양자 컴퓨팅 및 정보의 기초 Homework #4

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1. (a) $H^{\otimes t}|0\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\cdots\cdots\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}^t}\left(|0\rangle^{\otimes t} + |0\rangle^{\otimes t-1}|1\rangle + \cdots + |1\rangle^{\otimes t}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}^t}\left(|0\rangle + |1\rangle + \cdots + |2^t-1\rangle\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}^t}\sum_{j=0}^{2^t-1}|j\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}^t}\sum_{j\in\{0,1\}^t}|j\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

1. (b) $U_f|j\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = |j\rangle\left(\frac{|0\oplus f(j)\rangle - |1\oplus f(j)\rangle}{\sqrt{2}}\right) = \begin{cases} |j\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}} & (f(j)=\text{짝수}) \\ |j\rangle\frac{|1\rangle-|0\rangle}{\sqrt{2}} & (f(j)=\text{홀수}) \end{cases}$

$$= (-1)^{f(j)}|j\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$\therefore U_f\frac{1}{\sqrt{2}^t}\sum_{j\in\{0,1\}^t}|j\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}^t}\sum_{j\in\{0,1\}^t}U_f|j\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}^t}\sum_{j\in\{0,1\}^t}(-1)^{f(j)}|j\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

1. (c) $H|j\rangle = \frac{1}{\sqrt{2}}\sum_{l\in\{0,1\}}(-1)^{j\cdot l}|l\rangle$,

$$H^{\otimes t}|j_1\cdots j_t\rangle = H|j_1\rangle\otimes\cdots\otimes H|j_t\rangle$$

$$= \frac{1}{\sqrt{2}}\sum_{l_1\in\{0,1\}}(-1)^{j_1 l_1}|l_1\rangle\otimes\cdots\otimes\frac{1}{\sqrt{2}}\sum_{l_t\in\{0,1\}}(-1)^{j_t l_t}|l_t\rangle$$

$$= \frac{1}{\sqrt{2}^t}\sum_{l\in\{0,1\}^t}(-1)^{jl}|l\rangle,$$

$$\therefore H^{\otimes t}\left(\frac{1}{\sqrt{2}^t}\sum_{j\in\{0,1\}^t}(-1)^{f(j)}|j\rangle\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}^t}\sum_{j\in\{0,1\}^t}(-1)^{f(j)}\left(\quad\right)\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2^t}\sum_{l\in\{0,1\}^t}\left(\sum_{j\in\{0,1\}^t}(-1)^{f(j)+jl}\right)|l\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

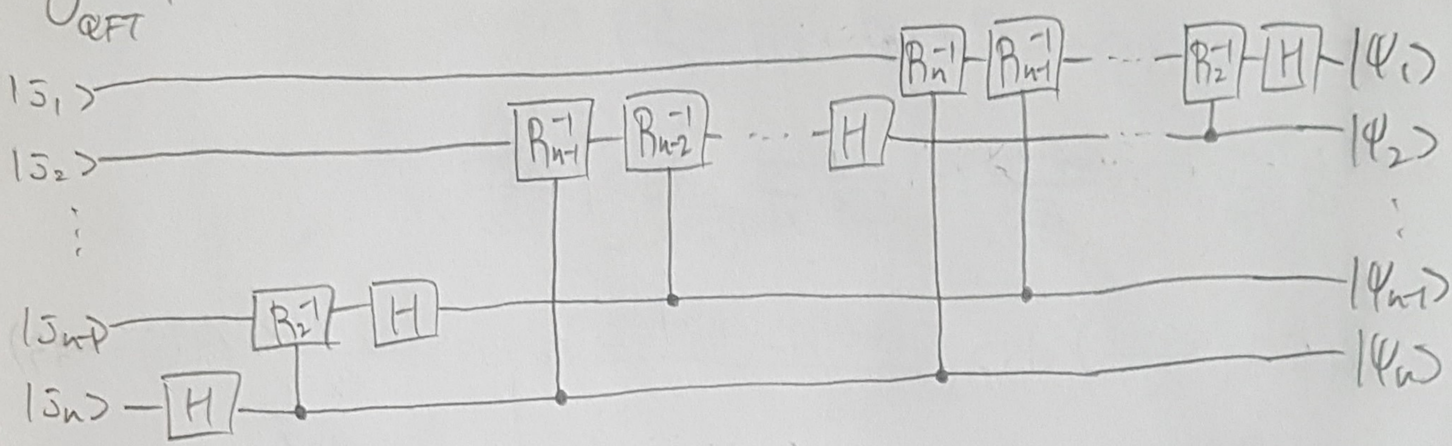
for constant → $|\ell\rangle = |00\cdots0\rangle$ 일 확률 $= |\pm 1|^2 = 1$

for balanced → $|\ell\rangle = |00\cdots0\rangle$ 일 확률 $= 0^2 = 0$

⇒ first register 의 측정결과로 f를 판단한다.

