2017-12451 컴퓨터공학부 이동학

양자 컴퓨팅 및 정보의 기초 Homework #3

1.A) $[\Omega, \Omega] = \Omega^2 - \Omega^2 = 0$

1.B) $[\Omega, \alpha] = \alpha\Omega - \alpha\Omega = 0$

1.C) $[\Omega, \Lambda\theta] = \Omega\Lambda\theta - \Lambda\theta\Omega =$

$\Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta = \cancel{\Lambda\Omega\theta} - \Lambda\theta\Omega + \Omega\Lambda\theta - \cancel{\Lambda\Omega\theta}$

1.D) $[\Lambda\Omega, \theta] = \Lambda\Omega\theta - \theta\Lambda\Omega =$

$\Lambda[\Omega, \theta] + [\Lambda, \theta]\Omega = \Lambda\Omega\theta - \cancel{\Lambda\theta\Omega} + \cancel{\Lambda\theta\Omega} - \theta\Lambda\Omega$

1.E) $[\Omega, \Lambda + \theta] = \Omega\Lambda + \Omega\theta - \Lambda\Omega - \theta\Omega$

$= [\Omega, \Lambda] + [\Omega, \theta]$

2.A) $[a, (a^\dagger)^2] = aa^\dagger a^\dagger - a^\dagger a^\dagger a$

$= aa^\dagger a^\dagger - a^\dagger(aa^\dagger - 1)$

$= (aa^\dagger - a^\dagger a)a^\dagger + a^\dagger = 2a^\dagger$

2.B) 귀납법, $[a, (a^\dagger)^n] = a(a^\dagger)^{n-1}a^\dagger - (a^\dagger)^{n-1}a^\dagger a$

$= a(a^\dagger)^{n-1}a^\dagger - (a^\dagger)^{n-1}(aa^\dagger - 1)$

$= (a(a^\dagger)^{n-1} - (a^\dagger)^{n-1}a)a^\dagger + (a^\dagger)^{n-1}$

$= \boxed{(n-1)(a^\dagger)^{n-2}}a^\dagger + (a^\dagger)^{n-1} = n(a^\dagger)^{n-1}$

$[a, (a^\dagger)^k] = k(a^\dagger)^{k-1}$

가정

2.C) $f(a^\dagger) = \sum t_n (a^\dagger)^n \rightarrow [a, f(a^\dagger)] = \sum [a, t_n (a^\dagger)^n]$

$= \sum n t_n (a^\dagger)^{n-1} = \frac{\partial}{\partial a^\dagger} f(a^\dagger)$

2.D) $f(a) = \sum t_n (a)^n \rightarrow [a, f(a)] = \sum [a, t_n (a)^n] = 0$

2.E) $[a, a^\dagger a + \frac{1}{2}] = [a, a^\dagger a] + \cancel{[a, \tfrac{1}{2}]}^{0} = a^\dagger \cancel{[a, a]} + [a, a^\dagger] a$

$= a$

2.F) $[a^\dagger, a^\dagger a + \frac{1}{2}] = [a^\dagger, a^\dagger a] + \cancel{[a^\dagger, \tfrac{1}{2}]}^{0} = a^\dagger [a^\dagger, a] + \cancel{[a^\dagger, a^\dagger]}^{0} a$

$= -a^\dagger$

3. $E(\Omega) = \langle \psi | \Omega | \psi \rangle$,

$[\Delta(\Omega)]^2 = E(\Omega^2) - E(\Omega)^2 = \langle \psi | \Omega^2 | \psi \rangle - \langle \psi | \Omega | \psi \rangle^2$

4. $(AB)^\dagger = B^\dagger A^\dagger = BA$, $\therefore \langle \psi | BA | \psi \rangle = x - iy$

$\rightarrow \langle \psi | [A, B] | \psi \rangle = 2iy$, $\langle \psi | \{A, B\} | \psi \rangle = 2x$,

by 코시-슈와르츠 부등식, $|\langle \psi | AB | \psi \rangle|^2 \le \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle$,

위의 결과 이용 → $|\langle \psi | [A, B] | \psi \rangle|^2 \le 4 \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle$,

$A = C - \langle C \rangle$, $B = D - \langle D \rangle$ 대입 → $|\langle \psi | [C, D] | \psi \rangle|^2 \le 4 \Delta(C)^2 \Delta(D)^2$,

$\therefore \Delta(C) \Delta(D) \ge \frac{1}{2} |\langle \psi | [C, D] | \psi \rangle|$

5. $X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

5. $I \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

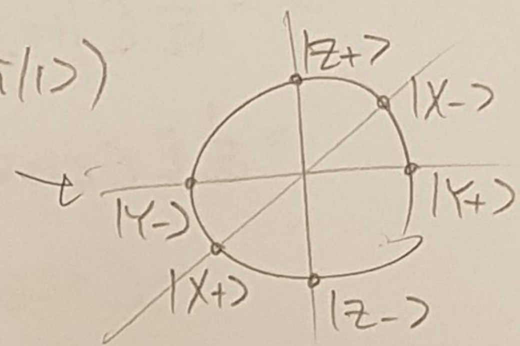
$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$[X \otimes Z, I \otimes X] \neq 0$, $[I \otimes X, X \otimes I] = 0$, $[X \otimes I, X \otimes Z] = 0$

