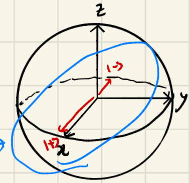
Prob. 2-1.

(A) $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$.

$|+\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \Rightarrow \theta = \frac{\pi}{2}, \varphi = 0.$

$|-\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \Rightarrow \theta = \frac{\pi}{2}, \varphi = \pi.$



(B),(C). $U_{\vec{n}}(\pi) = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$

$\Rightarrow \lambda = \pm 1$ +1 case: $(n_z - 1)a + b(n_x - in_y) = 0 \to |+\rangle_{\vec{n}} = \frac{1}{\sqrt{2-2n_z}}\begin{pmatrix} n_x - in_y \\ 1 - n_z \end{pmatrix}$

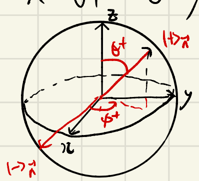
-1 case: $(n_z + 1)a + b(n_x - in_y) = 0 \to |-\rangle_{\vec{n}} = \frac{1}{\sqrt{2+2n_z}}\begin{pmatrix} n_x - in_y \\ -1 - n_z \end{pmatrix}$

if we define $n_x = \cos\phi\sin\theta$, $n_y = \sin\phi\sin\theta$, $n_z = \cos\theta$,

$$|+\rangle = \frac{1}{2\sin\frac{\theta}{2}}\begin{pmatrix} e^{-i\phi}\sin\theta \\ 1 - \cos\theta \end{pmatrix} = e^{-i\phi}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \Rightarrow \underline{\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle}$$

$$|-\rangle = \frac{1}{2\cos\frac{\theta}{2}}\begin{pmatrix} e^{-i\phi}\sin\theta \\ -(1+\cos\theta) \end{pmatrix} = e^{-i\phi}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix} = e^{-i\phi}\begin{pmatrix} \cos(\frac{\pi-\theta}{2}) \\ \sin(\frac{\pi-\theta}{2})e^{i(\phi+\pi)} \end{pmatrix}$$

$\Rightarrow \pi - \theta_+ = \theta_-$, $\pi + \phi_+ = \phi_-$ $\Rightarrow$ Two points intersects the origin



For Hadamard, $\underline{n_x = n_z = \frac{1}{\sqrt{2}}, n_y = 0.}$

$\therefore$ $\vec{n}\cdot\vec{\sigma}$'s eigenstates are really pointing $|\vec{n}\rangle, |-\vec{n}\rangle$ on Bloch sphere.

(D) $\forall\beta$, $\forall\vec{n}$, we can write arbitrary point on Bloch sphere

as $\cos\frac{\theta_{\vec{n}}}{2}|0\rangle_{\vec{n}} + e^{i\phi_{\vec{n}}}\sin\frac{\theta_{\vec{n}}}{2}|1\rangle_{\vec{n}}$, where $|0\rangle_{\vec{n}}$ & $|1\rangle_{\vec{n}}$

denotes +1 & -1 eigenstates of $\vec{n}\cdot\vec{\sigma}$.

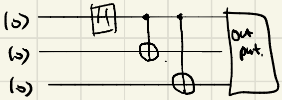
Then, $e^{-i\frac{\beta}{2}\vec{n}\cdot\vec{\sigma}}|\psi\rangle = e^{-i\frac{\beta}{2}\vec{n}\cdot\vec{\sigma}}\left(\cos\frac{\theta_n}{2}|0\rangle_{\vec{n}} + e^{i\phi_n}\sin\frac{\theta_n}{2}|1\rangle_{\vec{n}}\right)$

$= e^{-i\frac{\beta}{2}\cdot 1}\cos\frac{\theta_n}{2}|0\rangle_{\vec{n}} + e^{-i\frac{\beta}{2}(-1)}\sin\frac{\theta_n}{2}e^{i\phi_n}|1\rangle_n$

$= e^{-i\frac{\beta}{2}}\left(\cos\frac{\theta_{\vec{n}}}{2}|0\rangle_{\vec{n}} + \sin\frac{\theta_{\vec{n}}}{2}e^{i(\phi_{\vec{n}}+\beta)}|1\rangle_{\vec{n}}\right)$ ∴ $(\theta_{\vec{n}}, \phi_{\vec{n}}) \Rightarrow \underline{(\theta_{\vec{n}}, \phi_{\vec{n}}+\beta)}$

$\Rightarrow$ This is the rotation around $\vec{n}$ by angle β. Counter-clockwise.

