Seoul
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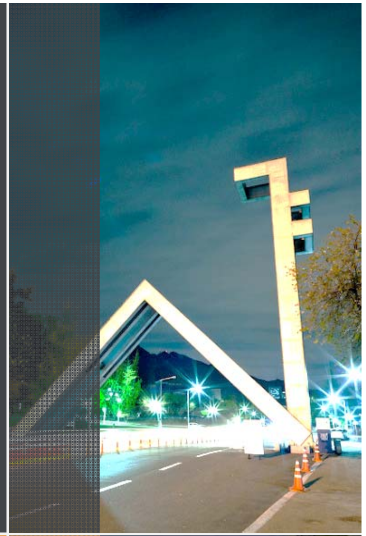
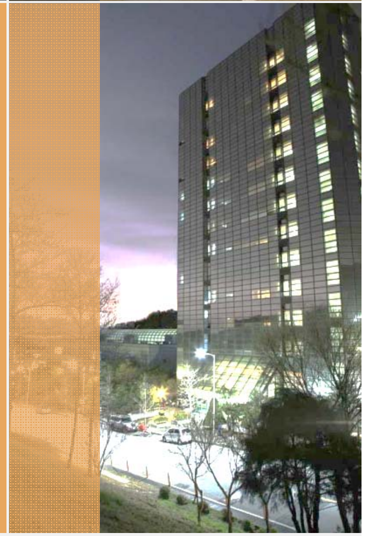




Conditional Probability

Name: Chong-kwon Kim

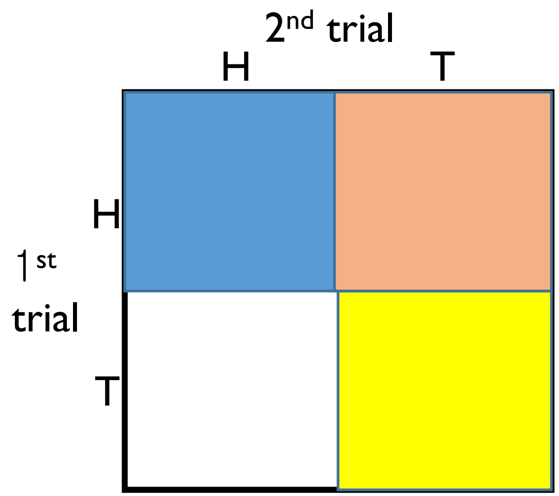
SCONE
Lab.

Independence

- Definition: Two events E and F are **independent** iff

$$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F)$$

- Example
  - Flip a coin twice
  - Event E: First trial is heads
  - Event F: Second trial is tails
  - $\Pr(E) = ½, \Pr(F) = ½ \rightarrow \Pr(E \cap F) = ¼$



What if the coin is unfair?

- Generally, If events $E_1, E_2, \dots E_n$ are mutually independent iff

$$\Pr(E_1 \cap E_2 \cap \dots \cap E_n) = \prod_{i=1}^{n} \Pr(E_i)$$

- Example
  - Baseball Matchup on a same day: Kia-Nx, Ss-LG, KT-SK, Hw-Do, NC-Lo
  - Prob. that Kia, Ss, KT, Do, NC win?

Independence

- Roll two dice, yielding values D1 and D2
- Events
  - E: D1 = 1
  - F: D2 = 1
  - What is Pr(E), Pr(F), and Pr(E ∩ F)?
  - Pr(E) = Pr(F) = 1/6, Pr(E∩F) = 1/36

    ➔ Pr(E ∩ F) = Pr(E) Pr(F) **Independent**

- Another event G: D1+D2 = 5 ➔ {(1,4), (2,3), (3,2), (4,1)}
  - What is Pr(E), Pr(G), and Pr(E ∩ G)?
  - Pr(E) = 1/6, Pr(G) = 4/36, Pr(E∩ G) = 1/36

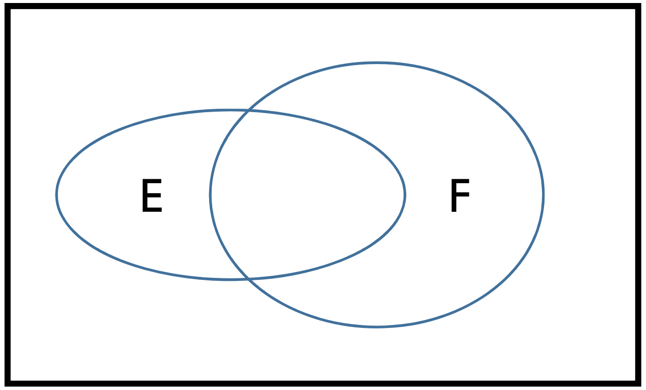
    ➔ Pr(E ∩ G) ≠ Pr(E) Pr(G) **Dependent**

Expectation of sum of rolling two dice?
Expectation of sum of rolling two dice given one is 1?

Conditional Probability

SCONE Lab.

