Prob. 4.2.

Proof

Suppose $|\tilde{\psi}_i\rangle = \sum_j u_{ij}|\tilde{\varphi}_j\rangle$ for some unitary u_{ij}. Then

$$\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_{ijk} u_{ij}u_{ik}^*|\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_k| \tag{2.168}$$

$$= \sum_{jk}\left(\sum_i u_{ki}^\dagger u_{ij}\right)|\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_k| \tag{2.169}$$

$$= \sum_{jk} \delta_{kj}|\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_k| \tag{2.170}$$

$$= \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j|, \tag{2.171}$$

which shows that $|\tilde{\psi}_i\rangle$ and $|\tilde{\varphi}_j\rangle$ generate the same operator.

Conversely, suppose

$$A = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j|. \tag{2.172}$$

Let $A = \sum_k \lambda_k|k\rangle\langle k|$ be a decomposition for A such that the states $|k\rangle$ are orthonormal, and the λ_k are strictly positive. Our strategy is to relate the states $|\tilde{\psi}_i\rangle$ to the states $|\tilde{k}\rangle \equiv \sqrt{\lambda_k}|k\rangle$, and similarly relate the states $|\tilde{\varphi}_j\rangle$ to the states $|\tilde{k}\rangle$. Combining the two relations will give the result. Let $|\psi\rangle$ be any vector orthonormal to the space spanned by the $|\tilde{k}\rangle$, so $\langle\psi|\tilde{k}\rangle\langle\tilde{k}|\psi\rangle = 0$ for all k, and thus we see that

$$0 = \langle\psi|A|\psi\rangle = \sum_i \langle\psi|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\psi\rangle = \sum_i |\langle\psi|\tilde{\psi}_i\rangle|^2. \tag{2.173}$$

Thus $\langle\psi|\tilde{\psi}_i\rangle = 0$ for all i and all $|\psi\rangle$ orthonormal to the space spanned by the $|\tilde{k}\rangle$. It follows that each $|\tilde{\psi}_i\rangle$ can be expressed as a linear combination of the $|\tilde{k}\rangle$, $|\tilde{\psi}_i\rangle = \sum_k c_{ik}|\tilde{k}\rangle$. Since $A = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$ we see that

$$\sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_{kl}\left(\sum_i c_{ik}c_{il}^*\right)|\tilde{k}\rangle\langle\tilde{l}|. \tag{2.174}$$

The operators $|\tilde{k}\rangle\langle\tilde{l}|$ are easily seen to be linearly independent, and thus it must be that

$\sum_i c_{ik}c_{il}^* = \delta_{kl}$. This ensures that we may append extra columns to c to obtain a unitary matrix v such that $|\tilde{\psi}_i\rangle = \sum_k v_{ik}|\tilde{k}\rangle$, where we have appended zero vectors to the list of $|\tilde{k}\rangle$. Similarly, we can find a unitary matrix w such that $|\tilde{\varphi}_j\rangle = \sum_k w_{jk}|\tilde{k}\rangle$. Thus $|\tilde{\psi}_i\rangle = \sum_j u_{ij}|\tilde{\varphi}_j\rangle$, where $u = vw^\dagger$ is unitary. □

Prob. 4-3.

(A) $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$

(B) $\rho = \frac{1}{3}\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00|+\langle 11|}{\sqrt{2}}\right) + \frac{1}{6}\left(\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00|-\langle 11|}{\sqrt{2}}\right) + \frac{1}{2}|00\rangle\langle 00|$

$= \frac{3}{4}|00\rangle\langle 00| + \frac{1}{12}|00\rangle\langle 11| + \frac{1}{12}|11\rangle\langle 00| + \frac{1}{4}|11\rangle\langle 11|$

Prob. 4-4.

($\Rightarrow$) if ρ is pure, $\exists\, |\psi\rangle$ s.t. $\rho = |\psi\rangle\langle\psi| \Rightarrow tr(\rho^2) = tr(|\psi\rangle\langle\psi|\psi\rangle\langle\psi|) = tr(\rho) = 1$.

($\Leftarrow$) if $tr(\rho^2) = 1$, $\exists\, \{(|\psi_i\rangle, p_i)\}$ s.t. $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $\{|\psi_i\rangle\}$ are orthonormal ($\because$ Hermitian)

$$\Rightarrow \rho^2 = \sum_{ij} p_{ij} |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| = \sum_i p_i^2 |\psi_i\rangle\langle\psi_i|$$

$$\Rightarrow tr(\rho^2) = \sum_i p_i^2 \leq \sum_i p_i = 1$$

The equality condition is when $p_i^2 = p_i$ for all i, which means

$p_i = 0$ or 1 $\Rightarrow$ only allows pure states.

Prob. 4-5.