2. $R_k^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi i/2^k} \end{bmatrix}$, $H^{-1} = H$,

$\therefore U_{QFT}^{-1}$



$$|\varphi_k\rangle = (|0\rangle + e^{-2\pi i\, 0.j_k\cdots j_n}|1\rangle)/\sqrt{2}$$

3. $e^{2\pi i(\varphi_1\cdots\varphi_t)} = \cancel{e^{2\pi i(\varphi_1\cdots\varphi_n)}} \cdot e^{2\pi i(0.\varphi_{n+1}\cdots\varphi_t)} = e^{2\pi i(0.\varphi_{n+1}\cdots\varphi_t)}$

$$\to U^{2^{t-1}}H|0\rangle \otimes \cdots \otimes U^{2^0}H|0\rangle = \frac{1}{\sqrt{2}^t}(|0\rangle + U^{2^{t-1}}|1\rangle)\otimes\cdots\otimes(|0\rangle + U^{2^0}|1\rangle)$$

$$= \frac{1}{\sqrt{2}^t}(|0\rangle + e^{2\pi i(2^{t-1}\varphi)}|1\rangle)\otimes\cdots\otimes(|0\rangle + e^{2\pi i(2^0\varphi)}|1\rangle)$$

$$= \frac{1}{\sqrt{2}^t}(|0\rangle + e^{2\pi i(0.\varphi_t)}|1\rangle)\otimes\cdots\otimes(|0\rangle + e^{2\pi i(0.\varphi_1\cdots\varphi_t)}|1\rangle)$$

$$= (U^{2^{t-1}}\otimes\cdots\otimes U^{2^0})H^{\otimes t}|0\rangle = \frac{1}{\sqrt{2}^t}\sum_{k\in\{0,1\}^t}(U^{2^{t-1}}\otimes\cdots\otimes U^{2^0})|k\rangle$$

$$= \frac{1}{\sqrt{2}^t}\sum_{k\in\{0,1\}^t} e^{2\pi i\varphi(k_1\cdots k_t)}|k_1\cdots k_t\rangle = \frac{1}{\sqrt{2}^t}\sum_{k=0}^{2^t-1} e^{2\pi i\varphi k}|k\rangle$$

4. $U_{QFT} : |j_n\rangle \to (|0\rangle + e^{2\pi i 0.j_n}|1\rangle)/\sqrt{2}$

$U_{QFT}^{-1} : (|0\rangle + e^{2\pi i 0.j_n}|1\rangle)/\sqrt{2} \to |j_n\rangle$

$\therefore U^{-1}_{QFT}\left(\frac{1}{\sqrt{2^t}}(|0\rangle + e^{2\pi i(0.\varphi_t)}|1\rangle)\otimes\cdots\otimes(|0\rangle + e^{2\pi i(0.\varphi_1\cdots\varphi_t)}|1\rangle)\right)$

$= |\varphi_t\cdots\varphi_1\rangle$ → 읽는 순서를 반대로 하면 $|\varphi_1\cdots\varphi_t\rangle$ 이므로

bit string을 알아내서 φ를 측정할 수 있다.

5. (1) $x = 8$, $\gcd(8,21) = 1$

(2) $8^0 \quad 8^1 \quad 8^2 \quad \cdots \to r = 2$

$\quad 1 \quad\; 8 \quad\; 1$

(3) r = 짝수, $8^{2/2+1} = 64 \not\equiv 0 \pmod{21}$

(4) $P_1 = \gcd(7,21) = 7$, $P_2 = \gcd(9,21) = 3$

6. (a) $U_{mod}|u_s\rangle = \frac{1}{\sqrt{r}}\sum_{k=0}^{r-1} e^{-\frac{2\pi i s k}{r}}(U_{mod}|x^k \bmod N\rangle)$

