Summary of previous lecture

- Properties of density matrix
  - $\mathrm{tr}(\rho^2) = \begin{cases} 1 & \Rightarrow \text{Pure state} \\ < 1 & \Rightarrow \text{Mixed state} \end{cases}$
  - (1) (Trace condition) $\mathrm{tr}(\rho) = 1$
  - (2) (Positivity condition) ρ is a positive operator
- Decomposition of density matrix is not unique
  - Possible decompositions are connected by unitary relation: $|\widetilde{\psi_i}\rangle = \sum_j u_{ij} |\widetilde{\phi_j}\rangle$
- Reduced density matrix
  - Partial trace: for $\rho^{AB} \equiv \sum_i p_i^{AB} |\psi_i^{AB}\rangle\langle\psi_i^{AB}|$ representing entire system, take trace only for part of the system $\rho^A \equiv \mathrm{tr}_B(\rho^{AB})$
  - Pure entangled state becomes mixed state when only part of the system is considered
  - During quantum teleportation, without the measurement result in Bell basis, the information stored in the destination qubit is equivalent to completely random quantum state.

Quantification of Entanglement

- Are the following states entangled or separable?
  - $|\psi_1\rangle = |1_A\rangle|2_B\rangle$
  - $|\psi_2\rangle = (|1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle)/\sqrt{2}$
  - $|\psi_3\rangle = (|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle + |2_A\rangle|1_B\rangle - |2_A\rangle|2_B\rangle)/2$
  - $|\psi_4\rangle = (2|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle + 2|2_A\rangle|2_B\rangle)/\sqrt{10}$
- Can we turn $|\psi_4\rangle$ into the form of $|\psi_2\rangle$?
  - $2|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle + 2|2_A\rangle|2_B\rangle$

    $\Rightarrow |a_A\rangle|\alpha_B\rangle - |b_A\rangle|\beta_B\rangle$ where $|a_A\rangle \& |b_A\rangle$ are orthogonal and $|\alpha_B\rangle \& |\beta_B\rangle$ are also orthogonal, respectively
  - If we define c_{ij} as $c_{11}|1_A\rangle|1_B\rangle + c_{12}|1_A\rangle|2_B\rangle + c_{21}|2_A\rangle|1_B\rangle + c_{22}|2_A\rangle|2_B\rangle$,

    $$c_{11} = \frac{2}{\sqrt{10}}, c_{12} = -\frac{1}{\sqrt{10}}, c_{21} = -\frac{1}{\sqrt{10}}, c_{22} = \frac{2}{\sqrt{10}} \Rightarrow C = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
  - Can we find unitary transform for A and B such that $C' = \begin{bmatrix} \blacksquare & 0 \\ 0 & \blacksquare \end{bmatrix}$?

Basis Transformation of a Matrix Representation

- Assume that $\mathbb{O}$ is a matrix representation of an operator Ω in some orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$, where each component of $\mathbb{O}$ is obtained by $\mathbb{O}_{ij} = \langle i|\Omega|j\rangle$.
- If we switch the basis from $|1\rangle, |2\rangle, \dots, |n\rangle$ to a new orthonormal basis $|I\rangle, |II\rangle, \dots, |N\rangle$, the new matrix representation $\mathbb{O}'$ of the same operator Ω in the new orthonormal basis can be obtained by $\mathbb{O}' = \mathbb{U}^\dagger \mathbb{O} \mathbb{U}$ or $\mathbb{O} = \mathbb{U}\mathbb{O}'\mathbb{U}^\dagger$.
- $\mathbb{U}$ is a matrix representation of $U = \sum_{m=1}^{n} |M\rangle\langle m|$ in the original basis $|1\rangle, |2\rangle, \dots, |n\rangle$.

Basis Transformation of a Vector Representation

- Assume that $\mathbb{v}$ is a column vector representation of a vector $|v\rangle$ in some orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$, where each component of $\mathbb{v}$ is obtained by $\mathbb{v}_k = \langle k|v\rangle$.
- Consider a new orthonormal basis vectors $|I\rangle, |II\rangle, \dots, |N\rangle$ that have column vector representation of
  $\begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix}, \begin{bmatrix} II_1 \\ \vdots \\ II_n \end{bmatrix}, \dots, \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix}$ with each component $J_i = \langle i|J\rangle$ in the original basis.
- If we switch the basis from $|1\rangle, |2\rangle, \dots, |n\rangle$ to a new orthonormal basis $|I\rangle, |II\rangle, \dots, |N\rangle$, the new column vector representation $\mathbb{v}'$ of the same vector $|v\rangle$ in the new orthonormal basis can be obtained by

$$\mathbb{v}'_k = \langle K|v\rangle = \langle K|\left\{\sum_{i=1}^{n} |i\rangle\langle i|\right\}|v\rangle = \sum_{i=1}^{n} \langle K|i\rangle\langle i|v\rangle = \sum_{i=1}^{n} \langle K|i\rangle\, \mathbb{v}_i = \sum_{i=1}^{n} \mathbb{W}_{ki}\, \mathbb{v}_i$$

