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Monte Carlo Method

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Derived Distribution

- Recall that
  - Let Y=g(X)
  - Pr(Y=y) = $\sum_{x \in G(x)=y} \Pr(X = x)$
  - Example:
    - Roll two dice
    - Let X_1, X_2 be numbers on the first/second dice
    - Let Y = $X_1 - X_2$
    - Pr(Y=0) = $\sum_{k=1}^{6} \Pr(X_1 = X_2 = k) = \frac{1}{6}$
- How about continuous case?
- Suppose we have $f_X(x)$ ($F_X(x)$), and aim to find $f_Y(y)$ ($F_Y(y)$) for Y=g(X)

  ➔ Use CDF

Derived Distribution

- Case: g(·) is a linear function
  - Let Y=aX+b
  - $F_Y(y) = \Pr(Y \le y) = \Pr(aX + b \le y)$
    $= \Pr\left(X \le \frac{y-b}{a}\right) = F_X(\frac{y-b}{a}))$
    ➔ $f_Y(y) = |\frac{1}{a}| \cdot f_X(\frac{y-b}{a})$

Case a > 0
How about a < 0 ?

  - Example:
    - Roll a die and let X be the number on the die
    - Let Y=2X−1
    - Pr(Y=1) = Pr(2X−1=1)=Pr(X=1) = 1/6
  - Example
    - X~Exp(λ), Y = aX+b
    - $f_X(x) = \lambda e^{-\lambda x}$
    - $f_Y(y) = |\frac{1}{a}| \cdot \lambda e^{-\lambda(y-b)/a}$

Derived Distribution

- Case $g(\cdot)$ is monotonic
  - Either strictly increasing or decreasing

    ➔ $g(x) < g(x')$ for all $x < x'$

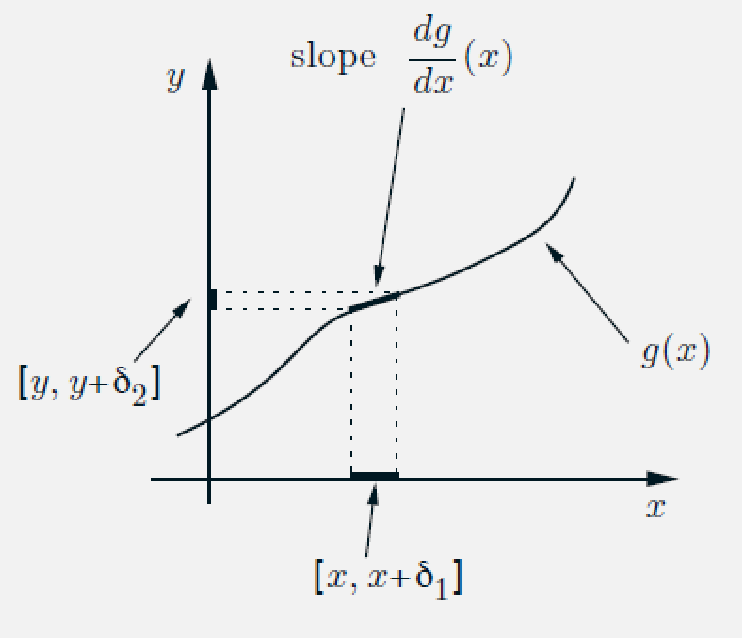
    ➔ $g(\cdot)$ is invertible and let its inverse function is $h(y)$
  - $y=g(x)$ ⇐ ⇒ $x=h(y)$
  - Example
    - $g(x)=ax+b$ ⇐ ⇒ $h(y) = (y-b)/a$
    - $g(x)=e^{ax}$ ⇐ ⇒ $h(y) = (\ln y)/a$

Monotonic Function Case

- Claim: Assume y= g(·) is monotonic,

  then $f_Y(y) = f_X(h(y)) \cdot |\frac{dh(y)}{dy}|$

- Proof
  - Case for increasing functions
  - $F_Y(y) = \Pr(g(X) \leq y) = \Pr(X \leq h(y)) = F_X(h(y))$
  - Differentiating the both sides,
    we obtain the result



