Chap 1. Matrices and Gauss Elimination

1.1 Introduction

Solving Linear Equations

$$\begin{cases} 1x + 2y = 3 \\ 4x + 5y = 6 \end{cases}$$

① Elimination:

$$\begin{cases} 1x + 2y = 3 \\ \quad\;\; -3y = -6 \end{cases} \Rightarrow y = 2 \Rightarrow x + 4 = 3$$

$$\therefore x = -1$$

② Determinants (Cramer's Rule)

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{-6}{-3} = 2.$$

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3}{-3} = -1$$

Remark:

- Cramer's Rule is inefficient for a large system of linear equations.
- Gauss Elimination is a good method that is constantly used to solve large systems of equations.

Four aspects of linear equations

① Geometry of planes:

two lines meeting at the point $(x,y)=(-1,2)$

② Matrix notation:

Factorization $A=\begin{bmatrix}1&2\\4&5\end{bmatrix}=\underbrace{\begin{bmatrix}1&0\\4&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1&2\\0&-3\end{bmatrix}}_{U}$

③ Singular case:

Two parallel lines $\begin{cases}1x+2y=3\\4x+8y=6\end{cases}\Rightarrow 0=-6$

No Solution!

Two overlapping lines: $\begin{cases}1x+2y=3\\4x+8y=12\end{cases}\Rightarrow 0=0$

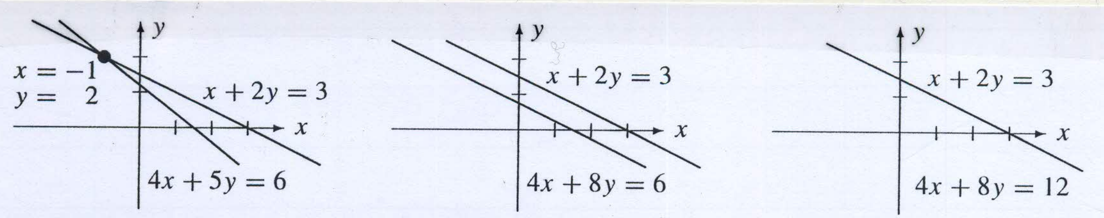
Infinitely many solutions!

④ The number of elimination steps:

o. The computing cost often dominates the accuracy.

100 equations requires $\approx\frac{1}{3}\times10^6$ steps,

$\Rightarrow$ the round off error could be quite significant.



One solution $(x, y) = (-1, 2)$ **Parallel: No solution** **Whole line of solutions**

Figure 1.1 The example has one solution. Singular cases have none or too many.

1.2 The Geometry of Linear Equations

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

By Rows:

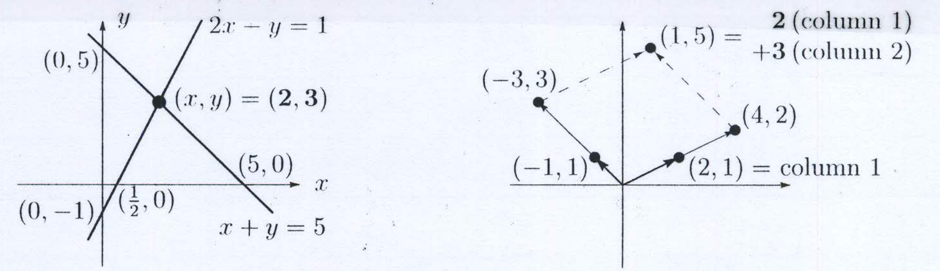
Two straight lines intersecting at $(x, y) = (2, 3)$

By Columns:

One vector equation

Column form: $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

To find the combination of the column vectors on the left that produces the vector on the right.



(a) **Lines meet at $x = 2, y = 3$**　　(b) **Columns combine with 2 and 3**

Figure 1.2 Row picture (two lines) and column picture (combine columns).

