M1522.002500 - 양자 컴퓨팅 및 정보의 기초

(Prof. Taehyun Kim)

Homework #2

Due date: May. 1, Fri 2020 12:00 pm

If you submit after the due date, your score will be deducted by 20%. No more submission will be accepted after May. 3, 2020. 12:00pm

To reduce the grading burden of TA, the homework will be graded all-or-nothing style. We will grade just a few problems randomly sampled from the homework and renormalize the total grading according to the relative weight of each problem. Also, within each problem, there may be several sub-problems, but we will grade only a few sub-problems within each problem, and the score of each problem will be determined by the graded sub-problems proportionally.

For example, if problem 1 is composed of 5 sub-problems, we will decide which sub-problems will be graded later, and if you solved that sub-problems correctly, you will get the full credit of problem 1. In the worst case, you might have solved all other sub-problems correctly, but got the wrong answers in all the graded sub-problems. That is an unfortunate situation, but the score for that entire problem will become 0. Without this policy, we cannot grade so much homework efficiently.

The homework should be *hand-written*, converted into a pdf file, and uploaded to ETL. The pdf file may either be a scanned-copy or camera-taken picture of your homework. If you solve the homework problems using digital pen on a tablet, it will be considered as your hand-writing, but make sure that your hand-writing is legible. Please make sure you denote the number of the problems correctly. We will post solutions for every homework and announce the problems and sub-problems to be graded after the hard deadline.

The homework should be written with your hand-writing either on a paper or a tablet. Computer-typed homework won't be accepted!

A. Unitary and normal operators

1. (10 points) Prove the following statements.

a) (5 points) $\text{The product of unitary operators is unitary.}$

b) (5 points) $\text{A is unitary} \implies \text{A's eigenvalues}$ $are\ complex\ numbers\ of\ unit\ modulus$.

B. Hermitian operators

2. (25 points) Prove the following statements. In the lecture, we proved the properties in (a) and (b) only when the Hermitian operators have non-degenerate eigenvalues. In (a) and (b), please assume that Hermitian operators might have degenerate eigenvalues. (Hint: Theorem 9 and 10 in the reference.)

a) (5 points) $\text{A is Hermitian} \implies \text{A}'s\ eigenvalues\ \text{are real.}$

b) (10 points) $\mathbb{A}\ \text{is a Hermitian matrix.} \implies \text{There exists a unitary matrix}\ \mathbb{U}$ such that $\mathbb{U}^{\dagger}\mathbb{A}\mathbb{U}$ is a diagonal matrix. (Hint: Two results must be included in the answer. 1) A has a matrix representation where it is composed of diagonal elements only. You may assume that the characteristic equation has at least one root. 2) If you are given $\mathbb{A}$ which is a matrix representation of A in a basis other than its eigenbasis, unitary transformation of $\mathbb{A}$ yields the diagonalized representation of A.)

c) (5 points) A is Hermitian $\Rightarrow$ e^{iA} is unitary (You may use $\text{e}^{\text{A}}e^{B} = e^{A+B}$ when $[A, B] = 0$)

d) (5 points) Consider the two commuting Hermitian operators A, B with non-degenerate eigenvalues.

$$A|i\rangle = p_i|i\rangle\ (i = 1,2)\ \text{and}\ B|\pm\rangle = q_{\pm}|\pm\rangle \implies \langle -|\text{A}|+\rangle = 0$$

3. (15 points) Consider Hermitian matrices $\mathrm{M}_1, M_2, M_3, M_4$ that obey

$$\mathrm{M_i} M_j + M_j M_i = 2\delta_{ij} I,$$

a) (5 points) Show that the eigenvalues of $\mathrm{M_i}$ are ± 1. (Hint: go to the eigenbasis of $\mathrm{M_i}$, and use the equation for i=j)

b) (5 points) By considering the relation

$$\mathrm{M_i} M_j = -M_j M_i \text{ for i} \neq \text{j}$$

Show that $\mathrm{tr}(\mathrm{M_i}) = 0$ (Hint: $\mathrm{tr(ACB)} = \mathrm{tr(CBA)}$)

c) (5 points) Show that they cannot be odd-dimensional matrices.

C. Basis transformation

4. (25 points) Assume that operator O has a matrix representation of matrix $\mathbb{O} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in an orthonormal basis $\{|0\rangle, |1\rangle\}$ and ket vectors $|v\rangle$ and $|w\rangle$ have column vector representations of $\mathbb{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbb{w} = \begin{bmatrix} c \\ d \end{bmatrix}$ in the same basis, respectively.

(a) (5 points) Please calculate $\mathbb{w}^\dagger \mathbb{O} \mathbb{v}$.

(b) (5 points) Find out the new column vector representations $\mathbb{v}', \mathbb{w}'$ of $|v\rangle$ and $|w\rangle$ in a new orthonormal basis $\left\{|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\right\}$.

(c) (5 points) What is the new matrix representation $\mathbb{O}'$ of an operator O with respect to the basis $\{|\pm\rangle\}$?

(d) (5 points) Please calculate $(\mathbb{w}')^\dagger \mathbb{O}' \mathbb{v}'$ and compare with (a). (This problem shows that even though the ket vectors and the linear operators have completely different vector and matrix representation in different basis, their inner product and linear transformation properties are invariant under basis transformation when the two bases are orthonormal within its set. Also note that if we write both (a) and (d) calculation in terms of bra and ket notation, they will be represented as $\langle w|O|v\rangle$ which is independent of basis choice.)

(e) (5 points) Construct the unitary transformation matrix that connects the diagonal basis of $\mathbb{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to the diagonal basis of $\mathbb{O}$.

D. Diagonalization, operator functions

5. (10 points) Two Hermitian operators anti-commute:

$$\{A, B\} = AB + BA = 0$$

Is it possible to have a simultaneous eigenbasis of A and B? Specify the conditions under which your argument holds.

6. (15 points) Consider $\mathbb{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbb{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

a) (7.5 points) Evaluate $e^{i\mathbb{X}}$, $e^{i\mathbb{Z}}$, $e^{i(sin\theta\, \mathbb{X} + cos\theta\, \mathbb{Z})}$.

b) (7.5 points) Calculate and compare $e^{i\mathbb{X}}e^{i\mathbb{Z}}$ and $e^{i(\mathbb{X}+\mathbb{Z})}$. Are they identical? (You may expand the terms up to orders that are sufficient enough to convince your result)