Summary of previous lecture

- Density matrix
  - When an ensemble of pure states $\{p_i, |\psi_i\rangle\}$ is given,

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

- Reformulation of postulate
  - Postulate 2: $\rho \overset{U}{\rightarrow} \sum_i p_i \left(U|\psi_i\rangle\langle\psi_i|U^\dagger\right) = U\rho U^\dagger$
  - Postulate 3
    - ρ ➔ $p(m) = \sum_i p_i \mathrm{tr}\left(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|\right) = \mathrm{tr}\left(M_m^\dagger M_m \rho\right)$
    - After measurement of m, ρ ➔ $\rho_m = \frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}$
- Properties of density matrix
  - $\mathrm{tr}(\rho^2) = \begin{cases} 1 & \Rightarrow \text{Pure state} \\ < 1 & \Rightarrow \text{Mixed state} \end{cases}$
  - When mixture of mixed states $\{p_i, \rho_i\}$ are given,

$$\rho = \sum_{i=1}^{n} p_i \rho_i$$

Ensembles of quantum states

- Assumption
  - The initial state of some quantum system is given as ρ_{init}.
  - Someone already measured this system, but the measurement result is not available to us.
- What is the best way to describe the quantum state of this quantum system?
  - If the measurement outcome is m, the probability of such case is $p(m) = \text{tr}\left(M_m^\dagger M_m \rho_{\text{init}}\right)$ and the density matrix of final state is $\rho_m = \frac{M_m \rho_{\text{init}} M_m^\dagger}{\text{tr}\left(M_m^\dagger M_m \rho_{\text{init}}\right)} = \frac{M_m \rho_{\text{init}} M_m^\dagger}{p(m)}$.
  - We don't know what m is. Therefore we need to consider for all possible m's.
  - Density matrix: $\sum_m p(m)\rho_m = \sum_m p(m) \frac{M_m \rho_{\text{init}} M_m^\dagger}{p(m)} = \sum_m M_m \rho_{\text{init}} M_m^\dagger$

Properties of density matrix

- Theorem 2.5: (Characterization of density operators) An operator ρ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ if and only if it satisfies the conditions:

  (1) (Trace condition) $\mathrm{tr}(\rho) = 1$

  (2) (Positivity condition) ρ is a positive operator

- Proof
  - $\mathrm{tr}(\rho) = \mathrm{tr}(\sum_i p_i |\psi_i\rangle\langle\psi_i|) = \sum_i p_i \mathrm{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$
  - For arbitrary $|v\rangle$, $\langle v|\rho|v\rangle = \langle v|(\sum_i p_i |\psi_i\rangle\langle\psi_i|)|v\rangle = \sum_i p_i \langle v|\psi_i\rangle\langle\psi_i|v\rangle = \sum_i p_i |\langle v|\psi_i\rangle|^2 \geq 0$
  - Conversely, assume ρ is any operator satisfying trace condition (1) and positivity condition (2).

    Due to positivity of ρ, $\rho = \sum_j \lambda_j |j\rangle\langle j|$ (spectral decomposition ← Any positive operator is Hermitian (Exercise 2.24))

    $|j\rangle$: orthonormal basis.

    λ_j are real, non-negative eigenvalues of ρ.

    $\sum_j \lambda_j = 1$ due to trace condition.

    Therefore, $\{\lambda_i, |j\rangle\}$ forms good ensemble for density operator.

Summary of QM in terms of DM

- Reformulation of quantum mechanics from page 102 of the textbook.

  - Postulate 1
    - Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *density operator*, which is a positive operator ρ with trace one, acting on the state space of the system. If a quantum system is in the state ρ_i with probability p_i, then the density operator for the system is

$$\sum_i p_i \rho_i$$

  - Postulate 2
    - The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$\rho' = U \rho U^\dagger$$

Summary of QM in terms of DM

- (Cont'd) Reformulation of quantum mechanics from page 102 of the textbook.
  - Postulate 3
    - Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = tr\left(M_m^\dagger M_m \rho\right)$$

    - The state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{\mathrm{tr}\left(M_m^\dagger M_m \rho\right)}$$

    - The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I$$

  - Postulate 4
    - The state space of a composite physical system is the tensor product of the state spaces of component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \cdots \rho_n$.

