2018 Engineering Mathematics II Midterm Solutions

Candidate Avg.		21.53
Max		35
Median		22
Min		3

EM2 TAs: Onyu Kang, Jiho Choi

If you have any question regarding your grade, please contact the following TA.

Total grade points **[37 points]**

- surname ㄱ ~ ㅅ: Onyu Kang,

- surname ㅇ ~ ㅎ: Jiho Choi

1. [2, 2, 3, 2 points]

1-A) [2 points] - Onyu Kang

1-B) [2 points] - Onyu Kang

1-C) [3 points] - Onyu Kang

1-D) [2 points] - Onyu Kang

2. [2, 2, 3 points]

2-A) [2 points] - Onyu Kang

2-B) [2 points] - Onyu Kang

2-C) [3 points] - Onyu Kang

3. [2, 3 points]

3-A) [2 points] - Onyu Kang

3-B) [3 points] - Onyu Kang

4. [2, 3, 3 points]

4-A) [2 points] - Jiho Choi

4-B) [3 points] - Jiho Choi

4-C) [3 points] - Jiho Choi

5. [2, 2, 4 points]

5-A) [2 points] - Jiho Choi

5-B) [2 points] - Jiho Choi

5-C) [4 points] - Jiho Choi

1. [2, 2, 3, 2 points]

1-A) [2 points] - Onyu Kang

Since the event $A \cap B_i$ (i=1,…,n) are disjoint and cover the entire sample space Ω, it follows that

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i) \quad \text{---[1 pt]}$$

$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i) = (0+0+0+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}) = \frac{1}{12}$ ---[1 pt]

Scores were specified for each.

1-B) [2 points] - Onyu Kang

$$\sum_y \Pr(Y=y) E[X|Y=y] = \sum_y \Pr(Y=y) \sum_x x Pr(X=x|Y=y)$$

$$= \sum_x \sum_y x Pr\,(X=x \mid Y=y)\ \Pr(Y=y)$$

$$= \sum_x \sum_y x Pr\,(X=x \cap Y=y)$$

$$= \sum_x x Pr\,(X=x) = E[X] \quad \text{---[1 pt]}$$

X$\sim B(n,p)$, $X_k \sim B(k,p)$,

$$E[X_k] = E[X_k|X_1=1] \cdot \Pr(X_1=1) + E[X_k|X_1=0] \cdot \Pr(X_1=0)$$

$$= (1 + \mathrm{E}[X_{k-1}]) \cdot \mathrm{p} + \mathrm{E}[X_{k-1}] \cdot (1-\mathrm{p})$$

$$= \mathrm{p} + \mathrm{E}[X_{k-1}]$$

$$= 2\mathrm{p} + \mathrm{E}[X_{k-2}]$$

$$\cdots$$

$$= (k\text{-}1)\mathrm{p} + \mathrm{E}[X_1]$$

$$= \mathrm{kp} \quad \text{---[1 pt]}$$

Scores were specified for each.

1-C) [3 points] - Onyu Kang

C: conditional event = {HT, TH, HH},

E={HH} ➔ Pr(E|C)=1/3 ---[1 pt]

Distinguish Heads into two types

H' (Heads and Big coin) and H (Heads and small coin)

C: Conditional event = {H'T, TH', H'H', H'H, HH'}, E={H'H', H'H, HH', HH} ➔ C ∩ E = { H'H', H'H, HH'}

Pr(H'T) = 1/100 * 1/2*1/2, Pr(H'H)= Pr(HH') = 1/00*1/2*1/2,

Ignore Pr(H'H') and let it be 0.

➔ Pr(E|C)=1/2 ---[2 pts]

Scores were specified for each.

1-D) [2 points] - Onyu Kang

X={-1,0,1}, Y=X^2

Y={1,0,1}, XY={-1,0,1} = X^3

Pr$(X = 1 \cap Y = 1)$ = $\frac{2}{3}$

Pr(X=1)Pr(Y=1)= $\frac{1}{3} \cdot \frac{2}{3}$ =$\frac{2}{9}$

These X and Y are dependent variables.

E[XY] = $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$

E[X] = $\frac{1}{3}$·(-1)+ $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$

E[Y]= $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$

E[X]E[Y] = 0·$\frac{2}{3}$ = 0

∴ E[XY] = E[X]E[Y]

No partial score. If you have a reasonable answer, you will get full score.

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2. [2, 2, 3 points]

2-A) [2 points] - Onyu Kang

D : has Disorder

TP : Test Positive

TN : Test Negative

$\Pr(D) = 0.001$

$\Pr(\bar{D}) = 0.999$

$\Pr(TP|D) = 0.99$

$\Pr(TP|\bar{D}) = 0.05$

$$\Pr(D|TP) = \frac{\left(Pr(TP|D) * Pr(D)\right)}{Pr(TP)} \quad \text{(Bayes' Theorem)}$$

$$= \frac{\left(Pr(TP|D) * Pr(D)\right)}{Pr(TP|D) * Pr(D) + \Pr(TP|\bar{D}) * \Pr(\bar{D})}$$

$$= \frac{0.99 * 0.001}{0.99 * 0.001 + 0.05 * 0.999}$$

= 99 / 5094

= 0.0194⋯

2-B) [2 points] - Onyu Kang

Let r.v. X = # Heads in m tosses, Y=Sequence number of the picked coin

E[X | Y=i] = m*(i/n)

Using $\sum_y E[X \ |Y = y] \cdot \Pr(Y = y)$, $\mathrm{E[X]} = \sum_i m \cdot \left(\frac{i}{n}\right) \cdot \left(\frac{1}{n}\right) \cong m$

No partial score.