- **Conditional probability, Pr(E|F)**
  - Probability that E occurs ***given*** that F has already occurred
  - "Conditioning on F"
- Sample space, Ω, shrinks to those elements in F (i.e. Ω ➔ Ω∩F)
- Event space, E, reduced to those elements coincide with F (i.e. E ∩F)



- With equally likely outcomes:

$$\Pr(E \mid F) = \frac{\#\,of\ outcomes\ in\ E\cap F}{\#\,of\ outcomes\ in\ F} = \frac{|E \cap F|}{|F|}$$

- In general

$$\Pr(E|F) = \frac{\Pr(E\cap F)}{\Pr(F)}$$ where Pr(F) > 0

Example

- Roll two dice, yielding values D_1 and D_2
- Let E be event: $D_1 + D_2 = 4$
- What is Pr(E)?
  - $|\Omega| = 36$, $E = \{(1, 3), (2, 2), (3, 1)\}$
  - $Pr(E) = 3/36 = 1/12$

- Let F be event: $D_1 = 2$
- Pr(E | F)?
  - $F = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)\}$
  - $E \cap F = \{(2, 2)\}$
  - $Pr(E \mid F) = 1/6$

Chain Rule

- From $\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}$,

  we obtain $\Pr(E \cap F) = \Pr(E \mid F) \cdot \Pr(F)$ **Chain Rule**

- If E and F are independent ➔ $\Pr(E|F) = \Pr(E)$
  - Example: Given the first coin flip is heads, the second coin flip is tails

- Generalized chain rule (Or Multiplication rule)

  $\Pr(E_1 \cap E_2 \cap \ldots \cap E_n)$

  $= \Pr(E_1)\Pr(E_2 \mid E_1)\Pr(E_3 \mid E_1 \cap E_2)\ldots\Pr(E_n \mid E_1 \cap E_2 \cap \ldots \cap E_{n-1})$

Properties of Conditional Probability

- Lemma:
  - $0 \leq Pr(E \mid F) \leq 1$
  - $Pr(\Omega \mid F) = 1$
  - For any sequence of pairwise mutually disjoint events $E_1, E_2, \cdots, E_n$

    $Pr(\bigcup_{i=1}^{n} E_i \mid F) = \sum_{i=1}^{n} Pr(E_i \mid F)$

Polynomial Identities: Revisit

SCONE Lab.

- Let $F(x) \neq G(x)$
- Randomized algorithm: Perform k trials and decide F(x)=G(x) if all trials claim F(x)=G(x)

- With replacement
  - Select r_i uniformly at random repeatedly from **R** = {1, 2, …, 100d}
  - Return r_i to **R** after the trial
  - Let Fi be an event that i-th trial fails ➔ $F(r_i) = G(r_i)$
  - $Pr(F_1) = Pr(F_2) = \ldots = Pr(F_k) \leq 1/100$
  - Pr(Randomized algorithm fails) = $Pr(F_1 \cap F_2 \cap \ldots \cap F_k) \leq (\frac{1}{100})^k$

Polynomial Identities: Revisit

- Without replacement
  - Discard r_i after the i-th trial
  - After the i-th trial, there are 100d-i elements in **R** and at most d-i roots in **R**

    ➔ $Pr(F_i \mid F_1 \cap F_2 \cap \ \dots \ \cap F_{i-1}) \leq \frac{d-(i-1)}{100d-(i-1)}$
  - Pr(Randomized algorithm fails)

    $= Pr(F_1 \cap F_2 \cap \dots \cap F_k)$

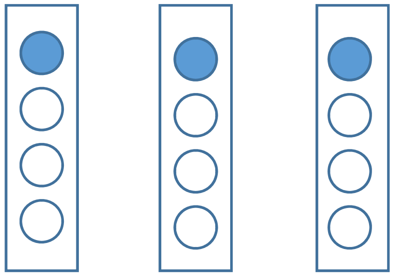
    $= Pr(F_1)Pr(F_2 \mid F_1)Pr(F_3 \mid F_1 \cap F_2)\dots Pr(F_k \mid F_1 \cap F_2 \cap \ \dots \ \cap F_{k-1})$

