M1522.002500 - 양자 컴퓨팅 및 정보의 기초

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Homework #4

Due date: 23:59, June. 11, 2019

Please hand in the homework before 6/12 in PDF or paper form.

Problem 4-1. Follow the procedure of proving Schmidt decomposition and give a summary or brief sketch of the proof. You can refer to any source including textbook, wiki, etc.

Problem 4-2. Prove the theorem 2.6 in lecture note 16.

Theorem 2.6: the sets $|\widetilde{\psi_i}\rangle$ and $|\widetilde{\varphi_j}\rangle$ generate the same density matrix if and only if $|\widetilde{\psi_i}\rangle = \sum_j u_{ij}\,|\widetilde{\varphi_j}\rangle$, where u_{ij} is a unitary matrix of complex numbers, with indices i and j, and we 'pad' whichever set of vectors $|\widetilde{\psi_i}\rangle$ and $|\widetilde{\varphi_j}\rangle$ with additional vectors 0 so that the two sets have the same number of elements.

Problem 4-3. Construct a density matrix that represents following state.

(A) A particle is in $|0\rangle$ with probability 3/4 and in $|1\rangle$ with probability 1/4.

(B) Alice throws a dice. If the result is 5 and 6, Alice sends $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$, if the result is 4, Alice sends $\frac{|00\rangle-|11\rangle}{\sqrt{2}}$, and if the result is 1, 2, and 3, Alice sends $|00\rangle$.

Problem 4-4. Purity of a density matrix ρ is defined as $tr(\rho^2)$. Prove that given state ρ is a pure state if and only if $tr(\rho^2) = 1$. Notice that when a unitary operator U is applied to ρ, the output state is $U\rho U^\dagger$.

Problem 4-5. Apply partial trace to density matrix of all 4 Bell states and show that the result is a mixed state. You can either choose first or second qubit.

Problem 4-6. In HW2, you mapped arbitrary two-level "pure" state on Bloch sphere. In the same way, you can map arbitrary two-level "density matrix" on and inside Bloch sphere.

(A) Represent arbitrary density matrix $\rho = \begin{pmatrix} a & b^* \\ b & 1-a \end{pmatrix}$ in $\{I, \sigma_x, \sigma_y, \sigma_z\}$ *basis, where* $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, only with real coefficient, i.e., $\rho = \frac{\alpha I + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z}{2}$. (n_x, n_y, n_z) is the point that maps given density matrix on and inside Bloch sphere.

(B) Prove that a state is "on" Bloch sphere if and only if the state is pure. Also, prove that a state is located at the origin of Bloch sphere if and only if the state is maximally mixed. (in other words, purity = 1/2)

Hint) You can show that the distance from the origin is square-root of purity.

(C) In reality, there are some unknown or inevitable process such as interaction with photon from environment that can cause stochastic evolution of given state. For some situation, we can model a density matrix with idle gate (no gate applied except time evolution) as $\rho(t) = \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3}e^{-\alpha t} \\ \frac{\sqrt{2}}{3}e^{-\alpha t} & \frac{1}{3} \end{pmatrix}, \alpha > 0$. Sketch the trajectory of the density matrix on and inside Bloch sphere and specify the initial & final state. This process is called "dephasing".

Problem 4-7. Prove that entropy of arbitrary density matrix is preserved under unitary quantum operation. Also, calculate the entropy of $\rho(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{-\alpha t} \\ \frac{1}{2}e^{-\alpha t} & \frac{1}{2} \end{pmatrix}, \alpha > 0$.

Problem 4-8. In general, Schrodinger's equation can be re-written under unitary map U that maps quantum states to other states, which corresponds to a frame change, i.e., $|\tilde{\psi}\rangle = U|\psi\rangle$ where $|\tilde{\psi}\rangle$ is the quantum state seen in new frame and $|\psi\rangle$ is in original frame. (ex: rotating reference frame, upside down, etc.) Prove that Schrodinger's equation still holds true in the new frame, in other words, $i\hbar\frac{\partial}{\partial t}|\tilde{\psi}\rangle = \widetilde{H}|\tilde{\psi}\rangle$, where $\widetilde{H} = UHU^\dagger - i\hbar U\frac{\partial U^\dagger}{\partial t}$.

Problem 4-9. Prove followings.

(A) Define $P_1 = \{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z, \pm iI, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$. Prove that this set is closed under multiplication. Also, prove that this set can be generated through matrix multiplication of elements in $\langle\sigma_x, \sigma_y, \sigma_z\rangle$

(B) Prove that $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ and $S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$ satisfies $H\sigma H^\dagger \in P_1$ and $S\sigma S^\dagger \in P_1$ for $\forall\sigma \in P_1$.

(C) Define $P_2 = \{\pm\sigma_m \otimes \sigma_n, \pm i\sigma_m \otimes \sigma_n, |m, n = 0(\sigma_0 = I), x, y, z\}$. Prove that CNOT(target = 0, control =1) gate satisfies $CNOT\sigma CNOT^\dagger \in P_2$ for $\forall\sigma \in P_2$.

Hint) Block matrix form would simplify the calculation.