Row Picture : Intersection of planes
Column Picture : Combination of columns

Three Planes

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v \quad\quad = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

three planes intersecting at (1, 1, 2)

Column Vectors and Linear Combinations

Column form : $u\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \mathbb{b}$

Linear Combination : $1\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

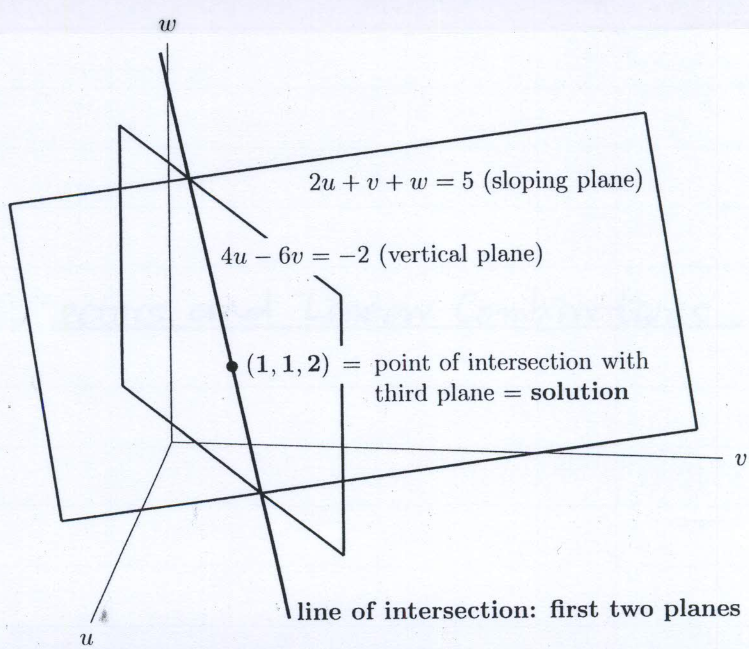


Figure 1.3 The row picture: three intersecting planes from three linear equations.

The Singular Case

(a) Two planes may be parallel

$$\left.\begin{cases} 2u+v+w=5 \\ 4u+2v+2w=11 \end{cases}\right] \text{Inconsistent}$$

parallel planes give no solution

(b) Every pair of planes intersects in a line, and those lines are parallel.

$$\begin{cases} u+v+w=2 \\ 2u \qquad +3w=5 \\ 3u+v+4w=6 \end{cases}$$

$u+v+w=2$ and $2u+3w=5$ add up to $\Rightarrow$ $3u+v+4w=7$

$3u+v+4w=6$ $\longrightarrow$ Inconsistent

The third plane is not parallel to the other planes, but it is parallel to their line of intersection.

(c) Another singular system has an <u>infinity of solutions</u>

$$\begin{cases} u+v+w=2 \\ 2u \qquad +3w=5 \\ 3u+v+4w=7 \end{cases}$$

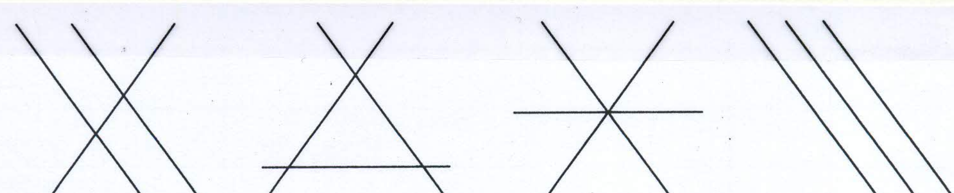
$u+v+w=2$ and $2u+3w=5$ add up to $\Rightarrow$ $3u+v+4w=7$

$3u+v+4w=7$ $\longrightarrow$ the same

too many solutions

(d) Three parallel planes:

No solution or a whole plane of solutions.



two parallel planes (a) no intersection (b) line of intersection (c) all planes parallel (d)

Figure 1.5 Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

The Singular Case : Column Picture

Three columns in the same plane

Solvable only for $\mathbb{b}$ in that plane

$$u\begin{bmatrix}1\\2\\3\end{bmatrix}+v\begin{bmatrix}1\\0\\1\end{bmatrix}+w\begin{bmatrix}1\\3\\4\end{bmatrix}=\mathbb{b}$$

- For $\mathbb{b}=(2,5,6)$, it is non-solvable.
- For $\mathbb{b}=(2,5,7)$, it is possible and there are too many solutions.

The three columns can be combined in infinitely many ways to produce $\mathbb{b}$.

When $u=3$, $v=-1$, $w=-2$,

$$3\begin{bmatrix}1\\2\\3\end{bmatrix}+(-1)\begin{bmatrix}1\\0\\1\end{bmatrix}+(-2)\begin{bmatrix}1\\3\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$(u,v,w)=(1,0,1)$ is a solution, and we can add any multiple of $(3,-1,-2)$ to $(u,v,w)=(1,0,1)$ to produce a solution.

