2018 Engineering Mathematics II Quiz 01 Solutions

Candidate Avg.		11.884
Max		18
Median		12
Min		2.5
Total Avg.		11.465

EM2 TAs: Onyu Kang, Jiho Choi

If you have any question regarding your grade, please contact the following TA.

1. [2, 2 points]

1-a) [2 points] - Jiho Choi

1-b) [2 points] - Jiho Choi

1-c) [2 points] - Onyu Kang

2. [2, 2 points]

2-①) [2 points] - Jiho Choi

2-②) [2 points] - Jiho Choi

3. [2, 2, 4 points]

3-a) [2 points] - Onyu Kang

3-b) [2 points] - Onyu Kang

3-c) [4 points] - Onyu Kang

Overall (total) grade points [18 points]

- surname ㄱ ~ ㅅ: Onyu Kang,

- surname ㅇ ~ ㅎ: Jiho Choi

1. [2, 2 points]

1-a) [2 points] Jiho Choi

[0.5 points] Show Part

Two events A and B are dependent, if $\Pr(\mathrm{A}) * \Pr(\mathrm{B}) \neq \Pr(\mathrm{A} \cap \mathrm{B})$

$\Pr(A) = \frac{6}{7}, \Pr(B) = \frac{1}{7}, \mathrm{A} \cap \mathrm{B} = \{\mathrm{Sa}\}$

$\Pr(\mathrm{A} \cap \mathrm{B}) = \frac{1}{7}$

$\therefore \Pr(\mathrm{A}) * \Pr(\mathrm{B}) \neq \Pr(\mathrm{A} \cap \mathrm{B})$

[0.5 points] Show Part

A|C and B|C are conditionally independent, if $\Pr(\mathrm{A}|\mathrm{C}) * \Pr(\mathrm{B}|\mathrm{C}) = \Pr(\mathrm{A} \cap \mathrm{B}|\mathrm{C})$

$$\Pr(A|C) = \frac{\Pr(A \cap \mathrm{C})}{\Pr(C)} = \frac{\frac{2}{7}}{\frac{2}{7}} = 1$$

$$\Pr(B|C) = \frac{\Pr(B \cap \mathrm{C})}{\Pr(C)} = \frac{\frac{1}{7}}{\frac{2}{7}} = \frac{1}{2}$$

$$\Pr(\mathrm{A} \cap \mathrm{B}|\mathrm{C}) = \frac{\Pr(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})}{\Pr(C)} = \frac{\frac{1}{7}}{\frac{2}{7}} = 1/2$$

$\therefore \Pr(\mathrm{A}|\mathrm{C}) * \Pr(\mathrm{B}|\mathrm{C}) = \Pr(\mathrm{A} \cap \mathrm{B}|\mathrm{C})$

[1.0 point] **Find Part**

Any reasonable set C with proof will get the full point.

$\mathrm{C} = \{\mathrm{M}, \mathrm{Tu}, \mathrm{Sa}, \mathrm{Su}\}, \{\mathrm{M}, \mathrm{Tu}, \mathrm{Sa}, \mathrm{F}\}, or \ldots$

$\Pr(A|C) = \Pr(\mathrm{B}|\mathrm{C}) = \frac{1}{2}$

$\Pr(A \cap \mathrm{B}|C) = \frac{1}{4}$

$\therefore \Pr(\mathrm{A}|\mathrm{C}) * \Pr(\mathrm{B}|\mathrm{C}) = \Pr(\mathrm{A} \cap \mathrm{B}|\mathrm{C})$

1-b) [2 points] Jiho Choi

Any reasonable proof or counter example will get the points.

Claim: If A and B are independent and B and C are independent, then A and C are independent.

Proof by contradiction

Let's say there is a fair dice and its possible outputs are {1, 2, 3, 4, 5, 6}

Let A be the events of even numbers come out: {2, 4, 6}

$$\Pr(\mathrm{A}) = \Pr(Even) = \frac{3}{6}$$

Let B be the events of numbers less than or equal to 4 come out. {1, 2, 3, 4}

$$\Pr(\mathrm{B}) = \Pr(N \leq 4) = \frac{4}{6}$$

$\Pr(\mathrm{A}) * \Pr(\mathrm{B}) = \frac{1}{3} = \Pr(\mathrm{A} \cap \mathrm{B}) = \frac{2}{6}$

