2018 Engineering Mathematics II Quiz 02 Solutions

Candidate Avg.		11.378
Max		19
Median		12
Min		0
Total Avg.		11.378

EM2 TAs: Onyu Kang, Jiho Choi

If you have any question regarding with your grade, please contact the following TA.

1. [2, 2, 2, 2, 2 points]

1-a) [2 points] Onyu Kang

1-b) [2 points] Onyu Kang

1-c) [2 points] Onyu Kang

1-d) [2 points] Onyu Kang

1-e) [2 points] Onyu Kang

2. [2, 2 points]

2-A) [2 points] Jiho Choi

2-B) [2 points] Jiho Choi

3. [3, 3 points]

3-A) [3 points] Jiho Choi

3-B) [3 points] Jiho Choi

Total grade points **[20 points]**

- surname ㄱ ~ ㅅ: Onyu Kang,

- surname ㅇ ~ ㅎ: Jiho Choi

1. [2 points each]

1-a) [2 points] - Onyu Kang

Let x be a Geometric random variable. Prove that E[X] = $\sum_{i=1}^{\infty} \Pr(X \geq i)$. Does the equality hold for random variables whose values are integers? Describe the conditions that make the equality holds true.

Solution)

$$\sum_{i=1}^{\infty} \Pr(X \geq i) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr(X = j)$$

$$= \sum_{j=1}^{\infty} \sum_{i=i}^{j} \Pr(X = j)$$

$$= \sum_{j=i}^{\infty} j\Pr(X = j)$$

$$= E[X].$$

-----[1 pts]

The condition that makes the equality hold true is that X be a discrete random variable that takes on only nonnegative integer values. -----[1 pts]

■

Scores were specified for each.

1-b) [2 points] - Onyu Kang

Prove or give a counter example for the following claim; if X and Y are independent random variables, then for any functions g(·) and f(·) E[g(X) · f(Y)] = E[g(X)] ·E[f(Y)]

Solution)

In continuous independent random variable case,

$$\mathrm{E}[\mathrm{g}(\mathrm{X}) \cdot \mathrm{f}(\mathrm{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)f(y)f_x(x)f_y(y)dxdy$$

$$= \int_{-\infty}^{\infty} f_y(y)f(y)\int_{-\infty}^{\infty} g(x)f_x(x)dxdy$$

$$= \int_{-\infty}^{\infty} f_y(y)f(y)\mathrm{E}[\mathrm{g}(\mathrm{X})]dy$$

$$= \mathrm{E}[\mathrm{g}(\mathrm{X})] \int_{-\infty}^{\infty} f_y(y)f(y)dy$$

$$= \mathrm{E}[\mathrm{g}(\mathrm{X})] \cdot \mathrm{E}[\mathrm{f}(\mathrm{Y})]$$

In discrete independent random variable case,

$$\mathrm{E}[\mathrm{g}(\mathrm{X}) \cdot \mathrm{f}(\mathrm{Y})] = \sum_x \sum_y g(x)f(y)f_x(x)f_y(y)$$

$$= \sum_x g(x)f_x(x)\sum_y f(y)f_y(y)$$

$$= \mathrm{E}[\mathrm{g}(\mathrm{X})] \cdot \mathrm{E}[\mathrm{f}(\mathrm{Y})]$$

■

If you prove one of the above cases, you will get 2 points.

1-c) [2 points] - Onyu Kang

The MGF of Bernoulli random variable with parameter p is $p \cdot e^t + 1 = p$. Using Theorem 4.3, show that the MGF of Binomial random variable X ~B(n,p) is $(p \cdot e^t + 1 - p)^{\mathrm{n}}$. Using the above MGF and the propositions, E[X]=$M_X^{(1)}(0)$ and E[X^2] = $M_X^{(2)}(0)$, derive expectation and 2nd moment of X.