Prob. 2-2



Prob. 2-3

$|\psi\rangle = \frac{1}{\sqrt{2}}(|H\rangle|V\rangle - |V\rangle|H\rangle)$ $|\alpha|^2+|\beta|^2=1$.

define $|0\rangle = \alpha|H\rangle + \beta|V\rangle$, $|1\rangle = \beta^*|H\rangle - \alpha^*|V\rangle$

$\Rightarrow \langle 0|1\rangle = (\langle H|\alpha^* + \langle V|\beta^*)(\beta^*|H\rangle - \alpha^*|V\rangle)$ ($\because$ H, V are orthonormal)

$= \alpha^*\beta^* - \beta^*\alpha^* = 0$

$\Rightarrow$

$|H\rangle = \alpha^*|0\rangle + \beta|1\rangle$ & $|V\rangle = \beta^*|0\rangle - \alpha|1\rangle$

$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left((\alpha^*|0\rangle+\beta|1\rangle)(\beta^*|0\rangle-\alpha|1\rangle) - (\beta^*|0\rangle-\alpha|1\rangle)(\alpha^*|0\rangle+\beta|1\rangle)\right)$

$= \frac{1}{\sqrt{2}}\left(\{|\beta|^2+|\alpha|^2\}|1\rangle\langle 0| - \{|\alpha|^2+|\beta|^2\}|0\rangle\langle 1|\right)$

$= \frac{1}{\sqrt{2}}\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right)$. $\Rightarrow$ as α, β are arbitrary,

Measurement in $\forall$ basis results in anti-correlation.

Prob 2-4.

(A) $|\psi\rangle_C = \alpha|0\rangle_C + \beta|1\rangle_C$

$$\Rightarrow |\psi\rangle_C |\psi^-_{\text{error}}\rangle_{AB} = \frac{1}{\sqrt{2}}\Big[\alpha(1+\epsilon)|001\rangle_{CAB} - \alpha\sqrt{1-2\epsilon-\epsilon^2}|010\rangle_{CAB} + \beta(1+\epsilon)|101\rangle_{CAB} - \beta\sqrt{1-2\epsilon-\epsilon^2}|110\rangle_{CAB}\Big)$$

$$= \frac{1}{2}\Big[\alpha(1+\epsilon)(\phi^+ + \phi^-)_{CA}|1\rangle_B - \alpha\sqrt{1-2\epsilon-\epsilon^2}(\psi^+ + \psi^-)_{CA}|0\rangle_B + \beta(1+\epsilon)(\psi^+ - \psi^-)_{CA}|1\rangle_B - \beta\sqrt{1-2\epsilon-\epsilon^2}(\phi^+ - \phi^-)_{CA}|0\rangle_B\Big)$$

$$= \frac{1}{2}\Big(\phi^+_{CA}\big[\alpha(1+\epsilon)|1\rangle_B - \beta\sqrt{1-2\epsilon-\epsilon^2}|0\rangle_B\big] + \phi^-_{CA}\big[\beta\sqrt{1-2\epsilon-\epsilon^2}|0\rangle_B + \alpha(1+\epsilon)|1\rangle_B\big]$$
$$+ \psi^+_{CA}\big[-\alpha\sqrt{1-2\epsilon-\epsilon^2}|0\rangle_B + \beta(1+\epsilon)|1\rangle_B\big] + \psi^-_{CA}\big[-\alpha\sqrt{1-2\epsilon-\epsilon^2}|0\rangle_B - \beta(1+\epsilon)|1\rangle_B\big]$$

(B) Fidelity $= |\langle\psi_{\text{correct}}|\psi_{\text{error}}\rangle_B| = \dfrac{|\alpha|^2\sqrt{1-2\epsilon-\epsilon^2} + |\beta|^2(1+\epsilon)}{\sqrt{|\alpha|^2(1-2\epsilon-\epsilon^2) + |\beta|^2(1+\epsilon)^2}} = \dfrac{|\alpha|^2\sqrt{1-2\epsilon-\epsilon^2} + |\beta|^2(1+\epsilon)}{\sqrt{1+(2\epsilon+\epsilon^2)(|\beta|^2-|\alpha|^2)}}$ //