  - If we define $W = \sum_{m=1}^{n} |m\rangle\langle M|$ and $\mathbb{W}$ as a matrix representation of W in the original basis $|1\rangle, |2\rangle, \dots, |n\rangle$, then it automatically satisfies $\mathbb{W}_{ki} = \langle K|i\rangle$ because

$$\mathbb{W}_{ki} = \langle k|W|i\rangle = \langle k|\left\{\sum_{m=1}^{n} |m\rangle\langle M|\right\}|i\rangle = \sum_{m=1}^{n} \delta_{km}\langle M|i\rangle = \langle K|i\rangle$$

- Conventionally, it is more common to define $U \equiv W^\dagger = \sum_{m=1}^{n} |M\rangle\langle m|$
  - Then $\mathbb{U}$ is a matrix representation of $U = \sum_{m=1}^{n} |M\rangle\langle m|$ in the original basis $|1\rangle, |2\rangle, \dots, |n\rangle$, and

$$\mathbb{U}_{ij} = \langle i|U|j\rangle = \langle i|\left\{\sum_{m=1}^{n} |M\rangle\langle m|\right\}|j\rangle = \langle i|J\rangle = J_i \quad \text{or} \quad \mathbb{U} = \begin{bmatrix} \langle 1|I\rangle & \cdots & \langle 1|N\rangle \\ \vdots & \ddots & \vdots \\ \langle n|I\rangle & \cdots & \langle n|N\rangle \end{bmatrix} = \begin{bmatrix} I_1 & \cdots & N_1 \\ \vdots & \ddots & \vdots \\ I_n & \cdots & N_n \end{bmatrix}$$

  - $\mathbb{W} \equiv \mathbb{U}^\dagger$ and $\mathbb{v}'_k = \sum_{i=1}^{n} \mathbb{U}^\dagger_{ki}\mathbb{v}_i \Rightarrow \mathbb{v}' = \mathbb{U}^\dagger\mathbb{v}$ or $\mathbb{v} = \mathbb{U}\mathbb{v}'$

Basis Transformation of a Vector Representation

- From the previous page,
  - $U = \sum_{m=1}^{n} |M\rangle\langle m|$ and $\mathbb{U}_{ij} = \langle i|J\rangle$ or $\mathbb{U} = \begin{bmatrix} \langle 1|I\rangle & \cdots & \langle 1|N\rangle \\ \vdots & \ddots & \vdots \\ \langle n|I\rangle & \cdots & \langle n|N\rangle \end{bmatrix} = \begin{bmatrix} I_1 & \cdots & N_1 \\ \vdots & \ddots & \vdots \\ I_n & \cdots & N_n \end{bmatrix}$

- Mapping to a new orthonormal basis

$$\sum_{i=1}^{n} \mathbb{U}_{ik}|i\rangle = \sum_{i=1}^{n} |i\rangle\langle i|K\rangle = |K\rangle \Leftrightarrow |i\rangle = \left\{\sum_{J=1}^{n} |J\rangle\langle J|\right\}|i\rangle = \sum_{J=1}^{n} |J\rangle\left(\sum_{k=1}^{n} \mathbb{U}_{kj}^{*}\langle k|i\rangle\right) = \sum_{J=1}^{n} \left(\mathbb{U}^{\dagger}\right)_{ji}|J\rangle$$

- Mapping to another new orthonormal basis
  - Define another new matrix $\mathbb{V} = \mathbb{U}^{*}$
  - Because $\mathbb{V}^{\dagger} = \mathbb{U}^{T}$ and $\mathbb{U}\mathbb{U}^{\dagger} = \mathbb{I}$, $\left(\mathbb{U}\mathbb{U}^{\dagger}\right)^{*} = \mathbb{U}^{*}\mathbb{U}^{T} = \mathbb{V}\mathbb{V}^{\dagger} = \mathbb{I}$ ➔ $\mathbb{V}$ is a unitary matrix
  - New basis defined as

