공지사항

- 강의 scripting 작업은 별도 공지할 예정임

- PC가 아닌 스마트폰이나 태블릿을 이용하여 ETL 동영상을 시청하는 경우 진도율이 업데이트가 안된다고 함.

➔ coursemos app 사용할 것. 자세한 내용은 ETL 공지사항 확인바람

➔ 출석은 진도율로 결정되므로 각자 반드시 주어진 기간내에 진도율이 90%이상인지 반드시 확인바람

➔ 만약 첫번째 강의를 시청했으나 위의 이슈로 진도율이 제대로 반영이 안된 경우에는 정다운 조교에게 메일로 알려줄 것

Summary of the Previous Lecture

- Definitions
  - (abstract) Linear vector space
  - Field
  - Linear (in)dependence
  - Dimension of vector space
  - Basis and components of vector for a given basis → uniqueness of expansion for the given basis

- Examples of (unusual) vector space
  - 2x2 matrices, functions with restrictions

Summary of the Previous Lecture

- Inner product space

- Generalized requirement for inner product
  - The result is a number (generally a complex)
  - $\langle V|W\rangle = \langle W|V\rangle^*$ (skew-symmetry)
  - $\langle V|V\rangle \geq 0$, 0 iff $|V\rangle = |0\rangle$ (positive semidefinite)
  - $\langle V|(a|W\rangle + b|Z\rangle) = a\langle V|W\rangle + b\langle V|Z\rangle$ (linearity in ket)

From Chap. 1 of "Principles of Quantum Mechanics", Ramamurti Shankar, 2nd ed. Plenum Press (1994)

Properties of Inner Product

- Notation: $a|W\rangle + b|Z\rangle = |aW + bZ\rangle$
- From the definition of the generalized inner product,
  $\langle V|(a|W\rangle + b|Z\rangle) = \langle V|aW + bZ\rangle = a\langle V|W\rangle + b\langle V|Z\rangle$
- $$\begin{aligned}\langle aW + bZ|V\rangle &= \langle V|aW + bZ\rangle^* \\ &= (a\langle V|W\rangle + b\langle V|Z\rangle)^* \\ &= a^*\langle V|W\rangle^* + b^*\langle V|Z\rangle^* \\ &= a^*\langle W|V\rangle + b^*\langle Z|V\rangle\end{aligned}$$

➔ Anti-linearity of the first factor (bra) in the inner product

From Chap. 1 of “Principles of Quantum Mechanics”, Ramamurti Shankar, 2nd ed. Plenum Press (1994)

Inner Product Spaces

- **Definition 8**: Two vectors are ***orthogonal*** or ***perpendicular*** if their inner product vanishes.

- **Definition 9**: $\sqrt{\langle V|V\rangle} = |V|$ will be referred as the ***norm*** or length of the vector

- **Definition 10**: A set of basis vectors, all of which are pairwise orthogonal and have unit norms, will be called an ***orthonormal basis***.

Inner Product Spaces

- **Theorem 3** *(Gram-Schmidt)*: For any linearly independent basis, we can always find an orthonormal basis by combining these basis vectors.
- If $|1\rangle, |2\rangle, \dots, |n\rangle$ are orthonormal basis:
  $\langle i|j\rangle = \begin{cases} 1 & for\ i = j \\ 0 & for\ i \neq j \end{cases} \equiv \delta_{ij}$ (Kronecker delta)
- We know that the components of a vector are uniquely determined for a given basis, but then how to find components v_i of a given vector $|V\rangle$ for orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$?
  - $v_i = \langle i|V\rangle$
- If two vectors $|V\rangle, |W\rangle$ are expanded in terms of orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$,

$$|V\rangle = \sum_i v_i |i\rangle$$
$$|W\rangle = \sum_j w_j |j\rangle$$

- $\langle V|W\rangle = \sum_i \sum_j v_i^* w_j \langle i|j\rangle = \sum_i v_i^* w_i$
  - Skew-symmetry guarantees that the norm is real and positive semidefinite.