2-C) [3 points] - Onyu Kang

Let C1, C2 be events to select coin C1 and C2, respectively. And E be an event of having k Heads among m tosses. We compute Pr(C1 | E).

$$\Pr(\mathrm{E}|\mathrm{C1}) = \binom{m}{k}(\frac{1}{4})^k \cdot (\frac{3}{4})^{m-k}$$

$$\Pr(\mathrm{E}|\mathrm{C2}) = \binom{m}{k}(\frac{3}{4})^k \cdot (\frac{1}{4})^{m-k}$$

Pr(C1 | E)

$$= \frac{\Pr(\mathrm{E}|\mathrm{C1})\Pr(C1)}{\Pr(E)}$$

$$= \frac{\Pr(\mathrm{E}|\mathrm{C1})\Pr(\mathrm{C1})}{\Pr(\mathrm{E}|C1)\Pr(\mathrm{C1}) + \Pr(\mathrm{E}|C2)\Pr(C2)}$$

$$= \frac{\binom{m}{k}\left(\frac{1}{4}\right)^k \cdot \left(\frac{3}{4}\right)^{m-k} \cdot \frac{1}{2}}{\binom{m}{k}\left(\frac{1}{4}\right)^k \cdot \left(\frac{3}{4}\right)^{m-k} \cdot \frac{1}{2} + \binom{m}{k}\left(\frac{3}{4}\right)^k \cdot \left(\frac{1}{4}\right)^{m-k} \cdot \frac{1}{2}}$$

$$= \frac{3^{m-k}}{3^{m-k}+3^k}$$

No partial score.

3. [2, 3 points]

3-A) [2 points] - Onyu Kang

X~Poi(λ). Consider arrivals of customers as events. Then we can model arrivals as trials with probability of success (female) is p. ➔ Y = # females among n is a Binomial r.v. with parameter n and p.

$$\Pr(Y=m \mid X=n) = \binom{n}{m} \cdot p^m (1-p)^{n-m}$$

No partial score.

3-B) [3 points] - Onyu Kang

$$\Pr(Y=m) = \sum_{n=m}^{\infty} \Pr(Y = m \mid X = n) \cdot \Pr(X=n)$$

$$= \sum_{n=m}^{\infty} \binom{n}{m} \cdot p^m (1-p)^{n-m} \cdot \frac{\lambda^n e^{-\lambda}}{n!}$$

No partial score.

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4. [2, 3, 3 points]

4-A) [2 points] - Jiho Choi

(1 point)

X_i is a Bernoulli [버눌리, 베르누이] random variable.

Since the area of heart is p and the probability that a dart lands on the heart is in proportion to its area, $\Pr(X_i = 1) = p$ and $\Pr(X_i = 0) = 1 - p$

c.f. Bernoulli distribution (of a random variable) is a special case of the binomial distribution where a single trial is conducted to asks a yes-no question; the question results in a Boolean-valued outcome. **Answer with Binomial R.V. will get 0.5 points**

(1 point)

X is a binomial random variable.

$$X = \sum_{i=1}^{n} X_i$$

$$\Pr(X_i = k) = \binom{n}{k} * p^k * (1-p)^{n-k}$$

X ~ B(n, p)

4-B) [3 points] - Jiho Choi

Let the area of the heart equals A.

K~B(n,A)

E(k)=n*A, Var(k)=n*A(1-A)

(2.5 points)

c.f. Chebyshev's Bound: $\Pr(|\mathrm{X}-\mathrm{E}[\mathrm{X}]| \geq a) \leq \frac{Var[X]}{a^2}$

$$\Pr\left(\left|A-\frac{k}{n}\right| \geq \delta\right) = \Pr(|\mathrm{An}-\mathrm{k}| \geq \delta n) = \Pr(|\mathrm{k}-\mathrm{An}| \geq \delta n) \leq \frac{Var(k)}{(\delta n)^2} = \frac{nA(1-A)}{\delta^2 n^2} = \frac{A(1-A)}{\delta^2 n}$$

$$\boldsymbol{\frac{A(1-A)}{\delta^2 n}} \leq \boldsymbol{\frac{1}{4}} * \boldsymbol{\frac{1}{\delta^2 n}} \quad (\because -\mathrm{A}^2+\mathrm{A} = -\left(\mathrm{A}-\frac{1}{2}\right)^2 + \frac{1}{4})$$