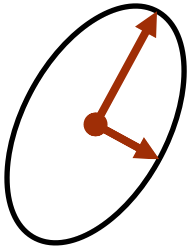
$$|L'\rangle \equiv \sum_{j=1}^{n} \mathbb{U}_{lj}^{\dagger}|j\rangle = \sum_{j=1}^{n} \mathbb{V}_{jl}|j\rangle$$

  are orthonormal because $\mathbb{V}$ is a unitary matrix.
  - $|i\rangle$ can be expanded in $|L'\rangle$ basis in the following way

$$|i\rangle = \left\{\sum_{L=1}^{n} |L'\rangle\langle L'|\right\}|i\rangle = \sum_{L=1}^{n} |L'\rangle\left(\sum_{j=1}^{n} \mathbb{V}_{jl}^{*}\langle j|\right)|i\rangle = \sum_{L=1}^{n} |L'\rangle\left(\sum_{j=1}^{n} \mathbb{U}_{jl}\langle j|i\rangle\right) = \sum_{L=1}^{n} \mathbb{U}_{il}|L'\rangle$$

Quantification of Entanglement

- For $|\psi_4\rangle = (2|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle + 2|2_A\rangle|2_B\rangle)/\sqrt{10}$
  - $C = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\mathbb{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
  - $\begin{cases} |I_A\rangle = (|1_A\rangle - |2_A\rangle)/\sqrt{2} \\ |II_A\rangle = (|1_A\rangle + |2_A\rangle)/\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} |1_A\rangle = (|I_A\rangle + |II_A\rangle)/\sqrt{2} \\ |2_A\rangle = (-|I_A\rangle + |II_A\rangle)/\sqrt{2} \end{cases}$
  - $\begin{cases} |I'_B\rangle = (|1_B\rangle - |2_B\rangle)/\sqrt{2} \\ |II'_B\rangle = (|1_B\rangle + |2_B\rangle)/\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} |1_B\rangle = (|I'_B\rangle + |II'_B\rangle)/\sqrt{2} \\ |2_B\rangle = (-|I'_B\rangle + |II'_B\rangle)/\sqrt{2} \end{cases}$
  - $|\psi_4\rangle = \frac{1}{\sqrt{10}}(3|I_A\rangle|I'_B\rangle + |II_A\rangle|II'_B\rangle)$
  - Meaning of Hermitian matrix

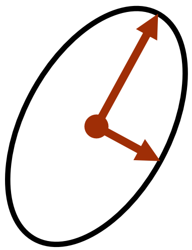


Quantification of Entanglement

- What about $|\psi\rangle = (|1_A\rangle|1_B\rangle + |1_A\rangle|2_B\rangle + |2_A\rangle|2_B\rangle)/\sqrt{3}$?
  - c_{ij} ➔ $\mathbb{C} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
  - c_{ij} is not Hermitian. What can we do?

- In general, we can have two parties that have different dimensions
  - Grouping of multiple qubits to test the separability of the qubits
  - $|\psi\rangle = (|1_A\rangle|1_B\rangle + |1_A\rangle|2_B\rangle + |1_A\rangle|3_B\rangle + |2_A\rangle|1_B\rangle - |2_A\rangle|2_B\rangle)/\sqrt{5}$
  - $\mathbb{C} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$
  - Can we transform it into the following shape?

$$\blacksquare|I_A\rangle|I_B\rangle + \blacksquare|II_A\rangle|II_B\rangle + \blacksquare|III_A\rangle|III_B\rangle$$



Singular Value Decomposition

- Singular Value Decomposition (SVD)
  - When a matrix $\mathbb{A}$ corresponding to a linear transformation is given, SVD allows us to find an orthogonal basis in the input vector space that will be transformed into another orthogonal basis in the output vector space
- Example
  - Assume that the input vector space is $\mathbf{R}^3$ and the output vector space is $\mathbf{R}^2$, and linear transformation is given in terms of matrix $\mathbb{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.
  - Do two orthogonal vectors (1,0,0), (0,0,1) in the input vector space get transformed into orthogonal vectors in the output space?
  - $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

  ➔ No!
- What will be the benefit if the orthogonal basis in the input space is transformed into another orthogonal basis in the output space?
  - For a linear transformation, we only need to multiply the corresponding eigenvalues to the coefficients of the basis.
  - Should we give up?

