Chap 1. First Order ODEs

1.1 Basic Concepts. Modeling

$\underline{F(x, y, y') = 0}$ or $\underline{y' = f(x, y)}$, where $y(x)$ is
(implicit form) (explicit form) a function of x.

Ex 1 (Verification of Solution)

$y = h(x) = c/x \ (x \neq 0)$ is a solution of $xy' = -y$

「pf) $y' = h'(x) = -c/x^2$, $xy' = -c/x = -y$, $\therefore xy' = -y$ 」

Ex 2 (Solution Curves)

$y' = dy/dx = \cos x \Rightarrow y = \int \cos x \, dx = \sin x + c$ → an arbitrary constant

(a family of solutions)

Each value of c gives one of these curves

Ex 3 (Exponential Growth. Exponential Decay)

① $y = ce^{3t} \Rightarrow y' = 3ce^{3t} = 3y$ (Exponential Growth)

② $y' = -0.2y \Rightarrow y = ce^{-0.2t}$ (Exponential Decay)

Initial Value Problem

$y' = f(x, y)$, $\underline{y(x_0) = y_0}$

Ex 4 (Initial Value Problem)

$y' = \frac{dy}{dx} = 3y$, $y(0) = 5.7$ → the initial value

$\Rightarrow y(x) = ce^{3x}$, $y(0) = c = 5.7$

$\therefore y(x) = 5.7e^{3x}$ → a particular solution

↳ the general solution

Modeling

Ex 5 (Radioactivity. Exponential Decay)

Step 1 : $\frac{dy}{dt} = ky$, $y(0) = 0.5 \rightarrow$ given amount of radioactive substance

Step 2 : $y(t) = ce^{kt}$, $y(0) = c = 0.5$

$y(t) = 0.5e^{kt}$.

Check the result : $\frac{dy}{dt} = 0.5ke^{kt} = k \cdot 0.5e^{kt} = ky$.

$y(0) = 0.5$

Step 3 : Interpretation of the result. $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

1.2 Geometric Meaning of $y' = f(x,y)$. Direction Fields.

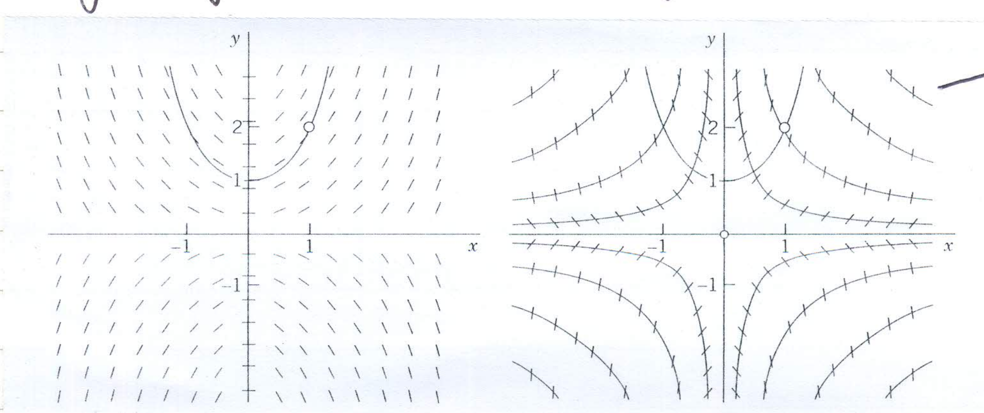
The derivative $y'(x)$ is the slope of $y(x)$

a solution curve of $y' = f(x,y)$ passing through (x_0, y_0)

must have at that point the slope $y'(x_0) = f(x_0, y_0)$.

Ex : $y' = xy$

Warning : Isoclines are Not the solution curves!



(a) By a CAS

(b) By isoclines

Using a Computer Algebra System, one can draw short line segments at the points of a square grid

Isoclines are the curves where $y' = f(x,y) = k = \text{const}$.

Ex. $y' = f(x,y) = xy = k = \text{const}$ (hyperbolas)

1.3 Separable ODEs. Modeling

$g(y) \cdot y' = f(x)$

$\int g(y) \cdot \underline{y' dx} = \int f(x) dx + c$

$\int g(y) \underline{dy} = \int f(x) dx + c$

Ex1 (a Separable ODE)

$y' = 1 + y^2 \Rightarrow \frac{dy}{1+y^2} = dx$, $\arctan y = x + c$

$\therefore y = \tan(x + c)$.

Modeling

Ex2 (Radioactive Dating) When did the Iceman live and die?