Let C be the event of odd numbers come out: {1, 3, 5}

$$\Pr(\mathrm{A}) = \Pr(Odd) = \frac{3}{6}$$

$\Pr(\mathrm{B}) * \Pr(\mathrm{C}) = \frac{1}{3} = \Pr(\mathrm{B} \cap \mathrm{C}) = \frac{2}{6}$

$\Pr(\mathrm{A}) * \Pr(\mathrm{C}) = \frac{1}{2} * \frac{1}{2}$

$\Pr(\mathrm{A} \cap \mathrm{C}) = 0$

$\therefore \Pr(\mathrm{A}) * \Pr(\mathrm{C}) \neq \Pr(\mathrm{A} \cap \mathrm{C})$

c.f.) Another counter example

With two fair coins A, B

S = {HH, HT, TH, TT}

A = {HH, HT}

B = {HT, TT}

C = {TH, TT}

$\Pr(\mathrm{A}) = \frac{1}{2}, \Pr(\mathrm{B}) = \frac{1}{2}, \Pr(\mathrm{A} \cap \mathrm{B}) = \frac{1}{4}, \Pr(\mathrm{B} \cap \mathrm{C}) = \frac{1}{4}, \Pr(\mathrm{A} \cap \mathrm{C}) = 0$

1-c) [2 points] Onyu Kang

Prove Theorem 1.6 (Law of Total Probability) [2 pts]

Theorem 1.6 [Law of Total Probability] : *Let* $E_1, E_2, \ldots, E_n$ *be mutually disjoint events in the sample space* Ω*, and let* $\bigcup_{i=1}^{n} E_i = \Omega$*. Then*

$$\Pr(B) = \sum_{i=1}^{n} \Pr(B \cap E_i) = \sum_{i=1}^{n} \Pr(B|E_i) \Pr(E_i)$$

Solution)

Since the events $B \cap E_i$ (i=1,...n) are disjoint and cover the entire sample space Ω, it follows that

$$\Pr(B) = \sum_{i=1}^{n} \Pr(B \cap E_i),$$ -----[1 pts]

Further,

$$\sum_{i=1}^{n} \Pr(B \cap E_i) = \sum_{i=1}^{n} \Pr(B|E_i) \Pr(E_i)$$

by the definition of conditional probability. -----[1 pts] ■

Scores were specified for each.

2. [2, 2 points]

2-①) [2 points] Jiho Choi

If you have an answer for *solution without the size limit of the teams*, please contact me. You might get more points than what you currently have.

[1.0 point]

$\Pr(E_1) = 1$

$\Pr(E_2) = \frac{9}{11}$ (two spots with team_1 and 9 spots with team_2, _3, _4)

[1.0 point]

$$\Pr(E_2|E_1) = \frac{\Pr(E_2 \cap E_1)}{\Pr(E_1)} = \frac{9}{11}/1 = \frac{9}{11}$$

2-②) [2 points] Jiho Choi

[1.0 point]

$\Pr(E_3|E_1 \cap E_2) = \frac{6}{10} = \frac{3}{5}$ (four spots with team_1, _2 and 6 spots with team_3, _4)

[1.0 point]

$\Pr(E_3) = \Pr(E_1 \cap \mathrm{E}_2 \cap \mathrm{E}_3) = \Pr(E_3|E_1 \cap \mathrm{E}_2)\,\Pr(E_1 \cap E_2) = \frac{3}{5} * \frac{9}{11} = \frac{27}{55}$

3. [2, 2, 4 points] Onyu Kang

3-a) [2 points] Onyu Kang

There are three coins A, B and C with probabilities of Heads are 1/4, 1/2 and 3/4, respectively. We select one coin randomly and toss the coin. Suppose the result is Heads. What are the probabilities that the selected coin is A, B and C, respectively? [2 pts]

Solution)

Let H_i be the event that the ith coin flipped is the Heads, and let T_i be the event that the ith coin flipped is the Tails. Before we flip the coins we have Pr(H | A) =1/4, Pr(H | B) = 1/2 and Pr(H | C) = 3/4.

