1. Very straightforward. Just expand LHS and RHS using $[\mathrm{A},\mathrm{B}] = AB - BA$ and compare them. If you have any questions, please contact the TA.

2.

(A) Use 1 (C)

(B) Use induction and 2 (A).

$$[a,\ (a^\dagger)^{n+1}] = (a^\dagger)^n[a,\ a^\dagger] + [a, (a^\dagger)^n]a^\dagger = (a^\dagger)^n + n(a^\dagger)^{n-1}a^\dagger = (n+1)(a^\dagger)^n$$

(C) $f(a^\dagger) = \sum_{n=0}^{\infty} c_n (a^\dagger)^n \Rightarrow RHS\text{:}\ \ [a, f(a^\dagger)] = \sum_{n=0}^{\infty} c_n [a(a^\dagger)^n - (a^\dagger)^n a]$

$= \sum_{n=1}^{\infty} c_n [a,\ (a^\dagger)^n]$ since $a - a = 0$ when $n = 0$

$= \sum_{n=1}^{\infty} c_n n (a^\dagger)^{n-1}$ from 2 (B)

$= \frac{\partial}{\partial a^\dagger} f(a^\dagger)$

(D) This is trivial since $[a,\ a] = 0$

(E) $a\left(a^\dagger a + \frac{1}{2}\right) - \left(a^\dagger a + \frac{1}{2}\right)a = aa^\dagger a - a^\dagger a a = (1 + a^\dagger a)a - a^\dagger a a = a$

(F) Similar to 2 (E)

3. Expand

$$\sigma^2 = \langle(\Omega - \langle\Omega\rangle)^2\rangle$$

Since $\langle\Omega\rangle$ is just a scalar, you can derive the variance just as you would for conventional statistics.

4. $\langle\psi|AB|\psi\rangle = x + iy \Rightarrow \langle\psi|BA|\psi\rangle = x - iy \Rightarrow \langle\psi|[A,B]|\psi\rangle = 2iy,\ \langle\psi|\{A,B\}|\psi\rangle = 2x$

By the Cauchy inequality for expectation values, we have

$$|\langle\psi|[A,B]|\psi\rangle|^2 \le 4|\langle\psi|AB|\psi\rangle|^2 \le 4|\langle\psi|A|\psi\rangle|^2|\langle\psi|B|\psi\rangle|^2$$

Substitute $\mathrm{A} = \mathrm{C} - \langle C\rangle, \mathrm{B} = \mathrm{D} - \langle D\rangle$. It can be easily shown that $[\mathrm{C} - \langle C\rangle, \mathrm{D} - \langle D\rangle] = [C, D]$. Using the relation proven in problem 3, we have

$$\sigma_C{}^2 = \langle\psi|(\mathrm{C}^2 - \langle C\rangle^2)|\psi\rangle, \qquad \sigma_D{}^2 = \langle\psi|(\mathrm{D}^2 - \langle D\rangle^2)|\psi\rangle$$

Plugging these values into the above inequality will yield the Heisenberg uncertainty principle.

5. $\mathbb{X}\otimes\mathbb{Z}=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix}, \mathbb{I}\otimes\mathbb{X}=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}, \mathbb{X}\otimes\mathbb{I}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$

Note that the Pauli matrices satisfy the commutation relations.

$$[\mathbb{X},\mathbb{Y}]=2i\mathbb{Z},\qquad [\mathbb{Y},\mathbb{Z}]=2i\mathbb{X},\qquad [\mathbb{Z},\mathbb{X}]=2i\mathbb{Y}$$

