Second Midterm

May 8, 2017

1. Consider the probability that every bin receives exactly one ball when *n* balls are thrown randomly into *n* bins. (2 pts each)

   (a) Give an upper bound on this probability using the Poisson approximation.

   (b) Determine the *exact* probability of this event.

   (c) Show that these two probabilities differ by a multiplicative factor that equals the probability that a Poisson random variable with parameter *//* takes on the value *n.*

2. Bounds (2 pts each)

   We can prove that if X is random variable with mean 0 and finite variance σ^2, then for any a >0, Pr[X >= a] <= σ^2/ (σ^2+a^2).

   a) Using the above fact, prove the one-sided Chebyshev inequality; if X has expectation μ and variance σ^2, then for any a > 0, Pr[X >= μ+a] <= σ^2/ (σ^2+a^2),

      and Pr[X <= μ-a] <= σ^2/ (σ^2+a^2).

   b) If the number of cars (let it be X) produced in a factory during a week is a random variable with mean 100 and variance 400, compute the upper bound of Pr[X >= 120] using the equation obtained in a). Also, compute the Markov bound.

   c) Let X be a Poisson with mean λ. We can obtain the Chernoff bound of Pr[X>=i] <= exp(λ$(e^t - 1)$ - it). Show that the bound is minimized at $e^t = i/\lambda$.

   d) Let X ~ Poi(20). Compute the Markov and Chernoff bounds of Pr[X>=26].

3. Markov Chain (2 pts each)

   A) Suppose that whether it rain or not today depends on previous weather conditions through the last two days. Show how this system may be modeled as a Markov chain. Particularly, show the state transitions satisfy the memoryless property. How many states are needed in your model?

   B) Suppose that if it has rained for the past two days, then it will rain today with probability 0.8; if it did not rain for any of the past two days, then it will rain today with probability

0.2, and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine the transition matrix of the system.

C) Draw the transition diagram of the system. Is the Markov chain irreducible? Pick (any) one state and show the state is positive recurrent.