$y' = ky \Rightarrow y(t) = y_0 \cdot e^{kt}$

$\ulcorner \because \frac{dy}{dt} = ky$, $\frac{dy}{y} = k dt$, $\ln|y| = kt + c$, $|y| = e^{kt} \cdot e^c$

$y = \underbrace{(\pm e^c)}_{y_0} \cdot e^{kt} \lrcorner$

Half-cycle of ${}_6C^{14}$ is $H = 5715$

$y_0 \cdot e^{k \times 5715} = \frac{1}{2} y_0$

$k = -(\ln 2)/5715 = -0.0001213$

In what year, the content will be 52.5%?

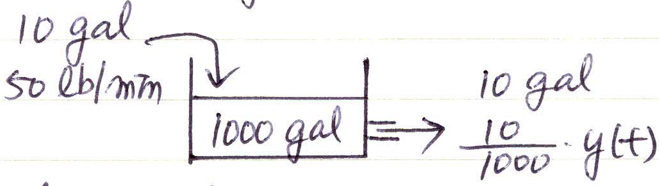
$y_0 \cdot e^{kt} = 0.525 y_0$

$e^{kt} = 0.525$

$t = \frac{\ln(0.525)}{-0.0001213} = 5312$

$\therefore$ The Iceman died about 5300 years ago.

Ex 3 (Mixing Problem)



$y(0) = 100$ lb

$$y'(t) = 50 - \frac{1}{100} y(t) = -0.01(y(t) - 5000), \quad y(0) = 100$$

$$\underline{100y' = 5000 - y, \quad y(0) = 100}$$

$$\frac{dy}{y-5000} = -0.01\,dt \Rightarrow \ln|y-5000| = -0.01t + c^*$$

$$y - 5000 = c \cdot e^{-0.01t}$$

$$100 - 5000 = c \cdot e^0 = c$$

$$\underline{y(t) = 5000 - 4900e^{-0.01t}}$$

Ex 4 (Heating an Office Building)

at 10 PM. $T = 70°F$ (The heating is shut off, $T_A = 50°F$)

2 AM $T = 65°F$

⋮

6 AM $T = ?$ when $T_A = 40°F$

the outside temperature

Step 1: $\frac{dT}{dt} = k(T - T_A)$

Step 2: We do not know T_A, thus take the average $T_A = 45°F$.

$$\frac{dT}{T-45} = k\,dt, \quad \ln|T-45| = kt + c^*, \quad T(t) = 45 + ce^{kt}$$

Step 3: $T(0) = 45 + c = 70$, $c = 25$, $T_p(t) = 45 + 25e^{kt}$

particular solution

Step 4: $T(4) = 65$

$$T_p(4) = 45 + 25e^{4k} = 65 \Rightarrow T_p(t) = 45 + 25e^{-0.056t}$$

Step 5: $T_p(8) = 45 + 25e^{-0.056 \times 8} = 61°F$ at $\underline{6\text{ AM}}$

morning glory

Extended Method: Reduction to Separable Form

$$y' = f\left(\frac{y}{x}\right)$$

Let $u = \frac{y}{x}$, $\Rightarrow$ $y = ux$, $y' = u'x + u$

$\Rightarrow$ $u'x + u = f(u)$

$$\frac{du}{f(u) - u} = \frac{dx}{x} \quad : \text{separable form}$$

Ex 6

$$2xy \cdot y' = y^2 - x^2$$

$$y' = \frac{y^2 - x^2}{2xy} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$

Let $u = \frac{y}{x}$ $\Rightarrow$ $y = ux$, $y' = u'x + u$

$$u'x + u = \frac{1}{2}\left(u - \frac{1}{u}\right)$$

$$u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\frac{u^2+1}{2u}$$

$$\frac{2u}{1+u^2}\,du = -\frac{dx}{x}$$

$$\ln(1+u^2) = -\ln|x| + c^*$$

$$1 + u^2 = \frac{c}{x}, \quad \text{where } c = \pm e^{c^*}$$

$$1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x}, \quad x^2 + y^2 = cx$$

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4} :$$

a family of circles passing through the origin and centers on the x-axis

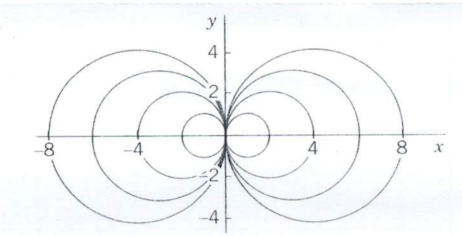


Fig. 12. General solution (family of circles) in Example 6

1.4 Exact ODEs. Integrating Factors

$M(x,y)dx + N(x,y)dy = 0$ is exact $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

That is, $\exists u(x,y)$ s.t $du = \underbrace{\frac{\partial u}{\partial x}}_{=M(x,y)}dx + \underbrace{\frac{\partial u}{\partial y}}_{=N(x,y)}dy$

$$du = 0$$

$$u(x,y) = c$$

$$\Rightarrow u = \int M dx + k(y) = \int N dy + l(x)$$

Ex1 (An Exact ODE)

