

Chap 5. Eigenvalues and Eigenvectors

5.1 Introduction (all matrices are now square.)

$$A\mathbf{x} = \lambda \mathbf{x} \text{ for some } \mathbf{x} \neq \mathbf{0}$$

Ex: (Initial Value Problem for ODE)

$$\begin{cases} \frac{dn}{dt} = 4n - 5w, & n=8 \text{ at } t=0 \\ \frac{dw}{dt} = 2n - 3w, & w=5 \text{ at } t=0 \end{cases}$$

$$\mathbf{u}(t) = \begin{bmatrix} n(t) \\ w(t) \end{bmatrix}, \quad \mathbf{u}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

Matrix form: $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ with $\mathbf{u} = \mathbf{u}(0)$ at $t=0$

[Single equation: $\frac{du}{dt} = au$ with $u=u(0)$ at $t=0$.]

Purely exponential solution: $u(t) = e^{at} u(0)$.

$$\Rightarrow \begin{cases} n(t) = e^{\lambda t} y \\ w(t) = e^{\lambda t} z \end{cases} \quad \text{or} \quad \underline{\mathbf{u}(t) = e^{\lambda t} \mathbf{x}} \quad \text{in vector notation}$$

Look for pure exponential solutions:

$$\begin{cases} \lambda e^{\lambda t} y = 4e^{\lambda t} y - 5e^{\lambda t} z \\ \lambda e^{\lambda t} z = 2e^{\lambda t} y - 3e^{\lambda t} z \end{cases} \quad) \quad \text{cancelling } e^{\lambda t}$$

$$\text{Eigenvalue: } \begin{cases} 4y - 5z = \lambda y \\ 2y - 3z = \lambda z \end{cases} \quad)$$

Problem
Eigenvalue Equation: $A\mathbf{x} = \lambda \mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$

The Solutions of $A\mathbf{x} = \lambda \mathbf{x}$

$(A - \lambda I)\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$

[SA] λ : eigenvalue of $A \Leftrightarrow A - \lambda I$ is singular
 $\det(A - \lambda I) = 0$: the characteristic equation
 Each λ is associated with eigenvectors \mathbf{x} :
 $(A - \lambda I)\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \lambda\mathbf{x}$.

Ex: (Continued) $A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}$

$$|A - \lambda I| = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0$$

$$\therefore \lambda = -1 \text{ or } 2$$

① $\lambda_1 = -1$: $(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② $\lambda_2 = 2$: $(A - \lambda_2 I)\mathbf{x} = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Pure exponential solutions to $d\mathbf{u}/dt = A\mathbf{u}$:

$$\mathbf{u}(t) = e^{\lambda_1 t} \mathbf{x}_1 = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad) \text{ special solutions}$$

$$\mathbf{u}(t) = e^{\lambda_2 t} \mathbf{x}_2 = e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Complete Solution: $\mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2$

Initial Solution: $\mathbf{u}(0) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$ or

$$\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \Rightarrow c_1 = 3, c_2 = 1$$

The solution to the original equation:

$$\mathbf{u}(t) = 3e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Writing the two components separately,

Solution: $\begin{cases} N(t) = 3e^{-t} + 5e^{2t}, & N(0) = 8 \\ W(t) = 3e^{-t} + 2e^{2t}, & W(0) = 5 \end{cases}$

Summary and Examples

Ex1: (Diagonal Matrix)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 3, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \lambda_2 = 2, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ex2: (Projection Matrix) $\Rightarrow \lambda = 1 \text{ or } 0$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \lambda_1 = 1, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \lambda_2 = 0, \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have $\lambda = 1$ when \mathbf{x} projects to itself, and $\lambda = 0$ when \mathbf{x} projects to the zero vector.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ has } \lambda = 1, 1, 0, 0$$

Ex3: (Triangular Matrix)

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = (1-\lambda)(\frac{3}{4}-\lambda)(\frac{1}{2}-\lambda)$$

- o. The Gaussian factorization $A = LU$ is not suited to the purpose of transforming A into a diagonal or triangular matrix without changing its eigenvalues.
- o. The eigenvalue problem is computationally more difficult than $A\mathbf{x} = \lambda b$.
- o. Normally, the pivots, diagonal entries, and eigenvalues are completely different. But ↴

5B The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of } A = \lambda_1 + \dots + \lambda_n = a_{11} + \dots + a_{nn}.$$

The product of the n eigenvalues equals the $\det(A)$.

