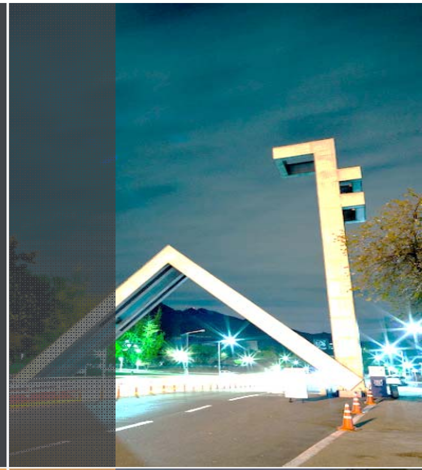

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Moments

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Key Definitions

- Definition: k–th moment of X $\equiv$ $\mathrm{E}[X^k]$

- Definition: Variance

$$\mathrm{Var}[X] = \mathrm{E}[(X - E[X])^2]$$
$$= \mathrm{E}[X^2] - E[X]^2$$

- Definition: Standard deviation

$$\sigma[X] = \sqrt{\mathrm{Var}[X]}$$

Variance: Binomial

- X: Binomial r v with n, p
- E$[X^2] = \sum_{j=0}^{n} \binom{n}{j} p^j \cdot (1-p)^{n-j} \cdot\ j^2$

$$= \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^j \cdot (1-p)^{n-j} \cdot ((j^2 - j) + j)$$

$$\vdots$$

$$= n(n-1)\ p^2 + np$$

- Var[X] $= E[X^2] - (E[X])^2$

$$= np(1-p)$$

- More simply, X is sum of n independent Bernoulli r v
- Var[X] = Var$[\sum_i$ Xi]

$$= \sum_i Var[X_i]$$

$$= np(1-p)$$

Generally,
Var$[\sum_i$ Xi] ?= $\sum_i \boldsymbol{Var[X_i]}$

Variance: Geometric R V

- Y: Geometric random variable
  - We know $E[Y] = 1/p$
  - From $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$, we obtain

    $(\frac{1}{1-x})^2 = \sum_{i=1}^{\infty} i \cdot x^{i-1}$

    $2 \cdot (\frac{1}{1-x})^3 = \sum_{i=2}^{\infty} i \cdot (i-1) \cdot x^{i-2}$

  - $E[Y^2] = \sum_{i=0}^{\infty} p(1-p)^{i-1} \cdot i^2$

    $= \frac{2-p}{p^2}$

$\sum i^2 \cdot x^i = \sum (i+1)(i+2)x^i - 3\sum_{i=0}^{\infty}(i+1)x^i + \sum_{i=0}^{\infty} x^i$

  - $Var[Y] = E[Y^2] - (E[Y])^2$

    $= \frac{1-p}{p^2}$

Independence

- Note that E[X+Y] = E[X] + E[Y] holds even if X and Y are dependent

- How about $E[X \cdot Y] \equiv E[X] \cdot E[Y]$?
  - True only if X and Y are independent
  - Counter example:
    - Flip two coins
    - X: Indicator function of first coin = heads
    - Y: Sum of heads in two coin flips
    - $E[X] = \frac{1}{2}$
    - $E[Y] = 1$
    - $E[X \cdot Y] = \sum_i \sum_j i \cdot j \Pr((X=i) \cap (Y=j))$
  - Independent
    - X: Indicator function of first coin = heads
    - Y: Indicator function of second coin = heads

Independence

- Theorem: If X and Y are independent,
  then $E[X \cdot Y] = E[X] \cdot E[Y]$

- Proof
  - $E[X \cdot Y] = \sum_i \sum_j i \cdot j \Pr((X=i) \cap (Y=j))$
    $= \sum_i \sum_j i \cdot j \Pr(X=i) \cdot \Pr(Y=j)$
    $=$

Independence

- Covariance of two r v X and Y

  Cov(X, Y) = E[(X−E[X])·(Y−E[Y])]

- Theorem: Var[X+Y] = Var[X] + Var[Y] + 2 · Cov(X,Y)
- Proof:
  - Var[X+Y] = E[$((X+Y)-(E[X]+E[Y]))^2$]

    = E[$(X-E[X])+(Y-E[Y])^2$]

- If X and Y are independent,

  then Cov(X, Y) = 0 and

  Var[X+Y] = Var[X] + Var[Y]

- Proof
  - Cov(X,Y) = $\mathrm{E}[(X-E[X])\cdot(Y-E[Y])]$

    $= \mathrm{E}[X\cdot Y - X\cdot E[Y] - Y\cdot E[X] + E[X]\cdot E[Y]]$

    = 0

Moment Generating Function

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- Function that generates moments

  $M_X(\mathrm{t}) = \mathrm{E}[\ e^{tX}\ ] = \sum_i e^{tx_i} \cdot \Pr(X = x_i)$

- $\mathrm{E}[\ X^n] = {M_X}^{(n)}(0)$
  - where ${M_X}^{(n)}(\mathrm{t})$ is n th derivative of $M_X(\mathrm{t})$

- Proof

  True! But omit the proof

  - If we (can) exchange expectation and differentiation operands
  - Then, ${M_X}^{(n)}(\mathrm{t}) = \mathrm{E}[\ X^n \cdot e^{tX}\ ]$
  - At t=0, ${M_X}^{(n)}(0) = \mathrm{E}[\ X^n\ ]$

MGF – Example

- Geometric Distribution, Pr(X=k) = $(1-p)^{k-1} \cdot p$
  - $M_X(\mathrm{t}) = \mathrm{E}[e^{tX}]$
    $= \sum_{k=1}^{\infty}(1-p)^{k-1} \cdot p \cdot e^{tk}$
    $= \frac{p}{1-p}(1-(1-p)e^t)^{-1} - 1$
  - $M_X{}^{(1)}(t) = p(1-(1-p)e^t)^{-2}e^t$
  - $M_X{}^{(2)}(t) = 2p(1-p)(1-(1-p)e^t)^{-3}e^{2t} + p(1-(1-p)e^t)^{-2}e^t$

Properties

- If two random variables X and Y have the same MGF, then X ≡ Y

- If X and Y are independent r.v., then

  $M_{X+Y}(\mathrm{t}) = M_X(\mathrm{t})\ M_Y(\mathrm{t})$

- Proof
  - $M_{X+Y}(\mathrm{t}) = \mathrm{E}\left[e^{t(X+Y)}\right]$

    $= E[e^{tX} \cdot e^{tY}]$

    $= M_X(\mathrm{t}) \cdot M_Y(\mathrm{t})$