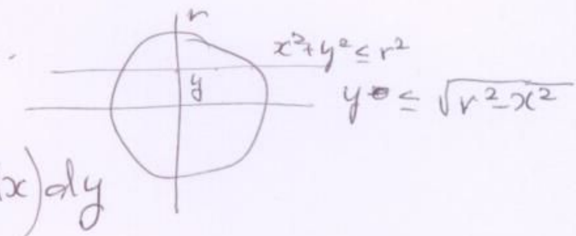
1

a) $f_{X,Y}(x,y) = \frac{1}{\pi r^2}, \quad x^2+y^2 \le r^2$

b) $\int_{-\infty}^{\infty} \frac{1}{\pi r^2}\, dx = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2} dx$

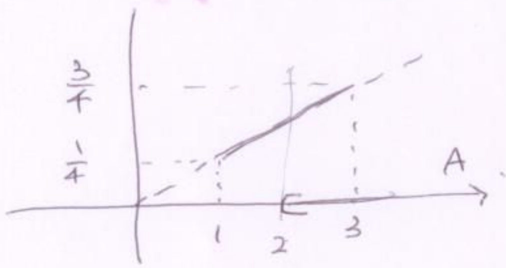


$E[Y] = \int_{-r}^{r} y \cdot \left(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2} dx \right) dy$

c)

$E[Y] = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} y \frac{1}{\pi r^2}\, dx\, dy$

2



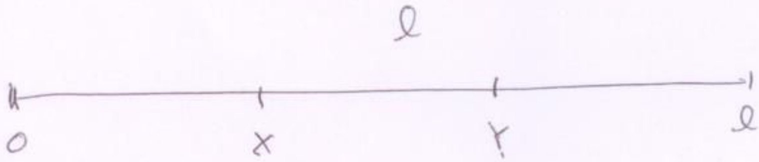
a) $E[X] = \int_1^3 \frac{x}{4} x\, dx, \quad P(A) = \int_2^3 \frac{x}{4} dx = \frac{5}{8}$

$f_{X|A}(x) = \frac{f_{X,X}(x)}{P_r(A)} = \frac{\frac{3}{4}x}{\frac{5}{8}} = \frac{24}{20}x, \quad 2 \le x \le 3$

$E[X|A] = \int_2^3 x \cdot \frac{24}{20} x\, dx$

b) $E[Y] = E[X^2] = \int x^2 \frac{x}{4} dx$

3



a) $Y \sim U[0,l]$ $\quad f_Y(y) = \frac{1}{l}$.

$X \sim U[0,Y]$ $\quad f_{X|Y}(x|y) = \frac{1}{y}$.

$f_{XY} = f_{X|Y} \cdot f_Y = \frac{1}{ly}$

b) $f_X = \int_{x}^{1} f_{XY}\, dy$ 0 ?

c)

d) Note Y and $\frac{X}{Y}$ are independent

$E[X] = E[Y] \cdot E[\frac{X}{Y}] \quad E[Y] = \frac{l}{2}, \quad E[\frac{X}{Y}] = \frac{1}{2}$.

[4] Refer to the class note.

[5] Definition of conditional density function

$$f_{X|Y} = \frac{f_{XY}}{f_Y}$$

Substitute $Y \leftarrow Y, Z$

$$\Rightarrow f_{X|Y,Z} = \frac{f_{XYZ}}{f_{YZ}}$$

[6] a) ~~$Pr(H) = \int_0^1 p \cdot p e^p \, dp =$~~

~~b) Instead of p use θ.~~

a) Instead of p use θ.

$$Pr(H) = \int_0^1 \theta \cdot \theta e^{\theta} \, d\theta = \int_0^1 \theta^2 e^{+\theta} \, d\theta.$$

b) $$f_{\Theta|H}(\theta|H) = \frac{\theta \cdot \theta e^{\theta}}{Pr(H)} = \frac{\theta^2 e^{\theta}}{Pr(H)}$$

c) $$Pr(H_2|H_1) = \frac{Pr(H_2 \cap H_1)}{Pr(H_1)}$$

$Pr(H_1) = Pr(H)$

$$Pr(H_2 \cap H_1) = \int_0^1 \theta^2 \, \theta e^{\theta} \, d\theta = \int_0^1 \theta^3 e^{\theta} \, d\theta.$$

[7] a)

$\boxed{\text{Tips: Think CDF when you deal with continuous R.V.}}$

$$\underset{F_{Z|X}(z|x)}{\underset{\parallel}{Pr(Z \le z | X=x)}} = Pr(X+Y \le z | X=x)$$
$$= Pr(Y \le z-x | X=x)$$
$$= F_{Y|X}(z-x | x)$$
$$= F_Y(z-x)$$

⇐ X, Y are independent

b) $X, Y \sim Exp(\lambda)$.
$Z = X+Y \sim Exp(2\lambda)$

$$f_{X|Z} = \frac{f_{XZ}}{f_Z} = \frac{f_{Z|X} \cdot f_X}{f_Z}$$

c) Same approach.