5.2 Diagonalization of a Matrix

The eigenvectors diagonalize a matrix.

SC Suppose the $n \times n$ matrix A has n lin. indep. eigenvectors. If these eigenvectors are the columns of a matrix S , then $S^{-1}AS$ is a diagonal matrix Λ . The eigenvalues of A are on the diagonal of Λ :

$$\text{Diagonalization } S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ 0 & & \ddots & \lambda_n \end{bmatrix}$$

the eigenvector matrix the eigenvalue matrix

<proof>

$$AS = A \begin{bmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{x}_1 & \dots & A\mathbf{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$AS = S\Lambda, \quad S^{-1}AS = \Lambda, \quad A = S\Lambda S^{-1}.$$

(S is invertible since its columns are lin. independent.)

Remarks:

① If A has no repeated eigenvalue, then the n eigenvectors are automatically independent. (See SD below.)

Any matrix with distinct eigenvalues can be diagonalized.

② The diagonalizing matrix S is not unique.

③ Other matrices S will not produce a diagonal Λ .

Suppose \mathbf{y} is the first column of S . Then $\lambda_1 \mathbf{y}$ is the first column of $S\Lambda$, which is $A\mathbf{y}$ (the first column of AS). Then \mathbf{y} must be an eigenvector: $A\mathbf{y} = \lambda_1 \mathbf{y}$. The order of the eigenvectors of S and the eigenvalues in Λ is the same.

Remark 4: Not all matrices possess n lin. indep. eigenvectors, so not all matrices are diagonalizable. An example is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0.$$

All eigenvectors of A are multiples of $(1, 0)$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$\lambda = 0$ is a double eigenvalue - its algebraic multiplicity is 2. But the geometric multiplicity is only 1 - there is only one independent eigenvector. We can't construct S .

Otherwise, since $\lambda_1 = \lambda_2 = 0$, Λ should be the zero matrix.

But, if $\Lambda = S^{-1}AS = 0$, then $A = S\Lambda S^{-1} = 0 \# \square$

The failure of diagonalization came from $\lambda_1 = \lambda_2$.

Ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 3$; $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1$.

These matrices are not singular. But the problem is the shortage of eigenvectors - which are needed for S .

Note:

- o. Diagonalizability of A depends on enough eigenvectors.
- o. Invertibility of A depends on nonzero eigenvalues.

Diagonalization can fail only if there are repeated eigenvalues. Even then, it does not always fail. $A = I$ has repeated eigenvalues $1, 1, \dots, 1$, but it is already diagonal!

For an eigenvalue that is repeated p times, we need to check whether there are p independent eigenvectors — i.e., whether $A - \lambda I$ has rank $n-p$. To complete that circle of ideas, we have to show that distinct eigenvalues present no problem.

SD

If eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ correspond to different eigenvalues $\lambda_1, \dots, \lambda_k$, then those eigenvectors are linearly independent.

Proof

$$\text{Suppose } c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}, \Rightarrow A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 = \mathbf{0}.$$

$$\Rightarrow c_1(\lambda_1 - \lambda_2)\mathbf{x}_1 = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 - \lambda_2(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = \mathbf{0}$$

Since $\lambda_1 \neq \lambda_2$ and $\mathbf{x}_1 \neq \mathbf{0}$, we have $c_1 = 0$

and similarly $c_2 = 0$.

