Continuous Distributions and Poisson Process

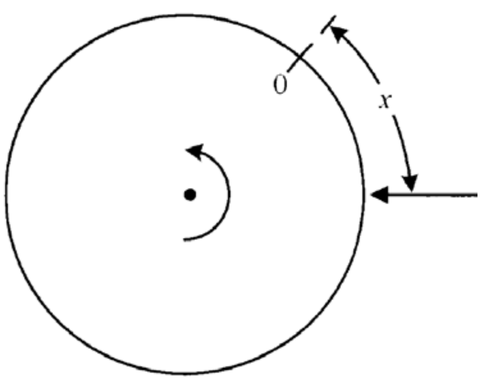
Name: Chong-kwon Kim

Seoul National University

SCONE Lab.

Discrete & Continuous R.V.

- **Discrete** R.V.
  - Defined over countable events
  - Each event may have a probability mass
- **Continuous** R.V.
  - Defined over uncountable continuous space
  - A point cannot have a probability mass
  - Axioms of probability hold
- Example
  - Roulette wheel with circumference of 1
  - Distance from 0 to the arrow
  - Sample space Ω: real number in a range [0,1)
  - Points cannot have a probability
    - For any $x \in [0, 1)$, $Pr(X=x) = 0$



Distribution & Density Functions

- Continuous **distribution function**
  - F(x) = Pr(X ≤ x)
- **Density function**
  - f(x) = F'(x)

  $F(x) = \int_{-\infty}^{x} f(t)dt$

  $Pr(a \le X \le b) = \int_{a}^{b} f(t)dt$

  $Pr(x < X \le x + dx) = F(x + dx) - F(x) \approx f(x)dx$

- Example:
  - Roulette

    F(x) = Pr(X < x) = x, for 0 ≤ x < 1

    f(x) = F'(x) = 1, for 0 ≤ x < 1

    $Pr(a \le X \le b) = \int_{a}^{b} f(t)dt$ = (b−a), for 0 ≤ a ≤ b < 1

Moments, etc

SCONE Lab.

- Mean, Variance, ⋯

  $\mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx$

  $\mathrm{E}[X^k] = \int_{-\infty}^{\infty} x^k \cdot f(x)dx$

  $\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - E[X])^2 \cdot f(x)dx$

  $= \int_{-\infty}^{\infty} x^2 \cdot f(x)dx - E[X]^2$

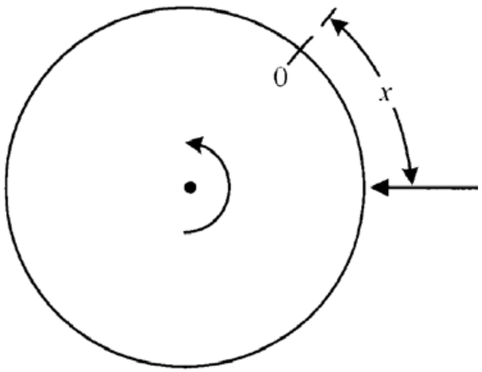
  $M_X(t) = \mathrm{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx$

- Example:
  - Roulette where f(x) = 1 for 0 ≤ x < 1

    $\mathrm{E}[X] = \int_0^1 x \cdot f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$

    $\mathrm{Var}[X] = \int_0^1 x^2\,dx - (\frac{1}{2})^2 = ?$

    $M_X(t) = \mathrm{E}[e^{tX}] = \int_0^1 e^{tx} \cdot x\,dx$



Expectation

- Lemma
  - Let X be a non-negative continuous r.v.
  - E[X]=$\int_{x=0}^{\infty} \Pr(X \geq a)\, dx$

- Proof
  - $\int_{x=0}^{\infty} \Pr(X \geq a)\, dx = \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(y) dy dx$

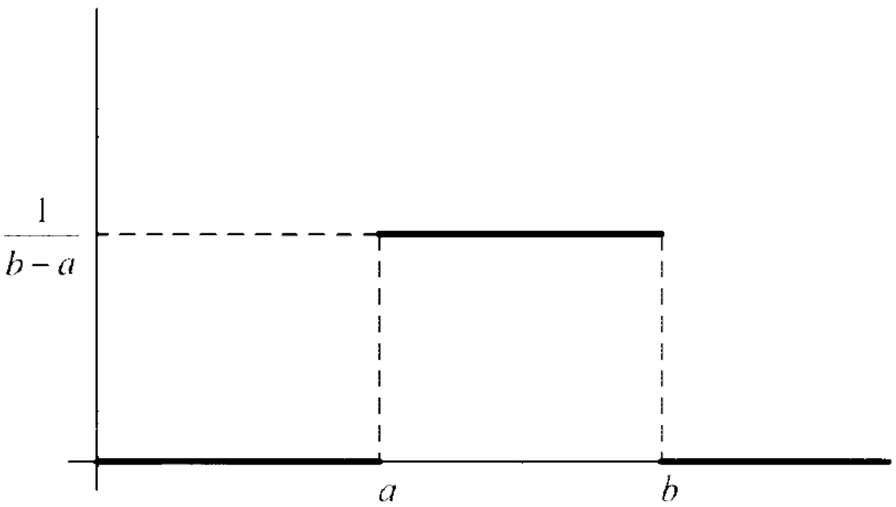
$$= \int_{y=0}^{\infty} \int_{x=0}^{y} f(y) dx dy$$

$$= \int_{y=0}^{\infty} y f(y) dy$$

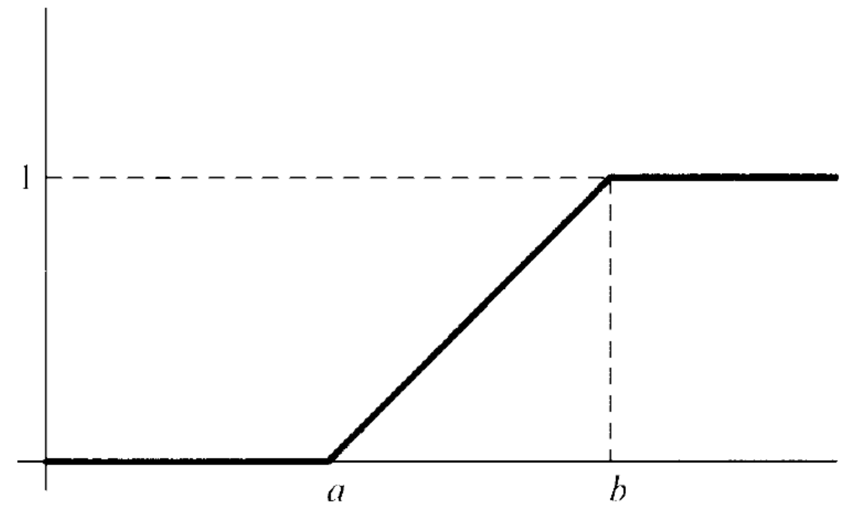
= E[X]