    $\leq \prod_{i=1}^{k} \frac{d-(i-1)}{100d-(i-1)} < (\frac{1}{100})^k$

Only **SLIGHTLY** better than with replacement algorithm

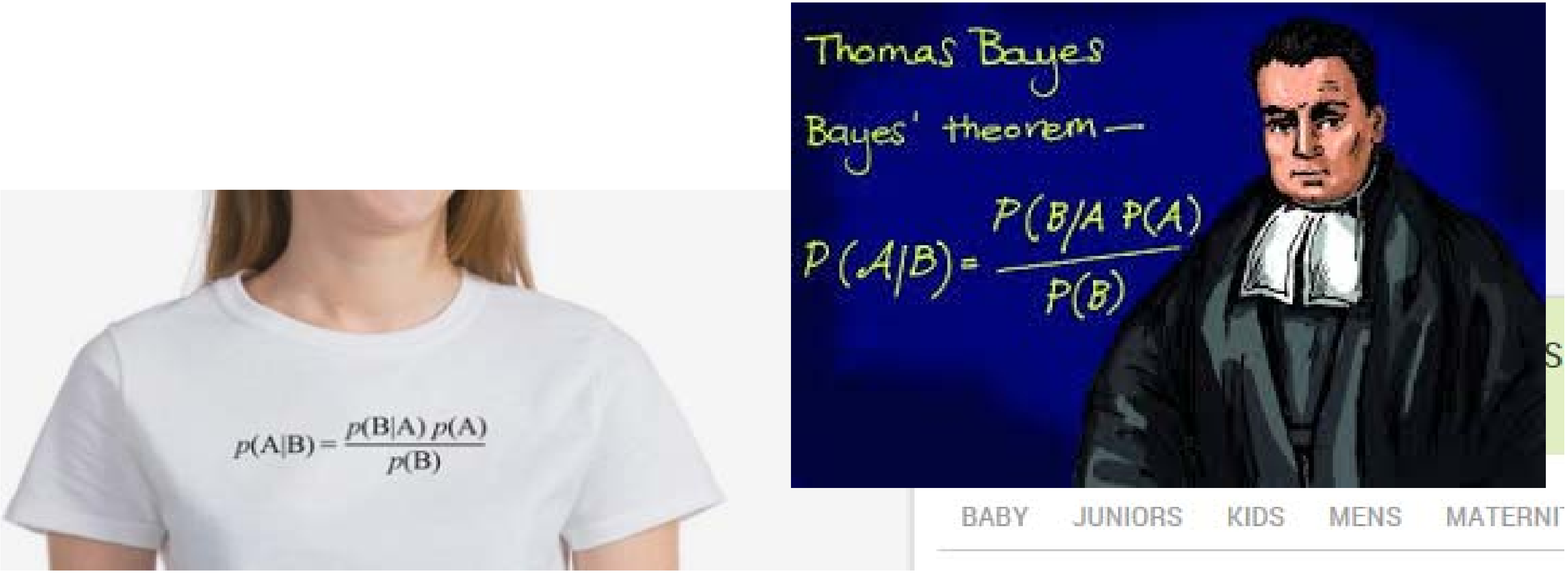
Example: Project Team

- With 12 students and make three project teams each of which consists of four randomly selected students
- Family name distribution: 3 Kim's (Let AKim, BKim, CKim) and 9 other surnames
- Probability that each team has exactly one Kim
- Solution
  - E1: AKim is in any one team
  - E2: AKim and BKim in different teams
  - E3: AKim, BKim and CKim in different teams
  - $\Pr(E3 \mid E1 \cap E2) = 4/10$



Bayes' Theorem (Law/Rule)

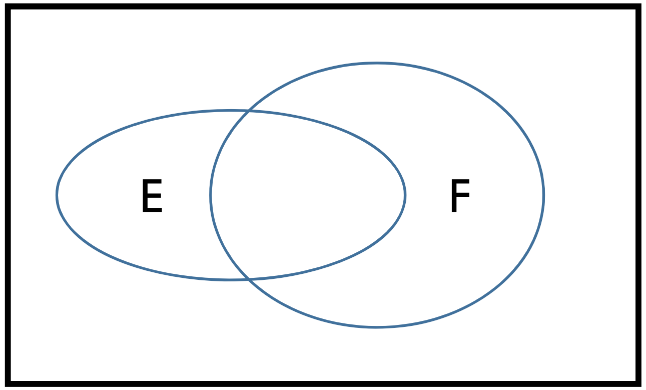
- Rev. Thomas Bayes (1702-1761) was a British minister



Bayes' Theorem

- $E = (E \cap F) \cup (E \cap \bar{F})$

Note $(E \cap F) \cap (E \cap \bar{F}) = \emptyset$



- $\Pr(E) = \Pr(E \cap F) + \Pr(E \cap \bar{F})$

$$= \Pr(E \mid F)\Pr(F) + \Pr(E \mid \bar{F})\Pr(\bar{F})$$

- $\Pr(F \mid E) = \dfrac{Pr(E \cap F)}{\Pr(E)} = \dfrac{Pr(E \mid F)\Pr(F)}{\Pr(E)}$

$$= \frac{Pr(E \mid F)\Pr(F)}{\Pr(E \mid F)\Pr(F) + \Pr(E \mid \bar{F})\Pr(\bar{F})}$$

- More generally,
  - Let $F_1, F_2, \ldots F_n$ be mutually exclusive and exhaustive events
  - Given E observed, want to determine which of F_j also occurred

$$\Pr(F_j \mid E) = \frac{Pr(E \mid F_j)\Pr(F_j)}{\sum_{i=1}^{n} Pr(E \mid F_i)\Pr(F_i)}$$