Decomposition of DM

- Does the eigenvalues and eigenvectors of a density matrix have some special significance with regard to the ensemble of quantum states represented by the given density matrix?
  - The answer is NO.
- Example
  - $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$
  - One decomposition
    - $|0\rangle$ with probability of $\frac{3}{4}$ and $|1\rangle$ with probability of $\frac{1}{4}$
  - Another decomposition
    - $|a\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$ with probability of $\frac{1}{2}$
    - $|b\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle$ with probability of $\frac{1}{2}$
    - $\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$

Decomposition of DM

- What class of ensembles gives rise to a particular density matrix?
  - Define un-normalized vector $|\widetilde{\psi_i}\rangle = \sqrt{p_i}|\psi_i\rangle$ so that $\rho = \sum_i |\widetilde{\psi_i}\rangle\langle\widetilde{\psi_i}|$
  - When two sets of vectors $|\widetilde{\psi_i}\rangle$ and $|\widetilde{\phi_j}\rangle$ are connected by unitary matrix, they generate the same operator ρ.
- Example
  - $\begin{cases} |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & \text{with probability of } 1/3 \\ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \alpha|1\rangle) & \text{with probability of } 1/3 \\ |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \alpha^*|1\rangle) & \text{with probability of } 1/3 \end{cases}$
  - $\alpha = e^{i\frac{2\pi}{3}}$ ➔ $\alpha^2 + \alpha + 1 = 0$ or $\alpha^* + \alpha + 1 = 0$ and $\alpha^* \cdot \alpha = 1$
  - $\rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2| + \frac{1}{3}|\psi_3\rangle\langle\psi_3| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
  - $\begin{cases} |\phi_1\rangle = |0\rangle & \text{with probability of } 1/2 \\ |\phi_2\rangle = |1\rangle & \text{with probability of } 1/2 \end{cases}$
  - Can we connect the set $\{p_i, |\psi_i\rangle\}$ to the set $\{p_j, |\phi_j\rangle\}$ with unitary transformation?

Decomposition of DM

- Example (continued)
  - $|\widetilde{\psi_i}\rangle$ set: $|\widetilde{\psi_1}\rangle = \sqrt{p_1}|\psi_1\rangle = \frac{1}{\sqrt{6}}(|0\rangle + |1\rangle)$, $|\widetilde{\psi_2}\rangle = \frac{1}{\sqrt{6}}(|0\rangle + \alpha|1\rangle)$, $|\widetilde{\psi_3}\rangle = \frac{1}{\sqrt{6}}(|0\rangle + \alpha^*|1\rangle)$
  - $|\widetilde{\phi_j}\rangle$ set: $|\widetilde{\phi_1}\rangle = \frac{1}{\sqrt{2}}|0\rangle$, $|\widetilde{\phi_2}\rangle = \frac{1}{\sqrt{2}}|1\rangle$
  - Number of vector elements in each set do not match

  ➔ Pad dummy vectors to the smaller set. That is, add $|\widetilde{\phi_3}\rangle = \sqrt{0}|v\rangle$ to $|\widetilde{\phi_j}\rangle$ set
  - $u_{ij} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^* \\ 1 & \alpha^* & \alpha \end{bmatrix}$ where $\alpha = e^{i\frac{2\pi}{3}}$
  - $|\widetilde{\psi_i}\rangle = \sum_j u_{ij}|\widetilde{\phi_j}\rangle$
- Theorem 2.6: (Unitary freedom in the ensemble for density matrix) The sets $|\widetilde{\psi_i}\rangle$ and $|\widetilde{\phi_j}\rangle$ generate the same density matrix if and only if

$$|\widetilde{\psi_i}\rangle = \sum_j u_{ij}|\widetilde{\phi_j}\rangle \ ,$$

where u_{ij} is a unitary matrix of complex numbers, with indices i and j, and we 'pad' whichever set of vectors $|\widetilde{\psi_i}\rangle$ or $|\widetilde{\phi_j}\rangle$ is smaller with additional vectors 0 so that the two sets have the same number of elements. ➔ Proof in page 104

Reduced density matrix

- Section 2.4.3
- Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}.
  - Example: $|0_A\rangle|0_B\rangle$ with probability of ¾, $|0_A\rangle|1_B\rangle$ with probability of ¼.
  - $\rho^{AB} = \frac{3}{4}|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + \frac{1}{4}|0_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 1_B|$
- Reduced density operator for system A is defined by

$$\rho^A \equiv \mathrm{tr}_B(\rho^{AB})$$

where tr_B is a map of operators known as the *partial trace* over system B.

