1 (3 points each) A) True, B) False C) True D) False E) True

A) True

$P(\overline{E}\cap\overline{F}) = P(\overline{E\cap F}) = 1 - P(E\cup F)$
$= 1 - P(E) - P(F) + P(E\cap F)$
$= 1 - P(E) - P(F) + P(E)*P(F)$
$= (1 - P(E))*(1 - P(F))$
$= P(\overline{E})P(\overline{F})$

B) False

Counterexample)

Let H1 = Samsung Lions goes to Playoffs in this season
H2 = I see the cat in this morning
H3 = Samsung Lions fails to go Playoffs in this season

H1 and H2 are independent, H2 and H3 are independent, but H1 and H3 are obviously dependent

Other example) Simply if H1 and H3 are the same events, we can counterprove it.

C) True (Proof is omitted)

D) False

Counterexample) if X ={-1,0,1} with uniform probability, and Y = {0,1} with probability {1/3, 2/3}, than E[X] = 0 and E[Y]=2/3

consider following table of $P(X\cap Y)$

Y \ X	-1	0	1
0	0	1/3	0
1	1/3	0	1/3

E[XY] = -1* 1/3 + 1*1/3 = 0

E[X] = 0 --> E[XY] = E[X] E[Y].

P(X=-1 and Y=1) = 1/3 -- (3)

P(X=-1) * P(Y=1) = 1/3 * 1/2 = 1/6 -- (4)

(3) is not same as (4) --> Counterproof is done

E) Prove $E[X^2] \geqq E[X]^2 \Leftrightarrow E[X^2] - E[X]^2 \geqq 0$

$E[X^2] - E[X]^2 = E[X^2] - 2*E[X]^2 + E[X]^2$
$= \sum_k k^2 P(X=k) - 2E(X)\sum_k kP(X=k) + E(X)^2\sum_k P(X=k)$
$= (\sum_k P(X=k)(k^2 - 2E(X)k + E(X)^2))$
$= \sum_k P(X=k)(k - E(X))^2 \geqq 0$

2 Bayes' Theorem (4,2 pts)

A)

F: flipping coin is fair coin

B: flipping coin is biased coin

Pr(H|F) = 1/2

Pr(H|B) = 1/3

$$\Pr(F|H) = \frac{\Pr(H|F)\Pr(F)}{\Pr(H|F)\Pr(F)+\Pr(H|B)\Pr(B)}$$
$$= \frac{(1/2)\times(1/2)}{(1/2)\times(1/2)+(1/3)\times(1/2)} = \frac{3}{5}$$

Pr(HH|F) = Pr(H|F) * Pr(H|F) = 1/4

Pr(HH|B) = Pr(H|B) * Pr(H|B) = 1/9

$$\Pr(F|HH) = \frac{\Pr(HH|F)\Pr(F)}{\Pr(HH|F)\Pr(F)+\Pr(HH|B)\Pr(B)}$$
$$= \frac{(1/4)\times(1/2)}{(1/4)\times(1/2)+(1/9)\times(1/2)} = \frac{9}{13}$$

B)

D: Have disorder

$\overline{D}$: Not have disorder

P: Positive result

Pr(P|D) = 0.999

Pr(P|$\overline{D}$) = 0.005

Pr(D) = 0.02

Pr($\overline{D}$) = 0.98

$$\Pr(D|P) = \frac{\Pr(P|D)\Pr(D)}{\Pr(P|D)\Pr(D)+\Pr(P|\overline{D})\Pr(\overline{D})}$$
$$= \frac{(0.999)\times(0.02)}{(0.999)\times(0.02)+(0.005)\times(0.98)} = \frac{999}{1244}$$

3 Geometric distribution and Others (2,3,2 pts)

A)

Let Ei : ith trial success

Pr(E1) = p, Pr($\overline{E1}$) = 1-p

E[X] = E[X|E1] * Pr(E1) + E[X|$\overline{E1}$] * Pr($\overline{E1}$)
= 1 * p + (1+E[X]) * (1-p)
= 1 + E[X] * (1-p)

p * E[X] = 1
E[X] = 1/p

B)

Pr(X=n+k) = Pr((n-1) events during (n+k-1) trials) * Pr((n+k) th trial success)

Pr(X=n+k) = $\binom{n+k-1}{n-1} p^{n-1} (1-p)^k p$
= $\binom{n+k-1}{n-1} p^n (1-p)^k$

Let Xn: number of trials until a certain event occur n times

E[Xn] = E[Xn|E1] * Pr(E1) + E[Xn|$\overline{E1}$] * Pr($\overline{E1}$)
= (1 + E[Xn-1]) * p + (1 + E[Xn]) * (1-p)
= p * E[Xn-1] + 1 + (1-p) * E[Xn]

p * E[Xn] = p * E[Xn-1] + 1

E[Xn] = E[X1] + (n-1)/p = n/p
(by (A), E[X1] = 1/p)

C)

Pr(Y = 0) = p^n
Pr(Y = 1) = 1- p^n

E[Y] = 1 * Pr(Y=1) + 0 * Pr(Y=0)

= 1 * (1-p^n) = 1- p^n

Var[Y] = E[Y^2] - $(E[Y])^2$ = 1- p^n - $(1-p^n)^2$

= p^n(1-p^n)

4. MGF (2 points each)

A) Using binomial theorem,

$MGFof\,X = E[e^{tx}] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{tk}$

$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$

$= (pe^t + 1 - p)^n$

Samely, $MGFof\,Y = E[e^{ty}] = \sum_{k=0}^{m} \binom{m}{k} p^k (1-p)^{m-k} e^{tk}$

$= \sum_{k=0}^{m} \binom{m}{k} (pe^t)^k (1-p)^{m-k}$

$= (pe^t + 1 - p)^m$

B) $MGFof\,X+Y = E[e^{t(x+y)}] = \sum_{l=0}^{m} \sum_{k=0}^{n} \binom{n}{k} \binom{m}{l} p^{k+l} (1-p)^{n+m-k-l} e^{t(k+l)} =$

$= (\sum_{l=0}^{m} \binom{m}{l} p^l (1-p)^{m-l} e^{tl}) (\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{tk})$

$= (\sum_{l=0}^{m} \binom{m}{l} (pe^t)^l (1-p)^{m-l}) (\sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k})$

$= (pe^t + 1 - p)^m \, (pe^t + 1 - p)^n$

$= (pe^t + 1 - p)^{m+n}$

= MGF of (X) * MGF of Y

also, X+Y ~ B(n+m,p)