Spam Email

- Frequently used words and phrases in spam email
  - "Dear Friend", "Prize", "Make Money Fast", "Hot", "Million", …
- 60% of all email is spam
  - 50% of spam has MMF
  - 10% of non-spam has MMF
- An email has MMF. What is the probability that the email is spam?
  - E: Email has MMF
  - F: Email is spam

$$\Pr(F|E) = \frac{Pr(E \mid F)\Pr(F)}{\Pr(E \mid F)\,\Pr(F) + \Pr(E \mid \bar{F})\Pr(\bar{F})}$$

Learn: Naïve Bayesian Filtering (NBF)

Another Example

- Three coins
  - Two of them are un-biased and one is biased such that Pr(heads) = 2/3
  - Flip three coins in a random order and found that first and second coins are heads and third is tails
  - Compute the probability that the first coin is the biased coin
- Solution
  - Observed event: (H,H,T)
  - F1: First coin is biased, (similarly F2, F3)
  - Pr(F1 | (H, H, T)) = ?

YAE: Mamma Mia

- Child is born with (A, a) gene pair (Event (A,a))
  - Mother has (A, A) gene pair
  - Two possible fathers:
    - Adam: (a,a), Bob: (A,a)
  - Mother's belief: Pr(Adam) = p, Pr(Bob) = (1-p)

  - What is probability that the father is Adam?
  - Pr(Adam | (A,a)) = ?

Probability Inference

SCONE Lab.

- Bayes' Theorem

Posterior — Likelihood — Prior

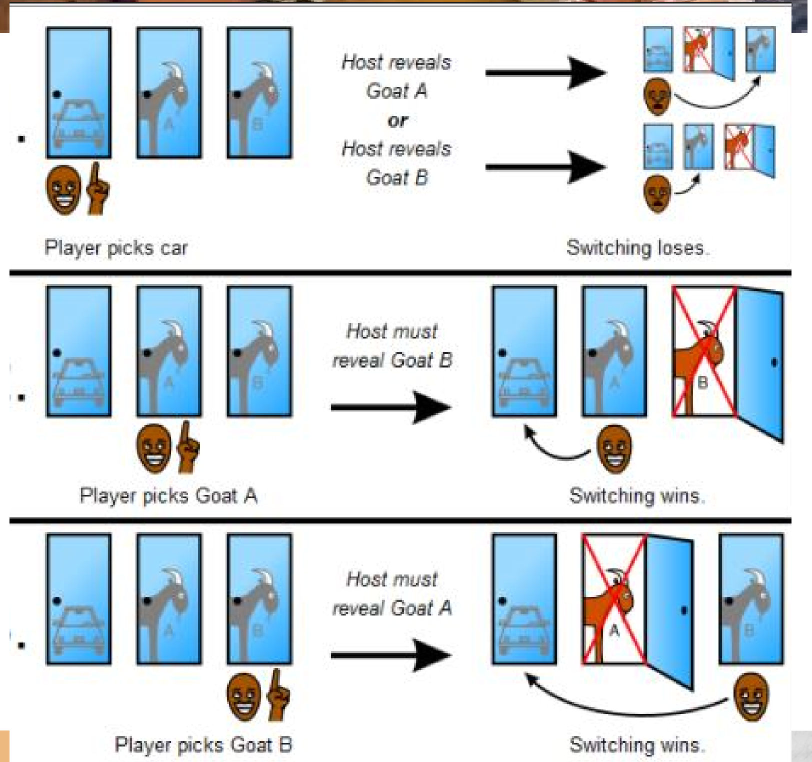
$$\Pr(F \mid E) = \frac{\Pr(E \mid F)\Pr(F)}{\Pr(E)}$$

- Probability changes after evidences (E) are observed

Monty Hall Problem

Let's Make a Deal

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?



Marilyn Savant
vs
Erdos

Monty Hall Problem

- Without loss of generality, assume the player picks door 1
- Define events
  - C1: Car is behind door 1 (Similarly C2, C3) → $Pr(C1) = 1/3$
  - X1: Player pick door 1
  - H3: Host open door 3
  - $Pr(H3 \mid C1 \cap X1) = ½$, $Pr(H3 \mid C2 \cap X1) = 1$, $Pr(H3 \mid C3 \cap X1) = 0$
- Probability of win after switching = $Pr(C2 \mid H3 \cap X1)$

  → Show that it is 2/3

Refer to Wikipedia

Random Bit Generator

- A random number bit generator produces a series of random bits, with probability p of producing a 1
  - Each bit generated is an independent trial
  - E: First n nits are all 1's, followed by a single 0

- Pr(E)?
  - Pr(first n 1's) = Pr(1st bit = 1)·Pr(2nd bit = 1) · · · Pr(n-th bit = 1)
    = p^n
  - Pr(E) = Pr(first n 1's)·Pr(n+1st bit = 0)
    = p^n(1-p)

