# Summary of the Previous Lecture

- Degeneracy
  - Having the same eigenvalues
- Hermitian
  - Real eigenvalue
  - Orthogonal eigenvector
  - Diagonalizable
- Unitary
  - Eigenvalues are unit modulus $u = e^{i\theta}$
  - Preserve norm & orthogonality of vectors
- Basis transformation
  - $\mathbb{O}$ : matrix representation of an operator $\Omega$ in orthonormal basis $|1\rangle, |2\rangle, \ldots, |n\rangle$
  - $\mathbb{O}'$ : new matrix representation of an operator $\Omega$ in the new orthonormal basis $|I\rangle, |II\rangle, \ldots, |N\rangle$
  - $\mathbb{U}$ : matrix representation of $U = \sum_{m=1}^{n} |M\rangle\langle m|$ in $|1\rangle, |2\rangle, \ldots, |n\rangle$
  - $\mathbb{O}' = \mathbb{U}^{\dagger}\mathbb{O}\mathbb{U}$

From Chap. 1 of "Principles of Quantum Mechanics", Ramamurti Shankar, 2nd ed. Plenum Press (1994)

# Diagonalization of Hermitian Matrices

- Assume that a Hermitian operator $\Omega$ is represented as a matrix $\mathbb{H}$ in some orthonormal basis $|1\rangle, |2\rangle, \ldots, |n\rangle$. If we trade this basis for the eigenbasis $|\omega_1\rangle, |\omega_2\rangle, \ldots, |\omega_n\rangle$, the new matrix $\mathbb{H}'$ representing $\Omega$ will become diagonal. ➔ $\mathbb{H}' = \mathbb{U}^\dagger \mathbb{H} \mathbb{U} = \mathbb{D}$
- Simultaneous diagonalization of two Hermitian operators
- **Theorem 13**: If $\Omega$ and $\Lambda$ are two **commuting Hermitian operators**, there exists (at least) a basis of **common eigenvectors** that diagonalizes them both.
  - When at least one of the operator is non-degenerate:
    - Assume $\Omega$ is non-degenerate and one of its eigenvector is $|\omega_i\rangle$ satisfying $\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle$, then $\Lambda|\omega_i\rangle$ is also an eigenvector with eigenvalue$\omega_i$. Proof) $\Omega(\Lambda|\omega_i\rangle) = \Lambda\Omega|\omega_i\rangle = \omega_i(\Lambda|\omega_i\rangle)$
    - Therefore $\Lambda|\omega_i\rangle = \lambda_i|\omega_i\rangle$ should be satisfied. ➔ $|\omega_i\rangle$ is also an eigenvector of $\Lambda$.
  - Full proof is in the reference from page 43 to 46.

# Functions of Operators

- Types of objects that can act on vectors
  - Scalar: commutes with both scalar and operators ➔ called c-numbers
  - Operator: generally do not commute with other operator ➔ called q-numbers
- Function of q-numbers
  - Analogy to function of c-numbers such as $\sin x$, $\log x$
  - Consider c-number function that can be written as a power series: $f(x) = \sum_{n=0}^{\infty} a_n x^n$
  - Define $f(\Omega) \equiv \sum_{n=0}^{\infty} a_n \Omega^n$
  - For example, most of the c-number functions can be expanded in power series via Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$
- Example of function of operator
  - Taylor series of $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ ➔ $e^\Omega = \sum_{n=0}^{\infty} \frac{1}{n!} \Omega^n$
  - All the above discussion will be valid only when the sum converges to a definite limit.

# Functions of **Hermitian** Operators

- Limit our discussion to the functions of **Hermitian** operator $\Omega$.
- By using eigenbasis of $\Omega$, $\Omega$ can be represented as diagonal matrix
  - $\mathbb{D} = \begin{bmatrix} \omega_1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_n \end{bmatrix}$
  - $\Omega^m = \mathbb{D}^m = \begin{bmatrix} \omega_1^m & & & \\ & \omega_2^m & & \\ & & \ddots & \\ & & & \omega_n^m \end{bmatrix}$
  - $\mathrm{e}^{\Omega} = \sum_{m=0}^{\infty} \frac{1}{m!} \Omega^m = \sum_{m=0}^{\infty} \frac{1}{m!} \mathbb{D}^m = \begin{bmatrix} \sum_{m=0}^{\infty} \frac{\omega_1^m}{m!} & & & \\ & \sum_{m=0}^{\infty} \frac{\omega_2^m}{m!} & & \\ & & \ddots & \\ & & & \sum_{m=0}^{\infty} \frac{\omega_n^m}{m!} \end{bmatrix} = \begin{bmatrix} e^{\omega_1} & & & \\ & e^{\omega_2} & & \\ & & \ddots & \\ & & & e^{\omega_n} \end{bmatrix}$
  - Generally, $f(\Omega) = \begin{bmatrix} f(\omega_1) & & & \\ & f(\omega_2) & & \\ & & \ddots & \\ & & & f(\omega_n) \end{bmatrix}$

