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Bounds

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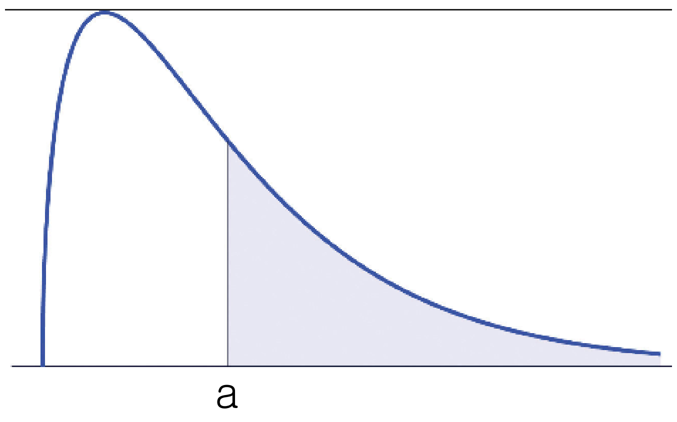
Bounds

- We are interested in "Tail Bound", like $\Pr(X \geq a)$



- Markov
  - Only E[X] is given
- Chebyshev
  - E[X] and Var[X] are known
- Chernoff
  - MGF based

Markov's Inequality

- Let X assumes only non-negative values.

  For any a > 0, $\Pr(X \geq a) \leq \frac{E[X]}{a}$

- Proof
  - $E[X] = \sum_i x_i \cdot \Pr(x_i)$

    $\geq \sum_{i:\, x_i<a} x_i \cdot \Pr(x_i) + \sum_{i:x_i \geq a} a \cdot \Pr(x_i)$

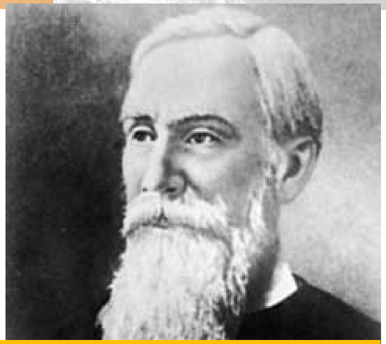
    $\geq \sum_{i:x_i \geq a} a \cdot \Pr(x_i) = a \cdot \Pr(X \geq a\,)$

Markov (1856–1922) was a
Russian Mathematician known for
Markov chain/process
Student of Chebyshev at St. Petersburg Univ.

- Example
  - X: # heads in n coin flips (note $X \geq 0$)
  - Probability of obtaining ≥3n/4 heads from n coin flips
  - E[X] = n/2
  - Pr(X ≥ 3n/4) ≤ (n/2) ÷ (3n/4) = 2/3

Is Markov bound tight? ➔ YES
Ex. 3.16

Chebyshev's Inequality



- Also known as Weak Law of Large Number
- For any a > 0,

  $$\Pr(|X - E[X]| \geq a) \leq \frac{Var[X]}{a^2}$$

- Proof
  - $\Pr(|X - E[X]| \geq a) = \Pr((X - E[X])^2 \geq a^2)$
  - Applying the Markov's Inequality, we obtain
  - $\Pr((X - E[X])^2 \geq a^2) \leq \frac{E[(X-E[X])^2]}{a^2} = \frac{Var[X]}{a^2}$
- Corollary: For any $t > 1$