$\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0$

<solution>

Step1 : $M = \cos(x+y)$, $N = 3y^2 + 2y + \cos(x+y)$

$$\frac{\partial M}{\partial y} = -\sin(x+y) = \frac{\partial N}{\partial x} \;:\; \text{exact!}$$

Step2 : $u = \int M dx + k(y) = \sin(x+y) + k(y)$

$$\frac{\partial u}{\partial y} = \cos(x+y) + k'(y) = N = 3y^2 + 2y + \cos(x+y)$$

$$k'(y) = 3y^2 + 2y$$

$$k(y) = y^3 + y^2 + c^*$$

$$\therefore u(x,y) = \sin(x+y) + y^3 + y^2 = c$$

Step3 : $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$= \cos(x+y)dx + (\cos(x+y) + 3y^2 + 2y)dy$$

$$= 0 \quad //$$

Ex 2 (an Initial Value Problem)

$$(\cos y \sinh x + 1)\,dx - \sin y \cosh x\,dy = 0, \quad y(1) = 2.$$

<solution>

Step 1: $\dfrac{\partial M}{\partial y} = -\sin y \sinh x = \dfrac{\partial N}{\partial x}$: exact

Step 2: $u = -\int \sin y \cosh x\,dy + l(x) = \cos y \cosh x + l(x)$

$$\frac{\partial u}{\partial x} = \cos y \sinh x + l'(x) = M = \cos y \sinh x + 1$$

$$l(x) = x + c^*$$

$$u(x,y) = \cos y \cosh x + x = c$$

$$u(1,2) = \cos 2 \cosh 1 + 1 = 0.358 = c \Leftarrow y(1)=2$$

$$\therefore u(x,y) = \cos y \cosh x + x = 0.358$$

Ex 3 Warning! Breakdown in the Case of Nonexactness

$-y\,dx + x\,dy = 0$ is not exact

since $\dfrac{\partial M}{\partial y} = -1 \neq 1 = \dfrac{\partial N}{\partial x}$

$$u = \int M\,dx + k(y) = -xy + k(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -x + k'(y) \neq x = N$$

$\therefore$ The present method does not work!

Reduction to Exact Form. Integrating Factors

$-y\,dx + x\,dy = 0$ is not exact

But, $\dfrac{-y\,dx + x\,dy}{x^2} = \underline{-\dfrac{y}{x^2}\,dx + \dfrac{1}{x}\,dy = d\left(\dfrac{y}{x}\right) = 0}$

exact!

Ex 4: $\dfrac{-y\,dx + x\,dy}{y^2} = -d\left(\dfrac{x}{y}\right)$, $\dfrac{-y\,dx + x\,dy}{xy} = -d\left(\ln\dfrac{x}{y}\right)$

$$\frac{-y\,dx + x\,dy}{x^2+y^2} = d\left(\arctan\frac{y}{x}\right)$$

Given $P(x,y)\,dx + Q(x,y)\,dy = 0$, multiply by $\underline{F(x,y)}$ so that $\underline{FP\,dx + FQ\,dy = 0}$ is exact.

an integrating factor

How to find $F(x,y)$?

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

$$F_y P + F P_y = F_x Q + F Q_x$$

function on one variable only

To make life easy: Let $F = F(x) \Rightarrow F_y = 0$

$$F P_y = F' Q + F Q_x$$

$$\boxed{\begin{gathered}\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = R(x)\\ F(x) = \exp\int R(x)\,dx\end{gathered}}$$

Similarly, if $F^* = F^*(y)$, then $F^* Q_x = F^*_y P + F^* P_y$

$\dfrac{1}{F^*}\dfrac{dF^*}{dy} = \dfrac{1}{P}\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) = R^*(y)$, $\underline{F^*(y) = \exp\int R^*(y)\,dy}$

Ex 5

$$(e^{x+y} + ye^{y})\,dx + (xe^{y} - 1)\,dy = 0 , \quad y(0) = -1$$

<solution>

Step 1 : Nonexactness

$$\frac{\partial P}{\partial y} = e^{x+y} + e^{y} + ye^{y} \neq e^{y} = \frac{\partial Q}{\partial x}$$

Step 2 : Integrating factor.

function in x only

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^{y}-1}\left(e^{x+y} + ye^{y}\right) \neq R(x)$$

$$R^{*} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^{y}}\left(e^{y} - e^{x+y} - e^{y} - ye^{y}\right) = -1$$

The integrating factor $F^{*} = \exp \int R^{*}(y)\,dy = e^{-y}$.

$$(e^{x} + y)\,dx + (x - e^{-y})\,dy = 0$$

$$u = \int (e^{x}+y)\,dx + k(y) = e^{x} + xy + k(y)$$

$$\frac{\partial u}{\partial y} = x + k'(y) = N = x - e^{-y} \Rightarrow k(y) = e^{-y} + c^{*}$$

$$u(x,y) = e^{x} + xy + e^{-y} = c$$: general solution

Step 3 : Particular solution

$$u(0,-1) = 1 + e = 3.72 = c$$

$$\therefore\ e^{x} + xy + e^{-y} = 3.72$$

Step 4 : Check by substitution that the answer satisfies the given equation as well as the initial condition.