Inner Product Spaces

- Vectors $|V\rangle, |W\rangle$ are **uniquely specified** by their components in **a given basis** ➔ Can be written as column vectors:
- $|V\rangle \quad \rightarrow \quad \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in this basis
- $|W\rangle \quad \rightarrow \quad \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ in this basis
- $\langle V|W\rangle = \sum_i v_i^* w_i = [v_1^* \quad \dots \quad v_n^*] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$
- Bra vector can be represented as a row vector with complex conjugation (transpose conjugation or ***adjoint operation***).
  - $\langle V| = [v_1^* \quad \dots \quad v_n^*]$

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Inner Product Spaces

- To take the **adjoint** of a linear equation relating kets (bras), replace every ket (bra) by its bra (ket) and complex conjugate all coefficients.
- $a|V\rangle \rightarrow \begin{bmatrix} av_1 \\ \vdots \\ av_n \end{bmatrix} \xrightarrow{\text{adjoint}} [a^* v_1^* \quad \ldots \quad a^* v_n^*] \rightarrow \langle V| a^*$
- $|V\rangle = \sum_i v_i |i\rangle \xrightarrow{\text{adjoint}} \langle V| = \sum_i \langle i| v_i^*$

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The Schwarz inequality

- **Theorem 5**: The Schwarz inequality
$$|\langle V|W\rangle| \leq |V||W|$$
  - Proof of the Schwarz inequality
  - We will use axiom $\langle Z|Z\rangle \geq 0$
  - $|Z\rangle = |V\rangle - \frac{|W\rangle\langle W|}{|W|^2}|V\rangle = |V\rangle - \frac{\langle W|V\rangle}{|W|^2}|W\rangle$
  - $\langle Z|Z\rangle = \left\langle V - \frac{\langle W|V\rangle}{|W|^2}W \middle| V - \frac{\langle W|V\rangle}{|W|^2}W \right\rangle$
$$= \left(\langle V| - \frac{\langle W|V\rangle^*}{|W|^2}\langle W|\right)\left(|V\rangle - \frac{\langle W|V\rangle}{|W|^2}|W\rangle\right)$$
$$= \langle V|V\rangle - \frac{\langle W|V\rangle}{|W|^2}\langle V|W\rangle - \frac{\langle W|V\rangle^*}{|W|^2}\langle W|V\rangle + \frac{\langle W|V\rangle^*}{|W|^2}\frac{\langle W|V\rangle}{|W|^2}\langle W|W\rangle \geq 0$$
  - $\langle V|V\rangle \geq \frac{\langle W|V\rangle}{|W|^2}\langle V|W\rangle \Rightarrow |V||W| \geq |\langle V|W\rangle|$
- **Theorem 6**: The triangular inequality
$$|V + W| \leq |V| + |W|$$

Subspace

- **Definition 11**: Given a vector space $\mathcal{V}$, a subset of its elements that form a vector space among themselves is called a **subspace**. A particular subspace i of dimensionality n_i will be denoted by $\mathcal{V}_i^{n_i}$.
- Example: orthogonal subspace with respective some vector $|W\rangle$: $\mathcal{V}_{\perp W}^{n-1}$

Linear Operators

- An **operator** Ω is an instruction for transforming any given vector $|V\rangle$ into another vector $|V'\rangle$ and this relation is written as $|V'\rangle = \Omega|V\rangle$.
- In this class, we will consider only the **linear operators** Ω that do not take us out of the vector space. ⇔ If $|V\rangle \in \mathcal{V}$, $\Omega|V\rangle \in \mathcal{V}$.
- Linear operator acting on bra is written as $\langle V|\Omega = \langle V'|$
- **Linear operator** should obey the following rules:
  - $\Omega\alpha|V_i\rangle = \alpha\Omega|V_i\rangle$
  - $\Omega\big(\alpha|V_i\rangle + \beta|V_j\rangle\big) = \alpha\Omega|V_i\rangle + \beta\Omega|V_j\rangle$
  - Same for the bra vectors