Singular Value Decomposition

- How can we always decompose a given matrix $\mathbb{A}$ into $\mathbb{A} = \mathbb{U}\mathbb{D}\mathbb{V}^\dagger$ form, where $\mathbb{U}, \mathbb{V}$ are unitary matrices, $\mathbb{D}$ is a diagonal matrix?
  - $\mathbb{A}^\dagger\mathbb{A} = \mathbb{V}\mathbb{D}^\dagger\mathbb{U}^\dagger\mathbb{U}\mathbb{D}\mathbb{V}^\dagger = \mathbb{V}\mathbb{D}^\dagger\mathbb{D}\mathbb{V}^\dagger = \mathbb{V}\begin{bmatrix} d_1^2 & & \\ & \ddots & \\ & & d_r^2 \end{bmatrix}\mathbb{V}^\dagger$
- Example
  - Input space $\mathbf{R}^3$, output space $\mathbf{R}^2$,
  - $\mathbb{A}^\dagger\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

  ➔ positive matrix! Therefore, we can diagonalize and find unitary matrix $\mathbb{V}$. Is this coincidence?
  - Eigenvalue & eigenvector:$\lambda = 3, 2, 0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Singular Value Decomposition

- (Cont'd) Example $\mathbb{A} = \mathbb{U}\mathbb{D}\mathbb{V}^\dagger$
  - $\mathbb{A}^\dagger\mathbb{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \mathbb{V}(\mathbb{D}^\dagger\mathbb{D})\mathbb{V}^\dagger$
  - Eigenvalue & eigenvector: $\lambda = 3, 2, 0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$
  - $\mathbb{V} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}, \mathbb{D}^\dagger\mathbb{D} = \begin{bmatrix} d_1^2 & & \\ & d_2^2 & \\ & & d_3^2 \end{bmatrix} = \begin{bmatrix} 3 & & \\ & 2 & \\ & & 0 \end{bmatrix}$
  - What should we do if the eigenvalue is negative?
  - $\mathbb{A}\,\mathbb{V} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{bmatrix}$

Singular Value Decomposition

- (Cont'd) Example $\mathbb{A} = \mathbb{U}\mathbb{D}\mathbb{V}^\dagger$
  - To find out $\mathbb{U}$, $\mathbb{A}\mathbb{A}^\dagger = \mathbb{U}\mathbb{D}\mathbb{V}^\dagger\mathbb{V}\mathbb{D}^\dagger\mathbb{U}^\dagger = \mathbb{U}\mathbb{D}\mathbb{D}^\dagger\mathbb{U}^\dagger = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$
  - Eigenvalue & eigenvector: $\lambda = 3, 2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - $\mathbb{U} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbb{D}\mathbb{D}^\dagger = \begin{bmatrix} d_1^2 & \\ & d_2^2 \end{bmatrix} = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}.$
    - Comparison with $\mathbb{D}^\dagger\mathbb{D} = \begin{bmatrix} 3 & & \\ & 2 & \\ & & 0 \end{bmatrix}$ ➔ $\mathbb{D}$: 2x3 matrix
  - $\mathbb{U}\mathbb{D}\mathbb{V}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & /\sqrt{6} & -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \mathbb{A}$
- SVD works for any types of matrix including the rectangular matrix
  - The number of the non-zero diagonal components is equal to the rank of $\mathbb{A}$ ➔ can be used for the determination of entanglement

Polar decomposition

- Section 2.1.10
- Theorem 2.3 (Polar decomposition): Let A be a linear operator on a vector space V. Then there exists unitary U and positive operator J and K such that

$$A = UJ = KU$$

where the unique positive operator J and K satisfying these equations are defined by $J \equiv \sqrt{A^\dagger A}$ and $J \equiv \sqrt{AA^\dagger}$. Moreover, if A is invertible then U is unique.

- $A = UJ$ is called left polar decomposition of A and $A = KU$ is called the right polar decomposition of A.
- Proof of left polar decomposition
  - $A^\dagger A$ is a positive operator ➔ $A^\dagger A$ can be diagonalized ➔ $J \equiv \sqrt{A^\dagger A}$ is a positive operator. ➔ $J = \sum_i \lambda_i |i\rangle\langle i|$ $(\lambda_i \geq 0)$.
  - Define $|\psi_i\rangle \equiv A|i\rangle$. ➔ $\langle\psi_i|\psi_i\rangle = \lambda_i^2$
  - Consider for now only those i for which $\lambda_i \neq 0$. ➔ define normalized vector $|e_i\rangle \equiv |\psi_i\rangle/\lambda_i$. ➔ If $i \neq j$, $\langle e_i|e_j\rangle = \langle i|A^\dagger A|j\rangle/\lambda_i\lambda_j = \langle i|J^2|j\rangle/\lambda_i\lambda_j = 0$.