$= \frac{1}{\sqrt{r}}\sum_{k'=1}^{r} e^{-\frac{2\pi i s(k'-1)}{r}}|x^{k'} \bmod N\rangle$

$= e^{\frac{2\pi i s}{r}} \cdot \frac{1}{\sqrt{r}}\sum_{k'=0}^{r-1} e^{-\frac{2\pi i s k'}{r}}|x^k \bmod N\rangle$

$= e^{\frac{2\pi i s}{r}}|u_s\rangle$

(b) $\sum_{s=0}^{r-1} e^{-\frac{2\pi i s n}{r}} = \begin{cases} \sum_{s=0}^{r-1} e^{-0} = r & (n=0) \\ (1-e^{-\frac{2\pi i \cancel{r} n}{\cancel{r}}})/(1-e^{-\frac{2\pi i n}{r}}) = 0 & (n \neq 0) \end{cases}$,

$\therefore \sum_{s=0}^{r-1} e^{-\frac{2\pi i s n}{r}} = r\delta_{n0}$

6.(c) $\frac{1}{\sqrt{r}}\sum_{s=0}^{r-1} e^{\frac{2\pi i s j}{r}}|u_s\rangle = \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1} e^{\frac{2\pi i s j}{r}}\left(\frac{1}{\sqrt{r}}\sum_{k=0}^{r-1} e^{-\frac{2\pi i s k}{r}}|x^k \bmod N\rangle\right)$

$= \frac{1}{r}\sum_{k=0}^{r-1}\left(\sum_{s=0}^{r-1} e^{-\frac{2\pi i s(k-j)}{r}}\right)|x^k \bmod N\rangle = \frac{1}{r}\sum_{k=0}^{r-1} r\,\delta_{(k-j)0}|x^k \bmod N\rangle$

$= |x^j \bmod N\rangle$

6.(d) $j=0 \to \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1} e^{\frac{2\pi i s 0}{r}}|u_s\rangle = \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}|u_s\rangle = |x^0 \bmod N\rangle = |1\rangle$

7.(b) $\frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1}|j\rangle\left(U_{mod}^j|1\rangle\right) = \frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1}|j\rangle\left(\frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}U_{mod}^j|u_s\rangle\right)$

$= \frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1}|j\rangle\left(\frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}e^{\frac{2\pi i s j}{r}}|u_s\rangle\right) = \frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1}|j\rangle|x^j \bmod N\rangle$

$= \frac{1}{\sqrt{2^t}\sqrt{r}}\sum_{s=0}^{r-1}\sum_{j=0}^{2^t-1} e^{\frac{2\pi i s j}{r}}|j\rangle|u_s\rangle$

7.(c) QFT: $|j\rangle \to \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{\frac{2\pi i j k}{2^n}}|k\rangle$

QFT^{-1}: $\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{\frac{2\pi i j k}{2^n}}|k\rangle \to |j\rangle$

$\to \frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1} e^{\frac{2\pi i j}{2^t}(2^t\cdot\frac{s}{r})}|j\rangle \to |2^t\frac{s}{r}\rangle,$

$\therefore \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}\left(U_{QFT}^{-1}\frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1}e^{\frac{2\pi i s j}{r}}|j\rangle\right)|u_s\rangle = \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}|2^t\frac{s}{r}\rangle|u_s\rangle$

8. $x=8,\ r=2,\ t=9$

(a) $\frac{1}{\sqrt{2^n}}\sum_{j=0}^{2^n-1}|j\rangle|1\rangle$

(b) $\frac{1}{\sqrt{2^n}}\sum_{j=0}^{2^n-1}|j\rangle|8^j \bmod 21\rangle = \frac{1}{\sqrt{2\cdot 2^9}}\sum_{s=0}^{1}\sum_{j=0}^{2^9-1} e^{\frac{2\pi i s j}{2}}|j\rangle|u_s\rangle$

(c) $\frac{1}{\sqrt{2}}\sum_{s=0}^{1}|2^9\frac{s}{2}\rangle|u_s\rangle = \frac{1}{\sqrt{2}}(|0\rangle|u_0\rangle + |256\rangle|u_1\rangle)$

$\frac{0}{2^9}=0,\ \frac{256}{2^9}=\frac{1}{2} \to \{\text{undetermined}, 2\},\ 8^2 \equiv 1 \pmod{21}$

