M1522.002500 - 양자 컴퓨팅 및 정보의 기초

(Prof. Taehyun Kim)

Homework #1

Due date: 6:00 P.M April. 12, 2019. offline at room 301-416.

Please scan or take a picture of your homework before submission.

Problem 1-1. Prove Gram-Schmidt theorem (Theorem 3). That is, given a linearly independent basis $|\mathrm{I}\rangle, |\mathrm{II}\rangle, |\mathrm{III}\rangle, |\mathrm{IV}\rangle, \ldots$, we can form linear combinations of these basis vectors to obtain an orthonormal basis $|1\rangle, |2\rangle, |3\rangle, |4\rangle, \ldots$.

Problem 1-2. Prove triangle inequality $|V+W| \leq |V| + |W|$. Show that the final inequality becomes equality only if $|V\rangle = a|W\rangle$ where a is a real positive scalar.

(Hint) Start with $|V+W|^2$ and use $Re(\langle V|W\rangle) \leq |\langle V|W\rangle|$ and Schwarz inequality.

Problem 1-3. Prove the identities of commutators:

(A) $[\Omega, \Omega] = 0$ ➔ The same operator commutes with itself.

(B) $[\Omega,\ \alpha] = 0$ where α is a scalar

(C) $[\Omega,\ \Lambda\Theta] = \Lambda[\Omega,\ \Theta] + [\Omega, \Lambda]\Theta$

(D) $[\Lambda\Omega,\ \Theta] = \Lambda[\Omega,\ \Theta] + [\Lambda, \Theta]\Omega$

(E) $[\Omega,\ \Lambda + \Theta] = [\Omega,\ \Lambda] + [\Omega,\ \Theta]$

Problem 1-4. Assume that $[a,\ a^\dagger]=1$. By using this relation and the identities in Problem 1-3, prove the following identities.

(A) $\left[a,\ \left(a^\dagger\right)^2\right]=2a^\dagger$

(B) $\left[a,\ \left(a^\dagger\right)^n\right]=n\left(a^\dagger\right)^{n-1}$

(C) If function $f(a^\dagger)$ can be expanded in a power series of $a^\dagger$, $[a, f(a^\dagger)]=\frac{\partial}{\partial a^\dagger}f(a^\dagger)$

(D) If function $f(a)$ can be expanded in a power series of a, $[a, f(a)]=0$

(E) $\left[a, a^\dagger a+\frac{1}{2}\right]=a$

(F) $\left[a^\dagger, a^\dagger a+\frac{1}{2}\right]=-a^\dagger$

Problem 1-5. Find eigenvectors of and diagonalize given matrix.

$$\begin{pmatrix} 0 & -1 & -2 \\ -2 & 1 & -2 \\ 3 & 1 & 5 \end{pmatrix}$$

Problem 1-6. Evaluate following functions in matrix form:

(A) $f(x)=e^{iAx}$, where $A=\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$

(B) $f(x)=\tan^{-1}(Ax)$, where $A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (Hint) $\tan^{-1}(x)=x-\frac{x^3}{3}+\frac{x^5}{5}\dots=\sum_i\frac{(-1)^i x^{2i+1}}{(2i+1)!}$

(C) $f(x)=e^{-iLx}$, where $L=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ (Hint) $L^3=\ ?$

Problem 1-7. Construct a corresponding operator or Hamiltonian for given device or physical system in each problem. If states are not specified, give an arbitrary labeling for each eigenstate. (Caution! The information is fully given)

(A) An atom with 3 energy levels: $\mathrm{E_0, E_1, E_2}$

(B) A device has its measurement output of -2 when we get “D” state and +1 when we get “A” state. It reads, “D & A are orthonormal”.

(C) An operator X has properties of $X|0\rangle = i|1\rangle, X|1\rangle = -i|2\rangle, X|2\rangle = i|0\rangle$.

Problem 1-8. Prove the general uncertainty principle between Hermitian operator Ω & Λ, such that $\sigma(\Omega) * \sigma(\Lambda) \geq \left|\frac{\langle[\Omega,\Lambda]\rangle}{2i}\right|$, where $\sigma(A) = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$, a standard deviation.

(Hint) Cauchy-Schwarz inequality.