  $$\Pr(|X - E[X]| \geq t \cdot \sigma[X]) \leq \frac{1}{t^2}$$

  $$\Pr(|X - E[X]| \geq t \cdot E[X]) \leq \frac{Var[X]}{t^2(E[X])^2}$$

Note: Non-negativity restriction on X is removed

Chebyshev (1821-1894) was a Russian Mathematician
One of Russian math. founders

Weak Law of Large Number
Ex. 3.25

Corollary

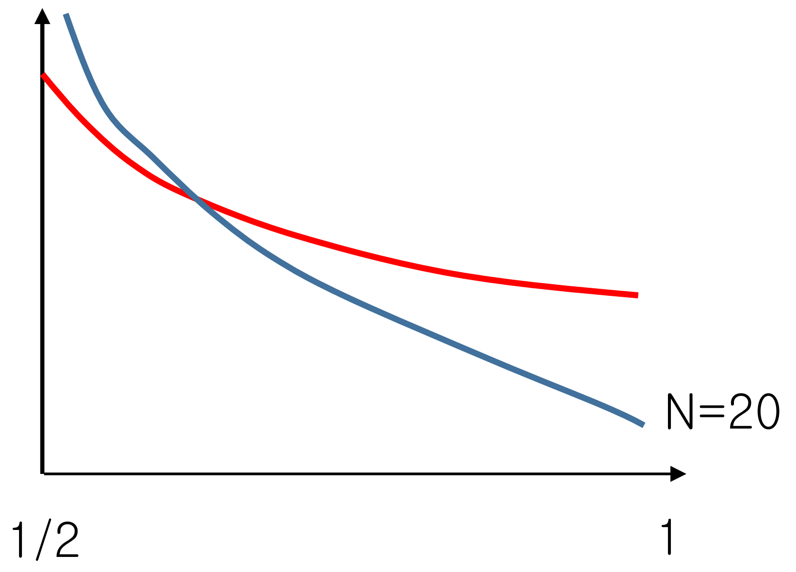
- Example
  - X: # heads in n coin flips
  - Probability of obtaining 3n/4 heads from n coin flips
  - E[X] = n/2, Var[X] = n/4
  - $\Pr(X \geq 3n/4) = \Pr(X-n/2 \geq n/4)$

$$\leq \Pr(|X-n/2| \geq n/4)$$

$$\leq \frac{Var[X]}{(n/4)^2} = 4/n$$

- Compare to the Markov bound (2/3)

As a function of k and n where $\Pr(X \geq kn)$



Chernoff Bounds

- Apply Markov inequality to e^{tX}
- From Markov inequality, for any t >0
  - Pr (X ≥ a) = Pr($e^{tX} \geq e^{ta}$) ≤ $\frac{E[\, e^{tX}]}{e^{ta}}$
  - In particular, Pr (X ≥ a) ≤ $\min_{t>0} \frac{E[\, e^{tX}]}{e^{ta}}$

MGF

Find appropriate t that minimizes the bound

- Similarly, for t < 0
  - Pr (X ≤ a) = Pr($e^{tX} \geq e^{ta}$) ≤ $\frac{E[\, e^{tX}]}{e^{ta}}$
  - Hence, Pr (X ≤ a) ≤ $\min_{t<0} \frac{E[\, e^{tX}]}{e^{ta}}$

Bound for L tail as well as R tail

Chernoff (1923~) is an American mathematician Professor at MIT & Harvard

Chernoff Bound for Poisson Trials

Bernoulli trial: Each experiment has the same distribution

- **Poisson trial**
  - A sequence of experiments(trials) each of which has different distribution
  - Let $X_1, X_2, \ldots, X_n$ be a sequence of **independent** Poisson trials with $\Pr(X_i=1) = p_i$
  - $X = X_1 + X_2 + .. + X_n$
  - Let $\mu = E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} p_i$
- Find the bounds of $\Pr(X \geq (1+\delta)\mu)$ and $\Pr(X \leq (1-\delta)\mu)$
- First derive $M_X(t)$
  - MGF of X_i
  - $M_{X_i}(t) = E[\, e^{tX_i}] = p_i \cdot e^t + (1-p_i) = 1 + p_i \cdot (e^t - 1)$

    $\leq e^{p_i \cdot (e^t - 1)}$

  - $M_X(t) = \prod_{i=1}^{n} M_{X_i}(t)$

    $\leq \prod_{i=1}^{n} e^{p_i \cdot (e^t - 1)} \quad = \exp\{\sum_{i=1}^{n} p_i \cdot (e^t - 1)\}$