So, using the tensor product algebra, we obtain

$$[\mathbb{X}\otimes\mathbb{Z},\mathbb{I}\otimes\mathbb{X}]=(\mathbb{X}\otimes\mathbb{Z})(\mathbb{I}\otimes\mathbb{X})-(\mathbb{I}\otimes\mathbb{X})(\mathbb{X}\otimes\mathbb{Z})=(\mathbb{X}\mathbb{I})\otimes(\mathbb{Z}\mathbb{X})-(\mathbb{I}\mathbb{X})\otimes(\mathbb{X}\mathbb{Z})$$
$$=(\mathbb{X})\otimes[(\mathbb{Z}\mathbb{X})-(\mathbb{X}\mathbb{Z})]=(\mathbb{X})\otimes(2i\mathbb{Y})=2i\,\mathbb{X}\otimes\mathbb{Y}$$

$$[\mathbb{I}\otimes\mathbb{X},\ \mathbb{X}\otimes\mathbb{I}]=(\mathbb{X}\otimes\mathbb{I})(\mathbb{I}\otimes\mathbb{X})-(\mathbb{I}\otimes\mathbb{X})(\mathbb{X}\otimes\mathbb{I})=(\mathbb{X})\otimes(\mathbb{X})-(\mathbb{X})\otimes(\mathbb{X})=\mathbb{O}_{4x4}$$

$$[\mathbb{X}\otimes\mathbb{I},\mathbb{X}\otimes\mathbb{Z}]=(\mathbb{X}\otimes\mathbb{I})(\mathbb{X}\otimes\mathbb{Z})-(\mathbb{X}\otimes\mathbb{Z})(\mathbb{X}\otimes\mathbb{I})=(\mathbb{X}\mathbb{X})\otimes(\mathbb{Z})-(\mathbb{X}\mathbb{X})\otimes(\mathbb{Z})=\mathbb{O}_{4x4}$$

The operators must be commutative in their respective vector spaces in order to be commutative as operators in a composite vector space.

6. $\hat{n}\cdot\vec{\sigma}=n_x\mathbb{X}+n_y\mathbb{Y}+n_z\mathbb{Z}=\begin{bmatrix}cos\theta & sin\theta e^{-i\varphi}\\ sin\theta e^{i\varphi} & -cos\theta\end{bmatrix}$

This matrix has the eigenvalues $\lambda=\pm1$ and eigenvectors $\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix},\begin{bmatrix}-\sin\left(\frac{\theta}{2}\right)\\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\end{bmatrix}$ that are parameterized by θ and φ which are the basis for the unit sphere. Therefore, they represent two orthogonal eigenstates on the Bloch sphere. Try $\theta=\varphi=0$. The two eigenstates just become $\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$.

7. Following the right definitions will lead you to the correct answers :D

8. From

$$|\theta_+\rangle=cos\theta\,|H\rangle+\sin\theta\,|V\rangle$$

$$|\theta_-\rangle=-\sin\theta\,|H\rangle+\cos\theta\,|V\rangle$$

we have,

$$|V\rangle=sin\theta\,|\theta_+\rangle+\cos\theta\,|\theta_-\rangle$$

$$|H\rangle=cos\theta\,|\theta_+\rangle-\sin\theta\,|\theta_-\rangle$$

Then the state entangled state $|\psi^-\rangle=\left(|H\rangle_A|V\rangle_B-|V\rangle_A|H\rangle_B\right)/\sqrt{2}$ can be expressed in terms of the basis $|\theta_+\rangle,|\theta_-\rangle$ as

$$|\psi^-\rangle=\left(|H\rangle_A|V\rangle_B-|V\rangle_A|H\rangle_B\right)/\sqrt{2}$$

$$= \frac{[(cos\theta\ |\theta_+\rangle_A - \sin\theta\ |\theta_-\rangle_B)(sin\theta\ |\theta_+\rangle_B + \cos\theta\ |\theta_-\rangle_B) - (sin\theta\ |\theta_+\rangle_A + \cos\theta\ |\theta_-\rangle_B)(cos\theta\ |\theta_+\rangle_B - \sin\theta\ |\theta_-\rangle_B)]}{\sqrt{2}}$$

$$= \frac{(|\theta_+\rangle_A\ |\theta_-\rangle_B - \ |\theta_-\rangle_A\ |+\rangle_B)}{\sqrt{2}}$$

Therefore, this relation is independent of the rotation in basis vectors.