Uniform Distribution

- U[a, b]: Uniform distribution over an interval [a, b]
  - $\Pr(c \le X \le c+\delta) = \Pr(d \le X \le d+\delta)$, for any proper c, d and δ



(a) $\boxed{f(x) = \frac{1}{b-a},\ a \le x \le b}$



(b) $\boxed{F(x) = \frac{x-a}{b-a},\ a \le x \le b.}$

- Mean, variance

  E[X] = $\int_a^b x \cdot \frac{1}{b-a} dx = (a+b)/2$

  Var[X] = ??

Uniform Distribution

- Lemma:

  Let $X_i \sim U[0,1], for\ i = 1,2, \ldots, n$

  Y_k is the k-th largest among $X_{1,} X_{2,} \ldots, X_n$

  Then E[Y_k] = k/(n+1)

- Proof:
  - Exercise 5
  - Prove that E[Y1] = 1/(n+1)
    - Y1=min{X1, X2, ⋯, Xn}
    - Y1 is less than a ←→ ?
    - Y1 is greater than a ←→ Each X_i is greater than a
    - Pr(Y1 > a) = $\prod_{i=1}^{n} \Pr(X_i > a)$

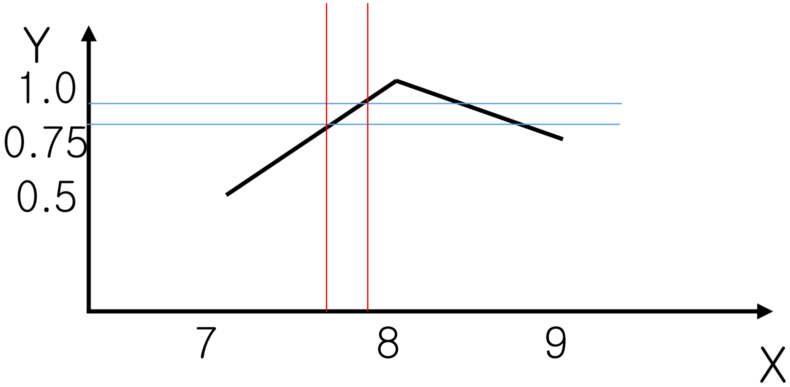
      = $(1-a)^n$

    → F(y) = 1−$(1-y)^n$

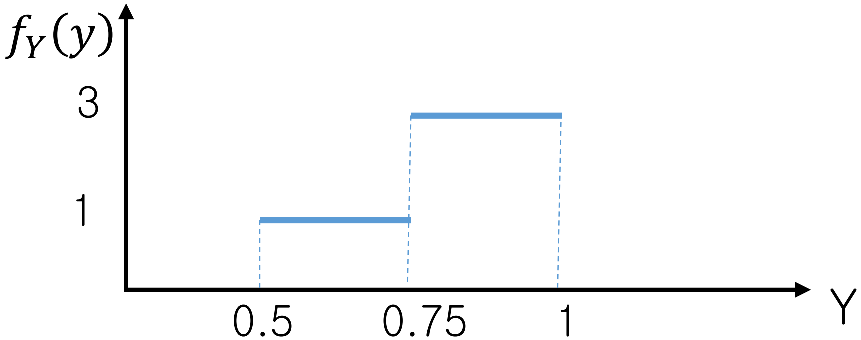
    → f(y) = n$(1-y)^{n-1}$

Example

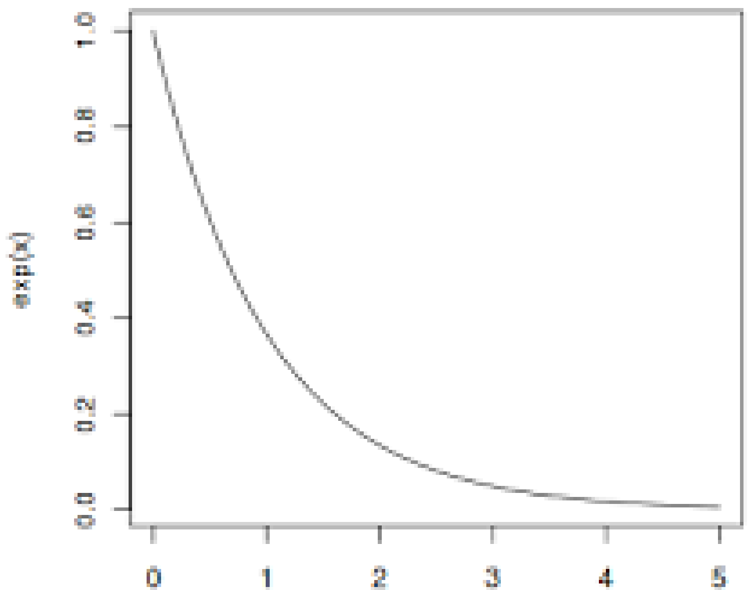
- Soo leaves home uniformly at random between 7:00 AM and 9:00 AM. Driving time to her work varies as follows.
  - Let X and Y be departure time and driving time, respectively
  - Y increases from 0.5 hr to 1.0 hr linearly as X increases from 7 to 8
  - Then, Y decreases from 1 to 0.75 linearly as X increases from 8 to 9