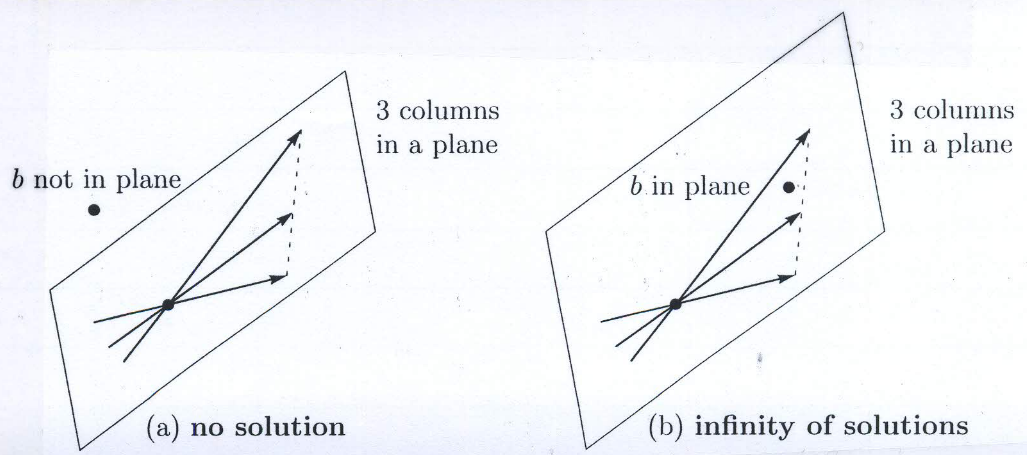


Figure 1.6 Singular cases: b outside or inside the plane with all three columns.

1.3 Example of Gaussian Elimination

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases} \Rightarrow \begin{cases} 2u + v + w = 5 \\ -8v - 2w = -12 \\ 8v + 3w = 14 \end{cases}$$

→ pivots

Forward Elimination

$$\Downarrow$$

→ pivots

$$\begin{cases} 2u + v + w = 5 \\ -8v - 2w = -12 \\ 1w = 2 \end{cases}$$

Triangular System

Backward Substitution

$w = 2,\ v = 1,\ u = 1$

Remark

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The right-hand side as an extra column

The Breakdown of Elimination

- Pivots cannot be zero
- If a zero appears in a pivot position,
  1. non-singular case, exchange rows (Sec 1.5)
  2. singular case, either no solution or infinitely many solutions (Chap 2 deals with the singular cases.)

Ex1 (Non-singular Case)

$$\begin{cases} u+v+w=\_ \\ 2u+2v+5w=\_ \\ 4u+6v+8w=\_ \end{cases} \Rightarrow \begin{cases} u+v+w=\_ \\ 3w=\_ \\ 2v+4w=\_ \end{cases}$$

$$\Downarrow$$

exchange equations 2 & 3

$$\begin{cases} u+v+w=\_ \\ 2v+4w=\_ \\ 3w=\_ \end{cases}$$

Ex2 (Singular Case)

$$\begin{cases} u+v+w=\_ \\ 2u+2v+5w=\_ \\ 4u+4v+8w=\_ \end{cases} \Rightarrow \begin{cases} u+v+w=\_ \\ 3w=\_ \\ 4w=\_ \end{cases}$$

(No exchange of equations can avoid zero in the second pivot position).

Case1: (No solution)

$$\begin{cases} u+v+w=\_ \\ 3w=6 \\ 4w=7 \end{cases}$$

unsolvable

Case2: (Infinitely many solutions)

$$\begin{cases} u+v+w=\_ \\ 3w=6 \\ 4w=8 \end{cases}$$

consistent ($w=2$)

But, we cannot decide u and v.