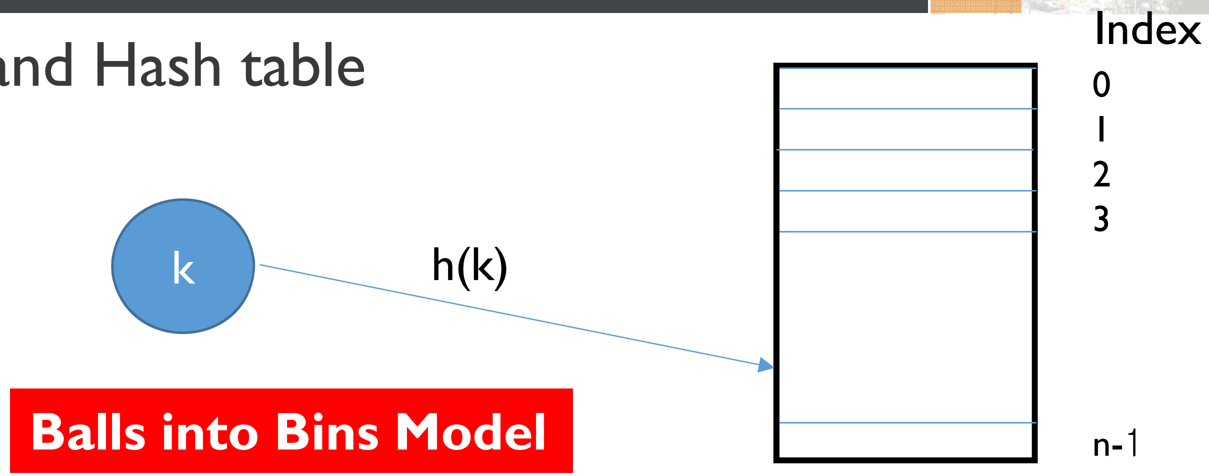
- Let F: k out of n random bits are 1
  - Pr(First k bits are 1, then n-k 0's) = $p^k(1-p)^{n-k}$
  - Pr(k out of n random bits are 1) = $\binom{n}{k} p^k(1-p)^{n-k}$

Search, Hashing and Bitcoin

- A fundamental operation in data analysis is to **find (search)** an object in a big dataset
- Many search algorithms
  - BST (Binary Search Tree)
  - Hashing
  - Usually, hashing is the most efficient and popular, yet simplest algorithm
    - Complexity = O(1)
- A hash function maps a large number to a smaller number, deterministically
  - One-way function
  - Given an input it is easy to compute its output, but the reverse is difficult
- Bitcoin
  - POW(Proof Of Work)
  - Given an output, find inputs that are close enough
  - SHA256 (256 bit Secure Hashing Algorithm)

Hash Table

- Key, Hash function, and Hash table



- Example: 주민번호
  - Each person has a unique key of 13 digits
    - Key space (K) = 10^{13}
  - There are < 10^8 unique keys
- Hash function h: K → (0, 1, …, n-1)
  - Simple uniform hash function: Each key is equally likely to hash to any of n slots (buckets)
- Collision
  - Two different keys are mapped to the same slot

Hash Table

- m keys are hashed into a hash table of n slots
  - Each key hashing is an independent trial
  - E: At least one key hashed to the first slot
  - Pr(E)?

Hint: Think out independent events. Then AND (intersection) of them.

- Solution
  - Fi: Key i not hashed to the first slot ($0 \leq i \leq m$)
  - Pr(Fi) = 1-1/n = (n-1)/n, for all $0 \leq i \leq m$
  - Pr(no keys hashed to the fist slot) = $Pr(F_1 \cap F_2 \cap \ldots \cap F_m)$
  - $P(E) = 1 - Pr(F_1 \cap F_2 \cap \ldots \cap F_m)$

    $= 1 - \left(\frac{n-1}{n}\right)^m$

- Similar to the *birthday problem*
  - Among m friends, at least one friend has the same birthday as you (n = 365)

Hash Table

- m keys are hashed into a hash table of n slots
- E: At least one of slots (1 to k) has keys hashed to it

- Solution
  - Ei: At least one key hashed into the i-th slot
  - Pr(E) = Pr(E1 U E2 U…U Ek)

$= 1\text{-Pr}(\overline{E1 \cup E2 \cup \ldots \cup Ek})$

$= 1\text{-Pr}(\overline{E1} \cap \overline{E2} \cap \ \ldots \cap \overline{Ek})$

Ei & Ej independent ?

