양자컴퓨팅 및 정보의 기초 Homework #2 2017-12451 이동학

(a) U, V = unitary $\rightarrow$ $UV(UV)^\dagger = UVV^\dagger U^\dagger = UU^\dagger = I$,

$\therefore UV$ = unitary

(b) $U|v\rangle = \lambda|v\rangle = |v'\rangle \rightarrow \langle v'|v'\rangle = \langle v|U^\dagger U|v\rangle$

$= |\lambda|^2 \langle v|v\rangle = \langle v|v\rangle$,

$\therefore |\lambda|^2 = 1$ ($\because \langle v|v\rangle \neq 0$)

(a) $\Omega|\omega\rangle = \omega|\omega\rangle \rightarrow \langle\omega|\Omega^\dagger|\omega\rangle = \langle\omega|\omega^*|\omega\rangle$

$= \langle\omega|\Omega|\omega\rangle = \langle\omega|\omega|\omega\rangle$,

$(\omega - \omega^*)\langle\omega|\omega\rangle = 0$, $\therefore \omega = \omega^*$ ($\because \langle\omega|\omega\rangle \neq 0$)

(b) characteristic eq. $\rightarrow$ least one root ω_1 and

non-zero eigen vector $|\omega_1\rangle$.

choose basis $|\omega_1\rangle$ and $\{|v_1\rangle, \cdots, |v_{n-1}\rangle\}$,

which are orthonormal and orthogonal to $|\omega_1\rangle$.

then $\Omega = \begin{bmatrix} \omega_1 & 0 \cdots & 0 \\ 0 & & \\ \vdots & \boxed{} & \\ 0 & & \end{bmatrix}$, and characteristic eq

$(\omega_1 - \omega)P^{n-1}(\omega) = 0$.

$P^{n-1}(\omega)$ generate $\omega_2, |\omega_2\rangle, \cdots$ repeat.

finally, $\Omega = \begin{bmatrix} \omega_1 & & \\ & \omega_2 & \\ & & \ddots \\ & & & \omega_n \end{bmatrix}$ $\rightarrow$ diagonal

(c) let $U = e^{i\Omega} \rightarrow U^{\dagger} = (e^{i\Omega})^{\dagger} = e^{-i\Omega^{\dagger}} = e^{-i\Omega}$,

$UU^{\dagger} = e^{i\Omega} e^{-i\Omega} = e^{0} = I$ ($\because [i\Omega, -i\Omega] = 0$),

$\therefore e^{i\Omega}$ = unitary.

(d) $\langle - | AB | + \rangle = \gamma_{+} \langle - | A | + \rangle$

$= \langle - | BA | + \rangle = \langle - | B^{\dagger} A | + \rangle = \gamma_{-} \langle - | A | + \rangle$,

$\therefore \langle - | A | + \rangle = 0$ ($\because \gamma_{+} \neq \gamma_{-}$)

(a) $i = j \rightarrow 2(M_i)^2 = 2I$, $\therefore (M_i)^2 | e_k \rangle = (w_{i,k})^2 | e_k \rangle = | e_k \rangle$

$\therefore w_{i,k} = \pm 1$

(b) $i \neq j \rightarrow M_i M_j = -M_j M_i$,

$Tr(M_i) = Tr(M_i M_j M_j) = -Tr(M_j M_i M_j)$

$= -Tr(M_i M_j M_j) = -Tr(M_i)$,

$\therefore Tr(M_i) = 0$

(c) $w_{i,k} = \pm 1$ & $Tr(M_i) = 0 \rightarrow$ num of 1 = num of -1

$\therefore$ num of eigen value = even $\rightarrow$ even-dimensional

(a) $w^{\dagger} O v = [c \;\; d] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = ad + bc$

(b) $v' = \frac{1}{\sqrt{2}} \begin{bmatrix} a+b \\ a-b \end{bmatrix}$, $w' = \frac{1}{\sqrt{2}} \begin{bmatrix} c+d \\ c-d \end{bmatrix}$

(c) $U = |0\rangle\langle +| + |1\rangle\langle -| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = U^+,$

$\therefore \sigma' = U^+ \sigma U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) $(w')^+ \sigma' v' = \frac{1}{2}\begin{bmatrix} c+d & c-d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a+b \\ a-b \end{bmatrix} = ad+bc$

& same as 4. (a)

(e) basis of $Z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix},$ basis of $O = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$

$\therefore U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$A|v\rangle = a|v\rangle$, $B|v\rangle = b|v\rangle$ & $(AB+BA)|v\rangle = 2ab|v\rangle = 0$

$\therefore$ to have a simultaneous eigenbasis of A and B, least one of eigenvalue of every eigenvector is 0 ($a=0$ or $b=0$)

(a) $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$

$\therefore e^{iX} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{i}+e^{-i} & e^{i}-e^{-i} \\ e^{i}-e^{-i} & e^{i}+e^{-i} \end{bmatrix}$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ & $e^{iZ} = \begin{bmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{bmatrix}$

(a) $i(\sin\theta X+\cos\theta Z)=\begin{bmatrix} i\cos\theta & i\sin\theta \\ i\sin\theta & -i\cos\theta \end{bmatrix}$ & $[\ ]^2=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$e^{i(\sin\theta X+\cos\theta Z)}=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{bmatrix} i\cos\theta & i\sin\theta \\ i\sin\theta & -i\cos\theta \end{bmatrix}^n$$

$$=\begin{bmatrix} \cos 1+\cos\theta\cdot i\sin 1 & \sin\theta\cdot i\sin 1 \\ \sin\theta\cdot i\sin 1 & \cos 1-\cos\theta\cdot i\sin 1 \end{bmatrix}$$

(b) $$e^{iX}e^{iZ}=\frac{1}{2}\begin{bmatrix} e^{i}+e^{-i} & e^{i}-e^{-i} \\ e^{i}-e^{-i} & e^{i}+e^{-i} \end{bmatrix}\begin{bmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} e^{2i}+1 & 1-e^{-2i} \\ e^{2i}-1 & 1+e^{-2i} \end{bmatrix}$$

$i(X+Z)=\begin{bmatrix} i & i \\ i & -i \end{bmatrix}$ & $[\ ]^2=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$,

$$e^{i(X+Z)}=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{bmatrix} i & i \\ i & -i \end{bmatrix}^n=\begin{bmatrix} \cos\sqrt{2}+\frac{1}{\sqrt{2}}\sin\sqrt{2} & \cos\sqrt{2}+\frac{1}{\sqrt{2}}\sin\sqrt{2} \\ \cos\sqrt{2}+\frac{1}{\sqrt{2}}\sin\sqrt{2} & \cos\sqrt{2}-\frac{1}{\sqrt{2}}\sin\sqrt{2} \end{bmatrix}$$

∴ not identical

$$\therefore e^{iX}e^{iZ}=\begin{bmatrix} \cos 1 & i\sin 1 \\ i\sin 1 & \cos 1 \end{bmatrix}\begin{bmatrix} \cos 1+i\sin 1 & 0 \\ 0 & \cos 1-i\sin 1 \end{bmatrix}$$

$$=\begin{bmatrix} \cos^2 1+i\cos 1\sin 1 & \sin^2 1+i\cos 1\sin 1 \\ -\sin^2 1+i\cos 1\sin 1 & \cos^2 1-i\cos 1\sin 1 \end{bmatrix}$$