1. A point is chosen at random (according to a uniform PDF) within a semicircle of the form {(x, y)$|x^2 + y^2 \leq r^2, y \geq 0\}$, for some given r > 0.
   (a) Find the joint PDF of the coordinates X and Y of the chosen point.
   (b) Find the marginal PDF of Y and use it to find E[Y].
   (c) Check your answer in (b) by computing E[Y] directly without using the marginal PDF of Y.

2. Let X be a random variable with PDF fx (x) = {X/4, if 1 <x $\leq$ 3, and let A be the event {X$\geq$ 2}.
   (a) Find E[X], P(A), $f_{X|A}$(X), and E[X|A].
   (b) Let Y = X^2. Find E[Y] and Var(Y).

3. We start with a stick of length *l*. We break it at a point which is chosen according to a uniform distribution and keep the piece of length Y, that contains the left end of the stick. We then repeat the same process on the piece that we were left with, and let X be the length of the remaining piece after breaking for the second time.
   (a) Find the joint PDF of Y and X.
   (b) Find the marginal PDF of X.
   (c) Use the PDF of X to evaluate E[X].
   (d) Evaluate E [X], by exploiting the relation X = Y·(X/ Y).

4. Let X and Y be two random variables that are uniformly distributed over the triangle formed by the points (0, 0), (1. 0), and (0, 2).
   (a) Find the joint PDF of X and Y.
   (b) Find the marginal PDF of Y.
   (c) Find the conditional PDF of X given Y.
   (d)Calculate E[X|Y=y], E[X] and E[Y].

5. Let X, Y. and Z be three random variables with joint PDF $f_{X,Y,Z}(x, y, z)$. Show the multiplication rule:
   $f_{X,Y,Z}(x, y, z) = f_{X|Y,Z}(x|y, z) \cdot f_{Y|Z}(y|z) \cdot f_Z(z)$

6. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF f_P(p) = $\mathrm{p} \cdot e^p, p \in [0,1]$
   A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.
   (a) Find the probability that a coin toss results in heads.
   (b) Given that a coin toss resulted in heads, find the conditional PDF of P.
   (c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the next toss.

7. Let X and Y be independent continuous random variables with PDFs $f_X, and\, f_Y$, respectively, and let Z = X + Y.

   (a) Show that $f_{Z|X}(z|x) = f_Y(z-x)$.

   (b) Assume that X and Y are exponentially distributed with parameter λ. Find the conditional PDF of X, given that Z = z.

   (c) Assume that X and Y are normal random variables with mean zero and variances σ_1^2, and σ_2^2, respectively. Find the conditional PDF of X, given that Z= z.

8. Soo. a student in a probability class. takes a multiple-choice test with 10 questions and 3 choices per question. For each question. there are two equally likely possibilities, independent of other questions: either she knows the answer, in which case she answers the question correctly. or else she guesses the answer with probability of success 1 /3.

   (a) Given that Soo answered correctly the first question, what is the probability that she knew the answer to that question?

   (b) Given that Soo answered correctly 6 out of the 10 questions, what is the posterior probability of the number of questions of which she knows the answer?

9. Students in a probability class take a multiple-choice test with 10 questions and 3 choices per question. A student who knows the answer to a question will answer it correctly, while a student that does not will guess the answer with probability of success 1/3. Each student is equally likely to belong to one of three categories i = 1, 2, 3: those who know the answer to each question with corresponding probabilities θ_i, where θ_1 = 0.3, θ_2 = 0.7, and θ_3= 0.95 (independent of other questions) . Suppose that a randomly chosen student answers k questions correctly.

   (a) For each possible value of k, derive the MAP estimate of the category that this student belongs to.

   (b) Let M be the number of questions that the student knows how to answer. Derive the posterior PMF, and the MAP and LMS estimates of M given that the student answered correctly 5 questions.

10. Professor May B. Hard, who has a tendency to give difficult problems in probability quizzes, is concerned about one of the problems she has prepared for an upcoming quiz. She therefore asks her TA to solve the problem and record the solution time. May's prior probability that the problem is difficult is 0.3, and she knows from experience that the conditional PDF of her TA 's solution time X, in minutes, is

    $f_{X|\theta}\left(\mathrm{x}\middle|\Theta = 1\right) = c_1 \cdot e^{-0.04x}$, $\mathrm{x} \in [5, 60]$,

    if a problem is difficult $(\Theta = 1)$, and is

    $f_{X|\theta}\left(\mathrm{x}\middle|\Theta = 2\right) = c_2 \cdot e^{-0.16x}$, $\mathrm{x} \in [5, 60]$,

    If a problem is not difficult $(\Theta = 2)$, where c_1, c_2, are normalizing constants. She uses the MAP rule to decide whether the problem is difficult.

(a) Given that the TA's solution time was 20 minutes, which hypothesis will she accept and what will be the probability of error?

(b) Not satisfied with the reliability of her decision, Professor May asks four more TAs to solve the problem. The TAs' solution times are conditionally independent and identically distributed with the solution time of the first TA. The recorded solution times are 10, 25, 15, and 35 minutes. On the basis of the five observations, which hypothesis will she now accept, and what will be the probability of error?

11. We have two boxes, each containing three balls: one black and two white in box 1; two black and one white in box 2. We choose one of the boxes at random, where the probability of choosing box 1 is equal to some given p, and then draw a ball.

(a) Describe the MAP rule for deciding the identity of the box based on whether the drawn ball is black or white.

(b) Assuming that $p = 1/2$, find the probability of an incorrect decision and compare it with the probability of error if no ball had been drawn.

12. A police radar always overestimates the speed of incoming cars by an amount that is uniformly distributed between 0 and 5 miles/hour. Assume that car speeds are uniformly distributed between 55 and 75 miles/hour. What is the LMS estimate of a car's speed based on the radar's measurement?

13. The number Θ of shopping carts in a store is uniformly distributed between 1 and 100. Carts are sequentially numbered between 1 and Θ. You enter the store, observe the number X on the first cart you encounter, assumed uniformly distributed over the range $1, \ldots, \Theta$, and use this information to estimate Θ. Find and plot the MAP estimator and the LMS estimator.

14. (a) Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables and let $Y = Y_1 + \ldots + Y_n$. Show that $E[Y_1 \mid Y] = Y/n$.

(b) Let Θ and W be independent zero-mean normal random variables, with positive integer variances k and m, respectively. Use the result of part (a) to find $E[\Theta|\Theta+W]$.

(c) Repeat part (b) for the case where Θ and W are independent Poisson random variables with integer means λ and μ, respectively.