$= 1\text{-}(\frac{n-k}{n})^m$

Odds

- **Odds** of an event (H) is defined as

$$\frac{\Pr(H)}{\Pr(\overline{H}\,)} = \frac{\Pr(H)}{1 - \Pr(H)}$$

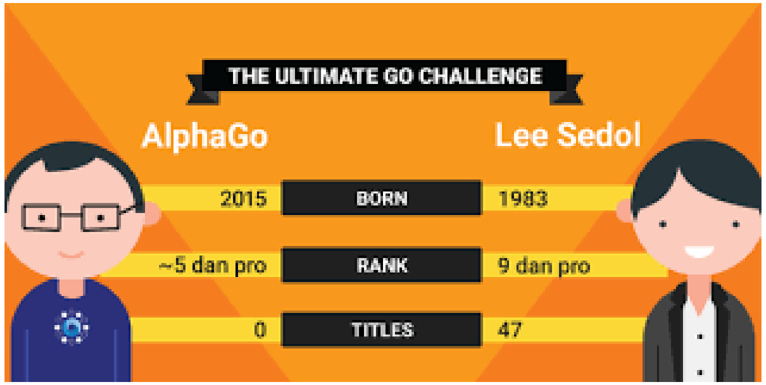
- Odds of H given evidence E

$$\frac{\Pr(H \mid E)}{\Pr(\overline{H} \mid E\,)} = \frac{\Pr(H)\Pr(E \mid H) \,/\, \Pr(E)}{\Pr(\overline{H})\Pr(E \mid \overline{H}) \,/\, \Pr(E)}$$

$$= \frac{\Pr(H)\Pr(E \mid H)}{\Pr(\overline{H})\Pr(E \mid \overline{H})}$$

- After observing E, update odds by $\frac{\Pr(E \mid H)}{\Pr(E \mid \overline{H})}$

Lee Sedol vs AlphaGo



- Let H: Lee is better than AG
- Before the match, Pr(H) = 0.9

- If Lee is better than AG, then Lee wins game with 0.8 probability
- If AG is better than Lee, then AG wins game with 0.9 probability
- E: AG won a game
- What is updated odds after the game?

- What if AG wins two games in a row?

Coins & Urns

- An urn contains 2 coins: A and B
  - A comes up heads with probability ¼
  - B comes up heads with probability ¾
  - Pick coin randomly and flip it, and it comes up heads

- What are the odds that A was picked?
  - Before the experiment Pr(A) = Pr($\bar{A}$) = Pr(B) = ½

    ➔ $\frac{\Pr(A)}{\Pr(\bar{A})} = \frac{\Pr(A)}{1-\Pr(A)}$ = 1
  - After the experiment
  - ➔ $\frac{\Pr(A \mid heads)}{\Pr(\bar{A} \mid heads)} = \frac{\Pr(A)\Pr(heads \mid A)}{\Pr(\bar{A})\Pr(heads \mid \bar{A})}$

    $= \frac{1/2}{1/2}\frac{1/4}{3/4}$ = 1/3

Verifying Matrix Multiplication

- Given three n×n matrices **A**, **B**, and **C**
- Want to verify that **AB** = **C**
- Complexity of matrix multiplication
  - $\Theta(n^3)$
  - $\Theta(n^{2.37})$ (Best Algorithm)
- Randomized algorithm
  - Select a vector $\bar{r} = (r_1, r_2, \ldots, r_n) \in \{0,1\}^n$
  - Compute **AB**$\bar{r}$ (First compute **B**$\bar{r}$ and then **A**(**B**$\bar{r}$), Complexity = $\Theta(n^2)$)
  - Compute **C**$\bar{r}$
  - Decision:
    - If **AB**$\bar{r}$ = **C**$\bar{r}$ → Conclude that **AB** = **C**
    - If **AB**$\bar{r} \neq$ **C**$\bar{r}$ → Conclude that **AB** $\neq$ **C**

Theorem

- If **AB** $\neq$ **C** and if $\bar{r}$ is chosen uniformly at random from $\{0,1\}^n$, then Pr(**AB**$\bar{r}$ = **C**$\bar{r}$) $\leq$ ½
- Proof
  - First, note that selecting $\bar{r}$ uniformly at random from $\{0,1\}^n$ is equivalent to select each r_i uniformly at random from $\{0,1\}$
  - Let **D** = **AB** – **C** $\neq$ **0**
  - From **AB**$\bar{r}$ = **C**$\bar{r}$, we know that **D**$\bar{r}$ = **0**
  - Because **D** $\neq$ **0**, there must be some non-zero elements in **D**
  - Let a non-zero element is d_{11}
  - $\sum_{j=1}^{n} d_{1j} \cdot r_j$=0

    ➔ $r_1 = -\frac{\sum_{j=2}^{n} d_{1j} \cdot r_j}{d_{11}}$ (1)
  - There is at most one choice of r_1 that satisfies Eq 1.
  - Because r_1 can be either 0 or 1, the probability that **AB**$\bar{r}$ = **C**$\bar{r}$ is at most ½

Randomized Algorithm

- Assume that **AB** $\neq$ **C**
- Repeat the test k times with $\bar{r}$ selected uniformly at random from $\{0, 1\}^n$.