The Cost of Elimination

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & & \\ a_{n1} & \cdots & a_{nn} & b_n \end{bmatrix} \xrightarrow{n(n-1)\text{ "multiply-subtract"}} \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{*} & & a_{2n}^{*} & b_2^{*} \\ \vdots & \vdots & & & \\ 0 & a_{n2}^{*} & \cdots & a_{nn}^{*} & b_n^{*} \end{bmatrix}$$

$$\xrightarrow{(n-1)(n-2)\text{ "multiply-subtract"}} \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{*} & \cdots & a_{2n}^{*} & b_2^{*} \\ 0 & 0 & a_{33}^{**} \cdots & a_{3n}^{**} & b_3^{**} \\ \vdots & \vdots & & & \\ 0 & 0 & a_{n3}^{**} \cdots & a_{nn}^{**} & b_n^{**} \end{bmatrix}$$

$\vdots$

$\xrightarrow{2\cdot 1\text{ "multiply-subtract"}}$

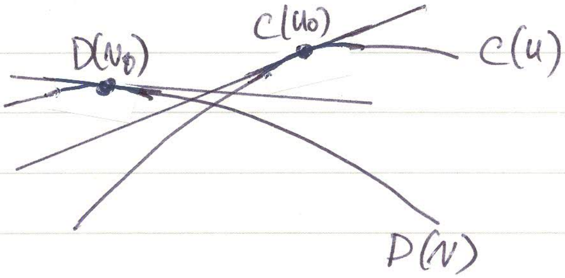
Total "multiply-subtract":

$$\sum_{k=1}^{n-1} (k+1)k = \frac{1}{3}(n-1)n(n+1) \approx \frac{1}{3}\cdot n^3$$

Linear Algebra for Curve-Curve Intersection

Given $C(u) = (x(u), y(u))$: two planar curves
$D(v) = (\alpha(v), \beta(v))$

Starting from
$C(u_0) = (x(u_0), y(u_0))$
$D(v_0) = (\alpha(v_0), \beta(v_0))$



We consider two tangent lines and their intersection

$$C(u_0) + s \cdot C'(u_0) = D(v_0) + t \cdot D'(v_0)$$

$$s \cdot C'(u_0) - t \cdot D'(v_0) = D(v_0) - C(u_0)$$

$$s \cdot \begin{bmatrix} x'(u_0) \\ y'(v_0) \end{bmatrix} + t \cdot \begin{bmatrix} -\alpha'(v_0) \\ -\beta'(v_0) \end{bmatrix} = \begin{bmatrix} \alpha(v_0) - x(u_0) \\ \beta(v_0) - y(u_0) \end{bmatrix}$$

$$\begin{bmatrix} x'(u_0) & -\alpha'(v_0) \\ y'(v_0) & -\beta'(v_0) \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \alpha(v_0) - x(u_0) \\ \beta(v_0) - y(u_0) \end{bmatrix}$$

Solve the linear equation for s and t

Let $u_1 = u_0 + s$, $v_1 = v_0 + t$,
Repeat the same procedure until satisfied.

1.4 Matrix Notation and Matrix Multiplication

$$\begin{cases} 2u+v+w=5 \\ 4u-6v \qquad = -2 \\ -2u+7v+2w=9 \end{cases}$$

Nine coefficients
Three unknowns
Three right-hand sides

$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$: The unknown; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$: The solution

$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$: Coefficient matrix

$A\mathbf{x} = \mathbf{b}$ Matrix form : $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

Two ways to multiply a matrix A by a vector $\mathbf{x}$

① $A\mathbf{x}$ by rows:

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 1\cdot 5 + 6\cdot 0 \\ 3\cdot 2 + 0\cdot 5 + 1\cdot 0 \\ 1\cdot 2 + 1\cdot 5 + 4\cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

② $A\mathbf{x}$ by columns:

$$2\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

$\boxed{1A}$ $A\mathbf{x}$ is a combination of the columns of A.
The coefficients are the components of $\mathbf{x}$.

o. The ith component of $A\mathbf{x}$ is $\sum_{j=1}^{n} a_{ij} x_j$, $\mathbf{x} = (x_j)_{j=1}^{n}$.

The Matrix Form of One Elimination Step

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} :$$ Elementary matrix that subtracts $2 \times$ (the first equation) from the second

$$E\mathbb{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$\boxed{1B}$ The elementary matrix E_{ij} subtracts l times row j from row i, where E_{ij} includes $-l$ in row i, column j.

Ex

$$E_{31}\mathbb{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - l b_1 \end{bmatrix}$$

Matrix Multiplication

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

"Associative Law"

Matrix multiplication: $\underline{(EA)\mathbb{x} = E(A\mathbb{x}) = EA\mathbb{x}}$

$\Rightarrow$ We may write $E(A\mathbb{x}) = E\mathbb{b}$ or $(EA)\mathbb{x} = E\mathbb{b}$.