1.5 Linear ODEs. Bernoulli Equation

$$y' + p(x)\,y = r(x)$$

Homogeneous Linear ODE: $y' + p(x)\,y = 0$

$$\frac{dy}{y} = -p(x)\,dx \Rightarrow \ln|y| = -\int p(x)\,dx + c^*$$

$$y(x) = c \cdot e^{-\int p(x)\,dx}, \quad \text{where } c = \begin{cases} e^{c^*} & \text{if } y>0, \\ -e^{c^*} & \text{if } y<0. \end{cases}$$

$y(x) \equiv 0$ is a trivial solution

Nonhomogeneous Linear ODE.

$$y' + p(x)\,y = r(x)$$

$$\underbrace{(py - r)}_{P}\,dx + \underbrace{1}_{Q}\cdot dy = 0$$

$$\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = p(x)$$

$\therefore F(x) = \exp \int p(x)\,dx$: Integrating factor

$$\to e^{\int p\,dx}\left[y' + py\right] = e^{\int p\,dx} \cdot r$$

$$e^{\int p\,dx} \cdot y = \int \left(e^{\int p\,dx} \cdot r\right) dx + c$$

$$\therefore y(x) = e^{-\int p\,dx}\left[\int \left(e^{\int p\,dx} \cdot r\right) dx + c\right]$$

__Ex1__ (First Order ODE, General Solution)

$$y' - y = e^{2x}$$

<solution>

$p(x) = -1, \quad r(x) = e^{2x}$

$h(x) = \int p(x)dx = -x + \overset{0}{\overset{\shortparallel}{\alpha}} = -x$ (we may take $\alpha = 0$)

$$y(x) = e^{-h}\left[\int e^{h}\cdot r\,dx + c\right]$$

$$= e^{x}\left[\int e^{-x}\cdot e^{2x}\,dx + c\right]$$

$$= ce^{x} + e^{x}\left[e^{x} + \beta\right]$$ we may take $\beta = 0$

$$= ce^{x} + e^{2x}$$

__Ex2__ (First Order ODE, Initial Value Problem)

$$y' + y\tan x = \sin 2x, \quad y(0) = 1$$

<solution>

$p(x) = \tan x, \quad r(x) = \sin 2x = 2\sin x\cos x$

$h(x) = \int p(x)dx = \int \tan x\,dx = \ln|\sec x|$

$e^{h} = \sec x, \quad e^{-h} = \cos x, \quad e^{h}\cdot r = 2\sin x$

$$y(x) = \cos x\left[\int 2\sin x\,dx + c\right]$$

$$= c\cos x - 2\cos^2 x.$$

$y(0) = c - 2 = 1 \Rightarrow c = 3$

$$\therefore y(x) = \underbrace{3\cos x}_{\text{the response to the initial data}} - \underbrace{2\cos^2 x}_{\text{the response to the input } \sin 2x}$$

Reduction to Linear Form. Bernoulli Equation

$$y' + p(x)\,y = g(x)\,y^{a} \quad : \text{Bernoulli Equation}$$

(a → any real number)

If $a=0$ or $a=1$, this equation is linear.
Otherwise, it is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}$$

$$u' = (1-a)\,y^{-a}\cdot y'$$

$$= (1-a)\,y^{-a}[g\cdot y^{a} - p\,y]$$

$$= (1-a)[g - p\,u]$$

$$\therefore\ \underline{u' + (1-a)p\,u = (1-a)\,g} \ : \text{Linear ODE}$$

Ex 4 (Logistic Equation)

$$y' = Ay - By^2$$

<solution>

$y' - Ay = -By^2$ with $a=2$ and $u = y^{1-a} = y^{-1}$.

$$u' = (1-2)\cdot y^{-2}\cdot y' = -y^{-2}(Ay - By^2) = -Ay^{-1} + B$$

$$u' + Au = B$$

$$u = c\,e^{-At} + B/A \ : \text{general solution}$$

$$y = \frac{1}{u} = \frac{1}{c\cdot e^{-At} + B/A}$$