  If all k test results are **AB**$\bar{r}$ = **C**$\bar{r}$, then conclude that **AB** = **C**

- Analysis
  - Fi: Event that i-th test fails
  - $Pr(F_1) = Pr(F_2) = \ldots = Pr(F_k) \leq 1/2$
  - $Pr(\text{Algorithm fails}) = Pr(F_1 \cap F_2 \cap \ldots \cap F_k) \leq 2^{-k}$

Application of Bayes' Theorem

- E: Event that **AB** = **C**
- At the beginning, we do not know if it is true or false
  ➔ Prior knowledge $\Pr(E) = \Pr(\bar{E}) = ½$
- B1: First test returns that the identity is correct
- $\Pr(E \mid B1) = \dfrac{\Pr(B1 \mid E)\cdot\Pr(E)}{\Pr(B1 \mid E)\cdot\Pr(E) + \Pr(B1 \mid \bar{E})\cdot\Pr(\bar{E})}$

  $\geq 2/3$
- B2: Second test returns that the identity is correct
- $\Pr(E \mid B2) \geq \dfrac{2/3}{2/3 + 1/3 \cdot 1/2} \geq 4/5$
- Assume that after i-th test, our belief is that $\Pr(E) \geq 2^i/(2^i+1)$
- $\Pr(E \mid Bi+1) \geq \dfrac{2^{i+1}}{2^{i+1}+1} = 1 - \dfrac{1}{2^i+1}$

Advanced Conditional Probability

- Insurance companies have been using probabilities to make different yet proper charges to customers
  - For example, customers who are more probable to incur costs are charged more than customers with less risks
- Car insurance company problem
  - There are two types of drivers: Careful (0.6) and Careless(0.4)
  - Probabilities that careful and careless customer have accidents in a year are 0.2 and 0.4, respectively
  - Events to have accidents in each year are independent (Depends only on the driver types)
  - Given that a new customer has accidents in the first year, What is the probability that the customer have accidents in the second year?
- Note $\Pr(E \mid F) = \Pr(E \mid G \cap F)\Pr(G \mid F) + \Pr(E \mid \bar{G} \cap F)\Pr(\bar{G} \mid F)$
  - $\Pr(E \mid F) = \Pr(E \cap G \mid F) + \Pr(E \cap \bar{G} \mid F)$

$$
\begin{aligned}
&= \Pr(E \cap G \cap F) / \Pr(F) + \Pr(E \cap \bar{G} \cap F) / \Pr(F) \\
&= \Pr(E \mid G \cap F)\Pr(G \cap F) / \Pr(F) + \Pr(E \mid \bar{G} \cap F)\Pr(\bar{G} \cap F) / \Pr(F) \\
&= \Pr(E \mid G \cap F)\Pr(G \mid F)\Pr(F) / \Pr(F) \\
&\quad + \Pr(E \mid \bar{G} \cap F)\Pr(\bar{G} \mid F)\Pr(F) / \Pr(F)
\end{aligned}
$$

Advanced Conditional Probability

- Solution
  - A2: Event that the customer have accidents in the second year
  - A1: Event that the customer have accidents in the first year
  - C: Event that customer is careful ($\bar{C}$: Careless)
  - $\Pr(E \mid F) = \Pr(E \mid G \cap F) \Pr(G \mid F) + \Pr(E \mid \bar{G} \cap F) \Pr(\bar{G} \mid F)$
  - $E \leftarrow A2$, $F \leftarrow A1$, $G \leftarrow C$
  - $\Pr(A2 \mid A1) = \Pr(A2 \mid A1 \cap C) \Pr(C \mid A1) + \Pr(A2 \mid A1 \cap \bar{C}) \Pr(\bar{C} \mid A1)$
  - Compute $\Pr(C \mid A1)$ using Bayes' Theorem
  - $\Pr(A2 \mid A1 \cap C)$??

- Suppose a customer have accidents in first and second years consecutively, what is the probability that the customer is a careful driver?

  → $\Pr(C \mid A1 \cap A2)$

AlphaGo W/W/W/L

Sequential Information Update

Sequential Information Update

SCONE Lab.

- A hypotheses H (such as a driver is careful driver) with an initial guess is given Pr(H is True) = p = 1- Pr(H is False)
- After an Event E is occurred, the conditional probability that H is True (Let this be T) is given as
  - $\Pr(T \mid E) = \dfrac{\Pr(E \mid T)\Pr(T)}{\Pr(E \mid T)\Pr(T) + \Pr(E \mid F)\Pr(F)}$
- Now, suppose we observed two successive (independent) events E1 and E2
  - $\Pr(T \mid E1 \cap E2) = \dfrac{\Pr(E1 \cap E2 \mid T)\Pr(T)}{\Pr(E1 \cap E2 \mid T)\Pr(T) + \Pr(E1 \cap E2 \mid F)\Pr(F)}$
- Can we consider E2 as E and Pr(T | E1) as Pr(T)?
- Solution
  - Yes, if E1 and E2 are conditionally independent given H

    ➔ $\Pr(E1 \cap E2 \mid H) = \Pr(E2 \mid H)\,\Pr(E1 \mid H)$
  - To show $\Pr(T \mid E1 \cap E2) = \dfrac{\Pr(E2 \mid T)\Pr(T \mid E1)}{\Pr(E2 \mid T)\Pr(T \mid E1) + \Pr(E2 \mid F)\Pr(F \mid E1)}$

Conditional Independence

- Let E and F are independent

  ➔ E given G and F given G are independent also?