Solution)

Since $\{x_1, x_2, \ldots, x_i\}$ are independent, X=$x_1 + x_2 + \ldots + x_i$.

$$\mathrm{M}_X(\mathrm{t}) = \prod_{i=1}^{n} \mathrm{M}_{x_i}(\mathrm{t})$$

$$= \prod_{i=1}^{n} (p \cdot e^t + 1 - p)$$

$$= (p \cdot e^t + 1 - p)^{\mathrm{n}}$$ -----[1 pt]

$$M_X^{(1)}(t) = n(p \cdot e^t + 1 - p)^{\mathrm{n}-1} \; (p \cdot e^t)$$

$$\therefore E[X] = M_X^{(1)}(0) = np$$ -----[0.5 pts]

$$M_X^{(2)}(t) = \mathrm{np} \cdot e^t \; (\mathrm{p} \; (\mathrm{n} \; e^t \; - \; 1) \; + \; 1) \; (1 \; + \; (-1 \; + \; e^t) \; \mathrm{p})^{n-2}$$

$$\therefore E[X^2] = M_X^{(2)}(0) = n^2p^2 - np^2 + np$$ -----[0.5 pts]

■

Scores were specified for each.

1-d) [2 points] - Onyu Kang

Continuation of C). Let X and Y are independent Binomial random variables; X~B(n,p) and Y~B(m,p). Define a new random variable Z=X+Y and derive the MGF of Z. Also using the uniqueness property of MGF (Theorem 4.2), show that Z~B(n+m, p).

Solution)

By theorem 4.3,

$$\begin{aligned} \mathrm{M}_Z(\mathrm{t}) &= \mathrm{M}_{X+Y}(\mathrm{t}) \\ &= \mathrm{M}_X(\mathrm{t})\mathrm{M}_Y(\mathrm{t}) \\ &= (p \cdot e^t + 1 - p)^{\mathrm{n}}(p \cdot e^t + 1 - p)^{\mathrm{m}} \\ &= (p \cdot e^t + 1 - p)^{\mathrm{n+m}} \end{aligned}$$

-----[1 pt]

W~B(n+m, p)
By theorem 4.2,

$$\mathrm{M}_W(\mathrm{t})=(p \cdot e^t + 1 - p)^{\mathrm{n+m}}=\mathrm{M}_Z(\mathrm{t})$$

Then W and Z have the same distribution.

∴ Z~B(n+m, p) -----[1 pt]

■

Scores were specified for each.

1-e) [2 points] - Onyu Kang

The MGF of a Geometric random variable with parameter p is $pe^t \div (1-(1-p)e^t)$. Let X and Y are two independent Geometric random variables with the same parameter p. Find the MGF of Z=X+Y. Also compute the expectation of Z using E[Z]= $M_Z^{(1)}(0)$.

Solution)

$$\mathrm{M}_Z(\mathrm{t}) = \mathrm{M}_{X+Y}(\mathrm{t})$$

$$= \mathrm{M}_X(\mathrm{t})\mathrm{M}_Y(\mathrm{t})$$

$$= \left(\frac{pe^t}{(1-(1-p)e^t)}\right)^2$$ -----[1 pt]

$$M_Z^{(1)}(\mathrm{t}) = \frac{2p^2e^{2t}}{\left((p-1)e^t+1\right)^3}$$

$$\therefore \; E[Z] = M_Z^{(1)}(0) = \frac{2}{p}$$ ----[1 pt]

Scores were specified for each.

