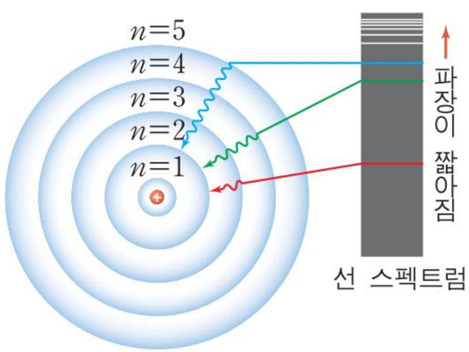
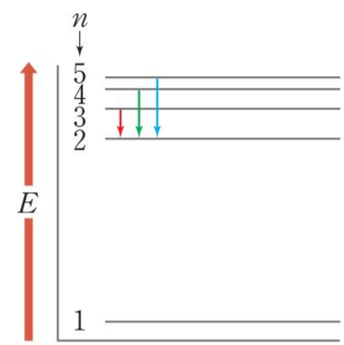
Postulate 1

- Postulate 1: the state of the particle is represented by a vector $|\psi(t)\rangle$ in a Hilbert space
  - However, the law of quantum mechanics doesn't tell us what the state space of Hilbert space should be.
  - Therefore state space should be found by experiment
  - Example space: space composed of $|0\rangle$ & $|1\rangle$
- Definition: a Hilbert space is a complete inner product space
  - Complete space: each Cauchy sequence is a convergent sequence
- Example
  - Two-level atom
  - Polarization of light

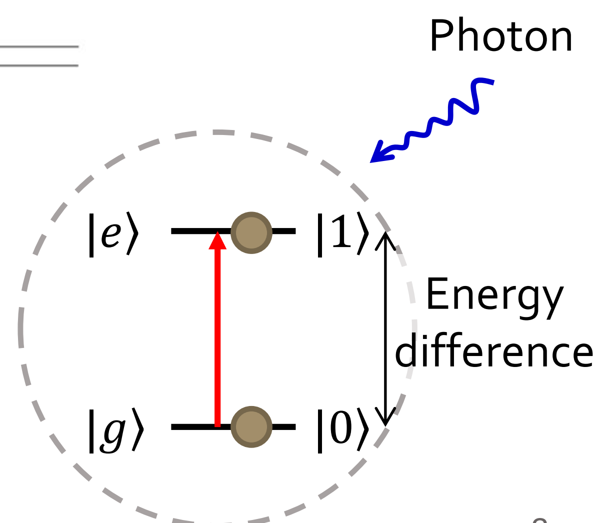
Examples of Quantum States

- Hydrogen atom
  - Electron inside an atom can take different energy levels
  - When the electron changes its energy state, the difference of the energy state will appear as a photon carrying the same amount of energy. ➔ Conservation of energy
  - In reverse, to move the electron from lower energy state to higher energy state, we need to provide energy in the form of electromagnetic wave.



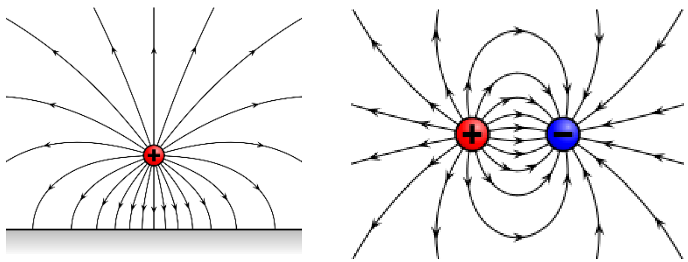


- Two-level atom (TLA)
  - Simplified model of multi-level atom
  - Label each level as $|g\rangle$ and $|e\rangle$ or $|0\rangle$ and $|1\rangle$
  - Arbitrary quantum state of an electron can be written as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

