Engineering Mathematics I

Midterm Exam, October 14, 2015

1. (25 points) Newton's law of cooling states that the time rate of change of the temperature $T(t)$ of a cooling body is proportional to the temperature difference between the body and its surroundings: $dT/dt = k(T - S)$, where $S(t)$ is the temperature of the surroundings and k is a constant.

   (a) (10 points) A body at a temperature of $100°$ is placed in a room of unknown temperature. The room temperature does not change. If after 10 min the body has cooled to $90°$ and after 20 min to $85°$, find the temperature of the surroundings.

$$
\begin{aligned}
\frac{dT}{T-S} &= kdt \\
\ln(T-S) &= kt + c \\
T - S &= \alpha e^{kt} \\
100 - S &= \alpha \\
90 - S &= \alpha e^{10k} \\
85 - S &= \alpha e^{20k} \\
(90 - S)^2 &= (85 - S)(100 - S) \\
5S - 400 = 0, &\quad S = 80
\end{aligned}
$$

   (b) (15 points) A body of temperature $100°$ is placed in water of temperature $50°$. After 10 min the temperature of the body is $80°$ and the temperature of the water is $60°$. Assuming all the heat lost by the body is absorbed by the water: $S(t)-S(0) = c(T(t)-T(0))$, for some constant c, find the temperature of the body and of the water at any time. Find the equilibrium temperature.

$$
\begin{aligned}
c = \frac{S(10) - S(0)}{T(10) - T(0)} &= \frac{60 - 50}{80 - 100} = -0.5 \\
S(t) = S(0) - 0.5(T(t) - T(0)) &= 100 - T(t)/2 \\
T(t) - S(t) = 3T(t)/2 - 100 &= \frac{3}{2}\left(T(t) - \frac{200}{3}\right) \\
\frac{dT}{T - \frac{200}{3}} &= \frac{3}{2}kdt \\
T(t) - \frac{200}{3} &= \alpha e^{\frac{3}{2}kt} \\
100 - \frac{200}{3} = \frac{100}{3} &= \alpha \\
80 - \frac{200}{3} = \frac{40}{3} &= \frac{100}{3}e^{15k} \\
k &= \frac{\ln 0.4}{15} \\
T(t) = 200/3 + (100/3)e^{(\frac{\ln 0.4}{15})t}& \\
S(t) = 200/3 - (50/3)e^{(\frac{\ln 0.4}{15})t}& \\
T(t), S(t) \to 200/3 \text{ as } t \to \infty&
\end{aligned}
$$

2. (15 points) Using the method of variation of parameters, solve the following system of equations:

$$\begin{aligned} y_1' &= -2y_1 + y_2 + 2e^{-t} \\ y_2' &= y_1 - 2y_2 + 3t \end{aligned}$$

$$\det(A - \lambda I) = (\lambda + 2)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0$$

$$\lambda_1 = -1,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = -3,\ \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-3t} \\ e^{-3t} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix}$$

$$\mathbf{u}'(t) = \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix}^{-1} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{t} & e^{t} \\ -e^{3t} & e^{3t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \begin{bmatrix} 1 + \frac{3}{2}te^{t} \\ -e^{2t} + \frac{3}{2}te^{3t} \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} t - \frac{3}{2}e^{t} + \frac{3}{2}te^{t} \\ -\frac{1}{2}e^{2t} - \frac{1}{6}e^{3t} + \frac{1}{2}te^{3t} \end{bmatrix}$$

$$\mathbf{Yu}(t) = \begin{bmatrix} e^{-t} & -e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \begin{bmatrix} t - \frac{3}{2}e^{t} + \frac{3}{2}te^{t} \\ -\frac{1}{2}e^{2t} - \frac{1}{6}e^{3t} + \frac{1}{2}te^{3t} \end{bmatrix} = \begin{bmatrix} t - \frac{4}{3} + \left(t + \frac{1}{2}\right)e^{-t} \\ 2t - \frac{5}{3} + \left(t - \frac{1}{2}\right)e^{-t} \end{bmatrix}$$

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \mathbf{Yu}(t) = \begin{bmatrix} c_1e^{-t} - c_2e^{-3t} + t - \frac{4}{3} + \left(t + \frac{1}{2}\right)e^{-t} \\ c_1e^{-t} + c_2e^{-3t} + 2t - \frac{5}{3} + \left(t - \frac{1}{2}\right)e^{-t} \end{bmatrix}$$

3. (10 points) Solve the following initial value problem by the power series method. Find the recurrence formula and find the first six nonzero terms in the series.