Multiplication by columns:

$$AB = A[\mathbb{b}_1\ \mathbb{b}_2\ \mathbb{b}_3] = [A\mathbb{b}_1\ A\mathbb{b}_2\ A\mathbb{b}_3]$$

$\boxed{1C}$ $A = (a_{ik})$, $B = (b_{kj})$

$\Rightarrow (AB)_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$.

$\boxed{1D}$

(i) Each entry of AB is the product of a row and a column:

$\underline{(AB)_{ij} = (\text{row } i \text{ of } A) \text{ times } (\text{column } j \text{ of } B)}$

(ii) Each column of AB is the product of a matrix and a column:

$\underline{\text{column } j \text{ of } AB = A \text{ times } (\text{column } j \text{ of } B)}$

(iii) Each row of AB is the product of a row and a matrix:

$\underline{\text{row } i \text{ of } AB = (\text{row } i \text{ of } A) \text{ times } B}$.

$\boxed{1E}$ Matrix multiplication is associative:

$(AB)C = A(BC) = ABC$

$\boxed{1F}$ Matrix operations are distributive:

$A(B+C) = AB + AC$ and

$(B+C)D = BD + CD$

$\boxed{1G}$ Matrix Multiplication is not commutative:

Usually $FE \neq EF$

<u>Ex4</u>: But $EF = FE$, for these two $\Rightarrow$ $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Ex1: $AB = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 0 \\ 4 & 8 & 0 \end{bmatrix}$

Ex2: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}$: row exchange matrix

Ex3: $IA = A$, $BI = B$ for I: Identity matrix

Ex4:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$EF = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = FE$$

Ex5:

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow GE \neq EG$$

$$GE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = EG$$

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad EFG = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The true order of elimination
$GFEA = U$ ← upper triangular

these numbers are from E, F, G

The right order for reversing the elimination steps.

1.5 Triangular Factors and Row Exchanges

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \mathbf{b}$$

Elementary matrices:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow$

$$GFEA\mathbf{x} = U\mathbf{x} = \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{(upper triangular)}}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = \mathbf{c}$$

$U\mathbf{x} = \mathbf{c}$ is solved by back-substitution

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = L$$

multipliers $l = 2, -1, -1$

$\boxed{1H}$ Triangular factorization $A = LU$ with no exchanges of rows.

L: lower triangular with 1s on the diagonal and multipliers l_{ij} below the diagonal

U: the upper triangular matrix which appears after forward elimination.

The diagonal entries of U are the pivots.

Ex1 $A=\begin{bmatrix}1&2\\3&8\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&2\\0&2\end{bmatrix}=LU$

Ex2 $A=\begin{bmatrix}0&2\\3&4\end{bmatrix}$ cannot be factored into $A=LU$

Ex3

$$A=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=LU$$

Ex4

$$A=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=LU$$

$$A=LU:\quad \begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}\text{row 1 of } U\\\text{row 2 of } U\\\text{row 3 of } U\end{bmatrix}=A$$

「$\because$ Apply the steps of elimination to both sides.
Left side = the result of forward elimination to the right-hand side 」

Ex5

$$A=\begin{bmatrix}1&-1&&\\-1&2&-1&\\&-1&2&-1\\&&-1&2\end{bmatrix}=\underset{L}{\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}}\underset{U}{\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}}$$

One Linear System = Two Triangular Systems

① Factor: $A = LU$

② Solve: $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$ $\left(\because A\mathbf{x} = L\underbrace{U\mathbf{x}}_{\mathbf{c}} = \mathbf{b}\right)$

⇒ The solution for any new right-hand side $\mathbf{b}$ can be found in only n^2 operations.
(Far below the $\frac{1}{3} \cdot n^3$ steps needed to factor A.)

