M1522.002500 - 양자 컴퓨팅 및 정보의 기초

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Homework #2

Due date: April. 22, 2019 (Mon)

Please hand in the homework on 4/22 regular class.

Problem 2-1. We will now see how to interpret a unitary gate in Bloch sphere representation. Follow the given procedure.

(A) Find the eigenstates and eigenvalues of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gate. Map the eigenstates on Bloch sphere and see that they form x-axis.

(B) Again, find the eigenstates and eigenvalues of $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ gate. Map the eigenstates on Bloch sphere and find out the axis of rotation.

(C) X gate and H gate are π-rotations along special axes on Bloch sphere. For unitary gates corresponding to rotation around an axis with angle π, we can write them in $U_{\vec{n}}(\pi) = \vec{n} \cdot \vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, where $\vec{n}$ is an arbitrary normal vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ *where* $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the eigenstates of $U_{\vec{n}}(\pi)$ and map them on Bloch sphere to find the axis of rotation.

(D) In the previous problems, we assumed that the eigenstates form the axis of rotation. Now, prove that if we arrange a unitary gate $e^{-\frac{i\pi}{2}\vec{n}*\vec{\sigma}}$, it is a rotation around $\vec{n}$ on Bloch sphere with amount π.

(Hint 1) $(n_x, n_y, n_z) = (cos(\varphi_n)\, sin(\theta_n), sin(\varphi_n)\, sin(\theta_n), cos(\theta_n))$, as we always see on unit sphere.

(Hint 2) We can rotate the Bloch sphere so that $\vec{n}$ becomes z-axis and in this regard, we can re-write arbitrary point on Bloch sphere in $cos(\frac{\theta'}{2})|0\rangle_n + e^{i\varphi'}sin(\frac{\theta'}{2})|1\rangle_n$, where $|0\rangle_n$ and $|1\rangle_n$ are pointing $\vec{n}$ and $-\vec{n}$, respectively. Apply the unitary gate in (D) to this state.

Problem 2-2. Construct a quantum circuit that takes input $|0\rangle^{\otimes 3}$ and produces output $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$.

Problem 2-3. See an example of quantum entanglement shown in p. 16 of Lecture 9 & 10 note. Prove that for any measurement basis, we can see correlation (For example, if the first qubit is A, then the second qubit must be D) between the measurement result of two qubits.

Problem 2-4. (Imperfect quantum teleportation) In the lecture 9 & 10, we talked about the "perfect" quantum teleportation, where the Bell state $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$ was "fully entangled". However, there must be some error ε due to some imperfect operation of devices or unknown stochastic procedure, i.e., $|\psi^-_{error}\rangle_{AB} = \frac{1}{\sqrt{2}}((1+\varepsilon)|01\rangle_{AB} - \sqrt{2-(1+\varepsilon)^2}|10\rangle_{AB})$.

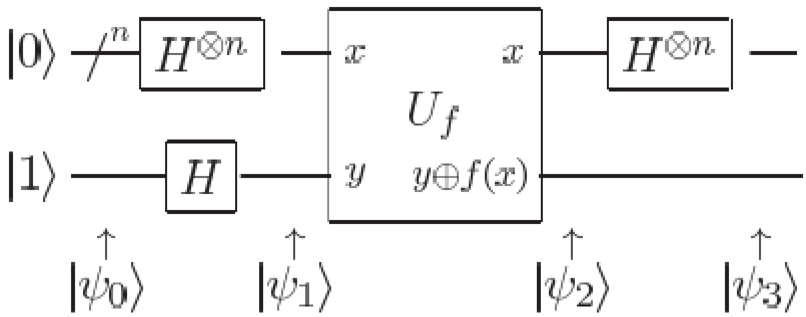
(A) Follow the procedure of quantum teleportation again, for this time, with $|\psi^-_{error}\rangle$. In this case, assume that the measurement output of "CA" system is always one of the Bell states. The unitary gates applied to "B" system after getting the measurement result also remain what we saw in the lecture.

(B) A simple metric to measure how similar two states are is "Fidelity", which is calculating $|\langle\beta|\alpha\rangle_B|$ if we have to compare the two qubits $|\alpha\rangle_B, |\beta\rangle_B$. Find the fidelity between the output state of "B" of "perfect" case and "imperfect" case given that we have measured $|\psi^-\rangle_{CA}$ in "CA" system. (Hint) You have to normalize the output state.

(C) Now, define the error by $1 - \text{fidelity}$. Expand the error in Taylor series of ε to first order to show that the error is at most $O(\varepsilon^2)$. This means that quantum teleportation is not easily affected by this type of error.

