M1522.002500 - 양자 컴퓨팅 및 정보의 기초

Homework #1

Due date: Apr. 21, 2020 12:00 pm

If you submit after the due date, your score will be deducted by 20%.

No more submission will be accepted after Apr. 23, 2020. 12:00pm

Student ID: ______________________________

Name: ______________________________

1	2	3	4	5	6	7	8	9	10

To reduce the grading burden of TA, the homework will be graded all-or-nothing style. We will grade just a few problems randomly sampled from the homework and renormalize the total grading according to the relative weight of each problem. Also, within each problem, there may be several sub-problems, but we will grade only a few sub-problems within each problem, and the score of each problem will be determined by the graded sub-problems proportionally.

For example, if problem 1 is composed of 5 sub-problems, we will decide which sub-problems will be graded later, and if you solved that sub-problems correctly, you will get the full credit of problem 1. In the worst case, you might have solved all other sub-problems correctly, but got the wrong answers in all the graded sub-problems. That is an unfortunate situation, but the score for that entire problem will become 0. Without this policy, we cannot grade so much homework efficiently.

The homework should be *hand-written*, converted into a pdf file, and uploaded to ETL. The pdf file may either be a scanned-copy or camera-taken picture of your homework. If you solve the homework problems using digital pen on a tablet, it will be considered as your hand-

writing, but make sure that your hand-writing is legible. Please make sure you denote the number of the problems correctly. We will post solutions for every homework and announce the problems and sub-problems to be graded after the hard deadline.

The homework should be written with your hand-writing either on a paper or a tablet. Computer-typed homework won't be accepted!

Properties of the linear vector space

1. (10 points) Prove the following properties only using definition 1 provided in lecture note 1, slide 4. (Hint: Exercise 1.1.1 in the reference)

a) (2.5 points) $|0\rangle$ is unique

b) (2.5 points) $0|V\rangle = |0\rangle$

c) (2.5 points) $|-V\rangle = -|V\rangle$

d) (2.5 points) $|-V\rangle$ is the unique additive inverse of $|V\rangle$

2. (10 points) [O, X problem] (a) – (c) Determine whether the following sets constitute subspaces of the given linear vector spaces respectively. (d) – (e) Determine whether the defined sets satisfy the definition of the linear vector space. We assume that vector additions are defined as component-wise addition.

a) (2 points) (   )There is a vector space $M_{2\times2}$ composed of 2x2 matrices defined over the real field. Does the set of 2x2 matrices with only positive real numbers as their components form a subspace of $M_{2\times2}$?

b) (2 points) (   ) There is a vector space $C_{2\times1}$ composed of 2x1 column vectors defined over the complex field. Is the set composed of only one vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ qualified as a subspace of $C_{2\times1}$?

c) (2 points) (   ) There is a vector space $M_{2\times2}$ composed of 2x2 matrices defined over the real field. Does the set of 2x2 matrices $\begin{bmatrix} a & b \\ c & 3 \end{bmatrix}$ form a subspace of $M_{2\times2}$ when a, b, c are real numbers?

d) (2 points) (   ) Does the set of 1x2 row vectors composed of $[r \quad 0]$ with r being real number satisfy the conditions of the linear vector space? Assume that that field is real.

e) (2 points) (   ) There is a set composed of 1x2 row vectors $[a \quad b]$ where a, b can be either 0 or 1, addition is defined as component-wise XOR, the multiplication is defined as AND, and the field is 0 or 1. Is this set a linear vector space?

3. (5 points) Given a basis set $\{|1\rangle, |2\rangle, \dots, |n\rangle\}$ that spans a vector space V defined over a field F, prove that any vector $v \in V$ has a unique representation with respect to the coefficients $v_i \in F$. (Hint: Theorem 2 in the reference)

$$v = \sum_{i=1}^{n} v_i |i\rangle$$

Properties of the inner product space

4. (10 points) $A, B \in M_{3\times3}$ where $M_{3\times3}$ is the vector space of 3×3 matrices defined over the real field R. Prove that the following map satisfies the definition of the inner product introduced in lecture note 1, slide 9.

$$A \cdot B = \mathrm{Tr}(A^T B)$$

5. (10 points) Prove the Gram-Schmidt procedure in the vector space R^n using the bra-ket notation (explicitly write down the resultant orthonormal vectors $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, \dots, |u_n\rangle\}$ constructed from a set of linearly independent vectors $\{|v_1\rangle, |v_2\rangle, |v_3\rangle, \dots, |v_n\rangle\}$). Then, apply it to find $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ given $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\} = \{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\}$.

(Hint: Page 14 in the reference, use induction for the proof)

6. (10 points) Prove the triangular inequality $|V + W| \leq |V| + |W|$ (Hint: Exercise 1.3.4 in the reference.)

Operators

7. (10 points) Answer the following problems for the following operators $\Omega, \Lambda, \Theta : H \to H$ where H is a Hilbert space. You may assume an N-dimensional Hilbert space, but please note that the following identities hold even when the Hilbert space is infinite-dimensional.

a) (2.5 points) Prove that $(\Omega^\dagger)^\dagger = \Omega$

b) (2 points) Prove that $(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$

c) (2.5 points) Show that $(\langle V|\Omega|W\rangle)^* = \langle W|\Omega^\dagger|V\rangle$

d) (2.5 points) Show that $\mathrm{Tr}(\Omega\Lambda) = \mathrm{Tr}(\Lambda\Omega)$

e) (2.5 points) Use D to prove $\mathrm{Tr}(\Omega\Lambda\Theta) = \mathrm{Tr}(\Lambda\Theta\Omega) = \mathrm{Tr}(\Theta\Omega\Lambda)$

Eigenvalue problem

8. (10 points) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$. Construct the matrix P such that $A_D = P^{-1}AP$ where A_D is the diagonalized matrix.

9. (10 points) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Is the matrix diagonalizable or not?

10. (15 points) Consider a three-dimensional complex ket space. Let the operators A and B be represented in some orthonormal basis as shown below. It is known that if two operators commute, there exists a simultaneously diagonalizing basis even when there is a degeneracy in the spectrum in the original basis. Find the basis that simultaneously diagonalizes A and B and represent A and B in that basis. Note that both operators A and B have degenerate eigenvalues respectively, but there exists only one set of eigenbasis which diagonalize them simultaneously.

$$A = \begin{bmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{bmatrix}, B = \begin{bmatrix} 0 & -ib & 0 \\ ib & 0 & 0 \\ 0 & 0 & b \end{bmatrix}$$