Polar decomposition

- Proof of left polar decomposition (continued)
  - From the previous page, $J = \sum_i \lambda_i |i\rangle\langle i|$ $(\lambda_i \geq 0)$. $|\psi_i\rangle \equiv A|i\rangle$. $|e_i\rangle \equiv |\psi_i\rangle/\lambda_i$ for $\lambda_i \neq 0$.
  - Using Gram-Schmidt procedure, extend $|i\rangle$ basis (for i with $\lambda_i \neq 0$) to $|j\rangle$ basis (for all j), and find the orthonormal basis set from $|e_i\rangle$ for arbitrary j.
  - Define a unitary operator $U \equiv \sum_i |e_i\rangle\langle i|$.
  - When $\lambda_i \neq 0$, $UJ|i\rangle = \lambda_i |e_i\rangle = A|i\rangle$.
  - When $\lambda_j = 0$, $\langle \psi_j | \psi_j \rangle = \langle j | A^\dagger A | j \rangle = 0$ and $UJ|j\rangle = \lambda_j |e_j\rangle = 0 = |\psi_j\rangle = A|i\rangle$.
  - From the above, the action of UJ and A on $|i\rangle$ are identical. ➔ $A = UJ$.
  - $A = UJ$ ➔ $A^\dagger = J^\dagger U^\dagger$ ➔ $A^\dagger A = J^2$ ➔ $J = \sqrt{A^\dagger A}$
  - If A is invertible, J is invertible, so $U = AJ^{-1}$ is uniquely determined.

Singular value decomposition

- Section 2.1.10
- Corollary 2.4 (Singular value decomposition): Let $\mathbb{A}$ be a square matrix. Then there exist unitary matrices $\mathbb{U}$ and $\mathbb{V}$, and a diagonal matrix $\mathbb{D}$ with non-negative entries such that

$$\mathbb{A} = \mathbb{U}\mathbb{D}\mathbb{V}$$

The diagonal elements of $\mathbb{D}$ are called the singular values of $\mathbb{A}$.

- Proof
  - By polar decomposition, $\mathbb{A} = \mathbb{S}\mathbb{J}$, for unitary $\mathbb{S}$, and positive $\mathbb{J}$. Because $\mathbb{J}$ is a positive matrix and therefore $\mathbb{J}$ is a Hermitian matrix, $\mathbb{J} = \mathbb{T}\mathbb{D}\mathbb{T}^{\dagger}$, for unitary matrix $\mathbb{T}$ and diagonal matrix $\mathbb{D}$ with non-negative entries. $\mathbb{U} = \mathbb{S}\mathbb{T}$ and $\mathbb{V} = \mathbb{T}^{\dagger}$.

Schmidt decomposition

- Section 2.5
- Theorem 2.7 (Schmidt decomposition): Suppose $|\psi\rangle$ is a pure state of a composite system, AB. Then there exist orthonormal state $|i_A\rangle$ for system A, and orthonormal state $|i_B\rangle$ for system B such that $|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$, where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt co-efficient.
- Sketch of proof (for complete proof, refer to page 109)
  - Assume A and B have the same dimension
  - Let $|j\rangle$ and $|k\rangle$ be any fixed orthonormal bases for system A and B, respectively. Then $|\psi\rangle$ can be written as $|\psi\rangle = \sum_j \sum_k a_{jk} |j\rangle|k\rangle$.
  - By the singular value decomposition, $a_{jk} = \mathbb{A} = \mathbb{UDV} = \sum_i u_{ji} d_{ii} v_{ik}$, $|\psi\rangle = \sum_j \sum_k \left(\sum_i u_{ji} d_{ii} v_{ik}\right)|j\rangle|k\rangle$.
  - By defining $|i_A\rangle \equiv \sum_j u_{ji}|j\rangle$, $|i_B\rangle \equiv \sum_k v_{ik}|k\rangle$, and $\lambda_i \equiv d_{ii}$,
  - $|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$

Schmidt decomposition

- $|i_A\rangle$, $|i_B\rangle$ are called the Schmidt bases
- The number of non-zero values λ_i is called the Schmidt number for the state $|\psi\rangle$.
- Quantification of entanglement (or degree of entanglement)
  - Schmidt number is preserved under unitary transformation
  - The Schmidt number of a product state is 1.
- Purification
  - When a mixed state ρ^A of system A is given, introduce a fictitious system R such that the composite system AR is in a pure state $|AR\rangle$ such that $\rho^A = \mathrm{tr}_R(|AR\rangle\langle AR|)$.
  - If ρ^A can be decomposed into $\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$, then assume that system R has the same dimension as A, with orthonormal basis $|i^R\rangle$, then we can define a pure state $|AR\rangle \equiv \sum_i \sqrt{p_i}|i^A\rangle|i^R\rangle$.