Chap 11. Fourier Analysis

11.1 Fourier Series

period

$f(x)$: periodic function $\Leftrightarrow$ $f(x+p) = f(x)$ for all x

the Fourier series of $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$: periodic function of period 2π

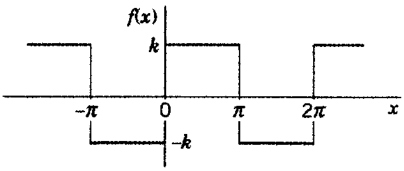
Euler Formulas
$$\begin{cases} a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx \\ a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n=1,2,\cdots \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n=1,2,\cdots \end{cases}$$

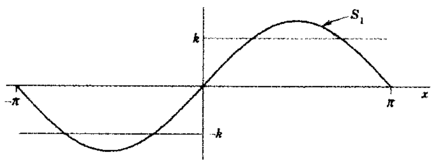
Fourier coefficients

Ex 1: $f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi, \end{cases}$ and $f(x+2\pi) = f(x)$

$$\Rightarrow \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \begin{cases} \dfrac{4k}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

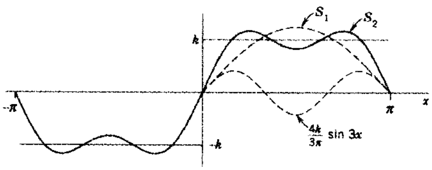


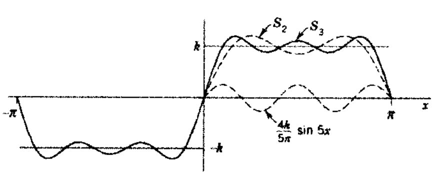


$S_1 = \frac{4k}{\pi}\sin x$

$S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$

$\vdots$





Orthogonality

$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$: Trigonometric system is orthogonal

$$\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = 0 \quad \text{if } m \neq n$$

$$\langle \sin mx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = 0 \quad \text{if } m \neq n$$

$$\langle \cos mx, \sin nx \rangle = \int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx = 0$$

11.2 Arbitrary Period ($p = 2L$)

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$\begin{cases} a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx \\ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx, & n = 1, 2, \cdots \\ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx, & n = 1, 2, \cdots \end{cases}$$

Ex1: $f(x) = \begin{cases} 0 & \text{if } -2 < x < -1, \\ k & \text{if } -1 < x < 1, \\ 0 & \text{if } 1 < x < 2, \end{cases}$ $\quad p = 2L = 4,$ $L = 2$

$$\Rightarrow \begin{cases} a_0 = \frac{1}{4} \int_{-2}^{2} f(x)\,dx = \frac{1}{4} \int_{-1}^{1} k\,dx = \frac{k}{2} \\ a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2}\,dx = \frac{1}{2} \int_{-1}^{1} k \cos \frac{n\pi x}{2}\,dx \\ \quad = \frac{2k}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 2k/n\pi & \text{if } n = 1, 5, 9, \cdots \\ -2k/n\pi & \text{if } n = 3, 7, 11, \cdots \end{cases} \\ b_n = 0 \end{cases}$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left(\cos \frac{\pi}{2} x - \frac{1}{3} \cos \frac{3\pi}{2} x + \frac{1}{5} \cos \frac{5\pi}{2} x - \cdots\cdots \right)$$

Even and Odd Functions

$f(x)$: even function $\Leftrightarrow f(-x)=f(x)$

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \quad : \text{Fourier cosine series}$$

(even function of period $2L$)

$$\begin{cases} a_0=\frac{1}{L}\int_0^L f(x)\,dx \\ a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n=1,2,\cdots \\ b_n=0 \end{cases}$$

$f(x)$: odd function $\Leftrightarrow f(-x)=-f(x)$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad : \text{Fourier sine series}$$

(odd function of period $2L$)

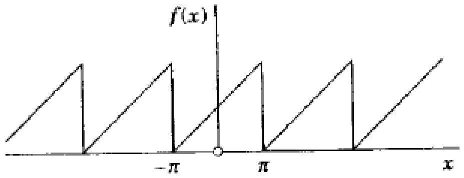
$$\begin{cases} a_n=0, \quad n=0,1,2,\cdots \\ b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,\cdots \end{cases}$$