(0.5 points)

$$\frac{1}{4} * \frac{1}{\delta^2 n} \leq 0.05$$

$$\frac{1}{4} * \frac{100}{5} * \frac{1}{\delta^2} \leq n$$

$$\therefore \mathrm{n} \geq \frac{5}{\delta^2}$$

c.f. $\boldsymbol{\frac{20\mathrm{A}(1-\mathrm{A})}{\delta^2}} \leq \boldsymbol{n}$, $Answer\ \ without\ \ the\ \ bound\ \ of\ \ A(1-A) \leq \frac{1}{4}\ \ also\ \ gets\ \ 2.5\ \ points$

4-C) [3 points] - Jiho Choi

c.f. Chernoff's Bound: $\Pr(X \geq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}}$

$\Pr(X \geq a) \leq \min_{t>0} \frac{E[e^{tX}]}{e^{ta}}$

➔ $\Pr(|X - \mu| \geq \delta \cdot \mu) \leq 2e^{-\frac{\mu \cdot \delta^2}{3}}$

Refer to Theorem 4.5 & Corollary 4.6

1)

Yes, with the MGF of k, we can apply Chernoff's inequality. (Apply Theorem 4.5)

$\Pr\left(\left|A - \frac{k}{n}\right| \geq \delta\right) = \Pr(|\mathrm{An} - \mathrm{k}| \geq \delta n) = \Pr(|X - \mu| \geq \delta \frac{\mu}{A}),\ \mu = A * n$

$\Pr\left(|X - \mu| \geq \delta \frac{\mu}{A}\right) \leq 2\mathrm{e}^{-\frac{\frac{\mu}{A}\delta^2}{3}}$ (Theorem 4.5 & Corollary 4.6)

2)

$\Pr(|\mathrm{An} - \mathrm{k}| \geq \delta n)$
$= \Pr(\mathrm{X} - \mathrm{nA} \geq \delta n) + \Pr(\mathrm{X} - \mathrm{nA} \leq -\delta n)$
$= \Pr(\mathrm{X} \geq \delta n + nA) + \Pr(\mathrm{X} \leq -\delta n + nA)$
$= \Pr\left(\mathrm{e}^{tX} \geq \mathrm{e}^{t(\delta n + nA)}\right) + \Pr\left(\mathrm{e}^{sX} \geq \mathrm{e}^{s(-\delta n + nA)}\right) \leq \frac{E[e^{tx}]}{e^{t(\delta n + nA)}} + \frac{E[e^{sx}]}{e^{s(-\delta n + nA)}}$ (t>0, s<0)

C.f. Answer "No" with reasonable approach will get some partial credit.

5. [2, 2, 4 points] Jiho Choi

5-A) [2 points] - Jiho Choi

Poisson Approximation

Poisson random variable Yi, Throw m=***2n*** **balls** into ***n*** **bins** $\mu = \frac{2n}{n} = 2$

$\mathrm{Y_i} \sim Poi(\frac{2n}{n})$

$$\Pr(\mathrm{Y_1} = 2, \mathrm{Y_2} = 2, \mathrm{Y_3} = 2, \dots, \mathrm{Y_n} = 2) = \prod_{i=1}^{n} \Pr(\mathrm{Yi} = 2) = \left(\frac{\mathrm{e}^{-2}2^2}{2!}\right)^{\mathrm{n}} = 2^n e^{-2n}$$

$$\Pr(\mathrm{X_1} = 2, \mathrm{X_2} = 2, \mathrm{X_3} = 2, \dots, \mathrm{X_n} = 2) \le \mathrm{e}\sqrt{2\mathrm{n}} * \prod_{i=1}^{n} \Pr(\mathrm{Yi} = 2) = \mathrm{e}\sqrt{2\mathrm{n}} * 2^n e^{-2n}$$

5-B) [2 points] - Jiho Choi

$\Pr(\mathrm{X_1} = 2, \mathrm{X_2} = 2, \mathrm{X_3} = 2, \dots, \mathrm{X_n} = 2) = \binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2} * \dots * \binom{2}{2} * \frac{\mathbf{1}}{\boldsymbol{n^{2n}}}$ (1.5 points)

$= \frac{(2\mathrm{n})!}{(2!)^n}\frac{1}{n^{2n}} = \frac{\frac{(2n)!}{(2!)^n}}{n^{2n}} = \frac{\boldsymbol{(2n)!}}{\boldsymbol{(2n^2)^n}}$ (0.5 points)

5-C) [4 points] - Jiho Choi

$\mathrm{E_n}$: A group consists of bin n~k are all non-empty

$$\Pr(\mathrm{Max.Cluster} \ge \mathrm{k}) = \Pr(\mathrm{E_1} \cup \mathrm{E_2} \cup \dots \cup \mathrm{E_{n-k+1}}) \le \Pr(\mathrm{E_1}) + \Pr(\mathrm{E_2}) + \dots + \Pr(\mathrm{E_{n-k+1}})$$

$$= \Pr(\mathrm{E_1}) + \Pr(\mathrm{E_2}) + \dots + \Pr(\mathrm{E_{n-k+1}}) \le 2 * \text{Probability of the Poisson Case}$$

$$= 2 * (\mathrm{n} - \mathrm{k} + 1) \cdot (1 - \frac{1}{e})^k$$

c.f. Answer with reasonable approach will get some partial credit.