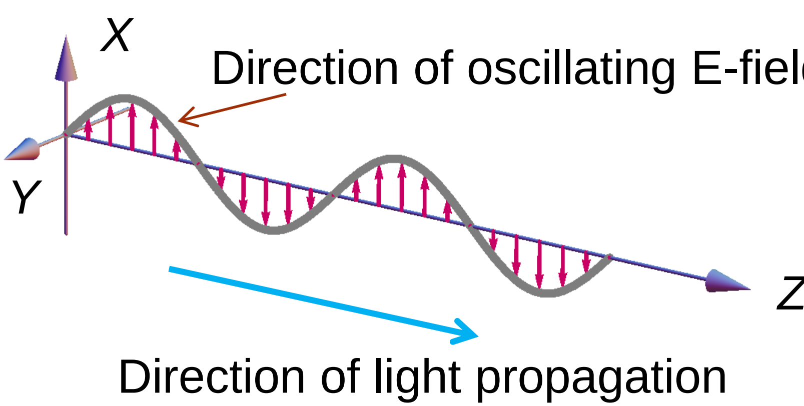


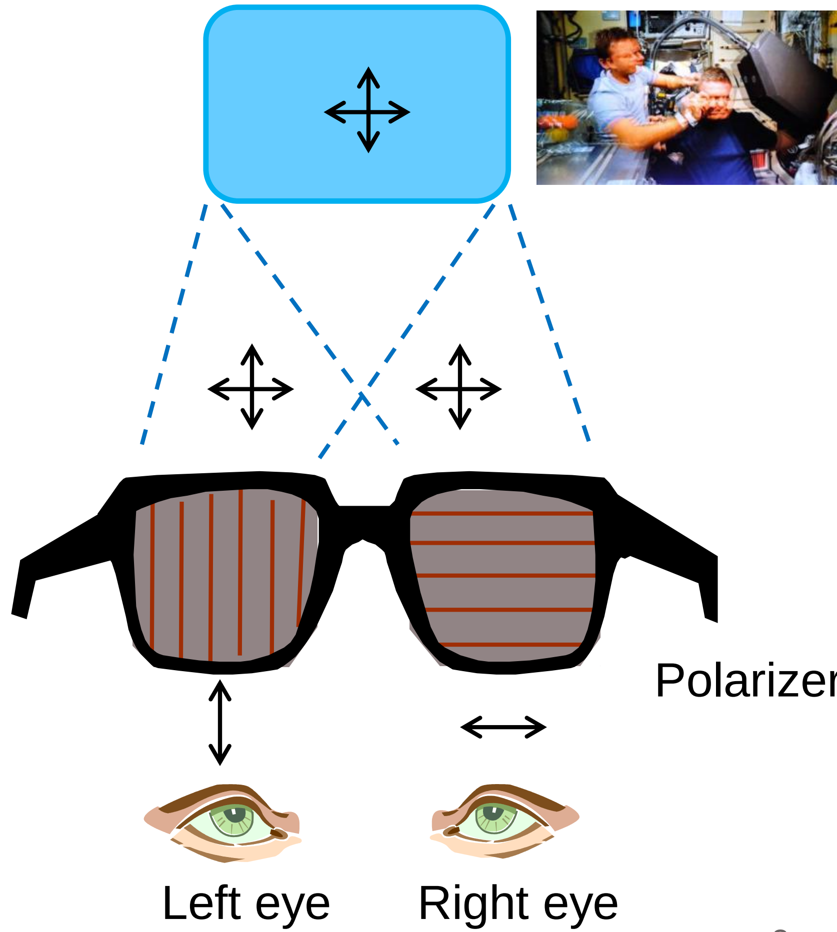
Examples of Quantum States

- Polarization of light
  - When an electromagnetic wave propagates through some medium, the electric field is oscillating along some axis.
- Example of **static** electric field