  - Example (not mathematically sound, but gives intuition):

    $\rho^A = \mathrm{tr}_B(\rho^{AB}) = \sum_{j=1}^{2}\langle j_B|\rho^{AB}|j_B\rangle$

    $= \langle 0_B|\rho^{AB}|0_B\rangle + \langle 1_B|\rho^{AB}|1_B\rangle = \frac{3}{4}|0_A\rangle\langle 0_A| + \frac{1}{4}|0_A\rangle\langle 0_A| = |0_A\rangle\langle 0_A|$

- Formally, the partial trace is defined by

$$\mathrm{tr}_B\left(|a_{1_A}\rangle\langle a_{2_A}| \otimes |b_{1_B}\rangle\langle b_{2_B}|\right) \equiv |a_{1_A}\rangle\langle a_{2_A}| \otimes \mathrm{tr}\left(|b_{1_B}\rangle\langle b_{2_B}|\right)$$
$$= |a_{1_A}\rangle\langle a_{2_A}| \left(\langle b_{2_B}|b_{1_B}\rangle\right)$$

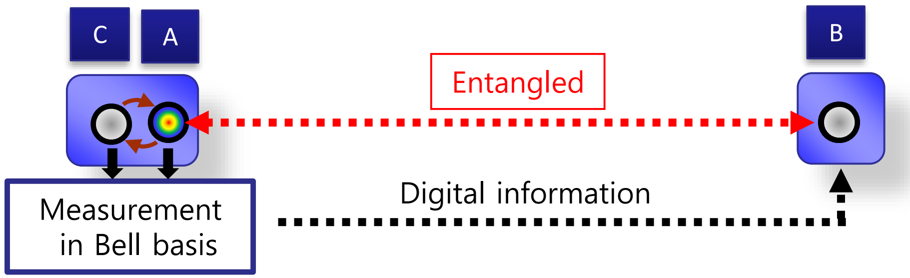
and it should be linear in its input.

Reduced density matrix

- Why are we interested in partial trace and reduced density matrix?
- Entangled state $|\psi^-\rangle = [|0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle]/\sqrt{2}$
  - $\rho^{AB} = \left(\frac{|01\rangle-|10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01|-\langle 10|}{\sqrt{2}}\right) = \frac{|01\rangle\langle 01|-|01\rangle\langle 10|-|10\rangle\langle 01|+|10\rangle\langle 10|}{2}$
- Partial trace on A:
  - $\rho^B = \mathrm{tr}_A(\rho^{AB}) = \frac{\mathrm{tr}_A(|0_A 1_B\rangle\langle 0_A 1_B|)-\mathrm{tr}_A(|0_A 1_B\rangle\langle 1_A 0_B|)-\mathrm{tr}_A(|1_A 0_B\rangle\langle 0_A 1_B|)+\mathrm{tr}_A(|1_A 0_B\rangle\langle 1_A 0_B|)}{2}$
    $= \frac{(|1_B\rangle\langle 1_B|)\langle 0_A|0_A\rangle-(|1_B\rangle\langle 0_B|)\langle 1_A|0_A\rangle-(|0_B\rangle\langle 1_B|)\langle 0_A|1_A\rangle+(|0_B\rangle\langle 0_B|)\langle 1_A|1_A\rangle}{2}$
    $= \frac{|1_B\rangle\langle 1_B|+|0_B\rangle\langle 0_B|}{2} = \frac{I_B}{2}$

- The same result as we obtained assuming actual measurements
- Note
  - The reduced density matrix represents mixed state because $\mathrm{tr}\left((\rho^B)^2\right) = 1/2 < 1$.
  - Consider that the entire state $|\psi^-\rangle$ is pure state, but if we only look at the partial system A, it appears as mixed state.
  - ➔ Signature of entanglement
  - Using partial trace, can you explain why communication faster than light with entangled state is impossible?

Review of quantum teleportation



- $|\Psi_C\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)$: arbitrary quantum state to teleport
- $|\Psi_C\rangle|\psi_{AB}^-\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)(|0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle)/\sqrt{2}$

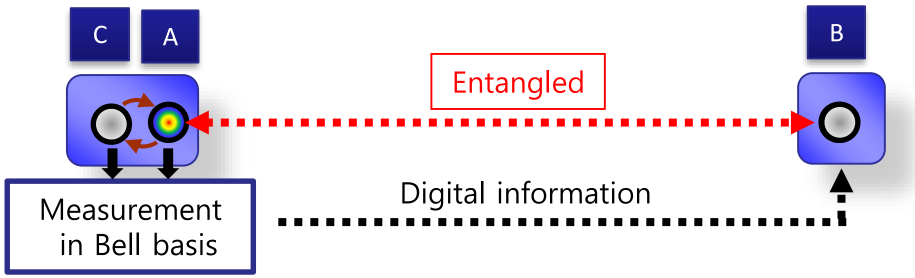
$$= [\alpha|0_C\rangle|0_A\rangle|1_B\rangle - \alpha|0_C\rangle|1_A\rangle|0_B\rangle + \beta|1_C\rangle|0_A\rangle|1_B\rangle - \beta|1_C\rangle|1_A\rangle|0_B\rangle]/\sqrt{2}$$