$|\phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle_{AB}}{\sqrt{2}} \Rightarrow \rho_{\phi^\pm} = \frac{1}{2}|00\rangle\langle 00| \pm \frac{1}{2}|00\rangle\langle 11| \pm \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$

$\Rightarrow tr_B(\rho_{\phi^\pm}) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

$|\psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle_{AB}}{\sqrt{2}} \Rightarrow \rho_{\psi^\pm} = \frac{1}{2}|01\rangle\langle 01| \pm \frac{1}{2}|01\rangle\langle 10| \pm \frac{1}{2}|10\rangle\langle 01| + \frac{1}{2}|10\rangle\langle 10|$

when $\langle 1|\rho|1\rangle$ when $\langle 0|\rho|0\rangle$

$\Rightarrow tr_B(\rho_{\psi^\pm}) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

Prob. 4-6.

(A) $\rho = \frac{1}{2}I + (a-\frac{1}{2})\sigma_z + \mathrm{Re}(b)\sigma_x + \mathrm{Im}(b)\sigma_y$

(B) ($\Rightarrow$) Suppose $|\vec{n}|^2 = 1 \Leftrightarrow (a-\frac{1}{2})^2 + (\mathrm{Re}(b))^2 + (\mathrm{Im}(b))^2 = 1/4$

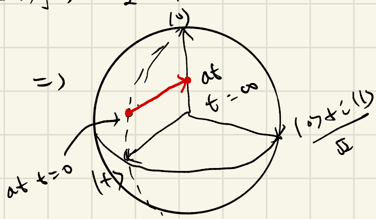
$\Rightarrow \mathrm{tr}(\rho^2) = \mathrm{tr}\left(\frac{1}{4}I + \frac{1}{2}(a-\frac{1}{2})\sigma_z + \frac{1}{2}\mathrm{Re}(b)\sigma_x + \frac{1}{2}\mathrm{Im}(b)\sigma_y + \left[(a-\frac{1}{2})^2 + (\mathrm{Re}(b))^2 + (\mathrm{Im}(b))^2\right]I\right)$

$= \mathrm{tr}\left(\frac{2}{4}I\right) = 1 \quad (\because \mathrm{tr}(\sigma_i) = 0)$

($\Leftarrow$) Suppose $\mathrm{tr}(\rho^2) = 1$

$\Rightarrow \mathrm{tr}\left(\left[\frac{1}{4} + (a-\frac{1}{2})^2 + (\mathrm{Re}(b))^2 + (\mathrm{Im}(b))^2\right]I\right) = \frac{1}{2} + 2\left[(a-\frac{1}{2})^2 + (\mathrm{Re}(b))^2 + (\mathrm{Im}(b))^2\right] = 1 \Leftrightarrow |\vec{n}|^2 = 1$ ▨

(C) $n_z = \frac{1}{3}$, $n_x = \frac{2\sqrt{2}}{3}e^{-\alpha t}$, $n_y = 0$ $\Rightarrow$



Prob. 4-7.

by spectral decomposition, $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ where $\{|\psi_j\rangle\}$ is orthonormal.

$\Rightarrow \rho \log_2 \rho = \left(\sum_j p_j |\psi_j\rangle\langle\psi_j|\right) \sum_i \log_2(p_i)|\psi_i\rangle\langle\psi_i| = \sum_i p_i \log_2(p_i)|\psi_i\rangle\langle\psi_i|$

$\Rightarrow \mathrm{tr}(\rho \log_2 \rho) = \mathrm{tr}\left(\sum_i p_i \log_2 p_i |\psi_i\rangle\langle\psi_i|\right).$

$\Rightarrow$ under $\forall$unitary U, $\tilde{\rho} = \sum_j p_j U|\psi_j\rangle\langle\psi_j|U^\dagger = \sum_j p_j |\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|$

$\Rightarrow \mathrm{tr}(\tilde{\rho}\log_2\tilde{\rho}) = \mathrm{tr}\left(\sum_i p_i \log_2 p_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\right)$

$= \mathrm{tr}\left(\sum_i p_i \log_2 p_i U|\psi_i\rangle\langle\psi_i|U^\dagger\right) = \mathrm{tr}(U^\dagger U \rho\log_2\rho) = \underline{\mathrm{tr}(\rho\log_2\rho)}$

check) at $t=0$, 0

at $t=\infty$, 1.

$\rho(t) = \frac{1+e^{-\alpha t}}{2}|+\rangle\langle+| + \frac{1-e^{-\alpha t}}{2}|-\rangle\langle-|$

$\Rightarrow$ entropy $= -\mathrm{tr}(\rho\log_2\rho) = -\left[\frac{1+e^{-\alpha t}}{2}\log_2\frac{1+e^{-\alpha t}}{2} + \frac{1-e^{-\alpha t}}{2}\log_2\frac{1-e^{-\alpha t}}{2}\right]$

Prob. 4-8.

define $|\tilde{\psi}\rangle = U|\psi\rangle \Rightarrow |\psi\rangle = U^\dagger|\tilde{\psi}\rangle \Rightarrow i\hbar\dfrac{\partial(U^\dagger|\tilde{\psi}\rangle)}{\partial t} = H(U^\dagger|\tilde{\psi}\rangle)$

$\Rightarrow i\hbar\dfrac{\partial U^\dagger}{\partial t}|\tilde{\psi}\rangle + i\hbar U^\dagger\dfrac{\partial|\tilde{\psi}\rangle}{\partial t} = HU^\dagger|\tilde{\psi}\rangle$