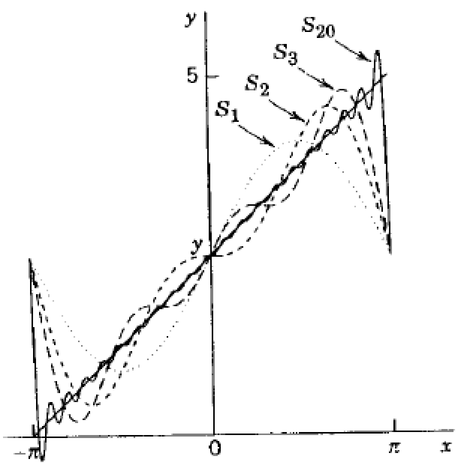
Ex2: $f(x)=x+\pi$ if $-\pi<x<\pi$, $f(x+2\pi)=f(x)$

$\Rightarrow f=f_1+f_2$, where $\underline{f_1(x)=x}$, $f_2(x)=\pi$

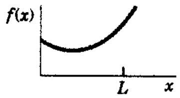
(odd function with $b_n=-\frac{2}{n}\cos n\pi$)

$$\Rightarrow f(x)=\pi+2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right)$$

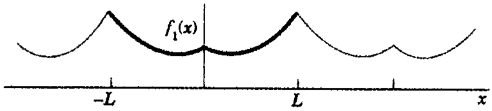




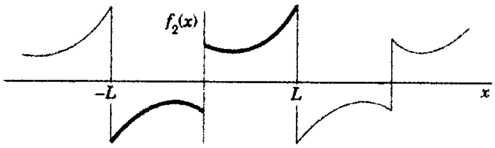
Half-Range Expansions



(0) The given function $f(x)$

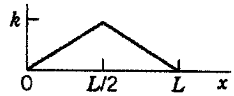


(a) $f(x)$ continued as an ***even*** periodic function of period $2L$



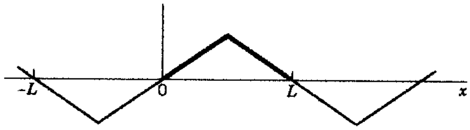
(b) $f(x)$ continued as an ***odd*** periodic function of period $2L$

Ex 6:



$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\[2ex] \dfrac{2k}{L}(L-x) & \text{if} \quad \dfrac{L}{2} < x < L. \end{cases}$$

(a) Even extension



(b) Odd extension

(Solution of Ex 6)

(a) Even periodic extension

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\,dx\right] = \frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx\right]$$

$$= \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

$$\Rightarrow f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right)$$

(b) Odd periodic extension

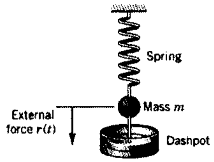
$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

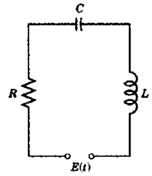
$$\Rightarrow f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \cdots\right)$$

11.3 Forced Oscillations

$$my'' + cy' + ky = r(t)$$

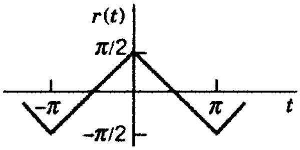
$$LI'' + RI' + \frac{1}{C} I = E'(t)$$





Ex1: Let $m = 1\,(g)$, $c = 0.05\,(g/sec)$, $k = 25\,(g/sec^2)$

$$y'' + 0.05y' + 25y = r(t)$$



$$r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \frac{\pi}{2} & \text{if } 0 < t < \pi, \end{cases} \qquad r(t + 2\pi) = r(t).$$

Representing $r(t)$ by a Fourier series:

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

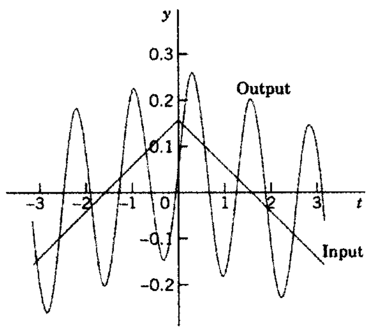
Consider $y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \quad (n = 1, 3, 5, \cdots)$

$$\Rightarrow \quad y_n = A_n \cos nt + B_n \sin nt,$$

$$A_n = \frac{4(25 - n^2)}{n^2 \pi D_n}, \quad B_n = \frac{0.2}{n \pi D_n},$$

where $D_n = (25 - n^2)^2 + (0.05n)^2$

$$\Rightarrow \quad y = y_1 + y_3 + y_5 + \cdots$$



11.4 Approximation by Trigonometric Polynomials

$f(x)$: a periodic function of period 2π that can be represented by a Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \quad : \text{Best Approximation ??}$$

Fourier coefficients

Let $F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$

$$\Rightarrow \quad E = \int_{-\pi}^{\pi} (f - F)^2 dx = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} f F \, dx + \int_{-\pi}^{\pi} F^2 dx$$

$$= \int_{-\pi}^{\pi} f^2 dx - 2\pi \left[2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n) \right] + \pi \left[2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2) \right]$$