Normalization $\longrightarrow$

(C) Error $=$ 1 $-$ Fidelity $= 1 - f(\epsilon)$, $\sqrt{1+x} \simeq 1 + \frac{x}{2} - \frac{1}{8}x^2$ $\quad \frac{1}{\sqrt{1+x}} \simeq 1 - \frac{x}{2} + \frac{3}{8}x^2$

$$f(\epsilon) \simeq \Big[|\alpha|^2(1-\epsilon-\epsilon^2) + |\beta|^2(1+\epsilon)\Big]\Big[1 - [|\beta|^2-|\alpha|^2](\epsilon+\tfrac{\epsilon^2}{2}) + \tfrac{3}{8}(2\epsilon+\epsilon^2)^2(|\beta|^2-|\alpha|^2)^2\Big]$$

$$\simeq \{1 + \epsilon[|\beta|^2-|\alpha|^2] - |\alpha|^2\epsilon^2\}\Big[1 - \epsilon(|\beta|^2-|\alpha|^2) + \epsilon^2\Big(\tfrac{3}{2}(|\beta|^2-|\alpha|^2)^2 - \tfrac{1}{2}(|\beta|^2-|\alpha|^2)\Big)\Big]$$

$$\Rightarrow \simeq 1 + \epsilon\big[|\beta|^2-|\alpha|^2 - (|\beta|^2-|\alpha|^2)\big] + \epsilon^2\Big[-|\alpha|^2 - [|\beta|^2-|\alpha|^2]^2 + \tfrac{3}{2}(|\beta|^2-|\alpha|^2)^2 - \tfrac{1}{2}(|\beta|^2-|\alpha|^2)\Big]$$

1st order coeff $= 0$

2nd order coeff $= \frac{1}{2}[|\beta|^2-|\alpha|^2]^2 - \frac{1}{2}[|\beta|^2-|\alpha|^2] - |\alpha|^2 \Rightarrow$ 1 $-$ fidelity $= O(\epsilon^2)$

For 2nd order to disappear, $[|\beta|^2-|\alpha|^2] = \dfrac{1-\sqrt{1+8|\alpha|^2}}{2} \Rightarrow 1-4|\alpha|^2 = -\sqrt{1+8|\alpha|^2} \Rightarrow 16|\alpha|^4 - 16|\alpha|^2 = 0$

$\Rightarrow \underline{|\alpha|^2 = 0 \text{ or } 1.}$

Prob. 2-5.

(A). $|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} |y\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

(B) $|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{f(y)} |y\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$, $|\psi_3\rangle = \frac{1}{2^n} \sum_{y,z\in\{0,1\}^n} (-1)^{f(y)} (-1)^{y\cdot z} |z\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

(C) Let A be a subset of $\{0, 1, \cdots, n-1\}$ s.t. $y_i = 1$. Then, for $x * y = 0$, $\Leftrightarrow \sum_{i\in A} x_i y_i = \sum_{i\in A} x_i \pmod 2 = 0$.

$\Rightarrow$ As each bit can alter the sum $\sum_{i\,(i\in A)} x_i = 0$ & each occurance is not correlated, we just have to consider a fixed $i_0 \in A$ to count the # of x. therefore, $2^n/2 = \underline{2^{n-1}}$ for each $\begin{smallmatrix}0\\1\end{smallmatrix}$ $\Rightarrow$ balanced.

(D) $|\langle 0^{\otimes n}|\psi_3\rangle|^2 = \left|\frac{1}{2^n} \sum_{y\in\{0,1\}^n} (-1)^{f(y)}\right|^2 = \begin{cases} 1 & \text{if constant} \\ 0 & \text{if balanced.} \end{cases} \Rightarrow$ iff.

Prob. 2-6.

(A) $|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} |y\rangle |0\rangle^{\otimes n}$

(B) $|\psi_2\rangle = \frac{1}{2^n} \sum_{y\in\{0,1\}^n} |y\rangle |f(y)\rangle$, $|\psi_3\rangle = \frac{1}{2^n} \sum_{y,z\in\{0,1\}^n} (-1)^{y\cdot z} |z\rangle |f(y)\rangle$

(C) First pick $x_0 \in \{0,1\}^n$ & put it in S_1. Next, put $x_0 \oplus s$ in S_1.