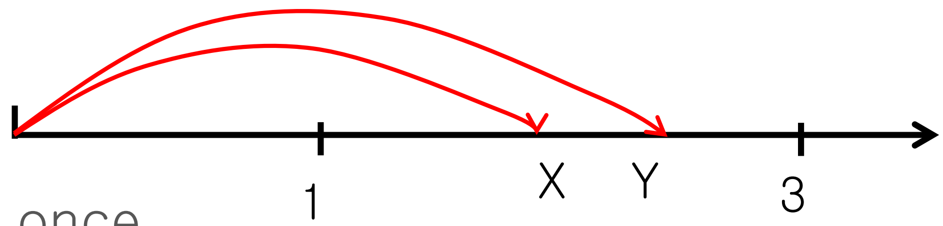
Example

- Example
  - X~U[0,1], Y=g(X)=X^2
  - h(y) = $\sqrt{y}$ ➔ $\frac{dh(y)}{dy} = \frac{1}{2\sqrt{y}}$
  - $f_X(h(y)) = f_X(\sqrt{y})\frac{dh(y)}{dy} = \frac{1}{2\sqrt{y}}$
  - $F_Y(y)$ = Pr(Y≤ y)

$$
\begin{aligned}
&= \Pr(X^2 \leq y) \\
&= \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y})
\end{aligned}
$$

Multiple R.V.

- $Z = g(X, Y)$
- Example



  - Two golfers each hit balls once

    Let X, Y be the driving distances of two golfers

    $X, Y \sim U[1, 3]$

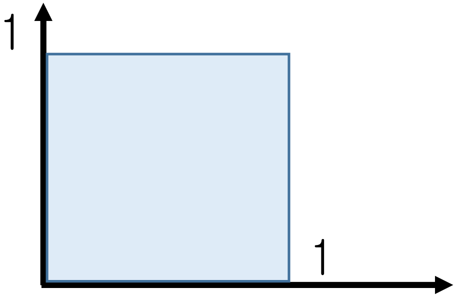
    Let $Z = \max\{X, Y\}$
  - Compute $f_Z(z) = ??$

  - Solutions
    - $$F_Z(z) = \Pr(Z \le z)$$
      $$= \Pr(X \le z, Y \le z)$$
      $$= \Pr(X \le z) \cdot \Pr(Y \le z) = \frac{z}{2} \cdot \frac{z}{2}$$

Example

- X, Y be independent r.v. uniformly distributed on the area [0,1]X[0,1]
  - Let Z=$\frac{Y}{X}$

    To show that



$$F_Z(z) = \Pr\left(\frac{Y}{X} \le z\right) = \begin{cases} \frac{z}{2}, & if\ 0 \le z \le 1, \\ 1 - \frac{1}{2z}, & if\ z > \\ 0, & ow \end{cases}$$

Convolution

- Sum of independent random variables
  - Z=X+Y
- Discrete case
  - Pr(Z=z) = Pr(X+Y=z) = $\sum_x$ Pr(X=x, Y=z−x)
- Continuous case
  - $\Pr(Z \leq z \mid X = x) = \Pr(X + Y \leq z \mid X = x)$

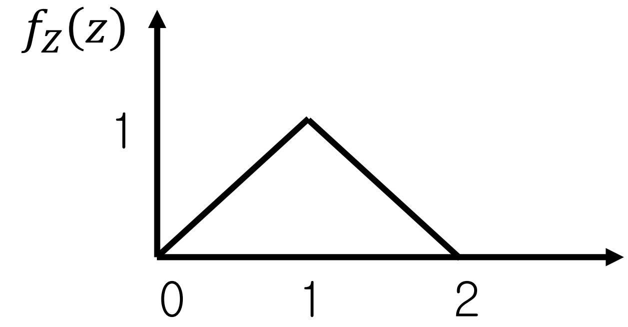
    $= \Pr(Y \leq z - x \mid X = x)$

    $= \Pr(Y \leq z - x)$

    → $f_{Z|X}(z|x) = f_Y(z - x)$
  - $f_{X,Z}(x, z) = f_X(x) \cdot f_{Z|X}(z|x) = f_X(x) \cdot f_Y(z - x)$
  - $f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(x, z)\, dx = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z - x)\, dx$

Example

- Let X, Y are independent and X=Y=U[0,1]
- Let Z = X+Y



- To show that
  - $f_Z(z) = \begin{cases} \min\{1, z\} - \max\{0, z-1\}, & 0 \le z \le 2 \\ 0, & ow \end{cases}$