By induction, we can extend this argument to any number of eigenvectors. \square

Examples of Diagonalization

$$\text{Ex1: } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, AS = S\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Ex2: } K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}: 90^\circ \text{ rotation} \Rightarrow \det(K - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$\lambda_1 = i: (K - iI)\mathbf{x}_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\lambda_2 = -i: (K + iI)\mathbf{x}_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \text{ and } S^{-1}KS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Powers and Products: A^k and AB

The eigenvalues of A^2 are exactly $\lambda_1^2, \dots, \lambda_n^2$, and every eigenvector of A is also an eigenvector of A^2 . By squaring $S^{-1}AS$, we have $S^{-1}A^2S = S^{-1}ASS^{-1}AS = \lambda^2$. The matrix A^2 is diagonalized by the same S , so the eigenvectors are unchanged. The eigenvalues are ~~squared~~.

[SE]

The eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$, and each eigenvector of A is still an eigenvector of A^k . When S diagonalizes A , it diagonalizes A^k :

$$\lambda^k = (S^{-1}AS)(S^{-1}AS) \cdots (S^{-1}AS) = S^{-1}A^kS.$$

If A is invertible, this rule also applies to its inverse ($k=1$).

The eigenvalues of A^{-1} are $1/\lambda_i$.

Ex: If $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x} = \lambda A^{-1}\mathbf{x}$ and $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$.

Ex3: $K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{90}^\circ \text{ rotation}} K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\lambda_1 = i, \lambda_2 = -i ; \quad \lambda_1^2 = -1, \lambda_2^2 = -1 ; \quad \frac{1}{\lambda_1} = -i, \frac{1}{\lambda_2} = i.$$

$$K^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \lambda^4 = \begin{bmatrix} (i)^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- For a product of two matrices, the eigenvalues of AB have no good answer. The eigenvalues of AB and $A+B$ have nothing to do with $\mu\lambda$ and $\lambda+\mu$, where λ and μ are eigenvalues of A and B , respectively.

Counter Example: $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 0$

But, both A and B have zero eigenvalue.

SH Diagonalizable matrices share the same eigenvector matrix $S \Leftrightarrow AB = BA$

<proof>

\Rightarrow If the same S diagonalizes both $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$

$$AB = S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1 \Lambda_2 S^{-1} \text{ and}$$

$$BA = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = S\Lambda_2 \Lambda_1 S^{-1}.$$

Since $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$ (diagonal matrices always commute), we have $AB = BA$.

\Leftarrow Suppose $AB = BA$. Starting from $A\mathbf{x} = \lambda\mathbf{x}$, we have

$$AB\mathbf{x} = BA\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x}.$$

Thus \mathbf{x} and $B\mathbf{x}$ are both eigenvectors of A , sharing the same λ (or else $B\mathbf{x} = \mathbf{0}$).

If we assume for convenience that the eigenvalues of A are distinct - the eigenspaces are all one-dimensional - then $B\mathbf{x}$ must be a multiple of \mathbf{x} . In other words, \mathbf{x} is an eigenvector of B as well as A .

The proof with repeated eigenvalues is a little longer. □

(Spectral Theorem)

Every real symmetric matrix A can be diagonalized by an orthogonal matrix Q (i.e. $Q^T Q = I$, $Q^T = Q^{-1}$):

$$Q^T A Q = \Lambda \quad \text{or} \quad A = Q \Lambda Q^T$$

The columns of Q contain orthonormal eigenvectors of A .

5.3 Difference Equations and Powers A^k

You invest \$1000 at 6% interest. Compounded once a year, the principal P is multiplied by 1.06. This is a difference equation $P_{k+1} = AP_k = 1.06 P_k$ with a time step of one year. After 5 years, the original $P_0 = 1000$ has been multiplied 5 times.

Yearly: $P_5 = (1.06)^5 P_0$ which is $(1.06)^5 1000 = \$1338$.