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# Functions of **Hermitian** Operators

- What if the **Hermitian** operator $\Omega$ is represented as **non-diagonal matrix** $\mathbb{H}$ in different basis $|1\rangle, |2\rangle, \cdots |n\rangle$?
  - Use unitary transformation $U = \sum_{m=1}^{n} |\omega_m\rangle\langle m|$ whose matrix representation in $|1\rangle, |2\rangle, \cdots |n\rangle$ is $\mathbb{U}$.
  - Then $\mathbb{U}^\dagger \mathbb{H} \mathbb{U} = \mathbb{D}$ will appear as diagonal matrix.
  - By using $\mathbb{H} = \mathbb{U}\mathbb{D}\mathbb{U}^\dagger$,

    $$\mathbb{H}^2 = \mathbb{U}\mathbb{D}\mathbb{U}^\dagger\mathbb{U}\mathbb{D}\mathbb{U}^\dagger = \mathbb{U}\mathbb{D}^2\mathbb{U}^\dagger, \ldots \ \mathbb{H}^m = \mathbb{U}\mathbb{D}^m\mathbb{U}^\dagger$$
  - $f(\mathbb{H}) = \sum_{m=0}^{\infty} a_m \mathbb{H}^m = \mathbb{U} \sum_{m=0}^{\infty} a_m \mathbb{D}^m \, \mathbb{U}^\dagger$

$$= \mathbb{U} \begin{bmatrix} f(\omega_1) & & & \\ & f(\omega_2) & & \\ & & \ddots & \\ & & & f(\omega_n) \end{bmatrix} \mathbb{U}^\dagger$$

# Derivatives of Operators w.r.t. Parameters

- Assume operator $\theta(\lambda)$ depends on a parameter $\lambda$.
- Derivative w.r.t. $\lambda$ is defined to be

$$\frac{d\theta(\lambda)}{d\lambda} \equiv \lim_{\Delta\lambda \to 0}\left[\frac{\theta(\lambda+\Delta\lambda)-\theta(\lambda)}{\Delta\lambda}\right]$$

- If $\theta(\lambda)$ is written as a matrix in some basis, the matrix representing $d\theta(\lambda)/d\lambda$ can be obtained by differentiating each matrix elements of $\theta(\lambda)$.
- Derivative of $\theta(\lambda) = e^{\lambda\Omega}$
  - Even when $\Omega$ is represented as non-diagonal matrix $\mathbb{H}$,

$$\frac{d}{d\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m \mathbb{H}^m}{m!} = \sum_{m=1}^{\infty}\frac{m\lambda^{m-1}\mathbb{H}^m}{m!} = \mathbb{H}\sum_{m=1}^{\infty}\frac{\lambda^{m-1}\mathbb{H}^{m-1}}{(m-1)!} = \mathbb{H}\sum_{n=0}^{\infty}\frac{\lambda^n \mathbb{H}^n}{n!} = \mathbb{H}e^{\lambda\mathbb{H}}$$

  - In other words, $d\theta(\lambda)/d\lambda = \Omega e^{\lambda\Omega} = e^{\lambda\Omega}\Omega = \theta(\lambda)\Omega$

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## Solution of Differential Equation

- How to solve differential equation $\frac{\partial}{\partial t}|\psi\rangle = i\Omega|\psi\rangle$?
  - When initial state $|\psi(0)\rangle$ is given, assume that $|\psi\rangle$ can be obtained by $|\psi(t)\rangle = U(t)|\psi(0)\rangle$.
  - Then we need to find out a condition for $U(t)$.
  - $\frac{\partial}{\partial t}U(t)|\psi(0)\rangle = i\Omega U(t)|\psi(0)\rangle$

  ➔ $\left(\frac{\partial}{\partial t}U(t) - i\Omega U(t)\right)|\psi(0)\rangle = 0$
  - Then $U(t)$ should satisfy the above equation for arbitrary initial state ➔ $\frac{\partial}{\partial t}U(t) - i\Omega U(t) = 0$
  - $U(t) = e^{t(i\Omega)}$
- When $\Omega$ is Hermitian, prove that $U(t) = e^{i\Omega t}$ is unitary.
  - Analogy: If $\omega$ is real, $u = e^{i\omega}$ is a number of unit modulus.