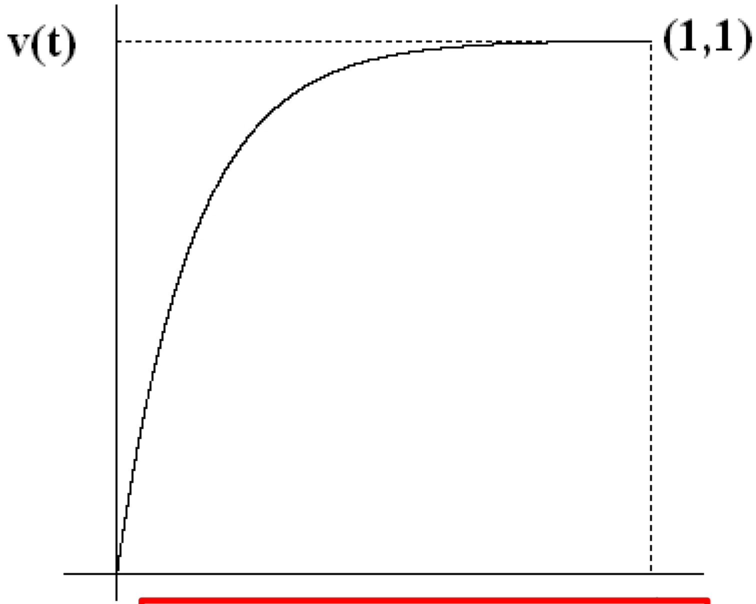
  - Find the pdf of Y
    - X~U[7,9]
    - $\Pr(y \le Y \le y+\delta) = (2\delta + 4\delta)/2 = 3\delta,\ 0.75 \le y \le 1.0$
    - $\Pr(y \le Y \le y+\delta) = (2\delta)/2 = \delta,\ 0.5 \le y \le 0.75$
    - $f_Y(y) = \begin{cases} 3, & 0.75 \le y \le 1.0 \\ 1, & 0.5 \le y \le 0.75 \\ 0, & ow \end{cases}$



Exponential Distribution



$f(x) = \theta e^{-\theta x}, x \geq 0$



$\mathrm{F}(x) = 1 - e^{-\theta x}, x \geq 0$

Random Variable X ~ Exp(θ)

E[X] = $\int_0^\infty x \cdot \theta e^{-\theta x}\, \mathrm{dx} = 1/\theta$

Var[X] = ??

Exponential Distribution

- **Memoryless property**

  Let X ~ Exp(θ)

  Then $\Pr(X > s + t \mid X > t) = \Pr(X > s)$

Memoryless Discrete r.v.
Geometric distribution

- Proof:
  - $$\Pr(X > s + t \mid X > t) = \frac{\Pr(X>s+t)}{\Pr(X>t)}$$
    $$= \frac{1-\Pr(X<s+t)}{1-\Pr(X<t)}$$
    $$= \frac{e^{-\theta(t+s)}}{e^{-\theta t}}$$
    $$= e^{-\theta s} = \Pr(X > s)$$

- Example:
  - Suppose you are waiting for a bus whose inter-arrival time follows Exp(0.1). Given that you already waited for five minutes, how long should you wait until the next bus on the average?

Exponential Distribution

- Lemma:

  Let $X_i \sim Exp(\theta_i), for\ i = 1,2,\dots,n,$ are independent

  And let Y = min($X_{1,} X_{2,} \dots, X_n$)

  ➔Then Y~ $Exp(\sum_{i=1}^{n} \theta_i)$ and Pr(Y=X_i) = $\frac{\theta_i}{\sum_{i=1}^{n} \theta_i}$

- Proof:
  - Prove for n=2
  - X1 and X2 are independent

    ➔ f(x1, x2) = $\theta_1 e^{\theta_1 x_1}\ \theta_2 e^{\theta_2 x_2}$
  - Pr(min(X1, X2) > x) = Pr(X1 > x)·Pr(X2 > x)

    $= e^{-\theta_1 x} \cdot e^{-\theta_2 x}$

  - Pr(X1 < X2) = $\int_{x2=0}^{\infty} \int_{x1=0}^{x2} f(x1, x2) dx1, dx2$

Normal Random Variable

- X ~ N(μ, σ^2): Normal distribution w/ mean = μ and variance = σ^2
- $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- Standard normal random variable
  - Normal random variable with mean = 0 and variance = 1
- Standardize
  - Suppose X ~ N(μ, σ^2), then Y = $\frac{1}{\sigma} \cdot (X - \mu)$ is a normal r.v. with mean = 0 and variance = 1
  - Y ~ N(0, 1)

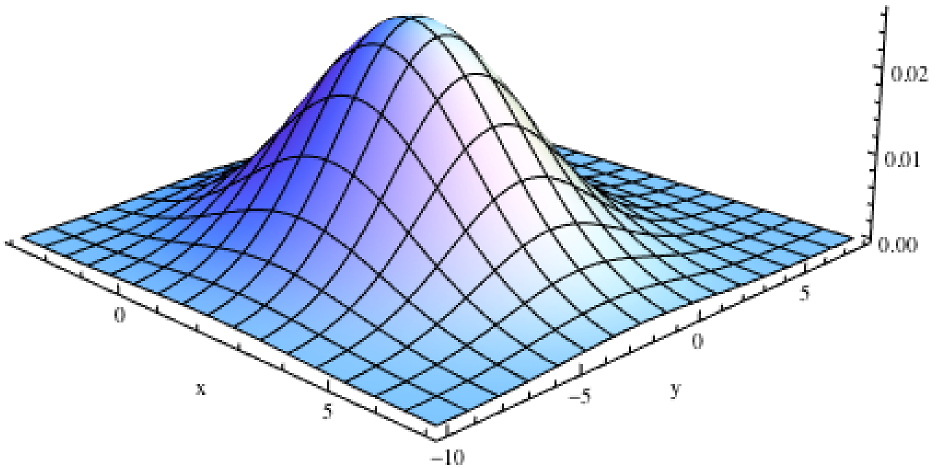
Joint & Marginal Dist.

- **Joint Distribution** of two continuous r.v. X and Y

  $F(x, y) = \Pr((X < x) \cap (Y < y))$

  $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$

  $F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v)\, du\, dv$



- **Marginal distribution/density** functions

  $F_X(x) = F(x, \infty) = \Pr((X < x) \cap (Y < \infty))$

  $f_X(x) = \frac{\partial}{\partial x} F_X(x)$

  $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

Notation: May use random variables as subscript such as $F_{X,Y}(x, y), f_X(x)$, to distinguish Joint & Marginal distributions

Joint & Marginal Dist.