$\boxed{8}$

a) ~~R.V.~~ Discrete R.V. $\Theta = \begin{cases} 1 & \text{Soo knows the answer} \\ 2 & \text{Soo does not } \text{"} \end{cases}$

R.V. ~~B~~ $P_1 = \begin{cases} 1, & \text{Soo answers correctly} \\ 2, & \text{" ~~ineo~~ incorrectly} \end{cases}$

$$Pr(\Theta=1 \mid P_1=1) = \frac{Pr(P_1=1 \mid \Theta=1)\cdot Pr(\Theta=1)}{Pr(P_1=1)}$$

$$\begin{cases} Pr(P_1=1 \mid \Theta=1) = 1 \\ Pr(P_1=1 \mid \Theta=2) = 1/3 \end{cases}$$

---

b) Let Θ : # of questions that Soo knows
X : # correct answers

~~Pr(~~

$$Pr(\Theta=k \mid X=6) = \frac{Pr(X=6 \mid \Theta=k)\cdot Pr(\Theta=k)}{Pr(X=6)}$$

$k \le 6$

$Pr(\Theta=k) = \binom{10}{k}\left(\frac{1}{2}\right)^k$

$Pr(X=6) \Rightarrow$ enumerate $k\ (0 \le k \le 6)$

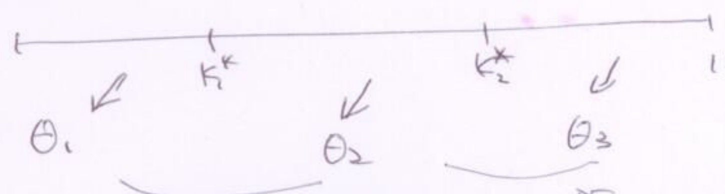
$$Pr(X=6 \mid \Theta=k) = \binom{10-k}{6-k}\left(\frac{1}{3}\right)^{6-k}\cdot\left(\frac{2}{3}\right)^4$$

If she knows k questions, then there are $10-k$ questions that she does not know. She successfully guess $6-k$ questions.

$\boxed{9}$

a) There will be thresholds k_1^*, k_2^* such that $0 \le k_1^* \le k_2^* \le 10$, and

$\boxed{\text{Note: Long calculations}}$



Compare posterior prob. ~~to determine k_1^*~~ $Pr(\Theta=\theta_1 \mid k)$, $Pr(\Theta=\theta_2 \mid k)$ to determine k_1^*

compare post. pro[b.] $Pr(\Theta=\theta_2 \mid k)$ & $Pr(\Theta=\theta_3 \mid k)$ to determine k_2^*.

$$Pr(\Theta=\theta_1 \mid k) = \frac{Pr(k \mid \Theta=\theta_1)\, Pr(\Theta=\theta_1)}{Pr(k)}$$

There are j ~~problems~~ questions among 10 questions that student type θ_1 knows

$$Pr(k \mid \Theta=\theta_1) = \sum_{j=0}^{k} Pr(k, J=j \mid \Theta=\theta_1) = \binom{10}{0}(0.3)^0(0.7)^{10}\cdot\binom{10}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{10-k} +$$

$j \le k$

$$\binom{10}{1}(0.3)^1(0.7)^9\cdot\binom{9}{\ } \left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right)^{10-k}$$

b) $\boxed{\text{Caution : Very long calculation}}$

$$Pr(M=m \mid X=5) = \sum_{i=1}^{3} Pr(M=m \mid X=5, \Theta=i) \cdot Pr(\Theta=i)$$

$$Pr(M=m \mid X=5, \Theta=1) = \sum_{k=0}^{m} \binom{10}{k} (0.3)^k (0.7)^{10-k} \cdot \binom{10-k}{5-k} \left(\frac{1}{3}\right)^{5-k} \left(\frac{2}{3}\right)^{10-k}$$

Type 1. k out 10 questions are known

(5-k) questions are correctly guessed.

Similarly, compute

$Pr(M=m \mid X=5, \Theta=2)$. and $\Theta=3$

k

10-k 5-k

⇒ Compute cases of $m=0, 1, \cdots 5$

and MAP selects one with largest posterior.

