Revisit of Postulate 3

- Previous description
  - If the particle is in a state $|\psi\rangle$, measurement of the variable (corresponding to) Ω will yield one of the eigenvalues ω_i with probability of $P(\omega_i) \propto |\langle\omega_i|\psi\rangle|^2$.
  - Then the state of the system will change from $|\psi\rangle$ to $|\omega_i\rangle$ as a result of measurement.
- Section 2.2.3
  - Quantum measurements are described by a set of *measurement operators* $\{M_m\}$. These operators act on the state space of the system being measured.
  - The index m refers to the measurement outcomes that may occur in the experiment.
  - If the state of the quantum system is $|\psi\rangle$ immediately before the measurement, then the probability that result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle\ ,$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}\ .$$

  - The measurement operators satisfy the completeness equation,

$$\sum_m M_m^\dagger M_m = I\ .$$

➔ $\sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = \langle\psi|\sum_m M_m^\dagger M_m\,|\psi\rangle = 1$

Example of revised Postulate 3

- Measurement of a qubit in computational basis
  - $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$: Hermitian
  - $M_0^\dagger M_0 = M_0^2 = M_0, M_1^\dagger M_1 = M_1^2 = M_1$

    ➔ $M_0^\dagger M_0 + M_1^\dagger M_1 = |0\rangle\langle 0| + |1\rangle\langle 1| = I$.
  - For the initial state $|\psi\rangle = a|0\rangle + b|1\rangle$
    - $p(0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle = |a|^2$
    - $p(1) = \langle\psi|M_1^\dagger M_1|\psi\rangle = \langle\psi|M_1|\psi\rangle = |b|^2$
  - The state after measurement in the two cases:
    - $\frac{M_0|\psi\rangle}{\sqrt{\langle\psi|M_0^\dagger M_0|\psi\rangle}} = \frac{M_0|\psi\rangle}{\sqrt{p(0)}} = \frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|}|0\rangle$ ➔ $\frac{a}{|a|}$ can be ignored
    - $\frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1^\dagger M_1|\psi\rangle}} = \frac{M_1|\psi\rangle}{\sqrt{p(1)}} = \frac{M_1|\psi\rangle}{|b|} = \frac{b}{|b|}|1\rangle$

Distinguishing quantum states

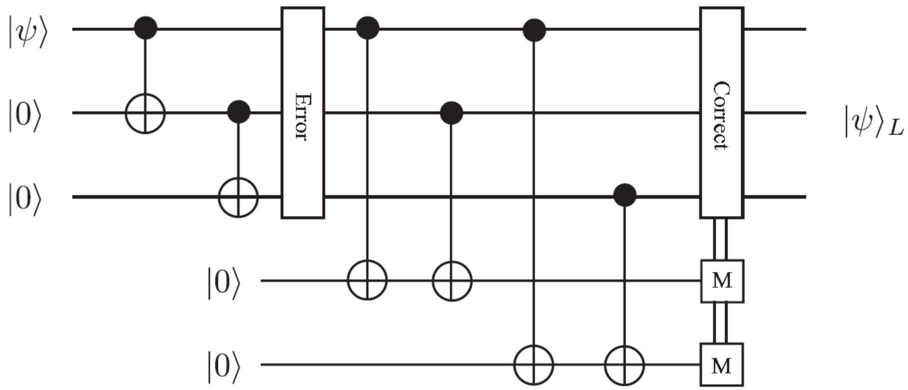
- Section 2.2.4
- Non-orthogonal quantum states cannot be distinguished with certainty.
  - Example: $|H\rangle$ vs $|D\rangle = \frac{|H\rangle+|V\rangle}{\sqrt{2}}$
- Distinguishability of orthonormal states
  - Suppose $|\psi_i\rangle$ are orthonormal for $i = 1 \cdots n$
  - Define measurement operators $M_i = |\psi_i\rangle\langle\psi_i|$ for $i = 1 \cdots n$
  - Define additional measurement operator M_0 as the positive square root of the operator $I - \sum_{i\neq 0}|\psi_i\rangle\langle\psi_i|$
  - Then $M_0, M_1, \cdots, M_n$ satisfies the completeness relation
  - If the state is prepared in $|\psi_i\rangle$ for some i, $p(i) = \langle\psi_i|M_i^\dagger M_i|\psi_i\rangle = 1$, so the result i occurs with certainty.

Distinguishing quantum states

- If the states $|\psi_i\rangle$ are not orthonormal, there is *no quantum measurement capable of distinguishing the states*.
- Sketch of proof
  - Assume there are such measurement operators M_j with outcome j capable of distinguishing the states
  - Then we need a mapping that will map outcome j to the index i of quantum state $|\psi_i\rangle$. That is, $i = f(j)$.
  - Assume we want to distinguish non-orthogonal $|\psi_1\rangle$ and $|\psi_2\rangle$ ➔ $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\phi\rangle$ where $\alpha \neq 0$.
  - Suppose k is a measurement outcome such that $\langle\psi_1|M_k^\dagger M_k|\psi_1\rangle \neq 0$ ➔ $f(k) = 1$
  - Because of $\alpha \neq 0$, $\langle\psi_2|M_k^\dagger M_k|\psi_2\rangle$ won't be zero in general ➔ When measurement outcome is k, we cannot distinguish between $|\psi_1\rangle$ and $|\psi_2\rangle$
- For more complete proof, refer to Box 2.3 on page 87.