$\Rightarrow i\hbar UU^\dagger\dfrac{\partial|\tilde{\psi}\rangle}{\partial t} = i\hbar\dfrac{\partial}{\partial t}|\tilde{\psi}\rangle = UHU^\dagger|\tilde{\psi}\rangle - i\hbar U\dfrac{\partial U^\dagger}{\partial t}|\tilde{\psi}\rangle = \underbrace{\left(UHU^\dagger - i\hbar U\dfrac{\partial U^\dagger}{\partial t}\right)}_{=\tilde{H}}|\tilde{\psi}\rangle$

$\Rightarrow i\hbar\dfrac{\partial|\tilde{\psi}\rangle}{\partial t} = \tilde{H}|\tilde{\psi}\rangle$, $\tilde{H} = UHU^\dagger - i\hbar U\dfrac{\partial U^\dagger}{\partial t}$

$\& (\tilde{H})^\dagger = \tilde{H}$; Hermitian.

$\left(\dfrac{\partial}{\partial t}(UU^\dagger) = 0 = \dfrac{\partial U}{\partial t}U^\dagger + U\dfrac{U^\dagger}{\partial t}\right)$

$\Rightarrow \left(i\hbar U\dfrac{\partial U^\dagger}{\partial t}\right)^\dagger = \left[i\hbar\left(-\dfrac{\partial U}{\partial t}U^\dagger\right)\right]^\dagger$

$= i\hbar U\dfrac{\partial U^\dagger}{\partial t}$

Prob. 4-9.

(A) $\sigma_i\cdot\sigma_j = i\epsilon_{ijk}\sigma_k \in P_1$, where $\epsilon_{ijk} = \begin{cases} 1 & \to \text{cyclic} \\ -1 & \to \text{acyclic}\end{cases}$

$\& \; I\cdot\sigma_i = \sigma_i \in P_1$, $\sigma_i^2 = I \in P_1$

$\& \pm\sigma_i \in P_1$, $\pm i\sigma_i \in P_1$ $(i = 0, x, y, z)$ $\Rightarrow$ closed under multiplication.

also, $\sigma_x\cdot\sigma_y = i\sigma_z$ $\sigma_y\sigma_x = -i\sigma_z$ $\sigma_x^2 = I$

$\sigma_y\cdot\sigma_z = i\sigma_x$ $\sigma_z\cdot\sigma_y = -i\sigma_x$ $(i\sigma_z)^2 = -I$

$\sigma_z\cdot\sigma_x = i\sigma_y$ $\sigma_x\cdot\sigma_z = -i\sigma_y$ $(\sigma_x i\sigma_x) = iI$

$-I\cdot iI = -iI$

$\Rightarrow$ All elements in P_1 can be generated by $\langle\sigma_x, \sigma_y, \sigma_z\rangle$

(B) $H\sigma_x H^\dagger = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \dfrac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \sigma_z \in P_1$

$H\sigma_y H^\dagger = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \dfrac{1}{2}\begin{pmatrix}i & -i\\ -i & -i\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} = -\sigma_y \in P_1$

$H\sigma_z H^\dagger = H\cdot H\cdot\sigma_x\cdot H\cdot H = \sigma_x \in P_1$

$S\sigma_x S^\dagger = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix} = \begin{pmatrix}0 & 1\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \sigma_y \in P_1$

$S\sigma_y S^\dagger = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix} = \begin{pmatrix}0 & -i\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} = -\sigma_x \in P_1$

$$S\sigma_z S^\dagger = \sigma_z \quad (\because [\sigma_z, S] = 0) \in P_1$$

Remaining cases are just phase multiplication

(C) $\mathrm{CNOT} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix}$

$\Rightarrow$ $\mathrm{CNOT}\, \sigma_x \otimes \sigma_i \,\mathrm{CNOT} = \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix}$

$$= \begin{pmatrix} 0 & \sigma_i\sigma_x \\ \sigma_x\sigma_i & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & -\sigma_x\sigma_i \\ \sigma_x\sigma_i & 0 \end{pmatrix} = -i\,\sigma_y \otimes \sigma_x\sigma_i \in P_2 \\ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (i = x) = \sigma_x \otimes I \in P_2 \end{cases}$$

$$\mathrm{CNOT}\, \sigma_y \otimes \sigma_i \,\mathrm{CNOT} = \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma_i\sigma_x \\ i\sigma_x\sigma_i & 0 \end{pmatrix}$$

$$= \begin{cases} i\sigma_x \otimes \sigma_x\sigma_i & (i \neq x) \\ \sigma_y \otimes I & (i = x) \end{cases} \in P_2$$

$$\mathrm{CNOT}\, \sigma_z \otimes \sigma_i \,\mathrm{CNOT} = \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} \sigma_i & \\ & -\sigma_i \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_x\sigma_i\sigma_x \end{pmatrix}$$

$$= \begin{cases} \begin{pmatrix} \sigma_x & \\ & -\sigma_x \end{pmatrix} = \sigma_z \otimes \sigma_x & (i = x) \\ \begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix} = I \otimes \sigma_i & (i \neq x) \end{cases} \in P_2$$

$\blacksquare$