2. [2, 2 points] Memoryless property & Other

2-A) [2 points] Jiho Choi

X is a geometric random variable with success probability p → $\Pr(x = n) = (1-p)^{n-1} * p$

(1 point)

$$\Pr(X > 3|X > 2) = \frac{\Pr(x > 3 \cap x > 2)}{\Pr(x > 2)} = \frac{\Pr(x > 3)}{\Pr(x > 2)} = \frac{\sum_{i=4}^{\infty}(1-p)^{i-1} * p}{\sum_{i=3}^{\infty}(1-p)^{i-1} * p} = \frac{(1-p)^3}{(1-p)^2} = (1-p)$$

(1 point)

$$\Pr(X > 1) = \sum_{i=2}^{\infty}(1-p)^{i-1}p = \frac{1-p}{1-(1-p)} * p = (1-\mathrm{p})$$

2-B) [2 points] Jiho Choi

$$\Pr(Y > 3|Y > 2) = \frac{\Pr(Y > 3 \cap Y > 2)}{\Pr(Y > 2)} = \frac{\Pr(Y > 3)}{\Pr(Y > 2)}$$

To find Y which satisfies $Pr(Y > 3|Y > 2) = \Pr(Y > 3)$, Y with the property $\Pr(Y > 2) = 1$ is needed.

ex.) Y={3}, {3, 4}, {3, 4, 5}

Any Y with the above property will get full points.

3. [3, 3 points]

3-A) [3points] Jiho Choi

X~B(n,p) → E[X]=np, Var[X]=np(1-p), k is integer ≥ 2

(1 point) Markov's Bound: $\Pr(\mathrm{x} \geq a) \leq \frac{\mathrm{E[X]}}{a}$

$$\Pr\left(\mathrm{X} \geq \frac{(\mathrm{k}-1)\mathrm{n}}{\mathrm{k}}\right) \leq \frac{\mathrm{E[X]}}{\frac{(k-1)n}{k}} = \frac{np}{\frac{(k-1)n}{k}} = \frac{k}{k-1}p$$

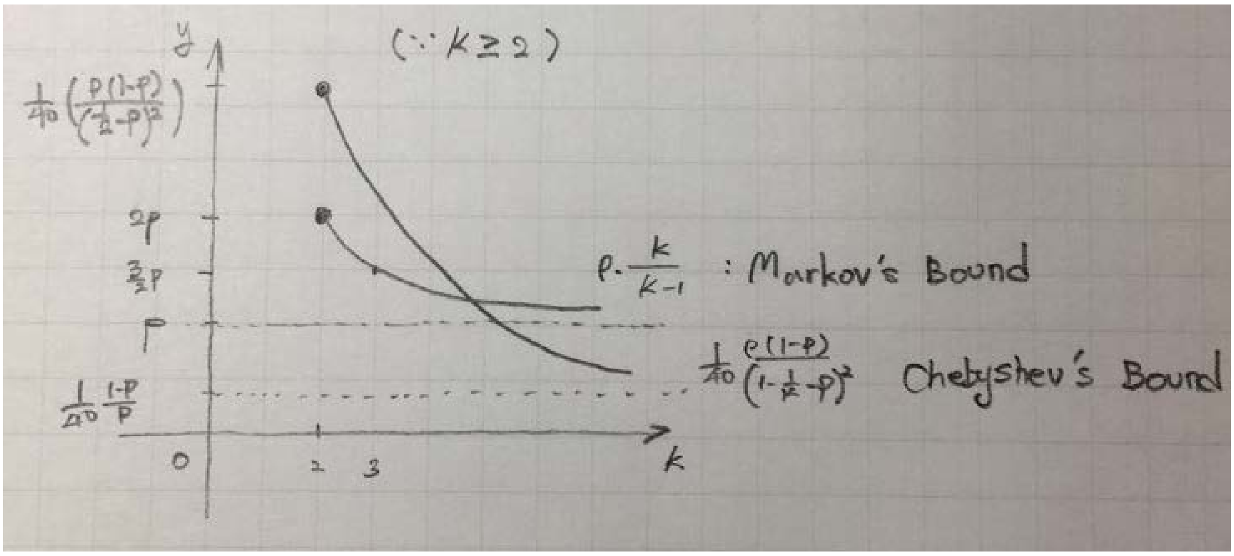
(1 point) Chebyshev's Bound: $\Pr(|\mathrm{X}-\mathrm{E[X]}| \geq a) \leq \frac{Var[X]}{a^2}$