- Example:
  - Let a, b are positives and let $F(x,y) = 1 - e^{-ax} - e^{-by} + e^{-(ax+by)}$, for x, y $\geq$ 0

    $f(x,y) = ab \cdot e^{-(ax+by)}$

    $F_X(x) = \mathrm{F}(\mathrm{x}, \infty) = 1 - e^{-ax}$

    OR, $F_X(x) = \int_0^x \int_0^\infty ab \cdot e^{-(au+bv)} du\, dv$

Independence

- Independence

  X and Y are independent if for all x and y,

  Pr((X < x) ∩ (Y < y)) = Pr(X < x) · Pr(Y < y)

  ➔ $F(x, y) = F_X(x) \cdot F_Y(y)$

  ➔ $\frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{\partial^2}{\partial x \partial y} F_X(x) \cdot F_Y(y) = f_X(x) \cdot f_Y(y)$

- Example

  - $F(x, y) = 1 - e^{-ax} - e^{-by} + e^{-(ax+by)}$, for x, y ≥ 0

    Are X, Y independent?

    Check if $F(x, y) = F_X(x) \cdot F_Y(y)$ (OR $f(x, y) = f_X(x) \cdot f_Y(y)$)

Conditional Distribution

- Note $\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}$

In case of discrete RV, Pr(F) > 0
How about continuous case?

- $\Pr(X \le 3 \mid Y=4) = \frac{\Pr((X \le 3) \cap (Y=4))}{\Pr(Y=4)}$

Both denominator and numerator are zero ➔ Take the limit

- $\Pr(X \le 3 \mid Y=4) = \lim_{\delta \to 0} \Pr(X \le 3 \mid 4 \le Y \le 4 + \delta)$

$$= \int_{u=-\infty}^{3} \frac{f(u,4)}{f_Y(4)} du$$

- Conditional density function: $f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

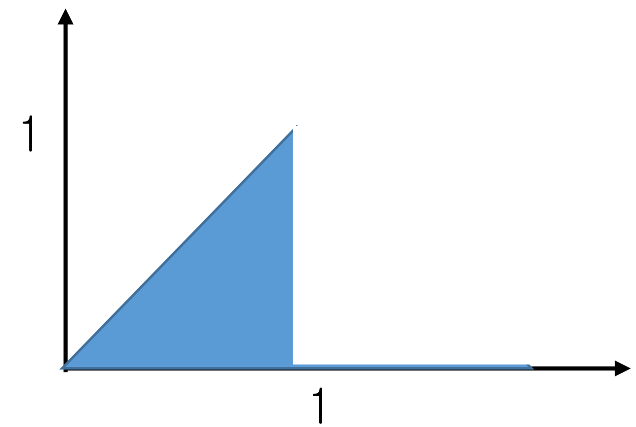
➔ $f_{X|Y}(x,y) \cdot f_Y(y) = f_{X,Y}(x,y) = f_{Y|X}(x,y) \cdot f_X(y)$

➔ $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{-\infty}^{\infty} f_{X|Y}(x,y) \cdot f_Y(y) dy$

Conditional Distribution

$$\boxed{\Pr(X \le x \mid Y=y) = \int_{u=-\infty}^{x} \frac{f(u,y)}{f_Y(y)} du}$$

- Example:
  - $F(x,y) = 1 - e^{-ax} - e^{-by} + e^{-(ax+by)}$
    - $\Pr(X \le 3 \mid Y=4) = \int_{u=-\infty}^{3} \frac{f(u,4)}{f_Y(4)} du = \int_{u=-\infty}^{3} \frac{abe^{-(au+4b)}}{be^{-4b}} du = 1 - e^{-3a}$
  - $f(x,y) = \begin{cases} c, & if\ 0 < x < 1, 0 < y < 1, y \le x \\ 0, & ow \end{cases}$
    - From $\int_0^1 \int_0^x c\, dydx = 1$ ➔ c=2
    - $f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
    - $f_Y(y) = \int_y^1 2\, dx = 2(1-y),\ \ for\ 0 \le y \le 1$
    - $f_{X|Y}(x,y) = \frac{2}{2(1-y)} = \frac{1}{(1-y)}$
    - $f_X(x) = ?$