Take $A_n = a_n$, $B_n = b_n$,

Then $$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$$

$$E - E^* = \pi \left\{ 2(A_0 - a_0)^2 + \sum_{n=1}^{N} \left[(A_n - a_n)^2 + (B_n - b_n)^2 \right] \right\} \geq 0$$

$\therefore E \geq E^*$

$E = E^* \Leftrightarrow A_0 = a_0, \; A_n = a_n, \; B_n = b_n, \; n = 1, 2, \cdots$

Since $E^* \geq 0$,

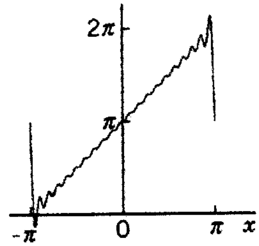
$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx \quad : \text{Bessel Inequality}$$

Ex1:

$f(x) = x + \pi \quad (-\pi < x < \pi)$

$$\Rightarrow F(x) = \pi + 2\left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - + \cdots + \frac{(-1)^{N+1}}{N} \sin Nx \right)$$

$$E^* = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi \left(2\pi^2 + 4 \sum_{n=1}^{N} \frac{1}{n^2} \right)$$



F with $N=20$

11.7 Fourier Integral

Ex1 (Square Wave)

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1, \\ 1 & \text{if } -1 < x < 1, \\ 0 & \text{if } 1 < x < L, \end{cases} \qquad f_L(x+2L) = f_L(x)$$

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since f_L is even, $b_n = 0$ for all n

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^1 \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}$$

↳ amplitude spectrum of f_L

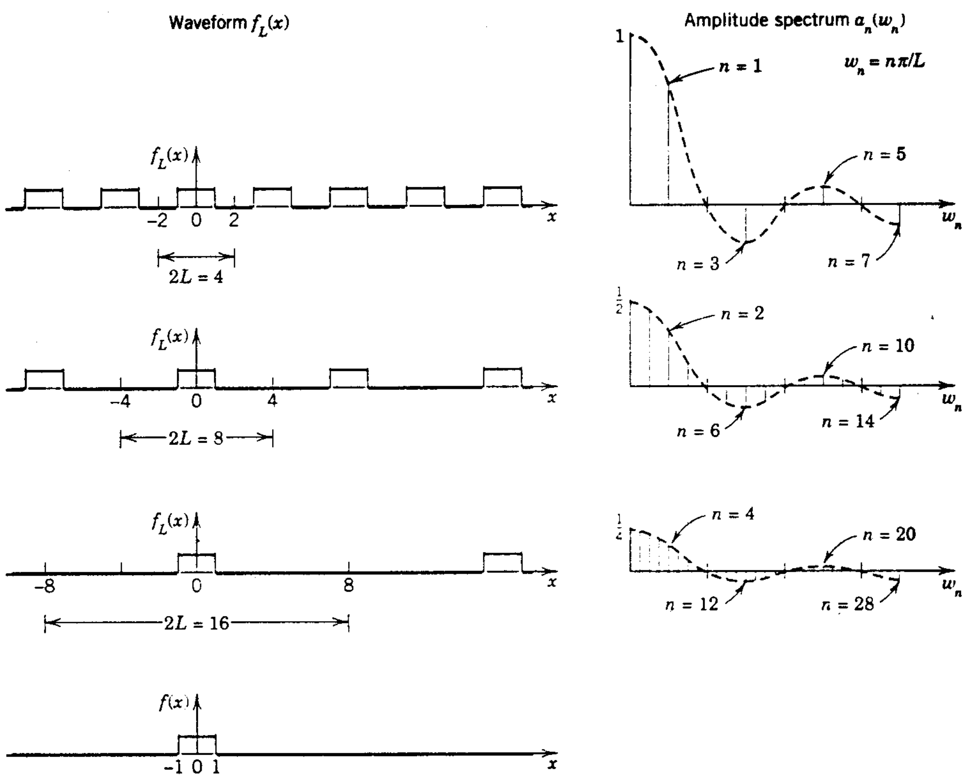


Fig. 254. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v\,dv\right]$$

Let $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$\Rightarrow \quad \frac{1}{L} = \frac{\Delta w}{\pi}$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right]$$

Assuming

$f(x) = \lim_{L\to\infty} f_L(x)$: absolutely integrable $\left(\text{i.e., } \int_{-\infty}^{\infty} |f(x)|\,dx < \infty\right)$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right]dw$$

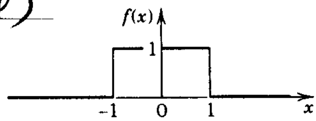
$$= \int_0^{\infty}\left[A(w)\cos wx + B(w)\sin wx\right]dw \quad : \text{Fourier Integral}$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