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Linear Operators

- Once the action of the **linear** operator Ω for all the basis vectors $|1\rangle, |2\rangle, \dots, |n\rangle$ is known, its action on any arbitrary vector is determined.
  - When $\Omega|i\rangle = |i'\rangle$ is known for all $i = 1 \dots n$, and an arbitrary vector $|V\rangle = \sum_i v_i|i\rangle$ is given,
  - $\Omega|V\rangle = \sum_i \Omega v_i|i\rangle = \sum_i v_i\Omega|i\rangle = \sum_i v_i|i'\rangle$

- Product of two operators
  - $\Lambda\Omega|V\rangle \equiv \Lambda(\Omega|V\rangle) = \Lambda|\Omega V\rangle$ ➔ we will use $|\Omega V\rangle$ notation to represent $\Omega|V\rangle$

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Linear Operators

- **Commutator**
  - Definition: $[\Omega, \Lambda] \equiv \Omega\Lambda - \Lambda\Omega$
  - **The order of the operators in a product is very important**, and generally $[\Omega, \Lambda] \neq 0$.
  - Useful identities of commutators
    - $[\Omega, \Lambda\Theta] = \Lambda[\Omega, \Theta] + [\Omega, \Lambda]\Theta$
    - $[\Lambda\Omega, \Theta] = \Lambda[\Omega, \Theta] + [\Lambda, \Theta]\Omega$
    - Looks similar to chain rule of derivative ➔ easy to memorize!
- Inverse of operator Ω
  - $\Omega\Omega^{-1} = \Omega^{-1}\Omega = \mathrm{I}$
  - Not every operator has an inverse.
  - Inverse of product of operators: $(\Omega\Lambda)^{-1} = \Lambda^{-1}\Omega^{-1}$ (Prove?)

Matrix Representation of Linear Operators

- Up to now, abstract vector can be represented by an n-tuple of numbers (called its components) for a given basis.
- Similarly, operator can be represented by a set of n^2 numbers for a given basis. The most convenient way for the linear operators is to use matrix shape, and these numbers will be called as its matrix elements in that basis.
- Recall the previous observation that the action of a linear operator is fully specified by its action on the basis vectors.
  - If the basis vector is transformed to a some vector by the linear operator by $\Omega|i\rangle = |i'\rangle$, then for any arbitrary vector $|V\rangle$, we can immediately calculate the result of transformation by $\Omega|V\rangle = \sum_i \Omega v_i |i\rangle = \sum_i v_i \Omega |i\rangle = \sum_i v_i |i'\rangle$.
  - To expand $|i'\rangle$ in terms of orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$, the components $c_{i',j}$ of $|i'\rangle$ for the given basis can be obtained by $\langle j|i'\rangle$.
  - $\langle j|i'\rangle = \langle j|\Omega|i\rangle \equiv \Omega_{ji}$ ➔ $\Omega|i\rangle = |i'\rangle = \sum_j \Omega_{ji}|j\rangle$
  - The n^2 numbers, Ω_{ji}, are called the matrix elements of Ω for the given orthonormal basis.

Matrix Representation of Linear Operators

- If the transformed vector $|V'\rangle = \Omega|V\rangle$ is expanded as $|V'\rangle = \sum_j v'_j|j\rangle$, the components v'_j of the $|V'\rangle$ can be obtained by
  - $v'_j = \langle j|V'\rangle = \langle j|\Omega|V\rangle = \langle j|\sum_i v_i\Omega|i\rangle = \sum_i v_i\langle j|\Omega|i\rangle = \sum_i \Omega_{ji}v_i$
  - Or $\begin{bmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_n \end{bmatrix} = \begin{bmatrix} \langle 1|\Omega|1\rangle & \langle 1|\Omega|2\rangle & \cdots & \langle 1|\Omega|n\rangle \\ \langle 2|\Omega|1\rangle & & & \\ \vdots & & & \vdots \\ \langle n|\Omega|1\rangle & & \cdots & \langle n|\Omega|n\rangle \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$