Now the time step is reduced to a month. The new difference equation is $P_{k+1} = (1 + 0.06/12) P_k$. After 5 years, or 60 months, you have \$11 more

Monthly: $P_{60} = \left(1 + \frac{0.06}{12}\right)^{60} P_0$ which is $(1.005)^{60} 1000 = \$1349$.

The next step is to compound every day, on 5×365 days.

Daily: $\left(1 + \frac{0.06}{365}\right)^{5 \times 365} 1000 = \1349.83

The interest is added at every instant. The bank could compound the interest N times a year, so $\Delta t = 1/N$.

Continuously: $\left(1 + \frac{0.06}{N}\right)^{5N} 1000 \rightarrow e^{0.30} 1000 = \1349.87

The bank can switch to a differential equation - the limit of the difference equation $P_{k+1} = (1 + 0.06 \Delta t) P_k$.

Discrete to: $\frac{P_{k+1} - P_k}{\Delta t} = 0.06 P_k$ approaches $\frac{dp}{dt} = 0.06 P$.

The solution is $p(t) = e^{0.06t} P_0$. After $t = 5$ years, this amounts to \$1349.87. The improvement over compounding every day is only 4 cents.

Fibonacci Numbers

Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, ...

Fibonacci equation: $F_{k+2} = F_k + F_{k+1}$

How can we find F_{1000} without starting at $F_0=0$ and $F_1=1$, and working all the way out to F_{1000} ?

$$\begin{cases} F_{k+2} = F_k + F_{k+1} \\ F_{k+1} = F_k \end{cases} \text{ becomes } \mathbf{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A \mathbf{u}_k.$$

The solution to a difference equation $\mathbf{u}_{k+1} = A \mathbf{u}_k$ is $\mathbf{u}_k = A^k \mathbf{u}_0$.
The real problem is to find some quick way to compute A^k .

56 If A can be diagonalized, $A = S \Lambda S^{-1}$, then $A^k = S \Lambda^k S^{-1}$,
 $\mathbf{u}_k = A^k \mathbf{u}_0 = S \Lambda^k S^{-1} \mathbf{u}_0$.

The columns of S are eigenvectors of A . Writing $S^{-1} \mathbf{u}_0 = \mathbf{c}$,

$$\mathbf{u}_k = S \Lambda^k \mathbf{c} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \lambda_1^k \mathbf{x}_1 + \cdots + c_n \lambda_n^k \mathbf{x}_n$$

After k steps, \mathbf{u}_k is a combination of the n pure solutions $\lambda^k \mathbf{x}$.

In a specific example, the first step is to find the eigenvalues.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \text{ has } \det(A - \lambda I) = \lambda^2 - \lambda - 1$$

Two eigenvalues $\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$

The eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\mathbf{u}_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } S^{-1} \mathbf{u}_0 = \mathbf{c}$$

$$S^{-1} \mathbf{u}_0 = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ gives } \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Fibonacci: $F_{1000} = c_1 \lambda_1^k + c_2 \lambda_2^k = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right]$.
Numbers

Markov Matrices

Each year $\frac{1}{10}$ of the people outside California move in, and $\frac{2}{10}$ of the people inside California move out. We start with y_0 people outside and z_0 inside. At the end of the first year the numbers outside and inside are y_1 and z_1 :

$$\text{Difference: } y_1 = 0.9y_0 + 0.2z_0 \quad \text{or} \quad \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$\text{equation} \quad z_1 = 0.1y_0 + 0.8z_0$$

Two essential properties of a Markov process:

1. The total number of people stay fixed.

2. The numbers outside and inside can never be negative.

We solve this Markov difference equation using $u_k = S^{-1}A^k S u_0$

$$A - \lambda I = \begin{bmatrix} 0.9 - \lambda & 0.2 \\ 0.1 & 0.8 - \lambda \end{bmatrix} \text{ has } \det(A - \lambda I) = \lambda^2 - 1.7\lambda + 0.7$$

$$\lambda_1 = 1 \quad \Rightarrow \quad A = S \lambda S^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 0.7$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1^k \\ 0.1^k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$= (y_0 + z_0) \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + (y_0 - z_0)(0.1)^k \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$$

