M1522.002500 - 양자 컴퓨팅 및 정보의 기초

(Prof. Taehyun Kim)

Homework #3

Due date: May. 16, Sat 2020 12:00 pm

If you submit after the due date, your score will be deducted by 20%. No more submission will be accepted after May. 18, 2020. 12:00pm

To reduce the grading burden of TA, the homework will be graded all-or-nothing style. We will grade just a few problems randomly sampled from the homework and renormalize the total grading according to the relative weight of each problem. Also, within each problem, there may be several sub-problems, but we will grade only a few sub-problems within each problem, and the score of each problem will be determined by the graded sub-problems proportionally.

For example, if problem 1 is composed of 5 sub-problems, we will decide which sub-problems will be graded later, and if you solved that sub-problems correctly, you will get the full credit of problem 1. In the worst case, you might have solved all other sub-problems correctly, but got the wrong answers in all the graded sub-problems. That is an unfortunate situation, but the score for that entire problem will become 0. Without this policy, we cannot grade so much homework efficiently.

The homework should be *hand-written*, converted into a pdf file, and uploaded to ETL. The pdf file may either be a scanned-copy or camera-taken picture of your homework. If you solve the homework problems using digital pen on a tablet, it will be considered as your hand-writing, but make sure that your hand-writing is legible. Please make sure you denote the number of the problems correctly. We will post solutions for every homework and announce the problems and sub-problems to be graded after the hard deadline.

The homework should be written with your hand-writing either on a paper or a tablet. Computer-typed homework won't be accepted!

Note)

1.The terminology *Pauli matrices* used throughout the homework refer to the $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ matrices introduced in the lectures.

2. Please read section 1.2 ~ section 1.3 and chapter 2 ~ section 2.2.5 except section 2.1.10 of the textbook (Quantum Computation and Quantum Information).

Commutator relations

1. (10 points) Prove the identities of commutators:

(A) (2 points) $[\Omega, \Omega] = 0$ ➔ The same operator commutes with itself.
(B) (2 points) $[\Omega,\ \alpha] = 0$ where α is a scalar
(C) (2 points) $[\Omega,\ \Lambda\Theta] = \Lambda[\Omega,\ \Theta] + [\Omega, \Lambda]\Theta$
(D) (2 points) $[\Lambda\Omega,\ \Theta] = \Lambda[\Omega,\ \Theta] + [\Lambda, \Theta]\Omega$
(E) (2 points) $[\Omega,\ \Lambda + \Theta] = [\Omega,\ \Lambda] + [\Omega,\ \Theta]$

2. (18 points) Assume that $[a,\ a^\dagger]$ =1. By using this relation and the identities in Problem 1, prove the following identities.

(A) (3 points) $[a,\ (a^\dagger)^2] = 2a^\dagger$

(B) (3 points) $[a,\ (a^\dagger)^n] = n(a^\dagger)^{n-1}$

(C) (3 points) If function $f(a^\dagger)$ can be expanded in a power series of $a^\dagger$, $[a, f(a^\dagger)] = \frac{\partial}{\partial a^\dagger} f(a^\dagger)$

(D) (3 points) If function $f(a)$ can be expanded in a power series of a, $[a, f(a)] = 0$

(E) (3 points) $\left[a, a^\dagger a + \frac{1}{2}\right] = a$

(F) (3 points) $\left[a^\dagger, a^\dagger a + \frac{1}{2}\right] = -a^\dagger$

Heisenberg's uncertainty principle

3. (5 points) Prove that standard deviation σ for some measurement on an observable Ω corresponds to,

$$\sigma^2 = \langle\psi|\Omega^2|\psi\rangle - \langle\psi|\Omega|\psi\rangle^2$$

4. (10 points) Suppose A and B are two Hermitian operators, and $|\psi\rangle$ is a quantum state. Suppose $\langle\psi|AB|\psi\rangle = x + iy$. Prove $\langle\psi|[A,B]|\psi\rangle = 2iy$ and $\langle\psi|\{A,B\}|\psi\rangle = 2x$ where $\{A,B\} = AB + BA$.