Ex6:

$$A\mathbf{x} = \mathbf{b}: \begin{cases} x_1 - x_2 = 1 \\ -x_1 + 2x_2 - x_3 = 1 \\ -x_2 + 2x_3 - x_4 = 1 \\ -x_3 + 2x_4 = 1 \end{cases} \Rightarrow$$

Splits into $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$

$$L\mathbf{c} = \mathbf{b}: \begin{cases} c_1 = 1 \\ -c_1 + c_2 = 1 \\ -c_2 + c_3 = 1 \\ -c_3 + c_4 = 1 \end{cases} \Rightarrow \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{c}: \begin{cases} x_1 - x_2 = 1 \\ x_2 - x_3 = 2 \\ x_3 - x_4 = 3 \\ x_4 = 4 \end{cases} \Rightarrow \mathbf{x} = \begin{bmatrix} 10 \\ 9 \\ 7 \\ 4 \end{bmatrix}$$

For these special tridiagonal matrices, the operation count drops from n^2 to $2n$.

unique ($L_1D_1U_1 = L_2D_2U_2 \Rightarrow L_1 = L_2, D_1 = D_2, U_1 = U_2$)

Remark: $A = LDU$, where L, U: 1s on the diagonal
D: the diagonal matrix of pivots

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
$= LDU$.

Row Exchanges and Permutation Matrices

$A\mathbf{x} = \mathbf{b}$: $\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

zero pivot

Permutation

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

(The unknowns u and v are not reversed!)

[iJ] ① In the nonsingular case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. Then $A\mathbf{x}=\mathbf{b}$ has a unique solution. With the rows reordered in advance, PA can be factored into LU.

② In the singular case, no P can produce a full set of pivots: elimination fails.

* P^{-1} is always the same as P^T
* In practice, we exchange rows (and also columns) when the pivot is near zero to reduce the roundoff error.

Ex)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U$$

row exchange, now $l_{31}=1$, $l_{21}=2$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \underline{PA = LU}$$

1.6 Inverses and Transposes

Inverse matrix. $\mathbb{b} = A\mathbb{x} \Rightarrow A^{-1}\mathbb{b} = \mathbb{x}$

Thus $A^{-1}A\mathbb{x} = \mathbb{x}$.

Not all matrices have inverses.

An inverse is impossible when $A\mathbb{x} = \mathbb{0}$ and $\mathbb{x} \neq \mathbb{0}$.

$\boxed{1K}$ The inverse of A is a matrix B s.t. $BA = I = AB$.

There is at most one such B, denoted by A^{-1}.

$$A^{-1}A = I, \quad AA^{-1} = I.$$

Notes

① The inverse exists if and only if elimination produces n pivots (row exchanges allowed ✓).

Elimination solves $A\mathbb{x} = \mathbb{b}$ without explicitly finding A^{-1}.

② $BA = I$ and $AC = I \Rightarrow \underline{B = C}$

[∵ $B(AC) = (BA)C \Rightarrow B = C$]

left-inverse, right-inverse

③ If A is invertible, the one and only one solution to $A\mathbb{x} = \mathbb{b}$ is $\underline{\underline{\mathbb{x} = A^{-1}\mathbb{b}}}$.

④ (Important!)

Suppose there is a nonzero vector $\underline{\mathbb{x} \neq \mathbb{0}}$ s.t. $A\mathbb{x} = \mathbb{0}$.

Then A cannot have an inverse. Contradiction

(∵ If A is invertible, then $A\mathbb{x} = \mathbb{0} \Rightarrow \underline{\mathbb{x} = A^{-1}\mathbb{0} = \mathbb{0}}$ #)

⑤ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Leftrightarrow$ $ad-bc \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant of A

∴ A matrix is invertible if its determinant is non-zero.

⑥ If $A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$

and $A^{-1}A = I$

$\boxed{1L}$ For invertible matrices A and B,
the inverse of AB : $(AB)^{-1} = B^{-1}A^{-1}$.

「$\because$ $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$
$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$」

<u>The Calculation of A^{-1} : The Gauss-Jordan Method</u>

x_i : columns of A^{-1} $\Rightarrow$ $Ax_i = e_i$ ← Solve for $i=1,\cdots,n$

$Ax_i = e_i$: $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Ex1</u> : (Gauss-Jordan)

$\begin{bmatrix} A & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$

The First Half
(Forward Elimination)

pivot = 2 → $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$

pivot = −8 → $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} U & L^{-1} \end{bmatrix}$

GFE

to be continued →

$\rightsquigarrow$ Second Half that takes U to I & L^{-1} to $U^{-1}L^{-1} = A^{-1}$, Creating zeros above the pivots, we reach A^{-1}

$$\begin{bmatrix} U & L^{-1} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

zeros above the pivots $\rightarrow \begin{bmatrix} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

(the first three columns are I, the last three are A^{-1})

The determinant is the product of pivots $(2)(-8)(1)$.

