
&\quad \sum_{j=0}^{k}(r-j)\binom{r}{j} P^{(j)}(m,k)\tilde{A}t^{m-k-1}(\ln t)^{r-j-1}\\
&= \sum_{j=0}^{k} \binom{r}{j}(m-k)P^{(j)}(m,k)\tilde{A}t^{m-k-1}(\ln t)^{r-j} +\\
&\quad \sum_{j=1}^{k+1}(r-j+1)\binom{r}{j-1} P^{(j-1)}(m,k)\tilde{A}t^{m-k-1}(\ln t)^{r-j}
\end{aligned}$$

$$= \binom{r}{0}(m-k)P(m,k)\tilde{A}t^{m-k-1}(\ln t)^r +$$
$$\sum_{j=1}^{k}\left[\binom{r}{j}(m-k)P^{(j)}(m,k) + (r-j+1)\binom{r}{j-1}P^{(j-1)}(m,k)\right]\tilde{A}t^{m-k-1}(\ln t)^{r-j}$$
$$+ (r-k)\binom{r}{k}P^{(k)}(m,k)\tilde{A}t^{m-k-1}(\ln t)^{r-k-1}$$
$$= \binom{r}{0}P(m,k+1)\tilde{A}t^{m-k-1}(\ln t)^r +$$
$$\sum_{j=1}^{k}\left[\binom{r}{j}(m-k)P^{(j)}(m,k) + j\binom{r}{j}P^{(j-1)}(m,k)\right]\tilde{A}t^{m-k-1}(\ln t)^{r-j} +$$
$$\binom{r}{k+1}(k+1)P^{(k)}(m,k)\tilde{A}t^{m-k-1}(\ln t)^{r-k-1} \quad \text{(using (2))}$$
$$= \binom{r}{0}P(m,k+1)\tilde{A}t^{m-k-1}(\ln t)^r + \sum_{j=1}^{k}\binom{r}{j}P^{(j)}(m,k+1)\tilde{A}t^{m-k-1}(\ln t)^{r-j} +$$
$$\binom{r}{k+1}P^{(k+1)}(m,k+1)\tilde{A}t^{m-k-1}(\ln t)^{r-k-1} \quad \text{(using (3) and (4))}$$
$$= \sum_{j=0}^{k+1}\binom{r}{j}P^{(j)}(m,k+1)\tilde{A}t^{m-k-1}(\ln t)^{r-j}.$$

Hence, by mathematical induction, the result follows. ☐

Define

$$Q(\lambda) = \sum_{i=0}^{n} a_i P(\lambda, i). \tag{6}$$

Then, $Q^{(j)}(m) = \sum_{i=0}^{n} a_i P^{(j)}(m,i)$.

Theorem 3.5. *Consider the n^{th} order non-homogeneous Euler-Cauchy equation with a monomial of degree m right-hand side:*

$$\sum_{i=0}^{n} a_i t^i y^{(i)} = At^m. \tag{7}$$

A particular solution for (7) is given by $y_p = \tilde{A}t^m(\ln t)^r$, where r is the multiplicity of $\lambda = m$ as a root of $Q(\lambda) = 0$, where Q as defined in (6) is the characteristic polynomial for the n^{th} order Euler-Cauchy equation in (7). Furthermore, the coefficient $\tilde{A}$ is given by $\tilde{A} = \dfrac{A}{Q^{(r)}(m)}$.

Proof. Setting $y = y_p$ in (7), we get

$$\sum_{i=0}^{n} a_i t^i y_p^{(i)} = \sum_{i=0}^{n} a_i t^i \sum_{j=0}^{i}\binom{r}{j}P^{(j)}(m,i)\tilde{A}t^{m-i}(\ln t)^{r-j} \quad \text{(using Theorem 3.4)}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \tilde{A} t^m \sum_{j=0}^{i} \binom{r}{j} a_i P^{(j)}(m,i)(\ln t)^{r-j} \\
&= \sum_{j=0}^{r} \tilde{A} t^m \binom{r}{j} \left[\sum_{i=0}^{n} a_i P^{(j)}(m,i) \right] (\ln t)^{r-j} \\
&\quad \text{(collecting terms with like powers of } \ln t) \\
&= \sum_{j=0}^{r} \tilde{A} t^m \binom{r}{j} Q^{(j)}(m)(\ln t)^{r-j} \\
&= \tilde{A} t^m Q^{(r)}(m).
\end{aligned}
$$

The last equality follows from the fact that $\lambda = m$ is a root of $Q(\lambda) = 0$ of multiplicity r and hence, $Q^{(j)}(m) = 0; \quad j = 0, \cdots, r-1$. Thus, y_p is a particular solution of (7) with $\tilde{A} = \dfrac{A}{Q^{(r)}(m)}$. □

Corollary 1. *Consider the n^{th} order non-homogeneous Euler-Cauchy equation with a polynomial of degree m right-hand side:*

$$
\sum_{i=0}^{n} a_i t^i y^{(i)} = \sum_{j=0}^{m} A_j t^j. \tag{8}
$$

A particular solution for (8) is given by $y_p = \sum_{j=0}^{m} \tilde{A}_j t^j (\ln t)^{r_j}$, where r_j is the multiplicity of $\lambda = j$ as a root of $Q(\lambda) = 0$, where Q as defined in (6) is the characteristic polynomial for the n^{th} order Euler-Cauchy equation in (8). Furthermore, the coefficients $\tilde{A}_j$ are given by $\tilde{A}_j = \dfrac{A_j}{Q^{(r_j)}(j)}$.

Proof. The result follows from applying Theorem 3.5 for each $j = 0, \cdots, m$, and the Superposition Principle. □

4. **Examples.** In this section, we apply the results derived above to solve the examples considered previously in Section 2. In each of these examples, we will use either Theorem 3.5 or Corollary 1, whichever is appropriate.

1. Compute a particular solution to the Euler-Cauchy equation:
   $t^2 y'' - 3ty' + 4y = 6t^2$.
   Solution:
   Here $Q(\lambda) = \lambda(\lambda - 1) - 3\lambda + 4 = \lambda^2 - 4\lambda + 4$.
   So, with $m = 2$, $r = 2$, and $A = 6$, Theorem 3.5 gives $y_p = 3t^2(\ln t)^2$, since $\tilde{A} = \dfrac{A}{Q^{(2)}(2)} = 3$.

2. Compute a particular solution to the Euler-Cauchy equation:
   $t^3 y''' - 3t^2 y'' + 6ty' - 6y = 2t + 7t^2$.
   Solution:
   Here $Q(\lambda) = \lambda(\lambda - 1)(\lambda - 2) - 3\lambda(\lambda - 1) + 6\lambda - 6 = \lambda^3 - 6\lambda^2 + 11\lambda - 6$. So,

with $r_1 = 1$, $r_2 = 1$, $A_1 = 2$, and $A_2 = 7$, Corollary 1 gives $y_p = t\ln t - 7t^2 \ln t$, since $\tilde{A}_1 = \dfrac{A_1}{Q^{(1)}(1)} = 1$ and $\tilde{A}_2 = \dfrac{A_2}{Q^{(1)}(2)} = -7$.

5. **Conclusion.** In this paper, we have presented a novel proof of the interesting form of the particular solution to Euler-Cauchy differential equations with polynomial right-hand sides, along with a general formula that can be used to determine the unknown coefficients in the particular solution. The formula presented here is useful in computing the particular solution in a much less tedious and time-consuming way than the standard approaches.

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