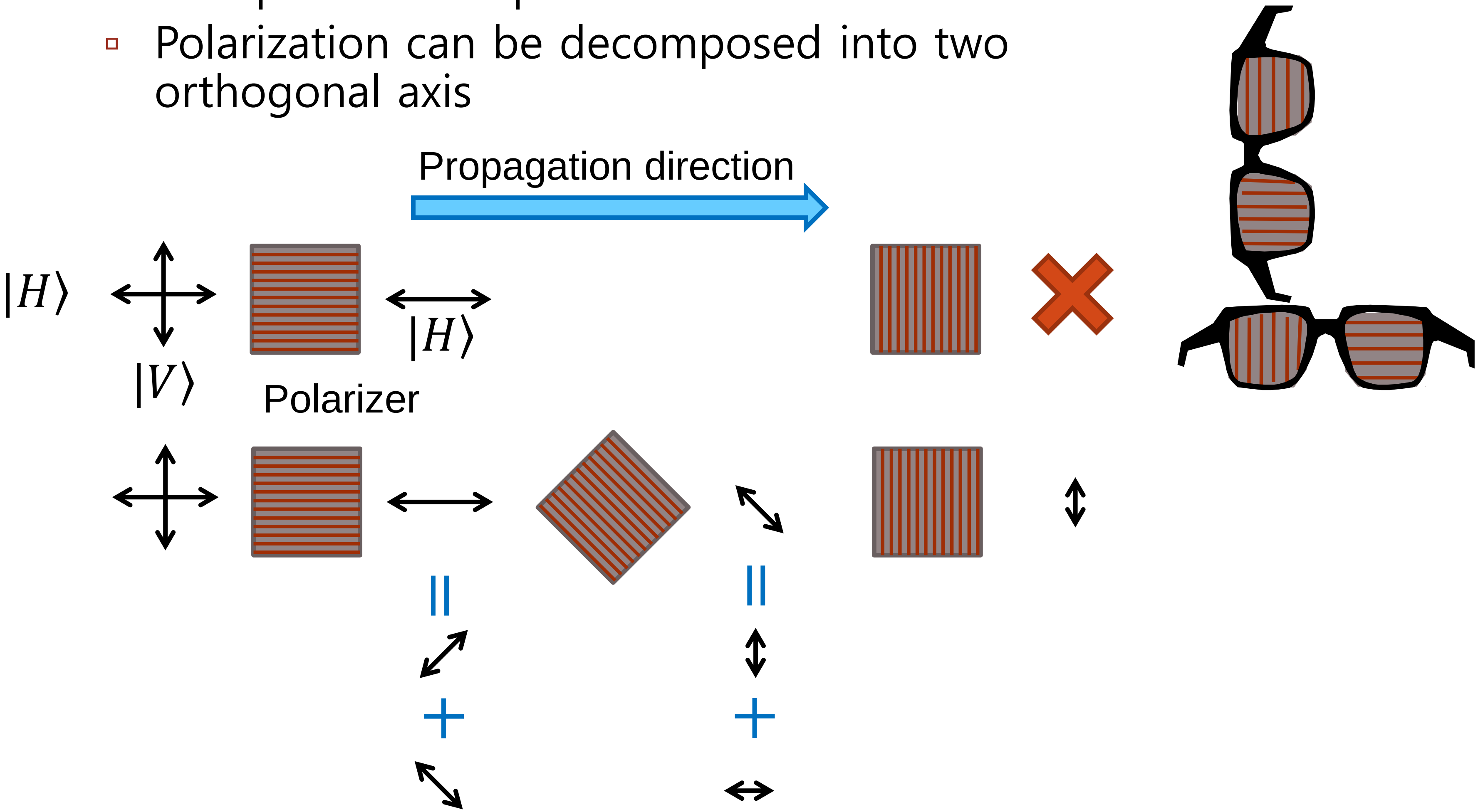
- Example of electromagnetic **wave**





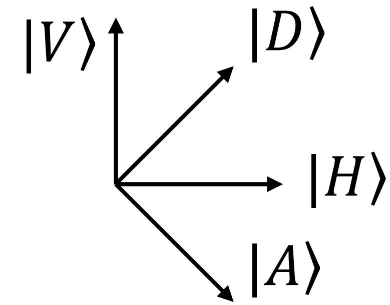
Examples of Quantum States

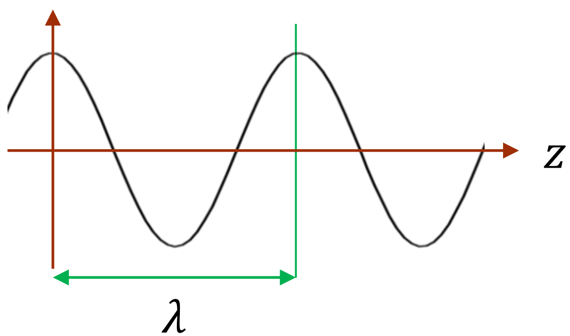
- Decomposition of polarization
  - Polarization can be decomposed into two orthogonal axis



Examples of Quantum States

- Sum of electric field follows the vector addition. Why?
  - Definition of electric field comes from electric force.
- Basis relation
  - $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$
  - $|A\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$
- What does phase of coefficient mean?

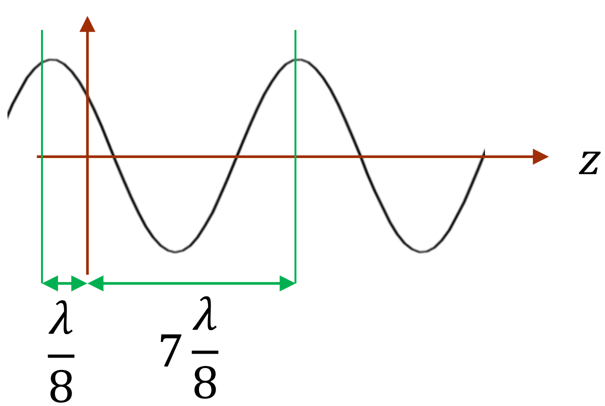




$$\cos\left(\frac{2\pi}{\lambda}z\right) = Re\left[\cos\left(\frac{2\pi}{\lambda}z\right) + i\sin\left(\frac{2\pi}{\lambda}z\right)\right]$$
$$= Re[\cos(kz) + i\sin(kz)] = Re\left[e^{ikz}\right]$$