If you expand the error further, you will find that for the second term to disappear, $(|\beta|^2 - |\alpha|^2)^2 - (|\beta|^2 - |\alpha|^2) - 2|\alpha|^2 = 0$. This is equivalent to $|\alpha|^2 = 0 \; or \; 1$, which is the same condition to get the exact fidelity of 1. Therefore, the minimum error occurs only when we send only digital signal $|0\rangle, |1\rangle$ with no superposition.

Problem 2-5. Deutsch-Jozsa problem is n-qubit version of Deutsch's problem. Let $f(x): \{0,1\}^n \rightarrow \{0,1\}$ be a function which is either "balanced" or "constant." "Balanced" means that the output of f is "0" for exactly half of the possible input string, i.e., for 2^{n-1} inputs, and "1" for the other. "Constant" means that f is always a constant value for all the inputs. The problem is to judge if f is balanced or constant. Follow the following procedure based on the figure given below. You can refer to the p. 34 of the textbook.



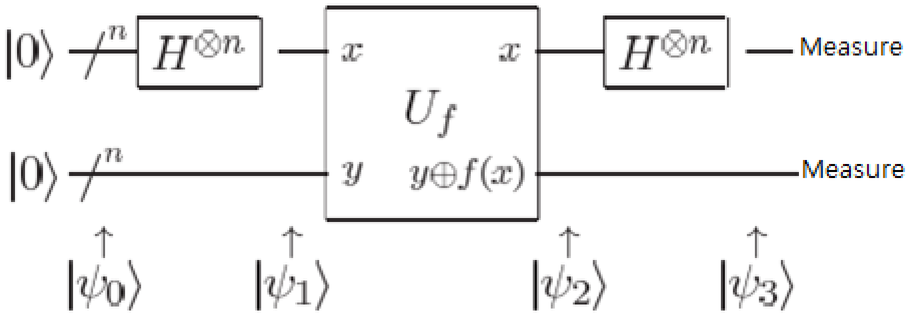
(A) Using the n-qubit Hadamard transform $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2}^n}\sum_{y\in\{0,1\}^n}(-1)^{x*y}|y\rangle$ where $x * y$ in this case is bitwise inner product (Boolean), $x * y = \sum_i x_i y_i$, write the (n+1)-qubit state $|\psi_1\rangle$.

(B) Based on (A), write the state $|\psi_2\rangle$ and $|\psi_3\rangle$.

(C) Prove that bitwise inner product $f_y(x) = x * y = \sum_i x_i y_i$ is a balanced function when y is not zero.

(D) Calculate the probability of measuring $|0\rangle^{\otimes n}$ when we measure $|\psi_3\rangle$. Prove that we can measure $|0\rangle^{\otimes n}$ if and only if the function is constant.

Problem 2-6. Simon's algorithm might be the most easy-to-understand case that shows "quantum advantage", even compared to classical randomized algorithm. The problem is that: Let $f: \{0,1\}^n \to \{0,1\}^n$ be a two-to-one function (Given an output, only two input produces the same output) with a condition that $\exists s \in \{0,1\}^n \ s.t. for\ \forall x \in \{0,1\}^n, f(x \oplus s) = f(x)$. $\oplus$ denotes bitwise XOR operation. (This kind of problem is called "promised") Finding the non-zero period s is the Simon's problem. Follow the following procedure based on the figure given below.



(A) Again, using the n-qubit Hadamard transform $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2}^n}\sum_{y\in\{0,1\}^n}(-1)^{x*y}|y\rangle$ where $x * y$ in this case is bitwise inner product (Boolean), $x * y = \sum_i x_i y_i$, write the (2n)-qubit state $|\psi_1\rangle$.

(B) Based on (A), write the state $|\psi_2\rangle$ and $|\psi_3\rangle$

(C) Prove that we can divide $\{0,1\}^n$ exactly in half and half with a given bit string $s \neq 0$, i.e., $\{0,1\}^n = S_1 \cup S_2, S_1 \cap S_2 = \emptyset, if\ \ x \in S_1\ \ then\ \ x \oplus s \in S_2\ \ \&\ \ if\ \ x \in S_2\ \ then\ \ x \oplus s \in S_1$. Show that under this condition, we can pair the states with the same output.

(D) Calculate the probability of measuring arbitrary $|y\rangle$ and show that the output is only one of those satisfying $s * y = 0$. As the period s is a n-bit string, it is enough to get n different y(s) to obtain s. Probabilistically, the query complexity to get n different y(s) is O(n). This speed cannot be obtained classically even with randomized algorithms, showing the "quantum advantage".

Problem 2-7. Prove that bell states cannot be written in product state, in other words, we cannot write bells states in $|\psi_1\rangle \otimes |\psi_2\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are 1-qubit state.

Problem 2-8. Calculate the output of the 3-qubit Quantum Fourier Transform circuit "Step by Step" on the input state $|+0+\rangle_{123}$. The circuit diagram is for general n-qubit QFT, however, you need to modify this to 3-qubit version.