<u>Remark1</u> (Don't Recommend it for solving $A\mathbf{x} = \mathbf{b}$!)

Two triangular steps are better!

$\mathbf{x} = A^{-1}\mathbf{b}$ separates into $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$.

$\frac{1}{2}n^2$ steps for each.

$\Rightarrow$

- It is a waste of time to form L^{-1} and U^{-1}
- The multiplication $A^{-1}\mathbf{b}$ would still take n^2 steps. It is the solution that we want, and not all the entries on the inverse.

<u>Remark2</u> (Operation count for A^{-1})

$$\frac{n^3}{6} + \frac{n^3}{3} + n\left(\frac{n^2}{2}\right) = n^3$$

$\hookrightarrow$ LU factorization

$\hookrightarrow$ back-substitutions

$\hookrightarrow$ Forward Eliminations: $\frac{1}{2}(n-j)^2$ for column $\mathbf{e}_j$.

<u>Invertible = Nonsingular (n pivots)</u>

Suppose A has a full set of n pivots

$AA^{-1} = I$ gives $\underline{Ax_i = e_i}$, x_i : columns of A^{-1}

solved by elimination or Gauss-Jordan

$A^{-1} = [x_1 \cdots x_n]$ is also a left-inverse.

(i.e. Solving $AA^{-1} = I$ also solved $A^{-1}A = I$).

Too. A 1-sided inverse of a squared matrix

is automatically a 2-sided inverse.

$\Leftarrow$) Every Gauss-Jordan step is a multiplication on the left by an elementary matrix:

① E_{ij}, ② P_{ij}, or ③ D (or D^{-1}) to divide by pivots.

$$\underbrace{(D^{-1} \cdots E \cdots P \cdots E)}_{\text{The left inverse of } A} A = I$$

If exists, it equals the right inverse by <u>Note 2</u>.

$\therefore$ Every nonsingular matrix is invertible.

$\Rightarrow$)

A cannot have a whole column of zeros.

Suppose elimination breaks down at some column

$$A \to \cdots \to A' = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \to \boxed{\text{This matrix cannot have an inverse.}}$$

Using column operations, i.e. reparametrization, the whole third column can be made zero.

Therefore, the original A was not invertible.

$$(A')^T = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ * & d_2 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}$$

$$A' = \underbrace{\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}}_{A''} \underbrace{\begin{bmatrix} 1 & l_{21} & l_{31} & l_{41} \\ 0 & 1 & l_{32} & l_{42} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$A\mathbf{x} = \mathbf{0} \Leftrightarrow A'\mathbf{x} = \mathbf{0} \Leftrightarrow A''\underbrace{U\mathbf{x}}_{\mathbf{y}} = \mathbf{0}$$

$$(\mathbf{x} \neq \mathbf{0} \Leftrightarrow \mathbf{y} \neq \mathbf{0})$$

$A''\mathbf{y} = \mathbf{0}$ for some $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \neq \mathbf{0}$

$\Rightarrow A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} = U^{-1}\mathbf{y} \neq \mathbf{0}$

\#

The Transpose Matrix

Entries of A^T : $(A^T)_{ij} = A_{ji}$

$\boxed{\text{TM}}$ ① $(AB)^T = B^T A^T$

② $(A^{-1})^T = (A^T)^{-1}$

$\ulcorner \because$ ① The first row of $(AB)^T$ is the first column of AB.

The columns of A are weighted by the first column of B.

Equivalently, the rows of A^T are weighted by the first rows of B^T

exactly the first row of $B^T A^T$.

The other rows of $(AB)^T$ and $B^T A^T$ also agree.

②

$AA^{-1} = I$ and $A^{-1}A = I$

$(A^{-1})^T A^T = I^T = I$ and $A^T (A^{-1})^T = I^T = I$

$\therefore (A^T)^{-1} = (A^{-1})^T$ $\lrcorner$

Symmetric Matrices

A : symmetric $\Leftrightarrow$ $A^T = A$ $\Rightarrow$ A is a square matrix

a symmetric matrix need not be invertible. (ex: zero matrix)

But, if A^{-1} exists, it is also symmetric.

$\ulcorner \because$ $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ $\Leftarrow$ $A^T = A$.