- No, Counter example
  - Roll two dice yielding values D1 and D2
  - E: D1=1
  - F: D2=6
  - G: D1+D2=7
  - E and F are independent, $Pr(E \cap F) = 1/36$ and $Pr(E)=1/6$, $Pr(F)=1/6$
  - $Pr(E \mid G)=1/6$, $Pr(F \mid G)=1/6$ and $Pr(E \cap F \mid G)=1/6$

- Events E and F are **conditionally Independent** given G iff

$$Pr(E \cap F \mid G) = Pr(E \mid G)\, Pr(F \mid G)$$

- Prove that if E and F are conditionally independent given G then $Pr(E \mid F \cap G) = Pr(E \mid G)$

Another Example

- 100 person in Bldg 302
  - 30 are in CS Dept. (Either students or faculty)
  - 20 are Faculty
  - There are 6 CS Faculty
  - Pr(CS)=0.3, Pr(F)=0.2 Pr(CS ∩ F)=0.06 ➔ CS and F are independent
  - Only the persons in CS Dept. or Faculty can use the DiningHall
  - CS given DiningHall and F given DiningHall are independent?
- Solution
  - D: DiningHall users = CS U F
  - |D| = 30 + 20 - 6 = 44
  - Pr(CS|D) = 30/44, Pr(F|D) = 20/44, Pr(CS ∩ F | D) = 6/44

    ➔ **Conditionally Dependent**

Independence & Conditioning

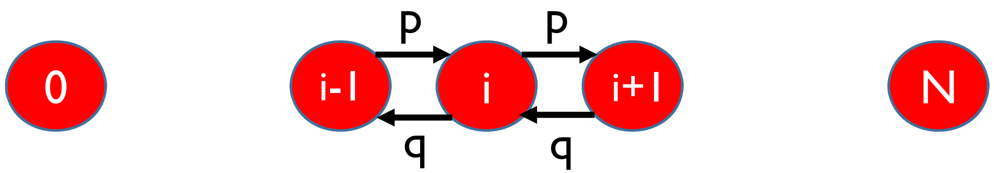
- Conditioning can make dependent events to independent?

- Yes, Example
  - Sample space: {M, Tu, W, Th, F, Sa, Su}
  - A: not Monday = {Tu, W, Th, F, Sa, Su}
  - B: Sa
  - C: {Sa, Su}
  - Pr(A)=6/7, Pr(B)=1/7 and Pr(A ∩ B)=1/7
    ➔ A and B are dependent

  - Pr(A|C)=1, Pr(B|C)=1/2, P(A ∩ B|C)=1/2
    ➔ A|C and B|C are independent

Gambler's Ruin Problem

- Game setting
  - Gambler A and B
  - Successive coin flips. If heads, A collect one unit from B. If tails, A give one unit to B
  - Pr(heads) = p = 1-Pr(tails)
  - A starts with i units and B starts with N-i units
  - Game finishes when one of gamblers collects all
  - Probability that A wins?



- Solution
  - E: A wins
  - H: first flip is heads
  - Pi = Pr(E) = Pr(E | H)Pr(H) + Pr(E | $\bar{H}$)Pr($\bar{H}$)

Recursion !!

Gambler's Ruin Problem

SCONE Lab.

- Solution
  - $P_i = Pr(E \mid H) \cdot p + Pr(E \mid \overline{H}) \cdot (1-p)$

    $= p \cdot P_{i+1} + q \cdot P_{i-1}$

    ➔ $p \cdot P_i + q \cdot P_i = p \cdot P_{i+1} + q \cdot P_{i-1}$

    ➔ $P_{i+1} - P_i = q/p\ (P_i - P_{i-1})$
  - Obviously, $P_0 = 0$ and $P_N = 1$

    $P_2 - P_1 = q/p\ (P_1 - P_0) = (q/p)\ P_1$

    $P_3 - P_2 = q/p\ (P_2 - P_1) = (q/p)^2\ P_1$

    .

    .

    $P_i - P_{i-1} = (q/p)^{i-1}\ P_1$

    ➔ $P_i - P_1 = P_1\ [\ (q/p) + (q/p)^2 + \ldots + (q/p)^{i-1}\ ]$

    ➔ $P_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)} \cdot P_1, & if\ p \neq 1/2 \\ i \cdot P_1, & if\ p = 1/2 \end{cases}$

Gambler's Ruin Problem

- Solution
  - From $P_N = 1$, we obtain

$$P_1 = \begin{cases} \dfrac{1-(q/p)}{1-(q/p)^N} & , if\ p \neq 1/2 \\ \dfrac{1}{N} & , if\ p = 1/2 \end{cases}$$

$$\rightarrow P_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & , if\ p \neq 1/2 \\ \dfrac{i}{N} & , if\ p = 1/2 \end{cases}$$