➔ Euler relation

➔ $k \equiv \frac{2\pi}{\lambda}$ is called wavenumber



$$\cos\left(\frac{2\pi}{\lambda}\left(z + \frac{\lambda}{8}\right)\right) = \cos\left(kz + \frac{2\pi}{\lambda}\cdot\frac{\lambda}{8}\right)$$
$$= Re\left[e^{i\left(kz+\frac{\pi}{4}\right)}\right] = Re\left[e^{ikz}e^{i\frac{\pi}{4}}\right] = Re\left[e^{ikz}e^{i\phi}\right]$$

Examples of Quantum States

- Polarization state of light
  - $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2} = \left(|H\rangle + e^{i0}|V\rangle\right)/\sqrt{2}$ ➔ diagonal
  - $|A\rangle = (|H\rangle - |V\rangle)/\sqrt{2} = \left(|H\rangle + e^{i\pi}|V\rangle\right)/\sqrt{2}$ ➔ anti-diagonal
  - $|R\rangle = (|H\rangle + i|V\rangle)/\sqrt{2} = \left(|H\rangle + e^{i(\pi/2)}|V\rangle\right)/\sqrt{2}$ ➔ Right-circular
  - $|L\rangle = (|H\rangle - i|V\rangle)/\sqrt{2} = \left(|H\rangle + e^{i(3\pi/2)}|V\rangle\right)/\sqrt{2}$ ➔ Left-circular
  - $\left(|H\rangle + e^{i\phi}|V\rangle\right)/\sqrt{2}$?
- Photon
  - A particle of light carrying a discrete bundle of electromagnetic energy
  - Evidence for quantized energy
    - Blackbody radiation: $\frac{\int_0^\infty E e^{-E/kT} dE}{\int_0^\infty e^{-E/kT} dE} \xrightarrow{\text{Planck replaced}} \frac{\sum_{n=0}^{\infty} nhf e^{-nhf/kT}}{\sum_{n=0}^{\infty} e^{-nhf/kT}}$
    - Photoelectric effect
    - Compton effect
    - Energy of a single photon: $E = hf = \left(\frac{h}{2\pi}\right)(2\pi f) \equiv \hbar\omega$ where $\hbar$ is Planck's constant, 1.054 $1.054 \times 10^{-34} (J \cdot s)$

Postulate 2

- Postulate 2: the evolution of a "closed" quantum system is described by a unitary transformation
  - $|\psi\rangle\ at\ t_1 \xrightarrow{\text{unitary transformation}} |\psi'\rangle\ at\ t_2$
  - Example: $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Z|0\rangle = |0\rangle$,$Z|1\rangle = -|1\rangle$

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

- Postulate 2' (continuous time version): the time evolution of the state of a "closed" quantum system is described by Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle$$

  - $\hbar$ is Planck's constant, 1.054 1.054×10^{-34} $(J \cdot s)$
  - $\mathcal{H}$ is called *Hamiltonian*. Hamiltonian describes how the system should evolve.

Postulate 2

- $i\hbar \frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle$
  - Hamiltonian $\mathcal{H}$ is Hermitian and represent the total energy of the system.
  - $\frac{d}{dt}|\psi\rangle = -i\frac{\mathcal{H}}{\hbar}|\psi\rangle$ ➔ $|\psi(t)\rangle = e^{-i\mathcal{H}(t-t_0)/\hbar}|\psi(t_0)\rangle$
  - $U(t-t_0) = e^{-i\mathcal{H}(t-t_0)/\hbar}$ is an unitary operator

Euler Relation

- $e^{ix} = \cos x + i \sin x$
- Proof
  - From Taylor expansion:
    - $f(x) = f(0) + \left.\frac{df}{dx}\right|_{x=0} x + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=0} x^2 + \frac{1}{3!}\left.\frac{d^3f}{dx^3}\right|_{x=0} x^3 + \cdots$
  - $\sin x = 0 + \frac{1}{1!}x + \frac{-0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
  - $\cos x = 1 + \frac{-0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$
  - $e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots = \sum_{n=0}^{\infty} \frac{z}{n!}$
  - If $z = ix$,
    - $$\begin{aligned} e^{ix} &= 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \ldots\right) \\ &= \cos x + i \sin x \end{aligned}$$