This means $|\langle\psi|[A,B]|\psi\rangle|^2 \le 4|\langle\psi|AB|\psi\rangle|^2$. Using Cauchy-Schwarz inequality, derive Heisenberg's uncertainty principle

$$\sigma_C\sigma_D \ge \frac{|\langle\psi|[C,D]|\psi\rangle|}{2}$$

(Hint: substitute $\mathrm{A} = \mathrm{C} - \langle C\rangle, \mathrm{B} = \mathrm{D} - \langle D\rangle$)

Tensor product

5. (5 points) Calculate the matrix representation of the tensor products of the Pauli operators (a) $\mathbb{X}\otimes\mathbb{Z}$, (b) $\mathbb{I}\otimes\mathbb{X}$, (c) $\mathbb{X}\otimes\mathbb{I}$. Are the three tensor products commutative in multiplication? That is, evaluate $[\mathbb{X}\otimes\mathbb{Z},\ \mathbb{I}\otimes\mathbb{X}]$, $[\mathbb{I}\otimes\mathbb{X},\ \mathbb{X}\otimes\mathbb{I}]$, $[\mathbb{X}\otimes\mathbb{I},\ \mathbb{X}\otimes\mathbb{Z}]$.

Single qubit gates

6. (10 points) Find the eigenvectors of the operator $\hat{n}\cdot\vec{\sigma} = n_x\mathbb{X} + n_y\mathbb{Y} + n_z\mathbb{Z}$ where $\hat{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions and $\vec{\sigma}$ denotes the three-component vector $(\mathbb{X},\mathbb{Y},\mathbb{Z})$ of Pauli matrices. Show that they represent states on the Bloch sphere.

(Hint: $\hat{n} = (n_x, n_y, n_z) = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta)$ is the radial unit vector in spherical coordinates. The Bloch sphere is constructed by mapping the two-dimensional Hilbert space to a three-dimensional sphere.)

7. (5 points) It is useful to be able to simplify circuits by inspection, using well-known identities. The following three identities are such examples. Prove them.

$$\mathrm{HXH} = \mathrm{Z};\quad \mathrm{HYH} = -\mathrm{Y};\quad \mathrm{HZH} = \mathrm{X}.$$

Quantum entanglement

8. (5 points) (Lecture 12 page 3) Prove that the same argument is valid for arbitrary basis

$$|\theta_+\rangle = cos\theta\ |H\rangle + \sin\theta\ |V\rangle$$

$$|\theta_-\rangle = -\sin\theta\ |H\rangle + \cos\theta\ |V\rangle$$

9. (10 points) (Lecture 12 page 6-7)

Suppose that $|\psi\rangle_C = \alpha|0\rangle + \beta|1\rangle$. We modify the typical Bell states as follows.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{-i\varphi}|11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - e^{-i\varphi}|11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + e^{-i\varphi}|10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - e^{-i\varphi}|10\rangle)$$

It is easily seen that these vectors form the basis of the space of the two-qubit system. Now consider quantum teleportation lectured in class. Instead of $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, we choose $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + e^{-i\varphi}|11\rangle)$ as the entangled state between A and B.

It was shown that the choice of measurement of the entangled state between C and A requires different operators to be acted on B in order to successfully teleport state $|\psi\rangle_C$ to $|\psi\rangle_B$. Find these operators (or matrix representations) for each measurement on $|\psi^+\rangle_{CA}, |\psi^-\rangle_{CA}, |\phi^+\rangle_{CA}, |\phi^-\rangle_{CA}$.

(Hint: The matrix of the form $S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega} \end{bmatrix}$ can be used to remove the relative phase between the two qubits. The resulting operator may be represented as a product of matrices.)

10. (15 points) Assume that qubit A on the Earth and qubit B on the Moon are entangled. At the same time, another qubit C on the Moon and the fourth qubit D on Mars are also entangled. To be specific, the qubits A and B are in state $|\varphi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$ and C and D are in state $|\varphi^+\rangle_{CD} = \frac{1}{\sqrt{2}}(|00\rangle_{CD} + |11\rangle_{CD})$. Now assume that we measure both qubits B and C on the Moon in the Bell basis.

a) (10 points) After the measurement, show that the qubit A on the Earth and the qubit D on the Mars form one of the Bell states. Note that the qubit A and the qubit D did not have any interaction before, but they are left in an entangled state after the measurement of the qubit B and C in the Bell basis. This phenomenon is called as entanglement swapping.

b) (5 points) Entanglement swapping can be also understood in terms of quantum teleportation. Show that it can be interpreted as the teleportation of the quantum state of qubit B by the help of the entangled state between the qubit C and the qubit D. This also shows that the quantum teleportation works not only for the qubit in a superposition state, but also when the qubit is part of the entangled state. Generally speaking, quantum teleportation works independent of the qubit status.