Applying Bayes' theorem, we have

$$\Pr(A|\mathrm{H}_1) = \frac{\Pr(H_1|\mathrm{A})\Pr(A)}{\Pr(H_1)} = \frac{\Pr(H_1|\mathrm{A})\Pr(A)}{\Pr(H_1|\mathrm{A})\Pr(A)+\Pr(H_1|\mathrm{B})\Pr(B)+\Pr(H_1|\mathrm{C})\Pr(C)}$$

$$= \frac{\frac{1}{4}\times\frac{1}{3}}{\frac{1}{4}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}+\frac{3}{4}\times\frac{1}{3}} = \frac{1}{6}$$

$$\Pr(B|\mathrm{H}_1) = \frac{2}{6},\ \Pr(C|\mathrm{H}_1) = \frac{3}{6}$$

If you answer incorrectly for each $\Pr(A|\mathrm{H}_1)$, $\Pr(B|\mathrm{H}_1)$ and $\Pr(C|\mathrm{H}_1)$ subtract 0.5 pts each.

3-b) [2 points] Onyu Kang

The second toss is Tails. What are the probabilities that the selected coin is A, B and C, respectively? [2 pts]

Solution)

Let H_i be the event that the ith coin flipped is the Heads, and let T_i be the event that the ith coin flipped is the Tails. Before we flip the coins we have Pr(H | A) =1/4, Pr(H | B) = 1/2 and Pr(H | C) = 3/4.

Applying Bayes' theorem, we have

$$\Pr(\mathrm{A}|(\mathrm{H}_1,\mathrm{T}_2)) = \frac{\Pr((\mathrm{H}_1,\mathrm{T}_2)|\mathrm{A})\Pr(A)}{\Pr((\mathrm{H}_1,\mathrm{T}_2))} = \frac{\Pr((\mathrm{H}_1,\mathrm{T}_2)|\mathrm{A})\Pr(A)}{\Pr(H_1|\mathrm{A})\Pr(A)+\Pr((\mathrm{H}_1,\mathrm{T}_2)|\mathrm{B})\Pr(B)+\Pr((\mathrm{H}_1,\mathrm{T}_2)|\mathrm{C})\Pr(C)}$$

$$= \frac{\frac{1}{4}\times\frac{3}{4}\times\frac{1}{3}}{\frac{1}{4}\times\frac{3}{4}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{3}+\frac{3}{4}\times\frac{1}{4}\times\frac{1}{3}} = \frac{3}{10}$$

$$\Pr(\mathrm{B}|(\mathrm{H}_1,\mathrm{T}_2)) = \frac{4}{10}, \Pr(\mathrm{C}|(\mathrm{H}_1,\mathrm{T}_2)) = \frac{3}{10}$$

If you answer incorrectly for each $\Pr(\mathrm{A}|(\mathrm{H}_1,\mathrm{T}_2))$, $\Pr(\mathrm{B}|(\mathrm{H}_1,\mathrm{T}_2))$ and $\Pr(\mathrm{C}|(\mathrm{H}_1,\mathrm{T}_2))$, subtract 0.5 pts each.

3-c) [4 points] Onyu Kang

Let X1 and X2 be random variables of the results after rolling a fair die twice. Define events as follows. [4 pts]

E: max{X1,X2} is odd (e.g max{X1, X2} is 1,3 or 5)

F: X1+X2 = 4

Compute following values. Pr(E), Pr[X1=2 | F], Pr(E | F), Pr[X1=2 | E]

Solution)

Let set of all possible outcomes of rolling a fair die twice Ω, $|\Omega|$ = 36

E = {(1,1), (1,3), (1,5), (2,3), (2,5), (3,1), (3,2), (3,3) (3,5), ...(5,4), (5,5)} , |E|= 15

F = {(1,3), (2,2), (3,1)} , |F|= 3

$\Pr(E) = \frac{15}{36}$ ----- [1 pts]

$\Pr(F) = \frac{3}{36}$

Pr[X1=2 | F] = $\frac{\Pr((X_1 = 2) \cap \mathrm{F})}{\Pr(F)} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$ -----[1 pts]

Pr(E | F) = $\frac{\Pr(\mathrm{E} \cap \mathrm{F})}{\Pr(F)} = \frac{\frac{2}{36}}{\frac{3}{36}} = \frac{2}{3}$ -----[1 pts]

Pr[X1=2 | E] = $\frac{\Pr((X_1 = 2) \cap \mathrm{E})}{\Pr(E)} = \frac{\frac{2}{36}}{\frac{15}{36}} = \frac{2}{15}$ -----[1 pts]

Scores were specified for each.