    $= \exp\{\mu \cdot (e^t - 1)\}$

For any x, $1+x \leq e^x$

Think it as x

Chernoff Bound for Poisson Trials

- Now prove
  1. For any δ > 0, $\Pr(X \geq (1+\delta)\mu) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$
  2. For 0 < δ ≤ 1, $\Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\mu\delta^2}{3}}$
  3. For R ≥ 6μ, $\Pr(X > R) \leq 2^{-R}$

- Proof
  - From Markov's Inequality,

$$\Pr(X \geq (1+\delta)\mu) = \Pr\left(e^{tX} \geq e^{t(1+\delta)\mu}\right)$$
$$\leq \frac{E[e^{tX}]}{e^{t(1+\delta)\mu}}$$
$$\leq \frac{\exp\{(e^{t}-1)\cdot\mu\}}{e^{t(1+\delta)\mu}}$$

  - For any $\delta > 0$, set $t = \ln(1+\delta) > 0$

➔ $\Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$

Find a proper t

Chernoff Bound for Poisson Trials

$$\Pr(X \geq (1+\delta)\mu) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

- Proof of 2 (For $0 < \boldsymbol{\delta} \leq 1$, $\Pr(X \geq (\mathbf{1} + \boldsymbol{\delta})\boldsymbol{\mu}) \leq e^{-\frac{\mu\delta^2}{3}}$)
  - For $0 < \delta \leq 1$, show that $\left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right) \leq e^{-\frac{\delta^2}{3}}$
  - Taking the logarithm to both sides and define $f(\delta)$ as

$$f(\delta) = \delta - (1+\delta)\ln(1+\delta) + \frac{\delta^2}{3}$$

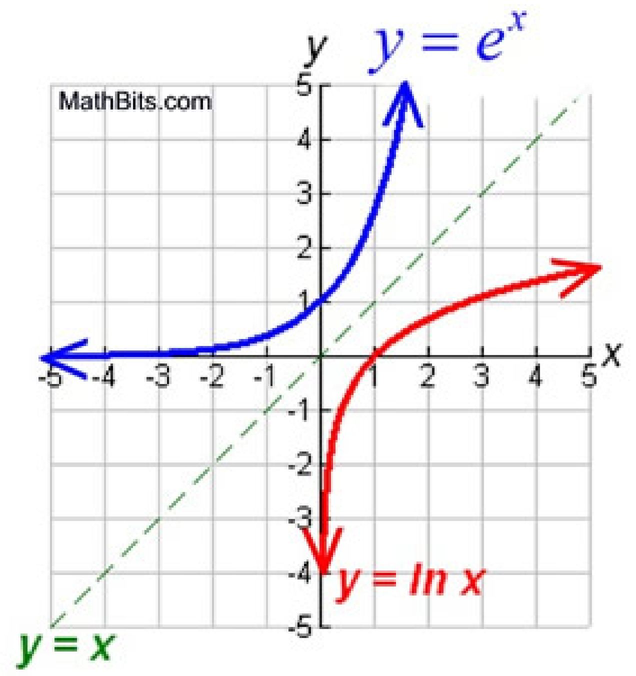
$$f'(\delta) = -\ln(1+\delta) + \frac{2}{3}\delta$$

$$f''(\delta) = -\frac{1}{1+\delta} + \frac{2}{3}$$

$f(0) = 0$
➔ $f(\delta) \leq 0$ for $0 \leq \delta \leq 1$

$f'(0) = 0, f'(1) < 0$
$f'(\delta) \leq 0$ for $0 \leq \delta \leq 1$

$f''(\delta) < 0$ for $0 \leq \delta < 1/2$,
$f''(\delta) > 0$ for $\delta > 1/2$



Chernoff Bound for Poisson Trials

- Proof of 3 (For $R \geq 6\boldsymbol{\mu}$, $\Pr(X > R) \leq \mathbf{2^{-R}}$)
  - $R = (1+\delta)\mu$
  - $R \geq 6\mu \Rightarrow \delta \geq 5$
  - $\Pr(X \geq (1+\delta)\mu) \leq (\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}})^{\mu}$

$$\leq (\frac{e}{1+\delta})^{(1+\delta)\mu}$$

$$\leq (\frac{e}{6})^{R}$$

$$\leq 2^{-R}$$