In the long run, the factor $(0.1)^k$ becomes extremely small. The solution approaches a limiting state $u_{\infty} = (y_{\infty}, z_{\infty})$

$$\text{Steady State: } \begin{bmatrix} y_{\infty} \\ z_{\infty} \end{bmatrix} = (y_0 + z_0) \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \quad \text{or} \quad Au_{\infty} = u_{\infty}$$

The steady state is the eigenvector of A corresponding to $\lambda = 1$.

[5L] A Markov matrix A has all $a_{ij} \geq 0$, with each column adding to 1.

- (a) $\lambda_1 = 1$ is an eigenvalue of A .
- (b) Its eigenvector \mathbf{x}_1 is nonnegative and $A\mathbf{x}_1 = \mathbf{x}_1$ (Steady state).
- (c) The other eigenvalues satisfy $|\lambda_i| \leq 1$.
- (d) If A or any power of A has all positive entries, these other $|\lambda_i|$ are below 1. The solution $A^k \mathbf{v}_0$ approaches a multiple of $\mathbf{x}_1 \Rightarrow$ the steady state \mathbf{u}_{∞} .

Remark

- a. Why is $\lambda=1$ always an eigenvalue? Each column of $A - I$ adds up to $1 - 1 = 0$. The rows of $A - I$ add up to the zero row. They are linearly dependent and $\det(A - I) = 0$.
- b. In the formula $\mathbf{u}_k = c_1 \lambda_1^k \mathbf{x}_1 + \dots + c_n \lambda_n^k \mathbf{x}_n$, no eigenvalue can be larger than 1. (Otherwise, \mathbf{u}_k would blow up.) If all other eigenvalues are strictly smaller than $\lambda_1 = 1$, then the first term in the formula will be dominant. The other λ_i^k go to zero, and $\mathbf{u}_k \rightarrow c_1 \mathbf{x}_1 = \mathbf{u}_{\infty}$ = steady state.

Stability of $\mathbf{u}_{kt} = A^k \mathbf{u}_t$

We want to study the behavior of $\mathbf{u}_{kt} = A^k \mathbf{u}_t$ as $k \rightarrow \infty$.

Assuming that A can be diagonalized, \mathbf{u}_k will be a combination of pure solutions: $\mathbf{u}_k = S \Lambda^k S^{-1} \mathbf{u}_0 = c_1 \lambda_1^k \mathbf{x}_1 + \dots + c_n \lambda_n^k \mathbf{x}_n$.

The growth of \mathbf{u}_k is governed by the λ_i^k . Stability depends on λ_i :

[5J] The difference equation $\mathbf{u}_{kt} = A^k \mathbf{u}_t$ is

stable if all eigenvalues $|\lambda_i| < 1$;

neutrally stable if some $|\lambda_i| = 1$ and all the other $|\lambda_i| < 1$;

unstable if at least one eigenvalue $|\lambda_i| > 1$.

c. In the stable case, A^k approaches zero and so does $\mathbf{u}_k = A^k \mathbf{u}_0$.

5.4 Differential Equations and e^{At}

Differential Equation: $\frac{du}{dt} = Au = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u$

The first step is to find the eigenvalues (-1 and -3) and the eigenvectors: $A[1] = (-1)[1]$ and $A[-1] = (-3)[-1]$.