Applications

Ex2 (Single Pulse, Sine Integral)



$$f(x)=\begin{cases}1 & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}$$

$$\Rightarrow A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv=\frac{2\sin w}{\pi w}$$

$$B(w)=\frac{1}{\pi}\int_{-1}^{1} 1\cdot \sin wv\,dv=0$$

$$\Rightarrow f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos wx\,\sin w}{w}\,dw$$

$$\Rightarrow \int_0^{\infty}\frac{\cos wx\,\sin w}{w}\,dw=\begin{cases}\pi/2 & \text{if } 0\leq x<1\\ \pi/4 & \text{if } x=1\\ 0 & \text{if } x>1\end{cases}$$

$$\left[\because \frac{1}{2}\big(f(1-0)+f(1+0)\big)=\frac{1}{2}\right]$$

If $x=0$, $\displaystyle\int_0^{\infty}\frac{\sin w}{w}\,dw=\frac{\pi}{2}$.

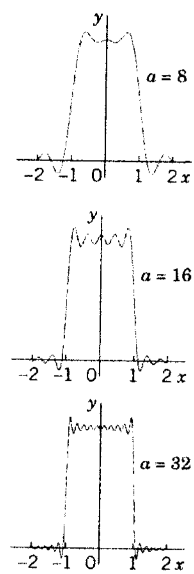
In general,

$$\int_0^{a}\frac{\cos wx\,\sin w}{w}\,dw$$

approximates

$$\int_0^{\infty}\frac{\cos wx\,\sin w}{w}\,dw$$

as $a\to\infty$



Fourier Cosine and Sine Integrals

If $f(x)$ is even, $B(w)=0$

$$A(w)=\frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$$

$$f(x)=\int_0^\infty A(w)\cos wx\,dw$$

If $f(x)$ is odd, $A(w)=0$

$$B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv$$

$$f(x)=\int_0^\infty B(w)\sin wx\,dw$$

11.8 Fourier Cosine and Sine Transforms

Fourier Cosine Transforms

If $f(x)$ is even,

$$f(x)=\int_0^\infty A(w)\cos wx\,dw,\quad A(w)=\frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$$

Let $\hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$: Fourier cosine Transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw$$

(Notation: $\mathcal{F}_c(f)=\hat{f}_c$)

Fourier Sine Transforms

If $f(x)$ is odd,

$$f(x)=\int_0^\infty B(w)\sin wx\,dw,\quad B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv$$

Let $\hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx$: Fourier sine Transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\sin wx\,dw$$

(Notation: $\mathcal{F}_s(f)=\hat{f}_s$)

Ex1 : $f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

$\Rightarrow \hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin aw}{w}\right)$

$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin wx\, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{1-\cos aw}{w}\right)$

Linearity, Transforms of Derivatives

(a) $\mathcal{F}_c(af+bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$

(b) $\mathcal{F}_s(af+bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$

Th1: $f(x)$: conti. and absolutely integrable on the x-axis

$f'(x)$: piecewise conti on each finite interval

$f(x) \to 0$ as $x \to \infty$

$\Rightarrow$ ⓐ $\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f(0)$,

ⓑ $\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}$

$\Rightarrow$ ⓐ $\mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0)$

ⓑ $\mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, w f(0)$

Ex3

$f(x) = e^{-ax}$, where $a > 0$

$\Rightarrow (e^{-ax})'' = a^2 e^{-ax}$

$f''(x) = a^2 f(x)$

$\Rightarrow a^2\mathcal{F}_c\{f(x)\} = \mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0)$

$(a^2+w^2)\mathcal{F}_c(f) = a\sqrt{\frac{2}{\pi}}$

$\therefore \mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right)$, $a > 0$

11.9 Fourier Transform

Complex Form of the Fourier Integral

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw$$

where
$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$$
$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)[\cos wv\cos wx + \sin wv\sin wx]\,dv\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\underbrace{\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\,dv}_{\text{even function of } w}\Big]dw$$

Note that
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\underbrace{\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\,dv}_{\text{odd function of } w}\Big]dw = 0$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)[\cos(wx - wv) + i\sin(wx - wv)]\,dv\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cdot e^{iw(x-v)}\,dv\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\cdot e^{-iwv}\,dv\Big]e^{iwx}\,dw$$

$\Rightarrow$ Let

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-iwx}\,dx \quad : \text{Fourier Transform}$$

Then
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\,e^{iwx}\,dw \quad : \text{Inverse Fourier Transform}$$