Example

- Syndrome measurement of 3-qubit repetition code



- Error-detection or syndrome diagnosis
  - $M_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111|$ no error
  - $M_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011|$ bit flip on qubit one
  - $M_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101|$ bit flip on qubit two
  - $M_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110|$ bit flip on qubit three
  - If the corrupted state is $a|100\rangle + b|011\rangle$, $\langle\psi|M_1^\dagger M_1|\psi\rangle = 1$

Projective measurement

- Section 2.2.5 Projective Measurements
  - Basically projective measurement is what we generally called as quantum measurement in this class up to now.
- A projective measurement is described by an *observable*, M, a Hermitian operator on the state space of the system being observed. The observable has a spectral decomposition,

$$M = \sum_m m P_m$$

where P_m is the projector onto the eigenspace of M with eigenvalue m. The possible outcomes of the measurement corresponds to the eigenvalues, m, of the observable.

  - Upon measuring the state $|\psi\rangle$, the probability of getting result m is given by

$$p(m) = \langle\psi|P_m|\psi\rangle \ .$$

  - Given that outcome m occurred, the state of the quantum system immediately after the measurement is

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}} \ .$$

POVM measurements

- Postulate 3 provides two types of information
  - Measurement statistics: probability to measure certain outcome
  - Quantum state after the measurement: state collapse
- POVM (Positive Operator-Valued Measure) formalism
  - Cares only about the probability, not about the quantum state after the measurement
  - Suppose measurement (M_m) is performed upon a quantum system in the state $|\psi\rangle$ ➔ $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$
  - Define $E_m \equiv M_m^\dagger M_m$ ➔ $\sum_m E_m = I$ and $p(m) = \langle\psi|E_m|\psi\rangle$
  - E_m is called as POVM element. ➔ $\{E_m\}$ vs $\{M_m\}$
  - The complete set of $\{E_m\}$ is known as POVM.
  - Projective measurement P_m can also be considered as an example of POVM.
  - P_m can be considered as either M_m or E_m .

POVM measurements

- Definition of positive operator (section 2.1.6)
  - An operator A such that for any vector $|v\rangle$, $\langle v|A|v\rangle$ is a real, non-negative number.
  - Special case of Hermitian operator
- Note that POVM element E_m is positive operator.
- If $\{E_m\}$ is some arbitrary set of positive operators such that $\sum_m E_m = I$, then there exists a set of measurement operators M_m. (Proof: $M_m \equiv \sqrt{E_m}$)

Example of POVM

- Suppose Alice gives Bob a qubit prepared in one of the two states, $|\psi_1\rangle = |0\rangle$ or $|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.
- Goal: Bob wants to perform a measurement which distinguishes the states *some of the time*, but *never* makes an error of mis-identification.
- POVM elements
  - $E_1 \equiv \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|$
  - $E_2 \equiv \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle-|1\rangle)(\langle 0|-\langle 1|)}{2}$
  - $E_3 \equiv I - E_1 - E_2$
  - $\sum_m E_m = I$
- If $|\psi_1\rangle = |0\rangle$ is given, $\langle\psi_1|E_1|\psi_1\rangle = 0$ so if E_1 is measured, it should be $|\psi_2\rangle$.
- Similarly, if E_2 is measured, it should be $|\psi_1\rangle$.
- If E_3 is measured, Bob doesn't know, but he does not make error.

Revisit of Postulate 2

- Recall Postulate 2: the evolution of a closed quantum system is described by a unitary transformation
  - $|\psi\rangle\ at\ t_1 \xrightarrow{\text{unitary transformation}} |\psi'\rangle\ at\ t_2$
- Postulates of quantum mechanics does not tell us how the open system will evolve ➔ We need to guess from the given postulates. ➔ Density Matrix

Density matrix

- Section 2.4 The density operator
- When two particles are entangled, if we measure one of the particles but don't know the measurement result, how can we represent the quantum state of the other particle?
  - $|\psi^-\rangle = [|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B]/\sqrt{2}$
  - If A measures $|0\rangle_A$, B remains in $|1\rangle_B$ state.
  - If A measures $|1\rangle_A$, B remains in $|0\rangle_B$ state.
  - From the above state $|\psi^-\rangle$, we know that $|0\rangle_B$ or $|1\rangle_B$ will remain with 50% of probability.
  - $\rho_B = \frac{1}{2}|0\rangle_{BB}\langle 0| + \frac{1}{2}|1\rangle_{BB}\langle 1| = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$
  - What happens if A was measured in $|D\rangle_A$ & $|A\rangle_A$ basis?
  - $\rho_B = \frac{1}{2}|A\rangle_{BB}\langle A| + \frac{1}{2}|D\rangle_{BB}\langle D| = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$
  - The same result will be obtained with measurements in other basis or even without any measurements.