Then, $[x_0 \oplus s] \oplus s = x_0 \in S_1$, which means that, two elements in $\{0,1\}^n$ are now seperated without any intervention with other elements.

Continuing this process for 2^{n-1} times would give us the desired S_1 & S_2.

$$\Rightarrow \frac{1}{2^n} \sum_{y,z\in\{0,1\}^n} (-1)^{y\cdot z} |z\rangle |f(y)\rangle = \frac{1}{2^n} \sum_{z,y\in S_1} |z\rangle \left((-1)^{y\cdot z} |f(y)\rangle + (-1)^{(y\oplus s)\cdot z} |f(y\oplus s)\rangle\right)$$

$$= \frac{1}{2^n} \sum_{z,y\in S_1} (-1)^{y\cdot z} \left[1 + (-1)^{y\cdot s}\right] |z\rangle |f(y)\rangle$$

(D) $$\left\| \frac{1}{2^n} \sum_{y\in S_1} (-1)^{y\cdot z} \left[1 + (-1)^{s\cdot z}\right] |f(y)\rangle \right\| = \frac{1}{2^{2n}} \left\{ \left[1 + (-1)^{s\cdot z}\right]^2 \underbrace{|S_1|}_{2^{n-1}} \right\}$$

$$= \begin{cases} 0 & (s\cdot z = 1) \\ \frac{1}{2^{n-1}} & (s\cdot z = 0) \end{cases}$$

Prob. 2-7.

$|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\psi_2\rangle = \delta|0\rangle + \gamma|1\rangle$

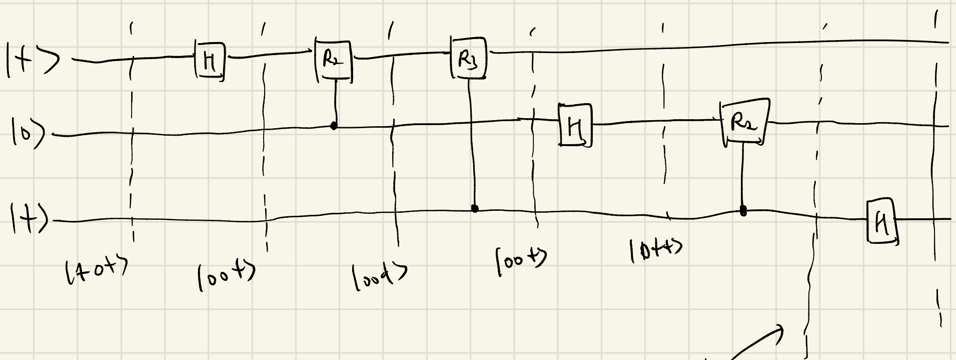
$\Rightarrow |\psi_1\rangle \otimes |\psi_2\rangle = \alpha\delta|00\rangle + \beta\delta|10\rangle + \alpha\gamma|01\rangle + \beta\gamma|11\rangle$

To become a Bell state,
$$\begin{cases} ① \ \beta\delta = \alpha\gamma = 0 \ \& \ |\alpha\delta| = |\beta\gamma| = \frac{1}{\sqrt{2}} \\ ② \ \alpha\delta = \beta\gamma = 0 \ \& \ |\beta\delta| = |\alpha\gamma| = \frac{1}{\sqrt{2}} \end{cases}$$

For both ① & ②, at least one of β and δ is zero $\Rightarrow$ $\beta\delta = 0$ $\quad \alpha\delta = 0$ or $\beta\gamma = 0 \rightarrow$ contradiction.

$\Rightarrow$ Bell states cannot be written in tensor product of single qubit states.

Prob. 2-8.



$\frac{1}{\sqrt{2}}\left(|0+0\rangle + \frac{1}{\sqrt{2}}|0\rangle(|0\rangle + i|1\rangle)|1\rangle\right)$

output $= \frac{1}{\sqrt{2}}|0++\rangle + \frac{1}{2}|0\rangle(|0\rangle + i|1\rangle)|-\rangle = \frac{1}{2\sqrt{2}}\left(2|000\rangle + (1+i)|010\rangle + (1-i)|011\rangle\right)$