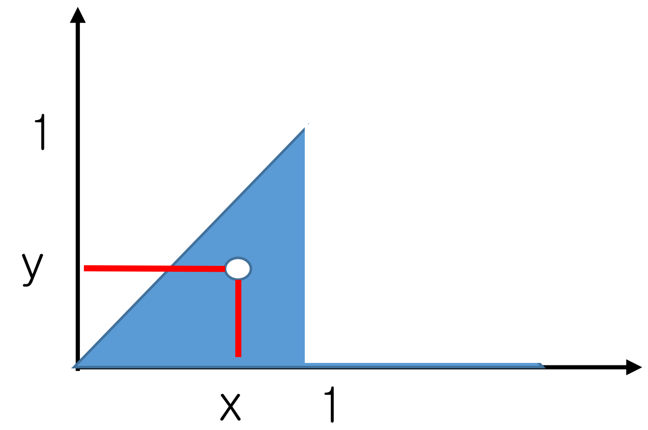


Conditional Distribution

- For $0 < x < 1, 0 < y < 1, x \le y$
- $F_{X,Y}(x,y) = \Pr(X \le x, Y \le y)$

$$= 2\left(\mathrm{y(x-y)} + \left(\frac{1}{2}\right) y^2\right)$$

$$= 2xy - y^2$$



- $F_X(x) = F_{X,Y}(x,\infty) = F(x,x)$

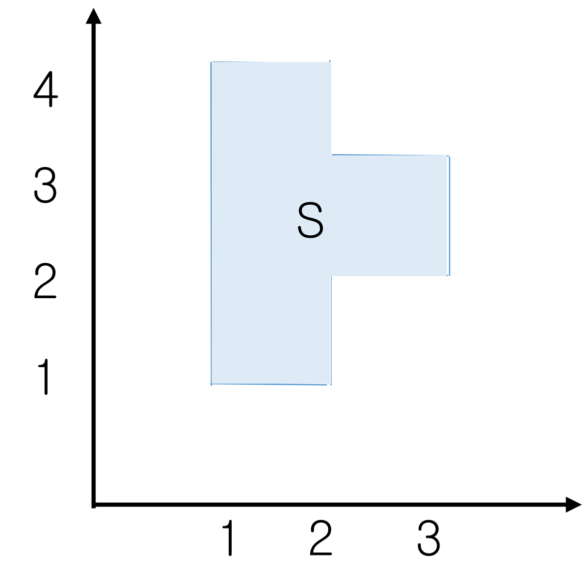
$$= \Pr(X \le x, Y \le x) = x^2$$

➔ $f_X(x) = 2x$

- $F_Y(y) =?$

Examples

- $f_{X,Y}(x,y)$ is uniform over an area S
  - Let $f_{X,Y}(x,y) = \mathrm{c}, \ \ \text{for } (\mathrm{x},\mathrm{y}) \in S$
  - Since $\int\int_{(x,y)\in S} f_{X,Y}(x,y) = 1$ ➔ c=1/4
  - $f_X(x) = \int_{-\infty}^{\infty} \frac{1}{4} dy = \begin{cases} \frac{1}{4}, & 2 \le x \le 3 \\ \frac{3}{4}, & 1 \le x \le 2 \\ 0, & ow \end{cases}$
  - $f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{4} dx = ?$
  - $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} 1, & 3 \le y \le 4 \\ \frac{1}{2}, & 2 \le x \le 3 \\ 1, & 1 \le x \le 2 \end{cases}$



$f_{Y|X=x}(y) = ?$

Conditional Expectation

- $E[X|Y=y]=\int_{-\infty}^{\infty} x \cdot f_{X|Y}(x)\, dx$

- Claim: $E[X]=\int_{-\infty}^{\infty} E[X\,|Y = y] \cdot f_Y(y)\, dy$
- Proof
  - $E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$$= \int_{-\infty}^{\infty} x \cdot \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \int_{-\infty}^{\infty} f_{X|Y}(x|y) \cdot f_Y(y) dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx\, f_Y(y) dy$$

$$= \int_{-\infty}^{\infty} E[X\,|Y = y] \cdot f_Y(y)\, dy$$

Bayes' Theorem

- Prior
  - Discrete / Continuous
- Likelihood(evidence, observation)
  - Discrete / Continuous
- Example of cases
  - DD: Find biased coin
    - Prior: Two coins, one fair and another biased(Pr(H)=2/3)
    - Likelihood: coin toss is H
  - DC: Signal (Radar) detection
    - Prior: Aircraft or Bird
    - Likelihood: Signal strength distributed as a Normal dist.
  - CD: Determine the Pr(H) ~ U[1/3, 2/3] of a biased coin
    - Prior: X ~ U[1/3, 2/3]
    - Likelihood: coin toss is H
  - CC: Engine lifetime (X), Time until an engine dies
    - X~Exp(x)
    - X~U[a, b]

Posterior

Likelihood

Prior

$$\underbrace{\Pr(F \mid E)}_{\text{Posterior}} = \frac{\overbrace{Pr(E \mid F)}^{\text{Likelihood}}\overbrace{\Pr(F)}^{\text{Prior}}}{\Pr(E)}$$

CC Case Example

- Pareuse engine lifetime
  - The engines are known to have an exponentially distributed lifetime, Y~Exp(x) ➔ $f_{Y|X}(y|\mathrm{x}) = x \cdot e^{-x \cdot y}$
  - X~U[10, 20] ➔ $f_X(\mathrm{x}) = \frac{1}{10}, \text{for } 10 \le x \le 20$
  - Suppose an engine dies at 18 (y = 18)
  - Compute $f_{X|Y}(x|\mathrm{y})$
    - $f_{X|Y}(x|\mathrm{y}) = \frac{f_{Y|X}(y|\mathrm{x}) \cdot f_X(\mathrm{x})}{f_Y(y)}$

$$= \frac{f_{Y|X}(y|\mathrm{x}) \cdot f_X(\mathrm{x})}{\int_{-\infty}^{\infty} f_{Y|X}(y|\mathrm{x}) \cdot f_X(\mathrm{x})\, dx}$$

$$= \frac{\left(\frac{1}{10}\right) x \cdot e^{-18x}}{\int_{10}^{20} \left(\frac{1}{10}\right) x \cdot e^{-18x} dx}$$

DC Case

- There are n possible causes
  - Pr(X=i) = p_i
- Observations Y~N(μ, σ^2)
  - $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
  - $f_{Y|X=i}(y|i) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu_i)^2}{2\sigma^2}}$
  - Bayes' Theorem
    - Pr(X=i | Y=y) $\approx \Pr(X = i | \, y \le Y \le y + \delta)$

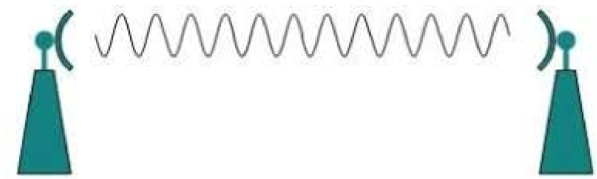
$$= \frac{\Pr(X=i)\cdot\Pr(y \le Y \le y+\delta \, | X=i)}{\Pr(y \le Y \le y+\delta)}$$

$$= \frac{\Pr(X=i) f_{Y|X=i}(y|x)\delta}{f_Y(y)\delta}$$

$$= \frac{\Pr(X=i) f_{Y|X=i}(y|x)}{f_Y(y)}$$

$$= \frac{\Pr(X=i) f_{Y|X=i}(y|x)}{\sum_i \Pr(X=i) f_{Y|X=i}(y)}$$

DC Case Example



- Signal transmission
  - A binary signal S (either ± 1) is transmitted
    - Pr(S=1) = p, Pr(S=−1) = (1−p)
  - The signal is attenuated and the received signal is Y=S+W where W~N(0, σ^2)

    ➔ Y ~ N($\pm$1, σ^2)
  - Assuming that Y=y, compute Pr(S=1|Y=y)