$$= [\alpha(|\phi_{CA}^+\rangle + |\phi_{CA}^-\rangle)|1_B\rangle - \alpha(|\psi_{CA}^+\rangle + |\psi_{CA}^-\rangle)|0_B\rangle$$
$$+\beta(|\psi_{CA}^+\rangle - |\psi_{CA}^-\rangle)|1_B\rangle + \beta(|\phi_{CA}^+\rangle - |\phi_{CA}^-\rangle)|0_B\rangle]/2$$

$$= [|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle)$$
$$+|\phi_{CA}^+\rangle(\quad \alpha|1_B\rangle + \beta|0_B\rangle) + |\phi_{CA}^-\rangle(\alpha|1_B\rangle - \beta|0_B\rangle)]/2$$

$$\begin{cases} |\psi_{CA}^+\rangle = [|0_C\rangle|1_A\rangle + |1_C\rangle|0_A\rangle]/\sqrt{2} \\ |\psi_{CA}^-\rangle = [|0_C\rangle|1_A\rangle - |1_C\rangle|0_A\rangle]/\sqrt{2} \\ |\phi_{CA}^+\rangle = [|0_C\rangle|0_A\rangle + |1_C\rangle|1_A\rangle]/\sqrt{2} \\ |\phi_{CA}^-\rangle = [|0_C\rangle|0_A\rangle - |1_C\rangle|1_A\rangle]/\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} |0_C\rangle|1_A\rangle = [|\psi_{CA}^+\rangle + |\psi_{CA}^-\rangle]/\sqrt{2} \\ |1_C\rangle|0_A\rangle = [|\psi_{CA}^+\rangle - |\psi_{CA}^-\rangle]/\sqrt{2} \\ |0_C\rangle|0_A\rangle = [|\phi_{CA}^+\rangle + |\phi_{CA}^-\rangle]/\sqrt{2} \\ |1_C\rangle|1_A\rangle = [|\phi_{CA}^+\rangle - |\phi_{CA}^-\rangle]/\sqrt{2} \end{cases}$$

Quantum teleportation with reduced DM



- $|\Psi_C\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)$: arbitrary quantum state to teleport

$$|\Psi_C\rangle|\psi_{AB}^-\rangle = [\ |\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle)$$
$$+|\phi_{CA}^+\rangle(\ \ \alpha|1_B\rangle + \beta|0_B\rangle) + |\phi_{CA}^-\rangle(\alpha|1_B\rangle - \beta|0_B\rangle)\ ]/2$$

- Density matrix of $|\Psi_C\rangle|\psi_{AB}^-\rangle$
  - $\rho^{CAB} = \frac{1}{4}[|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle) + |\phi_{CA}^+\rangle(\beta|0_B\rangle + \alpha|1_B\rangle) + |\phi_{CA}^-\rangle(-\beta|0_B\rangle + \alpha|1_B\rangle)]\ [\langle\psi_{CA}^+|(-\alpha^*\langle 0_B| + \beta^*\langle 1_B|) - \langle\psi_{CA}^-|(\alpha^*\langle 0_B| + \beta^*\langle 1_B|) + \langle\phi_{CA}^+|(\beta^*\langle 0_B| + \alpha^*\langle 1_B|) + \langle\phi_{CA}^-|(-\beta^*\langle 0_B| + \alpha^*\langle 1_B|)]$

- Reduced density matrix at B

$$\rho^B = \mathrm{tr}_{CA}(\rho^{CAB})$$
$$= \frac{1}{4}[(-\alpha|0_B\rangle + \beta|1_B\rangle)(-\alpha^*\langle 0_B| + \beta^*\langle 1_B|) + (\alpha|0_B\rangle + \beta|1_B\rangle)(\alpha^*\langle 0_B| + \beta^*\langle 1_B|)$$
$$+ (\beta|0_B\rangle + \alpha|1_B\rangle)(\beta^*\langle 0_B| + \alpha^*\langle 1_B|) + (-\beta|0_B\rangle + \alpha|1_B\rangle)(-\beta^*\langle 0_B| + \alpha^*\langle 1_B|)]$$
$$= \frac{1}{4}[(2\alpha\alpha^* + 2\beta\beta^*)|0_B\rangle\langle 0_B| + (2\beta\beta^* + 2\alpha\alpha^*)|1_B\rangle\langle 1_B|] = I_B/2$$