1. a) $(\mathrm{AB})^{\dagger}AB = B^{\dagger}A^{\dagger}AB = I$

   b) $\mathrm{Av} = \lambda\mathrm{v} \Rightarrow \mathrm{vA}^{\dagger}Av = |\lambda^2 v^2| \Rightarrow |\lambda| = 1$

2. a) $\mathrm{Av} = \lambda\mathrm{v} \Rightarrow \mathrm{v}^{\dagger}Av = \lambda|v|^2 = \lambda^*|v|^2 \Rightarrow \lambda = \lambda^*$

   b) Theorems 9 – 10 basically contain the essential logic.

   Let us understand this in terms of the Dirac notation. The unprimed basis vectors $|m\rangle$, $|n\rangle$ are the original basis vectors, while the primed basis vectors $|m'\rangle$, $|n'\rangle$ are the diagonalized basis vectors, or eigenvectors of $\mathbb{A}$. Then $\mathbb{A}_D|m'\rangle = \lambda_{m'}|m'\rangle$ or $\mathbb{A}_D = \sum_{m'}\lambda_{m'}|m'\rangle\langle m'|$ for eigenvalues $\lambda_{m'}$. $\mathbb{A}_D$ denotes that $\mathbb{A}$ is diagonalized.

   Recall that an orthonormal basis transformation is carried out by a unitary operator U in the following manner.

$$|m'\rangle = U|m\rangle = \sum_{m'}|m'\rangle\langle m'|m\rangle = \sum_{m'}U_{m'm}|m'\rangle$$

   The relation between $\mathbb{A}$ and $\mathbb{A}_D$ is then,

$$\langle m|\mathbb{A}|n\rangle = \langle m|U^{\dagger}U\mathbb{A}U^{\dagger}U|n\rangle = \langle m'|U\mathbb{A}U^{\dagger}|n'\rangle$$

$$= \sum_{i',j'}\langle m'|U|i'\rangle\langle i'|\mathbb{A}|j'\rangle\langle j'|U^{\dagger}|n'\rangle$$

$$= \sum_{i',j'}\langle m'|U|i'\rangle\lambda_{j'}\delta_{i'j'}\langle j'|U^{\dagger}|n'\rangle$$

$$= \langle m'|U(\sum_{i'}\lambda_{i'}|i'\rangle\langle i'|)U^{\dagger}|n'\rangle$$

$$= \langle m'|U\mathbb{A}_D U^{\dagger}|n'\rangle$$

   Therefore the matrices $\mathbb{A}$ and $\mathbb{A}_D$ evaluated with respect to their bases are related by $\mathbb{A} = U\mathbb{A}_D U^{\dagger}$.

   c) $(\mathrm{e}^{\mathrm{i}A})^{\dagger} = \frac{\Sigma(i^n A^n)^{\dagger}}{n!} = \frac{\Sigma -\mathrm{i}\mathrm{A}^{\mathrm{n}}}{n!} = e^{-iA}$

   $\mathrm{e}^{-\mathrm{i}A}e^{iA} = e^{(-i+i)A} = e^0 = I$

   d) Suppose all eigenvectors are unit vectors.

$$\langle -|[A,B]|+\rangle = \langle -|(AB - BA)|+\rangle = -q_-\langle -|A|+\rangle + q_+\langle -|A|+\rangle = (q_+ - q_-)\langle -|A|+\rangle = 0$$

   Since $q_+ - q_- \neq 0$, $\langle -|A|+\rangle = 0$.

3. a) $\mathrm{M}_{\mathrm{i}}^2 = I \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

   b)$\mathrm{tr}(\mathrm{M}_{\mathrm{i}}) = tr(IM_i) = tr(M_j^2 M_i) = tr(\mathrm{M}_{\mathrm{j}} M_i M_j) = tr\left(M_j(-M_j M_i)\right) = -tr(M_i) \Rightarrow tr(M_i) = 0$

   c) By a), b) $\mathrm{tr}(\mathrm{M}_{\mathrm{i}}) = \sum_{\mathrm{n}} \lambda_k = 0,\ \ \lambda = \pm 1 \Rightarrow \mathrm{n\ is\ even}$

4. a) $\mathrm{c}^* b + d^* a$

   b) $\mathrm{v}' = 1/\sqrt{2}(a+b, a-b),\ \ \mathrm{w}' = 1/\sqrt{2}(c+d, c-d)$

   c) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   d) $\frac{1}{2}(c^* a + c^* b + d^* b + d^* b + \mathrm{c}^* a - c^* b - d^* a + d^* b) = c^* b + d^* a$

   e) eigenbasis of Z = $\{(1,0),(0,1)\}$ eigenbasis of X=$\{(1,1),(1,-1)\}$

$$\mathrm{U} = 1/\sqrt{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   Note) If you did not take complex conjugate values in the inner product, -1.

5. Simultaneous diagonalizable $\Rightarrow$ [A,B] = 0 $\Rightarrow \mathrm{AB} = 0$ or at least one of the eigenvalues of either A or B must be zero.

   Note) This is the minimal condition.

   If you made a logical argument, but did not conclude with a statement relating A and B, -3.

   0 for a false answer.

6.

   a)

   $\mathrm{i)\ X} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

   $\mathrm{X}^{\mathrm{n}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (-1)^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

   $\mathrm{e}^{\mathrm{iX}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sum \frac{i^n}{n!} & 0 \\ 0 & \sum \frac{(-i)^n}{n!} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} cos1 & isin1 \\ isin1 & cos1 \end{bmatrix}$

   ii) Likewise, $\mathrm{e}^{\mathrm{iZ}} = \begin{bmatrix} e^i & 0 \\ 0 & e^{-i} \end{bmatrix}$

iii) $X\sin\theta + Z\cos\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$

$=> \exp(i(X\sin\theta + Z\cos\theta)) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$

b)

i) $e^{iX}e^{iZ}$ can be calculated from a)

ii) $X + Z = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}$

$=> \exp(i(X+Z)) = \begin{bmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix} \begin{bmatrix} e^{i\sqrt{2}} & 0 \\ 0 & e^{-i\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}$

It can be shown that $e^{iX}e^{iZ}$ and $e^{i(X+Z)}$ are not identical.

Note) If you gave an answer for only one of the two matrices $e^{iX}e^{iZ}$ or $e^{i(X+Z)}$, then -3.5.

If you obtained the correct matrices, but did not compare whether they are identical or not, -2.