Solution: $u(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Initial Condition: $u(0) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S \mathbf{c}$

$$\Rightarrow u(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} S^{-1} u(0).$$

Fundamental formula $Se^{At}S^{-1}u(0)$ solves the differential eq:

$$u(t) = Se^{At}S^{-1}u(0) \text{ with } \lambda = \begin{bmatrix} -1 & \\ & -3 \end{bmatrix} \text{ and } e^{\lambda t} = \begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}$$

Matrix Exponential: $e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots$

The series always converges, and its sum e^{At} has ^{the} properties

$$\textcircled{1} \quad (e^{As})(e^{At}) = e^{A(s+t)} \text{ and } (e^{At})(e^{-At}) = I.$$

$$\textcircled{2} \quad \frac{d}{dt}(e^{At}) = Ae^{At} \Rightarrow u(t) = e^{At}u(0) \text{ solves the ODE.}$$

$$\textcircled{3} \quad \text{This solution must be the same as the form } Se^{\lambda t}S^{-1}u(0).$$

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots$$

$$= I + S\lambda S^{-1}t + \frac{1}{2!}S\lambda^2 S^{-1}t^2 + \frac{1}{3!}S\lambda^3 S^{-1}t^3 + \dots$$

$$= S(I + \lambda t + \frac{1}{2!}\lambda^2 t^2 + \frac{1}{3!}\lambda^3 t^3 + \dots)S^{-1}$$

$$= S e^{\lambda t} S^{-1}$$

$$\text{Ex 1: } A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$e^{At} = S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-3t} - e^{-t} & e^{-t} + e^{-3t} \end{bmatrix}$$

At $t=0$, we get $e^0 = I$. The infinite series e^{At} gives the result for all t , but a series can be hard to compute. The form $S e^{\Lambda t} S^{-1}$ gives the same answer when A can be diagonalized; it requires n independent eigenvectors in S . This simpler form leads to a combination of n exponentials $e^{\lambda_1 t} x_1 + \dots + e^{\lambda_n t} x_n$ — which is the best solution of all.

(5L)

If A can be diagonalized, $A = S \Lambda S^{-1}$, then $du/dt = Au$ has the solution: $u(t) = e^{At} u(0) = S e^{\Lambda t} S^{-1} u(0)$.

The columns of S are the eigenvectors x_1, \dots, x_n of A .

$$u(t) = [x_1 \dots x_n] \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} S^{-1} u(0)$$

$$= c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n = \text{combination of } e^{\lambda_i t} x_i$$

The constants that match the initial conditions $u(0)$ are $C = S^{-1} u(0)$.

o. e^{At} is never singular ($\because \det e^{At} = e^{\lambda_1 t} \dots e^{\lambda_n t} = e^{\text{trace}(At)}$)

Quick proof: Just recognize e^{At} as its inverse.

$$o. [e^{At} v_1 \dots e^{At} v_n] = e^{At} [v_1 \dots v_n]$$

\Rightarrow If n solutions are linearly independent at $t=0$, they remain linearly independent forever.

5.6 Similarity Transformations

The matrices A and $M^{-1}AM$ are similar. Two questions:

- ① What do the similar matrices $M^{-1}AM$ have in common?
- ② With a special choice of M , what special form can be achieved by $M^{-1}AM$? \Rightarrow The Jordan form.

o. The combinations $M^{-1}AM$ arise in a differential or difference equation, when a change of variables $u = Mv$ introduces v .

$$\frac{du}{dt} = Au \text{ becomes } M \frac{dv}{dt} = AMv \text{ or } \frac{dv}{dt} = M^{-1}AMv.$$

$$u_{n+1} = Au_n \text{ becomes } Mv_{n+1} = AMv_n \text{ or } v_{n+1} = M^{-1}AMv_n.$$

In the special case $M = S$, the system is uncoupled because $\Lambda = S^{-1}AS$ is diagonal. Other M 's are also useful.

[SP]

Suppose that $B = M^{-1}AM$. Then A and B have the same eigenvalues.

Every eigenvector x of A corresponds to an eigenvector $M^{-1}x$ of B .

$$\text{Also } Ax = \lambda x \Rightarrow A = MBM^{-1} \text{ and } MBM^{-1}x = \lambda x \Rightarrow BM^{-1}x = \lambda(M^{-1}x).$$

o. In fact, $A - \lambda I$ and $B - \lambda I$ have the same determinant.