6. $|X_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|X_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|Y_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|Y_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$|Z_+\rangle = |0\rangle$, $|Z_-\rangle = |1\rangle$



$a|0\rangle + b|1\rangle \quad (a = \cos\theta, \ b = e^{i\varphi}\sin\theta)$

$\hat{n} \cdot \vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta \, e^{-i\varphi} \\ \sin\theta \, e^{i\varphi} & -\cos\theta \end{pmatrix}$

7. $H^2=I$ 고 $HX=ZH$, $HY=-YH$ 를 증명하면 됨

$$HX=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$$

$$ZH=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$$

$=$

$$HY=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}i&-i\\-i&-i\end{pmatrix}$$

$$-YH=\frac{1}{\sqrt{2}}\begin{pmatrix}0&i\\-i&0\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}i&-i\\-i&-i\end{pmatrix}$$

$=$

8. $|H\rangle=\cos\theta|\theta+\rangle-\sin\theta|\theta-\rangle$

$|V\rangle=\sin\theta|\theta+\rangle+\cos\theta|\theta-\rangle$

$$|\psi-\rangle=\frac{1}{\sqrt{2}}\left[|H\rangle_A|V\rangle_B-|V\rangle_A|H\rangle_B\right]$$

$$=\frac{1}{\sqrt{2}}\Big[(\cos\theta|\theta+\rangle_A-\sin\theta|\theta-\rangle_A)(\sin\theta|\theta+\rangle_B+\cos\theta|\theta-\rangle_B)$$
$$-(\sin\theta|\theta+\rangle_A+\cos\theta|\theta-\rangle_A)(\cos\theta|\theta+\rangle_B-\sin\theta|\theta-\rangle_B)\Big]$$

$$=\frac{1}{\sqrt{2}}\Big[\cancel{\cos\theta\sin\theta|\theta+\rangle_A|\theta+\rangle_B}-\sin^2\theta|\theta-\rangle_A|\theta+\rangle_B$$
$$+\cos^2\theta|\theta+\rangle_A|\theta-\rangle_B-\cancel{\sin\theta\cos\theta|\theta-\rangle_A|\theta-\rangle_B}$$
$$-\cancel{\sin\theta\cos\theta|\theta+\rangle_A|\theta+\rangle_B}-\cos^2\theta|\theta-\rangle_A|\theta+\rangle_B$$
$$+\sin^2\theta|\theta+\rangle_A|\theta-\rangle_B+\cancel{\cos\theta\sin\theta|\theta-\rangle_A|\theta-\rangle_B}\Big]$$

$$=\frac{1}{\sqrt{2}}\left[|\theta+\rangle_A|\theta-\rangle_B-|\theta-\rangle_A|\theta+\rangle_B\right]$$

9. $|\psi\rangle_c |\phi^+\rangle_{AB} = (\alpha|0\rangle_c + \beta|1\rangle_c)\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + e^{-i\varphi}|1\rangle_A|1\rangle_B)$

$= \frac{1}{\sqrt{2}}(\alpha|0\rangle_c|0\rangle_A|0\rangle_B + \alpha e^{-i\varphi}|0\rangle_c|1\rangle_A|1\rangle_B + \beta|1\rangle_c|0\rangle_A|0\rangle_B + \beta e^{-i\varphi}|1\rangle_c|1\rangle_A|1\rangle_B)$

$\Big(|00\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\phi^-\rangle),\ |11\rangle = \frac{e^{i\varphi}}{\sqrt{2}}(|\phi^+\rangle - |\phi^-\rangle),$
$|01\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle),\ |10\rangle = \frac{e^{-i\varphi}}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle) \Big)$

$\to\ = \frac{1}{2}(\alpha(|\phi^+\rangle_{CA} + |\phi^-\rangle_{CA})|0\rangle_B + \alpha e^{-i\varphi}(|\psi^+\rangle_{CA} + |\psi^-\rangle_{CA})|1\rangle_B$
$+ \beta e^{i\varphi}(|\psi^+\rangle_{CA} - |\psi^-\rangle_{CA})|0\rangle_B + \beta(|\phi^+\rangle_{CA} - |\phi^-\rangle_{CA})|1\rangle_B$

$= \frac{1}{2}(|\psi^+\rangle_{CA}(\alpha e^{-i\varphi}|1\rangle_B + \beta e^{i\varphi}|0\rangle_B)$
$+ |\psi^-\rangle_{CA}(\alpha e^{-i\varphi}|1\rangle_B - \beta e^{i\varphi}|0\rangle_B)$
$+ |\phi^+\rangle_{CA}(\alpha|0\rangle_B + \beta|1\rangle_B)$
$+ |\phi^-\rangle_{CA}(\alpha|0\rangle_B - \beta|1\rangle_B)),$

$\therefore\ |\psi^+\rangle_{CA}:\ X\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$

$|\psi^-\rangle_{CA}:\ -Y\begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$

$|\phi^+\rangle_{CA}:\ I$

$|\phi^-\rangle_{CA}:\ Z$