    - Pr(S=1 | Y=y) = $\frac{\Pr(S=1) f_{Y|S}(y|S=1)}{f_Y(y)}$

$$= \frac{\Pr(S=1) f_{Y|S}(y|S=1)}{\Pr(S=1) f_{Y|S}(y|S=1) + \Pr(S=-1) f_{Y|S}(y|S=-1)}$$

$$= \frac{\frac{p}{\sqrt{2\pi}\sigma} e^{-\frac{(y-1)^2}{2\sigma^2}}}{\frac{p}{\sqrt{2\pi}\sigma} e^{-\frac{(y-1)^2}{2\sigma^2}} + \frac{(1-p)}{\sqrt{2\pi}\sigma} e^{-\frac{(y+1)^2}{2\sigma^2}}} =$$

CD Case Example

- Estimating the probability of Heads

  - Random variable X
    - A coin is known to have prob. of Heads ~ U[1/3, 2/3]

    ➔ $f_X(x) = 3,\ 1/3 \le x \le 2/3$

  - Flip the coin two times and the result is HH
    - $f_{X|HH}(x|HH) = \frac{f_X(x)\Pr(HH|X=x)}{\Pr(HH)}$

      $= \frac{f_X(x)\Pr(HH|X=x)}{\int_{1/3}^{2/3} f_X(x)\Pr(HH|X=x)\,dx}$

      $=$

Poisson Process

- Definition: Stochastic counting process
  - $\{N(t);\ t \geq 0\}$ is stochastic counting process

    ① Non-negative: $N(t) \geq 0$

    ② Integer

    ③ Increasing: $N(t) \geq N(s)$ for all $t \geq s \geq 0$

- Definition: Poisson Process
  - A Poisson Process with rate λ, $\{N(t);\ t \geq 0\}$, has the following properties

    ① Stationary: For any s, t > 0, $\Pr(N(t+s) - N(s) = a) = \Pr(N(t) = a)$

    ② Independent: For any disjoint intervals, $[t_1, t_2]\ and\ [t_3, t_4]$, the distribution of $N(t_2) - N(t_1)$ is independent of the $N(t_4) - N(t_3)$

    ③ $\lim_{h \to 0} \frac{\Pr(N(s+h) - N(s) = 1)}{h} = \lambda$

    ➔Prob. an event occurs in a short interval t is approximately $\lambda \cdot t$

    ④ $\lim_{h \to 0} \frac{\Pr(N(s+h) - N(s) \geq 2)}{h} = 0$

Poisson Process

- Theorem:
  - Let {N(t); t ≥ 0} be a Poisson Process with rate λ

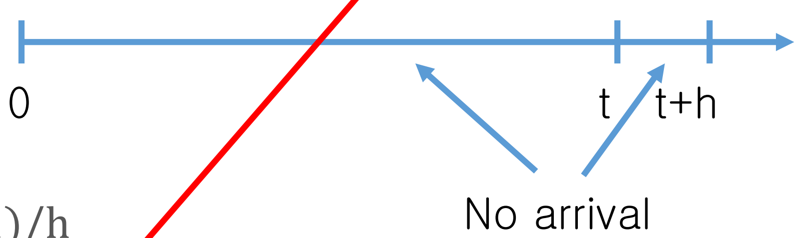
    ➔ Then for any s, t > 0 and for any integer n,

    $$P_n(t) = \Pr(N(s+t) - N(s) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

$$③\ \lim_{h\to 0}\frac{\Pr(N(h)=1)}{h} = \lambda$$

$$④\ \lim_{h\to 0}\frac{\Pr(N(h)\geq 2)}{h} = 0$$

- Proof:
  - We will use the properties ③ and ④
  - First derive $P_0(t) = e^{-\lambda t}$
    - $P_0(t+h) = P_0(t)\cdot P_0(h)$
    - $P_0(t+h) - P_0(t) = P_0(t)\cdot(P_0(h) - 1)$
    - $(P_0(t+h) - P_0(t))/h = P_0(t)\cdot(P_0(h) - 1)/\mathrm{h}$
    - $\lim_{h\to 0}(P_0(t+h) - P_0(t))/h = \lim_{h\to 0} P_0(t)\cdot(P_0(h) - 1)/\mathrm{h}$
    - $P_0'(t) = P_0(t)\cdot(-\lambda)$
    - $\frac{P_0'(t)}{P_0(t)} = -\lambda$
    - Taking the integration on both sides, $ln\,P_0(t) = -\lambda t$+c
    - Taking exponentiation, $P_0(t) = e^{-\lambda t + c}$ ➔ $P_0(t) = e^{-\lambda t}$ because $P_0(0) = 1$



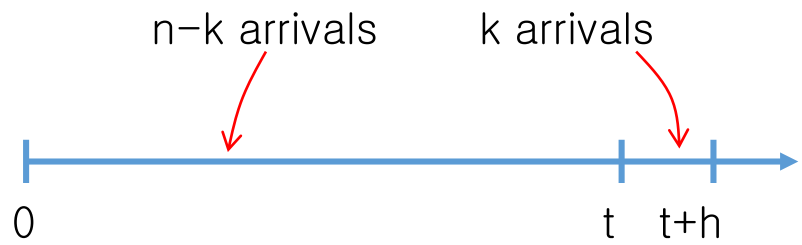
Poisson Process

SCONE Lab.

- Proof – Cont.
  - Now prove that $P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$
    - $P_n(t+h) = \sum_{k=0}^{n} P_{n-k}(t) \cdot P_k(h)$
    - $P_n(t+h) = P_n(t) \cdot P_0(h)$
      $+P_{n-1}(t) \cdot P_1(h) + \cdots$
    - $P_n(t+h) - P_n(t) = P_n(t) \cdot (P_0(h) - 1) + P_{n-1}(t) \cdot P_1(h) + \cdots$
    - $\lim_{h\to 0} \frac{P_n(t+h)-P_n(t)}{h} = \lim_{h\to 0} P_n(t)\frac{P_0(h)-1}{h} + \lim_{h\to 0} P_{n-1}(t)\frac{P_1(h)}{h} + \cdots$
    - $P_n'(t) = -\lambda \cdot P_n(t) + \lambda \cdot P_{n-1}(t) + 0 + \cdots$
    - $e^{\lambda t}\left(P_n'(t) + \lambda \cdot P_n(t)\right) = \lambda e^{\lambda t} \cdot P_{n-1}(t)$
    - $\frac{d}{dt} e^{\lambda t} \cdot P_n(t) = \lambda e^{\lambda t} \cdot P_{n-1}(t)$
    - At n=1, $\frac{d}{dt} e^{\lambda t} \cdot P_1(t) = \lambda e^{\lambda t} \cdot P_o(t) = \lambda e^{\lambda t} \cdot e^{-\lambda t} = \lambda$