LMS computes $E[M \mid X] = \sum_{m=0}^{5} m \cdot Pr(M=m \mid X=5)$

$\boxed{10}$ a) $Pr(\Theta=1)=0.3$ & $Pr(\Theta=2)=0.7$

$X = 20$.

$$Pr(\Theta=1 \mid X=20) = \frac{f_{X|\Theta=1}(20 \mid \Theta=1) \cdot Pr(\Theta=1)}{f_X(x=20)}$$

$$= \frac{\text{''}}{f_{X|\Theta=1}(20 \mid \Theta=1) \cdot P_r(\Theta=1) + f_{X|\Theta=2}(20 \mid \Theta=2) Pr(\Theta=2)}$$

$$= \frac{0.3 \cdot c_1 e^{-0.04(20)}}{0.3 \cdot c_1 \cdot e^{-0.04 \cdot (20)} + 0.7 \cdot c_2 \cdot e^{-0.16 \cdot (20)}}$$

Need to compute c_1 & c_2 also.

$Pr(\Theta=2 \mid X=20) = \cdots$

MAP selects one with larger posterior.

b) $X = (20, 10, 25, 15, 35)$ $\quad X = (X_1, X_2, \cdots X_5)$

Vector

$f_{X|\Theta} = f_{X_1|\Theta} \cdot f_{X_2|\Theta} \cdots f_{X_5|\Theta}$ (independent).

$f_{X|\Theta=1} = c_1 e^{-0.04(20)} \cdot c_1 e^{-0.04(10)} \cdots c_1 \cdot e^{-0.04(35)}$

$= c_1^5 \cdot e^{-0.04(20+10+25+15+35)}$.

b) Conti.

Compute $Pr(\Theta=1 \mid X=(20,10,25,15,35))$ &

$Pr(\Theta=2 \mid X = \quad " \quad)$

and pick one with larger posterior

$\boxed{\text{Error computation}}$

suppose MAP selects $\Theta=1$ (OR $\Theta=2$)

$\Rightarrow$ then compute $Pr(\Theta=2 \mid X=(20,10,\cdots 35))$ OR

$Pr(\Theta=1 \mid X=(20,\cdots 35))$

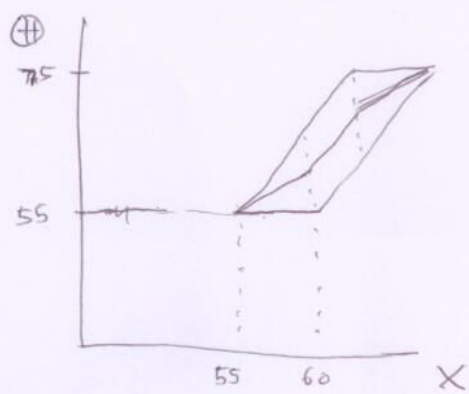
$\boxed{11}$

~~Left~~ Easy & Omitted.

$\boxed{12}$

Real car speed : $\Theta \sim U[55,75]$

$X = \Theta + W$ where $W \sim U[0,5]$



~~Similar~~ Same as the one explained in the class.

$\boxed{13}$

Θ : $Pr(\Theta=i)=\frac{1}{100}$, $1 \le i \le 100$.

~~X : $Pr(X=j)=$~~

X : $Pr(X=j \mid \Theta=i)=\frac{1}{i}$ $\quad 1 \le j \le i$.

$$Pr(\Theta=i \mid X=j)=\frac{Pr(X=j \mid \Theta=i)\cdot Pr(\Theta=i)}{Pr(X=j)}=\cdots$$

$\boxed{14}$

a) ~~$E[\ ] = E[Y/Y] = \sum_{i=1}^{n} E[\frac{Y_i}{Y}] = n\cdot E[\frac{Y_i}{Y}] \Rightarrow E[\frac{Y_i}{Y}] = \frac{1}{n}$~~

b) ~~$\Theta \sim N(0,k)$, $W \sim N(0,m)$~~

~~$\Theta + W \sim N(0, k+m)$~~

14

a)

$$E[Y_i \mid Y] = E[Y_j \mid Y].$$

$$E[(Y_1 + \cdots + Y_n) \mid Y] = n \cdot E[Y_i \mid Y]$$

$$\shortparallel$$

$$E[Y \mid Y]$$

$$\shortparallel$$

$$Y \quad \Rightarrow \quad E[Y_i \mid Y] = \frac{Y}{n}$$

b) $H \sim N(0, k)$

$W \sim N(0, \cancel{k}m)$

Let S_i be a standard normal $\Rightarrow S_i \sim N(0,1)$

$H = S_1 + S_2 + \cdots + S_k$

$W = \cancel{S_1 + S_2 + \cdots + S_m}$

$S_{k+1} + S_{k+2} + \cdots + S_{k+m}$

From a) $\quad E[S_i \mid H + W] = \dfrac{H + W}{k+m}$

$$E[H \mid H + W] = E[S_1 + S_2 \cdots + S_k \mid H + W]$$

$$= \frac{k(H + W)}{k+m}$$

c)

$H \sim Poi(\lambda)$

Let $S_i \sim Poi(1) \Rightarrow H = S_1 + \cdots + S_\lambda$

integer

Similar approach as b)