Product of matrices $B - \lambda I = M^{-1}AM - \lambda I = M^{-1}(A - \lambda I)M$.

Product rule $\det(B - \lambda I) = \det M^{-1} \det(A - \lambda I) \det M = \det(A - \lambda I)$

The polynomials $\det(A - \lambda I)$ and $\det(B - \lambda I)$ are equal.

Their roots — the eigenvalues of A and B — are the same.

Ex 1:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has $\lambda=1$ and 0. Each B is $M^{-1}AM$.

If $M = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$: triangular with $\lambda=1$ and 0.

If $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$: projection with $\lambda=1$ and 0.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $B = \text{arbitrary matrix with } \lambda=1 \text{ and } 0$.

Change of Basis = Similarity Transformation

Every linear transformation is represented by a matrix.

The matrix depends on the choice of basis! If we change the basis by M, we change the matrix A to a similar matrix B.

Suppose we have a basis v_1, \dots, v_n . The jth column of A comes from $Tv_j = a_{1j}v_1 + \dots + a_{nj}v_n$: combination of v_i 's.

For a new basis v_1, \dots, v_n , the new matrix B is constructed in the same way: $Tv_j = b_{1j}v_1 + \dots + b_{nj}v_n$: combination of v_i 's

Each v must be a combination of v_i 's: $v_j = \sum m_{ij}v_i$

o. The matrix is really representing the Identity transformation when the only thing happening is the change of basis (T is I).

The inverse matrix M^{-1} also represents the identity transform when the basis is changed from the v_i 's back to the V_i 's

[SQ]

The matrices A and B that represent the same transformation T with respect to two different bases are similar:

$$[T]_{V \rightarrow V} = [I]_{V \rightarrow V} [T]_{V \rightarrow V} [I]_{V \rightarrow V}$$

$$B = M^{-1} A M$$

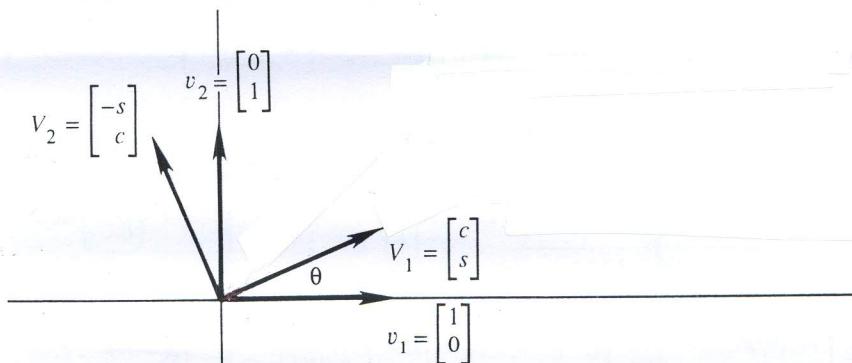


Fig. 5.5. Change of basis to make the projection matrix diagonal.

$T\mathbf{v}_1 = \mathbf{v}_1$ and $T\mathbf{v}_2 = \emptyset$. The matrix is diagonal

Eigenvector basis $B = [T]_{V \rightarrow V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathbf{v}_1 = (\cos \theta) \mathbf{v}_1 + (\sin \theta) \mathbf{v}_2 \text{ and } \mathbf{v}_2 = (-\sin \theta) \mathbf{v}_1 + (\cos \theta) \mathbf{v}_2.$$

Change of basis $M = [I]_{V \rightarrow U} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

The inverse matrix M^{-1} goes from U to V .