$$\Pr\left(\mathrm{X} \geq \frac{(\mathrm{k}-1)\mathrm{n}}{\mathrm{k}}\right)$$

$$\Pr\left(\mathrm{X}-\mathrm{E[x]} \geq \frac{(\mathrm{k}-1)\mathrm{n}}{\mathrm{k}} - E[X]\right) = \frac{1}{2} * \Pr\left(|\mathrm{X}-\mathrm{E[x]}| \geq \frac{(\mathrm{k}-1)\mathrm{n}}{\mathrm{k}} - E[X]\right)$$

$$= \frac{1}{2} * \Pr\left(|\mathrm{X}-\mathrm{E[x]}| \geq \frac{(\mathrm{k}-1)\mathrm{n}}{\mathrm{k}} - np\right) \leq \frac{1}{2}\frac{Var[X]}{\left(\mathrm{n}(1-\frac{1}{\mathrm{k}}-p)\right)^2} = \frac{1}{2}\frac{np(1-p)}{n^2\left(1-\frac{1}{k}-p\right)^2} = \frac{1}{2}\frac{p(1-p)}{n\left(1-\frac{1}{k}-p\right)^2}$$

(1 point)
n=20, Markov's Bound: $\frac{k}{k-1}p$, Chebyshev's Bound: $\frac{1}{40}\frac{p(1-p)}{\left(1-\frac{1}{k}-p\right)^2}$



3-B) [3 points] Jiho Choi

$$\Pr(\mathrm{X} - \mathrm{E[X]} \geq \mathrm{t} * \sigma[\mathrm{x}]) \leq \frac{1}{1+t^2}$$

Using Chebyshev's Bound:

$\Pr(\mathrm{X} - \mathrm{E[X]} \geq \mathrm{t} * \sigma[\mathrm{x}]) = \Pr(\mathrm{X} - \mathrm{E[X]} + \mathrm{z} \geq \mathrm{t} * \sigma[\mathrm{x}] + z) = \Pr((\mathrm{X} - \mathrm{E[X]} + \mathrm{z})^2 \geq (\mathrm{t} * \sigma[\mathrm{x}] + z)^2)$

$\leq \dfrac{E[(X - E[X] + z)^2]}{(t\sigma + \mathrm{z})^2}$ $(\because \mathit{Chebyshev's\ \ Inequality})$

$= \dfrac{\sigma^2 + \mathrm{z}^2}{(t\sigma + \mathrm{z})^2}$

$f'(z) = 2z(t\sigma + \mathrm{z})^{-2} - 2(\sigma^2 + z^2)(t\sigma + \mathrm{z})^{-2}$

$\mathrm{F(z)}\ \ is \min at\ \ z = \frac{\sigma}{\mathrm{t}'}\ \frac{\sigma^2+\mathrm{z}^2}{(t\sigma+\mathrm{z})^2} = \frac{1}{1+t^2}$

Cf) Common Incorrect Answer

Using Chebyshev's Bound: $\Pr(|\mathrm{X} - \mathrm{E[X]}| \geq a) \leq \frac{Var[X]}{a^2}$ **not correct for $\mathbf{0 < t < 1}$**

$\downarrow \text{need to be symmetric}$

$$\Pr(\mathrm{X} - \mathrm{E[X]} \geq \mathrm{t} * \sigma[\mathrm{x}]) = \frac{1}{2}\Pr(|\mathrm{X} - \mathrm{E[X]}| \geq \mathrm{t} * \sigma[\mathrm{x}]) \leq \frac{1}{2}\frac{Var[X]}{(\mathrm{t} * \sigma[\mathrm{x}])^2} = \frac{1}{2 * t^2}$$

$$\frac{1}{2 * t^2} \leq \frac{1}{1+t^2}\ \ (\text{for}\ \ 1 \leq \mathrm{t})$$