2018 Engineering Mathematics II Quiz 03 Solutions

Candidate Avg.		8.19
Max		17.5
Median		8.0
Min		0.0

EM2 TAs: Onyu Kang, Jiho Choi

If you have any question regarding your grade, please contact the following TA.

Total grade points [22 points]

- surname ㄱ ~ ㅅ: Onyu Kang,

- surname ㅇ ~ ㅎ: Jiho Choi

1. [2, 2, 2, 2, 2 points]

1-A) [2 points] – Jiho Choi

1-B) [2 points] – Jiho Choi

1-C) [2 points] – Jiho Choi

1-D) [2 points] – Jiho Choi

1-E) [2 points] – Jiho Choi

2. [3, 3, 3, 3 points]

2-A) [3 points] - Onyu Kang

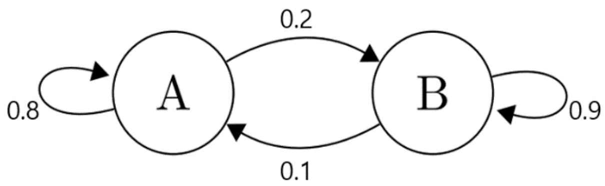
2-B) [3 points] - Onyu Kang

2-C) [3 points] - Onyu Kang

2-D) [3 points] - Onyu Kang

1. [2, 2, 2, 2, 2 points]

1-A) [2 points] – Jiho Choi



Using Markov chains

$(500, 500)\begin{bmatrix}0.8 & 0.2\\0.1 & 0.9\end{bmatrix} = (\mathbf{450}, \mathbf{550})$

Cf.) Other solutions with the correct computation will also get full points.

Population(A): 500 - 100 + 50

Population(B): 500 - 50 + 100

Or $\begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix}\binom{500}{500} = (450, 550)$

1-B) [2 points] – Jiho Choi

$P = \begin{bmatrix}0.8 & 0.2\\0.1 & 0.9\end{bmatrix}$

$0.8 * \pi_A + 0.1 * \pi_B = \pi_A$

$0.2 * \pi_A + 0.9 * \pi_B = \pi_B$

$2 * \pi_A = \pi_B, \pi_A + \pi_B = 1$

$\therefore \pi_A = \frac{1}{3}, \pi_B = \frac{2}{3}$

Region A: 333, Region B: 667

Cf.)

Other solutions such as $\boldsymbol{\pi * P^{50}}$ without the approximation will not get any point.

$\boldsymbol{\pi * P^{50}}$ with a reasonable approximation may get 0.5 points.

1-C) [2 points] – Jiho Choi

$$\mathrm{h}_{0,0} = 0.5 * 1 + 0.5 * 3 = 2$$

$$\mathrm{h}_{1,1} = 1.0 * 2 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2 + \frac{1}{1-\frac{1}{2}} = 4$$

$$\mathrm{h}_{2,2} = 1.0 * 1 + \frac{1}{1-\frac{1}{2}} + 1.0 * 1 = 4$$

$$\mathbf{h_{0,0} = 2, h_{1,1} = 4, h_{2,2} = 4}$$

Cf.) Having two correct values will get 1 point.

1-D) [2 points] – Jiho Choi

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\pi_0 = 0.5 * \pi_0 + \pi_2$$

$$\pi_1 = 0.5 * \pi_0$$

$$\pi_2 = \pi_1$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\therefore \pi_0 = \frac{1}{2}, \pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{4}$$

Cf.) Using expected # of transitions to revisit, $\mathrm{h}_{i,\mathrm{i}} = \frac{1}{\pi_\mathrm{i}}$ will also get full points.

1-E) [2 points] – Jiho Choi

$$\pi_B = \pi_C = \frac{1}{2}\pi_A$$
$$\pi_D = \pi_E = \frac{1}{2}\pi_B = \frac{1}{4}\pi_A$$
$$\pi_F = \pi_G = \frac{1}{2}\pi_C = \frac{1}{4}\pi_A$$

$$\pi_H = \frac{1}{2}\pi_D + \frac{1}{2}\pi_E = \frac{1}{4}\pi_A$$

$$\left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right)\pi_A = 1$$

$$\boldsymbol{\pi_A = \frac{4}{13}, \pi_B = \frac{2}{13}, \pi_H = \frac{1}{13}}$$

2. [3, 3, 3, 3 points]

2-A) [3 points] - Onyu Kang

$F_X(X) = Pr(0 \leq X \leq \frac{t}{2}) = \frac{2X}{t}$

$F_\theta(\theta) = Pr(0 \leq \theta \leq \frac{\pi}{2}) = \frac{2\theta}{\pi}$

1.5 pts for each

2-B) [3 points] - Onyu Kang

F(X,θ) = $F_X(X) \cdot F_\theta(\theta) = \frac{2X}{t} \cdot \frac{2\theta}{\pi} = \frac{4X\theta}{t\pi}$ $(0 \leq X \leq \frac{t}{2}, 0 \leq \theta \leq \frac{\pi}{2})$

E[X· θ] = $\int_0^{\frac{t}{2}} \int_0^{\frac{\pi}{2}} \frac{4}{t\pi} \cdot x\theta d\theta dx = \frac{4}{t\pi} \int_0^{\frac{t}{2}} \left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{2}} xdx = \frac{\pi}{2t} \int_0^{\frac{t}{2}} xdx = \frac{\pi}{2t} \left[\frac{x^2}{2}\right]_0^{\frac{t}{2}} = \frac{\pi t}{16}$

1.5 pts for each

2-C) [3 points] - Onyu Kang

Let d be the horizontal distance between the tip of the needle and the center of the needle. The distance between the center of the needle and the tip of the needle is $\frac{\ell}{2}$.

$$d = \frac{\ell}{2} \cos\theta$$

Therefore, if X, the distance between the center of the needle and the vertical line, is less than d, the needle crosses the vertical line.

Such probability is

Pr$(X \leq \frac{\ell}{2}\cos\theta) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\ell}{2}\cos\theta} f(x,\theta) \; d\theta dX = \frac{4}{t\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\ell}{2}\cos\theta} \; d\theta dx$

$$= \frac{4}{t\pi} \int_0^{\frac{\pi}{2}} \frac{\ell}{2} \cos\theta \, d\theta = \frac{2\ell}{t\pi} \; [\sin\theta]_0^{\frac{\pi}{2}} = \frac{2\ell}{t\pi}$$

2-D) [3 points] - Onyu Kang

The probability of a needle crossing a line is

$$\frac{2\ell}{t\pi}$$

This means that if you drop lots of needle randomly and count how many cross the parallel lines, you can calculate what π is by rearranging the formula:

$$\pi \approx \frac{2\ell n}{tk}$$

k ~ B (n, $\frac{2\ell}{t\pi}$)

For error probability γ and targeted accuracy δ,

Pr($|k - E[k]| \geq n\delta$) $\leq \frac{var[k]}{n^2\delta^2} = \frac{n \cdot \frac{2\ell}{t\pi}\left(1- \frac{2\ell}{t\pi}\right)}{n^2\delta^2} = \frac{\frac{2\ell}{t\pi}\left(1- \frac{2\ell}{t\pi}\right)}{n \cdot \delta^2}$ by chebyshev inequality.

Pr($\left|\frac{k}{n} - \frac{2\ell}{t\pi}\right| \geq \delta$) $\leq \frac{\frac{2\ell}{t\pi}\left(1- \frac{2\ell}{t\pi}\right)}{n^2\delta^2} = \gamma$