$$y'' + x^2 y = 0, \quad y(0) = 12,\ y'(0) = 20.$$

$$\begin{aligned}
y &= \sum_{m=0}^{\infty} a_m x^m, \quad x^2 y = \sum_{m=0}^{\infty} a_m x^{m+2} = \sum_{s=2}^{\infty} a_{s-2}\ x^s \\
y' &= \sum_{m=1} m a_m x^{m-1} = \sum_{s=0} (s+1) a_{s+1}\ x^s \\
y'' &= \sum_{m=2} m(m-1) a_m x^{m-2} = \sum_{s=0} (s+1)(s+2) a_{s+2}\ x^s \\
y'' + x^2 y &= 2a_2 + 6a_3\ x + \sum_{s=2}^{\infty} \left[a_{s-2} + (s+1)(s+2)a_{s+2}\right]\ x^s \ = \ 0 \\
a_n &= -\frac{a_{n-4}}{n(n-1)}, \ \text{for } n = 4, 5, 6, \cdots
\end{aligned}$$

$$a_0 = y(0) = 12,\ a_1 = y'(0) = 20,\ a_2 = 0,\ a_3 = 0,$$
$$a_4 = -1,\ a_5 = -1,\ a_6 = 0,\ a_7 = 0,\ a_8 = \frac{1}{56},\ a_9 = \frac{1}{72}$$

$$y(x) = 12 + 20x - x^4 - x^5 + \frac{1}{56}x^8 + \frac{1}{72}x^9 + \cdots$$

4. (15 points) Using Laplace transforms, solve the following initial value problem:

$$y'' - 3y' + 2y = e^{-t}, \quad y(1) = 1,\ y'(1) = 0.$$

$$\tilde{t} = t - 1, \quad \tilde{y}'' - 3\tilde{y}' + 2\tilde{y} = e^{-\tilde{t}-1}, \quad \tilde{y}(0) = 1,\ \tilde{y}'(0) = 0.$$

$$s^2\tilde{Y} - s - 3\left(s\tilde{Y} - 1\right) + 2\tilde{Y} = \frac{e^{-1}}{s+1}$$

$$(s^2 - 3s + 2)\tilde{Y} = s - 3 + \frac{e^{-1}}{s+1}$$

$$\begin{aligned}
\tilde{Y}(s) &= \frac{s-3}{(s-1)(s-2)} + \frac{e^{-1}}{(s+1)(s-1)(s-2)} \\
&= \frac{2}{s-1} - \frac{1}{s-2} + \frac{e^{-1}}{6}\left(\frac{1}{s+1} - \frac{3}{s-1} + \frac{2}{s-2}\right)
\end{aligned}$$

$$\tilde{y}(\tilde{t}) = 2e^{\tilde{t}} - e^{2\tilde{t}} + \frac{e^{-1}}{6}\left(e^{-\tilde{t}} - 3e^{\tilde{t}} + 2e^{2\tilde{t}}\right)$$

$$y(t) = \frac{1}{6}e^{-t} + \left(2 - \frac{1}{2e}\right)e^{t-1} + \left(\frac{1}{3e} - 1\right)e^{2(t-1)}$$

5. (15 points) Using Laplace transforms, show that

$$\int_0^x \left[\int_0^t f(u)\ du\right] dt = \int_0^x f(t)(x-t)\ dt$$

$$\begin{aligned}
g(t) &= \int_0^t f(u)\ du, \quad G(s) = \frac{1}{s}\cdot F(s) \\
\mathcal{L}\left[\int_0^x g(t)dt\right] &= \frac{1}{s}\cdot G(s) = \frac{1}{s^2}\cdot F(s) \\
\int_0^x g(t)dt &= x * f(x) = f(x) * x
\end{aligned}$$

6. (10 points) Find a function $f(t)$, if it exists. Otherwise, explain why there is no such solution.

(a) (3 points) $\quad t * f(t) = t^4$

$$\frac{1}{s^2}\cdot F(s) = \frac{F(s)}{s^2} = \frac{4!}{s^5}, \quad F(s) = \frac{4!}{s^3}, \quad f(t) = 12t^2$$

(b) (3 points) $\quad 1 * 1 * f(t) = \frac{1}{2}t^2$

$$\frac{1}{s}\cdot\frac{1}{s}\cdot F(s) = \frac{F(s)}{s^2} = \frac{1}{s^3}, \quad F(s) = \frac{1}{s}, \quad f(t) = 1$$

(c) (4 points) $\quad 1 * f(t) = 1$

$$\int_0^t f(\tau)d\tau = 1, \text{ for all } t \geq 0.$$

$$\text{For } t = 0,\ 0 = \int_0^0 f(\tau)d\tau = 1\ \#.$$

There is no such a solution $f(t)$.

7. (10 points) Using Laplace transforms, solve the following initial value problem:

$$y'' + 2y' + 2y = \delta(t-\pi), \quad y(0) = 0,\ y'(0) = 0.$$

Solution:

$$\begin{aligned}
s^2Y + 2sY + 2Y &= e^{-\pi s} \\
(s^2 + 2s + 2)Y &= e^{-\pi s} \\
Y(s) &= \frac{e^{-\pi s}}{(s+1)^2 + 1}
\end{aligned}$$

$$\text{Let } F(s) = \frac{1}{(s+1)^2+1}, \text{ then } f(t) = e^{-t}\sin t$$

$$y = f(t-\pi)u(t-\pi) = e^{-(t-\pi)}\sin(t-\pi)u(t-\pi)$$