$\therefore A^{-1}$ is symmetric $\lrcorner$

Symmetric Products R^TR, RR^T, LDL^T

$(R^TR)^T = R^T(R^T)^T = R^TR$

RR^T is also symmetric, but it is different from R^TR even if R and R^T are square matrices.

$\boxed{\text{IN}}$ Suppose $A = A^T$ can be factored into $A = LDU$ without row exchanges. Then $U = L^T$.

Pf: $A = LDU$, $A^T = U^TD^TL^T$.

$LDU = U^TD^TL^T$

By the uniqueness of the LDU factorization (Problem 20 of page 53), $L = U^T$ ⌟

Ex:

$$\begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^T$$

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ 0 & d - \frac{b^2}{a} & e - \frac{bc}{a} \\ 0 & e - \frac{bc}{a} & f - \frac{c^2}{a} \end{bmatrix}$$

The smaller matrices stay symmetric as elimination proceeds, and we can work with half the matrix.

$\Rightarrow$ The work of elimination is reduced from $\frac{1}{3}n^3$ to $\frac{1}{6}n^3$.

1.7 Special Matrices and Applications

2nd Order O.D.E. $\begin{cases} -u''(x) = f(x), \quad 0 \le x \le 1. \\ u(0) = u(1) = 0 \ : \text{boundary condition.} \end{cases}$

$$u''(x) \approx \frac{1}{h^2}\left(u(x+h) - 2u(x) + u(x-h)\right)$$

Let $u_i = u(ih)$, $i = 0, 1, \cdots, n+1$, where $h = \frac{1}{n+1}$.

$\Rightarrow$ Difference Equation : $-u_{j+1} + 2u_j - u_{j-1} = h^2 f(jh)$,
for $j = 1, \cdots, n$

Matrix Equation
$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}$$

tridiagonal, symmetric, and positive definite
(the pivots are positive)

$$A = \underbrace{\begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & & & \\ & 1 & -\frac{2}{3} & & \\ & & 1 & -\frac{3}{4} & \\ & & & 1 & -\frac{4}{5} \\ & & & & 1 \end{bmatrix}}_{U (= L^T)}$$

- L, U : bidiagonal
- The pivots $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$ are all positive.
  Their product is the determinant of A, $\det A = 6$.
- Elimination needs only $2n$ operations (instead of $n^3/3$)

$\Rightarrow$ Tridiagonal system $A\mathbf{x} = \mathbf{b}$
can be solved in $O(n)$ time!

a band matrix A has $a_{ij}=0$ if $|i-j| \geq w$

half band width

- For each column, elimination requires $w(w-1)$ operations
- Elimination on the n columns of a band matrix requires about $w^2 n$ operations $\to$ roughly n^3 for $w=n$

too much!

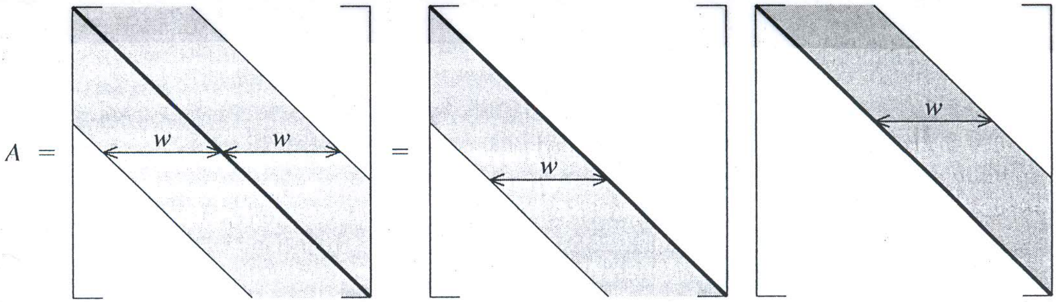


Figure 1.8 A band matrix A and its factors L and U.

The precise number of divisions and multiplication-subtractions that produce L, D, and U (without assuming a symmetric A) is

$$P=\frac{1}{3}w(w-1)(3n-2w+1)$$

For $w=n$, $P=\frac{1}{3}n(n-1)(n+1)$: a whole number

In solving $A\mathbf{x}=\mathbf{b}$ for a band matrix $w=2$, $4n$ operations are sufficient for solving $L\mathbf{c}=\mathbf{b}$ and $U\mathbf{x}=\mathbf{c}$