Standard basis $A = MBM^{-1} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$

- o. The way to diagonalize A is to find its eigenvectors. They go into the columns of M (or S) and $M^{-1}AM$ is diagonal.
- o. In the language of linear transformation: Choose a basis consisting of eigenvectors. The standard basis led to A , which was not simple. The right basis led to B : diagonal.
- o. Note: $M^{-1}AM$ does not arise in solving $Ax = b$. There the basic operation was to multiply A (on the left) by a matrix that preserves the nullspace and row space of A . It normally changes the eigenvalues.
- o. Eigenvalues are actually calculated by a sequence of simple similarities. The matrix goes gradually toward a triangular form, and the eigenvalues gradually appear on the main diagonal (see more details in Chap 7). This is much better than trying to compute $\det(A - \lambda I) = 0$. For a large matrix, it is numerically impossible to concentrate all that information into the polynomial and then get it out again.

The Jordan Form

If A has a full set of eigenvectors, we take $M=S$ and arrive at $J=S^{-1}AS=\Lambda$. Then the Jordan form coincides with the diagonal Λ . This is impossible for a defective (nondiagonalizable) matrix. For every missing eigenvector, the Jordan form will have a 1 just above its main diagonal. The eigenvalues appear on the diagonal because J is triangular. And distinct eigenvalues can always be decoupled. It is only a repeated λ that may (or may not) require an off-diagonal 1 in J .

5u

If A has s independent eigenvectors, it is similar to a matrix with s blocks:

Jordan form : $J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$

Each Jordan block J_i is a triangular matrix that has only a single eigenvalue λ_i and only one eigenvector:

Jordan block $J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$

The same λ_i will appear in several blocks, if it has several independent eigenvectors. Two matrices are similar if and only if they share the same Jordan form J .

- o Not all matrices are diagonalizable, and the Jordan form is the most general case. Its construction is both technically and extremely unstable. A slight change in A can put back all the missing eigenvectors, and remove the off-diagonal 1 s.

Ex4:

all lead to

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

These four matrices have eigenvalues 1 and 1 with only one eigenvector – so J consists of one block.

The determinants all equal 1. The traces are 2.

The eigenvalues satisfy $1+1=1$ and $1+1=2$.

The eigenvalues are on the diagonal. These matrices are similar – they belong to the same family.

$$(T) M^{-1}TM = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J$$

$$(B) P^{-1}BP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J$$

$$(A) U^{-1}AU = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T$$

and then $M^{-1}TM = J$.

$$\underline{\text{Ex5: }} A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

There can be a single 3×3 block, or a 2×2 block and a 1×1 block, or three 1×1 blocks. Three possible Jordan forms are

$$J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The only eigenvector of A is $(1, 0, 0)$. Its Jordan form has only one block, and A must be similar to J_1 . The matrix B has the additional eigenvector $(0, 1, 0)$, and its Jordan form is J_2 with two blocks. As for $J_3 = 0$, it is in a family by itself; the only matrix similar to J_3 is $M^{-1}OM = 0$.

Ex 6: Application to difference and differential equations (powers and exponentials). If A can be diagonalized, the powers of $A = S \Lambda S^{-1}$ are easy: $A^k = S \Lambda^k S^{-1}$.

In every case we have Jordan's similarity: $A = M J M^{-1}$, so now we need the powers of J : $J^k = M J^k M^{-1}$.

J is block-diagonal, the powers $(J_i)^k$ can be taken separately

$$(J_i)^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}.$$

This block J_i will enter when λ is a triple eigenvalue with a single eigenvector. Its exponential is in the solution to the corresponding differential equation:

$$\text{Exponential } e^{J_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

The third column of this exponential comes directly from solving

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ starting from } u_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This can be solved by back-substitution.

These powers and exponentials of J are a part of the solutions u_k and $u(t)$. The other part is the M that connects A to J :

If $u_{kt} = A u_k$, then $u_k = A^k u_0 = M J^k M^{-1} u_0$.

If $du/dt = Au$, then $u(t) = e^{At} u(0) = M e^{Jt} M^{-1} u(0)$.