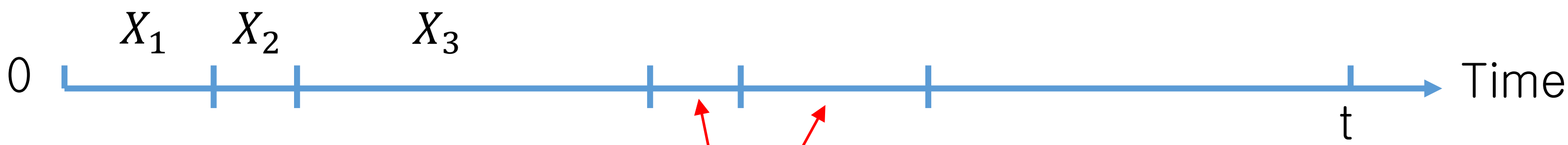
      ➔ $P_1(t) = (\lambda t + c) \cdot e^{-\lambda t}$

      Finally, since $P_1(0) = 0, P_1(t) = \lambda t \cdot e^{-\lambda t}$
    - By induction, we obtain $P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$



As $h \to 0$, only k=0 & 1 matter

Conditional Inter-arrival Time



Inter-arrival time of a Poisson process with rate λ

➔ Exponential distribution

➔ **How about the distribution of the inter-arrival time GIVEN that N(T)=n**

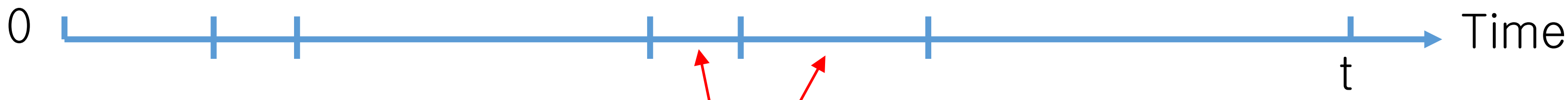
- X_1: time to the first event of a Poisson process with rate λ
- X_n: Time between (n-1)-th and n-th events

Interarrival Time Distribution

- Theorem: X_1 has an exponential distribution with parameter λ
- Proof
  - Pr(X_1 > t) = Pr(N(t)=0) = $e^{-\lambda t}$
  - F(X_1) = 1− Pr(X_1 > t) = 1− $e^{-\lambda t}$

- The random variables X_i are iid exponential random variables with parameter λ
- Proof
  - Pr($X_i > \mathrm{t}\ |(X_1{=}t_1) \cap (X_2{=}t_2) \cap (X_{i-1}{=}t_{i-1})$)
    = Pr(N($\sum_{k=1}^{i-1} t_k + t$) − N($\sum_{k=1}^{i-1} t_k$) =0)
    = $e^{-\lambda t}$

Conditional Inter-arrival Time



Inter-arrival time of a Poisson process with rate λ

➔ Exponential distribution

➔ How about the distribution of the inter-arrival time GIVEN that N(T)=n

- Surprisingly, n arrivals occur uniformly over the interval [0, t) if conditioned on N(t) = n
  ➔From Lemma 8.3, expected interval between two consecutive arrivals is 1/(n+1)

- For n=1, will show that probability that the only arrival occurs in 0 < s < t is s/t
  - $\Pr(X_1 < s \mid N(t) = 1) = \frac{\Pr((X_1 < s) \cap (N(t)=1))}{\Pr(N(t)=1)}$

$$= \frac{\Pr((N(s)=1) \cap (N(t)-N(s)=0))}{\Pr(N(t)=1)} = \frac{s}{t}$$

Distribution function of U[0, t]

Conditional Inter-arrival Time

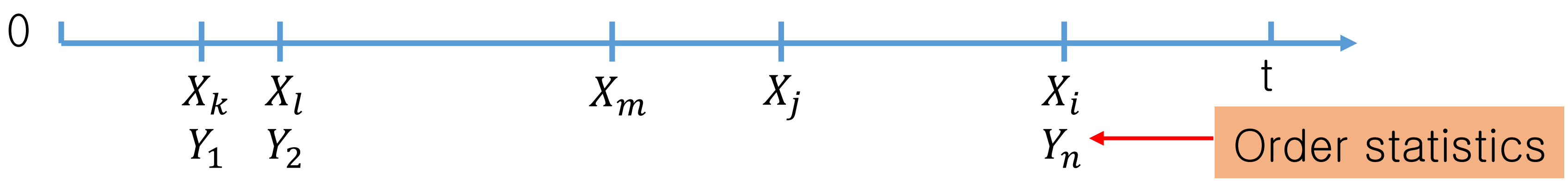
- Now, to prove that for general n
- Theorem:
  - Given that N(t) = n, the n arrival times have the same distribution of random variables of U[0, t]
- Proof: Sketch of the proof
  - Consider two sets of random variables
  - $\{X_i \sim \mathrm{U}[0,\mathrm{t}], \mathrm{for\ i} = 1,2,\ldots,\mathrm{n}\}$
  - $\{Z_i: \mathrm{arrival\ time\ of\ Poisson\ process, for\ i} = 1,2,\ldots,\mathrm{n},\mathrm{n}+1\}$
  - To show, Distribution of n Uniform random variable, U[0, t] — Not ordered

    =

    Distribution of n arrival times of Poisson process given N(t) = n — Ordered

Conditional Inter-arrival Time

- Note that the joint distribution of n R.V. of U[0, t] is $1/t^n$



Sum over all permutations

- $\Pr(Y_1 < s_1, Y_2 < s_2, \cdots, Y_n < s_n)$

  $= \sum \Pr(X_1 < s_1, X_2 < s_2, \cdots, X_n < s_n \mid X_1 < X_2 < \cdots < X_n)$

  $= \sum \Pr(X_1 < s_1, X_1 < X_2 < s_2, \cdots, X_{n-1} < X_n < s_n)$

- Now, consider arrival times of a Poisson process

  Let $Z_1, Z_2, .., Z_{n+1}$ be the first n+1 arrival times

  Prove that $\Pr(Z_1 < s_1, Z_2 < s_2, \cdots, Z_n < s_n \mid N(t)=n)$
  $= \Pr(Y_1 < s_1, Y_2 < s_2, \cdots, Y_n < s_n)$