9. From

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle\ + e^{-i\varphi}|11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - e^{-i\varphi}|11\rangle)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle +\ e^{-i\varphi}|10\rangle)$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - e^{-i}\ \ |10\rangle)$$

We have

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle\ + |\Phi^-\rangle)$$

$$|11\rangle = \frac{1}{\sqrt{2}}e^{i\varphi}(|\Phi^+\rangle - |\Phi^-\rangle)$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle +\ |\psi^-\rangle)$$

$$|10\rangle = \frac{1}{\sqrt{2}}e^{i\varphi}(|\psi^+\rangle - |\psi^-\rangle)$$

Following the same process in Lecture 12 page 6-7 will result in

$|\psi\rangle_C|\Phi^+\rangle_{AB} = (\alpha|0\rangle_C\ + \beta|1\rangle_C)\frac{1}{\sqrt{2}}(|00\rangle_{AB}\ + e^{-i\varphi}|00\rangle_{AB})$

$= \frac{1}{2}[|\Phi^+\rangle_{CA}(\alpha|0\rangle_B\ + \beta|1\rangle_B)\ +\ |\Phi^-\rangle_{CA}(\alpha|0\rangle_B - \beta|1\rangle_B) +\ e^{i\varphi}|\psi^+\rangle_{CA}(\beta|0\rangle_B + e^{-i2\varphi}\alpha|1\rangle_B) +$
$e^{i\varphi}|\psi^-\rangle_{CA}(-\beta|0\rangle_B + e^{-i2\varphi}\alpha|1\rangle_B)]$

Then for each of the Bell states in the CA system, the operators that we must act on the state in B in order to reconstruct the state $|\psi\rangle_B = \alpha|0\rangle_B\ + \beta|1\rangle_B$ are,

$|\Phi^+\rangle_{CA}$: $\mathbb{I}$

$|\Phi^-\rangle_{CA}$: $\mathbb{Z}$

$|\psi^+\rangle_{CA}$: $\mathbb{XP}$

$|\psi^-\rangle_{CA}$: $\mathbb{ZXP}$

Where $\mathbb{P} = \begin{bmatrix}1 & 0\\0 & e^{i2\varphi}\end{bmatrix}$

10.

(a) $|\varphi^+\rangle_{AB}|\varphi^+\rangle_{CD} = \frac{1}{\sqrt{2}}(|00\rangle_{\mathrm{AB}} + |11\rangle_{\mathrm{AB}})\ \frac{1}{\sqrt{2}}(|00\rangle_{CD} + |11\rangle_{CD})$

$$= \frac{1}{2}(|\varphi^+\rangle_{BC}|\varphi^+\rangle_{AD} + |\varphi^-\rangle_{BC}|\varphi^-\rangle_{AD} + |\psi^+\rangle_{BC}|\psi^+\rangle_{AD} + |\psi^-\rangle_{BC}|\psi^-\rangle_{AD})$$

We have used

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle\ + |\Phi^-\rangle)$$

$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle +\ |\psi^-\rangle)$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle)$$

Measurement in the BC system causes the composite qubits in the AD system to collapse to one of the Bell states.

(b) It is evident that the differently prepared entangled states in AB and CD lead to different entangled states in AD and BC. But the relation between B and D in this particular state $|\varphi^+\rangle_{AB}|\varphi^+\rangle_{CD}$ is peculiar in the sense that they are initially completely irrelevant states but somehow there is a mapping between the systems AD and BC such that they are in the same Bell basis. This can be understood in terms of quantum teleportation where the measured state in BC is exactly reproduced in the state in AD.

Let us see how this would work sequentially. Suppose qubit D on Mars and C on the moon are in the entangled state $|\varphi^+\rangle_{CD}$. On Earth we entangle qubits A and B into the state $|\varphi^+\rangle_{AB}$ and transport qubit B to the moon. Then we somehow make a measurement on the composite qubit system BC in the Bell basis. This state will be reproduced in the composite qubit system AD. Loosely speaking, qubits A and D act like a sender and receiver, while qubits B and C behave like a message and mediator in this process. (This description is not trivial to picture because of the entanglement between qubits. Rather than to focus on the individual qubits, try to understand the mechanism in terms of the relations between composite systems.)