9. (a) $|\alpha\rangle = \sum_{x'} |x'\rangle$, $|\beta\rangle = \sum_{x''} |x''\rangle$ $(x' = \text{sol} \times,\ x'' = \text{sol})$

$\therefore O(a|\alpha\rangle + b|\beta\rangle) = aO(|\alpha\rangle) + bO(|\beta\rangle) = a\sum_{x'} O|x'\rangle + b\sum_{x''} O|x''\rangle$

$= a\sum_{x'} (-1)^{f(x')}|x'\rangle + b\sum_{x''}(-1)^{f(x'')}|x''\rangle$

$= a\sum_{x'}|x'\rangle + b\sum_{x''}(-1)|x''\rangle = a|\alpha\rangle - b|\beta\rangle$

9. (b) $|\gamma\rangle = |\gamma\rangle_{\psi_{//}} + |\gamma\rangle_{\psi_\perp} = (|\psi\rangle\langle\psi|)|\gamma\rangle + (I - |\psi\rangle\langle\psi|)|\gamma\rangle$,

$|\gamma\rangle_{\psi\text{대칭}} = |\gamma\rangle_{//} - |\gamma\rangle_{\psi_\perp} = (2|\psi\rangle\langle\psi| - I)|\gamma\rangle$

$\rightarrow$ reflection $= 2|\psi\rangle\langle\psi| - I = 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n} - I$

$= H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n}$

9. (c) $\cos\frac{\theta}{2} = \langle\alpha|\psi\rangle = \frac{\sqrt{N-M}}{\sqrt{N}}$, $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{N-2M}{N}$,

$\cos\theta \fallingdotseq 1 - \frac{\theta^2}{2}$ $(\theta \ll 1)$, $\therefore 1 - \frac{\theta^2}{2} = 1 - \frac{2M}{N}$, $\theta \fallingdotseq 2\sqrt{\frac{M}{N}}$

10. (a) $|x\rangle = H^{\otimes 2}|00\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

10. (b) $|y\rangle = O|x\rangle = \frac{1}{2}(O|00\rangle + O|01\rangle + O|10\rangle + O|11\rangle)$

$= \frac{1}{2}((-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle)$

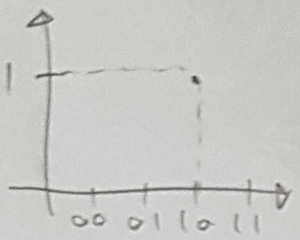
$= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$

10. (c) $|\psi\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$\langle\psi|y\rangle = \frac{1}{4}(\langle 00| + \langle 01| + \langle 10| + \langle 11|)(|00\rangle + |01\rangle - |10\rangle + |11\rangle) = \frac{1}{2}$,

$(2|\psi\rangle\langle\psi| - I)|y\rangle = |\psi\rangle - |y\rangle = |10\rangle$

10. (d)



$P(|10\rangle) = 1$

$P(\text{나머지}) = 0$

11. $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_k |k\rangle \Rightarrow \langle\psi|k'\rangle = \left(\frac{1}{\sqrt{N}}\sum_k \langle k|\right)|k'\rangle = \frac{1}{\sqrt{N}}\sum_k \delta_{kk'} = \frac{1}{\sqrt{N}}$,

$\therefore (2|\psi\rangle\langle\psi| - I)\sum_k a_k|k\rangle = \sum_k a_k(2\langle\psi|k\rangle|\psi\rangle - |k\rangle)$

$= \sum_k \left(2\frac{a_k}{\sqrt{N}}|\psi\rangle - a_k|k\rangle\right) = \sum_k \left(2\frac{a_k}{N}\sum_{k'}|k'\rangle - a_k|k\rangle\right)$

$= \sum_{k'}|k'\rangle\, 2\langle\alpha\rangle - \sum_k a_k|k\rangle = \sum_k (-\alpha_k + 2\langle\alpha\rangle)|k\rangle$

12. $|p_3p_2p_1p_0\rangle = |q_2q_1q_0q_3\rangle = |2 \times q_3q_2q_1q_0 \bmod 15\rangle$, $\therefore$ A=2

$|p_3p_2p_1p_0\rangle = |\bar{q}_2\bar{q}_1\bar{q}_0\bar{q}_3\rangle = |1111\rangle - |q_2q_1q_0q_3\rangle$

$= |(15-2) \times q_3q_2q_1q_0 \bmod 15\rangle$, $\therefore$ B=13