Continuous Time Markov Process

- Discrete Time MC: Transition occurs at discrete points
- Continuous Time Markov Process: Transitions occur at any time

- Definition: A continuous time random process $\{X_t \mid t \geq 0\}$ is Markovian if for all $s, t > 0$

$$\Pr(X(s+t) = x \mid X(u), 0 \leq u \leq t) = \Pr(X(s+t) = x \mid X(t))$$

The transition probability is independent of states before t

- Observations
  - Time to stay at state i before transit to other state has an exponential distribution Exp(θ_i)
  - P=(p_{ij}) : Transition probability matrix of Embedded MC
    - Splitting of Poisson process: Given that a transition occurs from state i, randomly transit to state j with probability p_{ij}

Stationary Distribution of CTMC

- As DTMC where stationary distribution $\pi_i = \lim_{n\to\infty} P_{ji}^n$, we define stationary distribution in CTMC $\pi_i = \lim_{t\to\infty} P_{ji}\ (t)$
- From the following equation
  - $P'_{ji}(t) = \lim_{h\to 0} \frac{P_{ji}(t+h)-P_{ji}(t)}{h}$

$$= \lim_{h\to 0} \frac{\sum_k P_{jk}(t)\cdot P_{ki}(h)-P_{ji}(t)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sum_{k\neq i} P_{jk}(t)\cdot \boxed{P_{ki}(h)}}{h} - \frac{1-P_{ii}(h)}{h} P_{ji}(t)\right)$$

$$\boxed{\theta_k \cdot h \cdot p_{ki}}$$

$$\boxed{\begin{aligned} 1 - P_{ii}(h) &= \sum_k p_{ik}(h) - P_{ii}(h) \\ &= \theta_i h - \theta_i h p_{ii} \end{aligned}}$$

$$= \sum_k P_{jk}(t) \cdot \theta_k \cdot p_{ki} - \theta_i \cdot P_{ji}(t)$$

  - Taking limit on both sides & $\lim_{t\to\infty} P'_{ji}(t) = 0$

$$\boldsymbol{\pi_i \theta_i = \sum_k \pi_k \theta_k p_{ki}}$$

Rate of transition out of state i

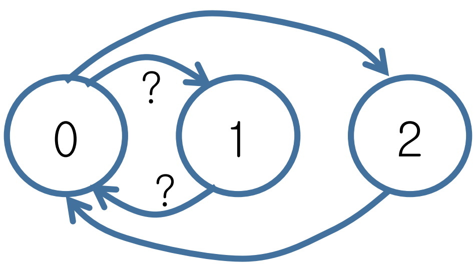
Rate of transition into state i

Markovian Queues

- Queueing system notation A/B/n
- A: Arrival process characteristics
  - M: Memoryless(Markovian), D: Deterministic, G: General, ..
- B: Departure process, Service process characteristics
- n: Number of servers
- State: Number of customers in the system
  - State changes from j to j+1 when a new customer arrives
  - State changes from j to j−1 if the customer who is receiving the service completes her service
- Queueing system is Markovian if both arrival and departure processes are Poisson processes
  - Combining of two Poisson processes is Poisson

M/M/1 Queue

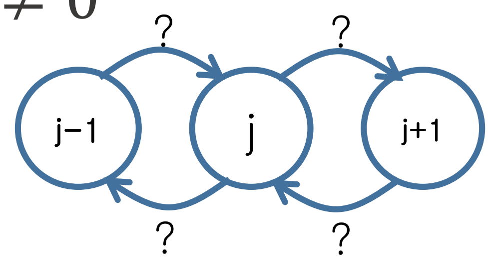
- Arrival is Poisson process with rate λ
- Service time has Exponential distribution, Exp(μ)
- M(t): # customers in the system
  - # customers waiting for their turns + One who is receiving the service
- $\{M(t): t \geq 0\}$, CTMC
- At j = 0,



  - Transition from 0 to 2: Two arrivals in a short time *h*
  - Transition from 2 to 0: Two service completions in a short time *h*
  - Transition from 0 to 1: Arrival according to Poisson with rate λ
  - Transition from 1 to 0: Service completion (Poisson) with rate μ

M/M/1 Queue

- At j ≠ 0



  - Combining of two Poisson processes, arrival and departure
    - Rate = (λ + μ)
    - Probability that event is arrival = λ/(λ + μ)

- Apply to equation $\pi_i\theta_i = \sum_k \pi_k\theta_k p_{ki}$
  - At $i = 0$,

    $\pi_0\theta_0 = \pi_0\theta_0 p_{00} + \pi_1\theta_1 p_{10}$

  - At $i \neq 0$,

    $\pi_i\theta_i = \pi_{i-1}\theta_{i-1}p_{i-1,i} + \pi_i\theta_i p_{ii} + \pi_{i+1}\theta_{i+1}p_{i+1,i}$

M/M/1 Queue

- From the equations, we obtain $\pi_k = (1 - \lambda/\mu)(\lambda/\mu)^k$
- N: Expected number of customers in the system
  - $N = \sum_{k=0}^{\infty} k \cdot \pi_k$
    $= \frac{\lambda}{\mu - \lambda}$

  Note that N is independent of scheduling algorithms such as FIFO, FILO

- T: Expected time spent in the system for FIFO scheduling
  - L(k): Event that the new customer finds k customers are already in the system
  - $T = \sum_{k=0}^{\infty} E[T \,|L(k)] \cdot \Pr(L(k))$
    $= \sum_{k=0}^{\infty} \frac{k+1}{\mu} \cdot (1 - \lambda/\mu)(\lambda/\mu)^k$
    $= N/\lambda$

Pr(L(k)) = π_k
PASTA: Poisson Arrivals See Time Average
Wolff, Operations Research, 1982

Little's Theorem

- N= λT
- Powerful tool
  - Can be applied to any queueing system regardless of arrival/departure processes, service disciplines, …
- Sketch of the proof

For large observation time, T

$$N = \frac{1}{T} \cdot \int_{t=0}^{T} N(t)dt \approx \frac{1}{T} \cdot (T_1 + T_2 + \cdots + T_{A(T)})$$