Prob 1.

(A) We can write $\hat{A} = \sum_j \lambda_j |e_j\rangle\langle e_j|$ & $|\psi\rangle = \sum_i c_i |e_i\rangle$, where $c_i = \langle e_i|\psi\rangle$,

as eigenvectors $\{|e_j\rangle_j\}$ form orthonormal basis of Hilbert space

Therefore,

$$\langle\psi|\hat{A}|\psi\rangle = \sum_{ijk} c_i^* c_k \langle e_i|e_j\rangle\langle e_j|e_k\rangle \lambda_j = \sum_{ijk} c_i^* c_k \delta_{ij}\delta_{jk}\lambda_j$$

$$= \sum_i |c_i|^2 \lambda_j = \underline{\sum_i P(\lambda_j)\lambda_j} = \langle A\rangle.$$

According to Copenhagen Interpretation,

$|c_j|^2$ is understood as the probability of finding $|\psi\rangle$ in $|e_j\rangle$, which gives λ_j as the measurement result and $|e_j\rangle$ as the collapsed state. Therefore, $\langle\psi|\hat{A}|\psi\rangle = \langle A\rangle$.

(B) If we change a basis to another one with Unitary Matrix, the operator becomes $\underline{UAU^\dagger}$.

$\Rightarrow tr(UAU^\dagger) = tr(U^\dagger UA) = tr(A) \Rightarrow$ trace is invariant under Unitary transformation.

($\because tr(AB) = tr(BA)$)

Prob. 2.

$$|\psi\rangle_{init} = |\psi\rangle_{ABCD} = \frac{(|00\rangle_{AB} + |11\rangle_{AB})(|00\rangle_{CD} + |11\rangle_{CD})}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle\right)_{ABCD}.$$

As $|00\rangle = \frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}}$, $|11\rangle = \frac{|\Phi^+\rangle - |\Phi^-\rangle}{\sqrt{2}}$, $|01\rangle = \frac{|\Psi^+\rangle + |\Psi^-\rangle}{\sqrt{2}}$, $|10\rangle = \frac{|\Psi^+\rangle - |\Psi^-\rangle}{\sqrt{2}}$,

$$|\psi\rangle_{ABCD} = \frac{1}{2}\Big(|00\rangle_{AD}\frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}}_{BC} + |01\rangle_{AD}\Big(\frac{|\Psi^+\rangle + |\Psi^-\rangle}{\sqrt{2}}\Big)_{BC} + |10\rangle_{AD}\Big(\frac{|\Psi^+\rangle - |\Psi^-\rangle}{\sqrt{2}}\Big)_{BC}$$
$$+ |11\rangle_{AD}\frac{|\Phi^+\rangle - |\Phi^-\rangle}{\sqrt{2}}_{BC}\Big)$$
$$= \frac{1}{2}\Big(|\Phi^+\rangle_{BC}\Big(\frac{|00\rangle_{AD} + |11\rangle_{AD}}{\sqrt{2}}\Big) + |\Phi^-\rangle_{BC}\Big(\frac{|00\rangle_{AD} - |11\rangle_{AD}}{\sqrt{2}}\Big) + |\Psi^+\rangle_{BC}\Big(\frac{|01\rangle_{AD} + |10\rangle_{AD}}{\sqrt{2}}\Big)$$
$$+ |\Psi^-\rangle_{BC}\Big(\frac{|01\rangle_{AD} - |10\rangle_{AD}}{\sqrt{2}}\Big)\Big) = \underline{\frac{1}{2}\Big(|\Phi^+\rangle_{BC}|\Phi^+\rangle_{AD} + |\Phi^-\rangle_{BC}|\Phi^-\rangle_{AD} + |\Psi^+\rangle_{BC}|\Psi^+\rangle_{AD} + |\Psi^-\rangle_{BC}|\Psi^-\rangle_{AD}\Big)}$$

Therefore, if we measure BC in Bell basis,

Qubit AD collapses to the same Bell basis.

Prob. 3. (1) Char. eq. $\det(A-\lambda I)=0$

$$\Rightarrow \det\begin{pmatrix} 2-\lambda & i & 1 \\ -i & 1-\lambda & i \\ 1 & -i & 2-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda)(2-\lambda)-1-1-(1-\lambda)-(2-\lambda)-(2-\lambda)$$

$$= -\lambda^3+5\lambda^2-8\lambda+4-7+3\lambda$$

$$= -\lambda^3+5\lambda^2-5\lambda-3 = -(\lambda-3)(\lambda^2-2\lambda-1)=0$$

$\Rightarrow \lambda_1=3, \quad \lambda_2 = 1+\sqrt{2}, \quad \lambda_3 = 1-\sqrt{2}.$

(2) eigenvectors.

① $\lambda_1=3 \Rightarrow \begin{pmatrix} -1 & i & 1 \\ -i & -2 & i \\ 1 & -i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \to (e_1)$

② $\lambda_2 = 1+\sqrt{2} \Rightarrow \begin{pmatrix} 1-\sqrt{2} & i & 1 \\ -i & -\sqrt{2} & i \\ 1 & -i & 1-\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{cases} (3-2\sqrt{2})a + (1-\sqrt{2})ib + (1-\sqrt{2})c = 0 \\ a - ib + (1-\sqrt{2})\cancel{c=0} \\ a - \sqrt{2}ib - c = 0 \end{cases} \Rightarrow (\sqrt{2}+1)ib = (\sqrt{2}-2)c, \quad \begin{cases} (2-2\sqrt{2})a + (2-\sqrt{2})ib = 0 \\ \end{cases}$

$\Rightarrow \quad a=1, \quad b = i\cdot\frac{2-2\sqrt{2}}{2-\sqrt{2}} = i\cdot(\sqrt{2}-2)(\sqrt{2}+1) = -\sqrt{2}i, \quad c=-1$

$\Rightarrow \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix} \to (e_2)$

③ $\lambda_3 = 1-\sqrt{2} \Rightarrow \begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix} \to (e_3)$

(3) diagonalize. $\Rightarrow \quad D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & & \\ & 1+\sqrt{2} & \\ & & 1-\sqrt{2} \end{pmatrix}$

Diagonalize X $\Rightarrow$ -2

Prob 4.

(A) $\exists$ eigenvector set & eigenvalues set $\{|e_j\rangle\}_j$ for both $\hat{A}$ & $\hat{B}$.

$\{\lambda_j^A\}_j, \{\lambda_j^B\}_j$

$$\Rightarrow \hat{A} = \sum_j \lambda_j^A |e_j\rangle\langle e_j| , \quad \hat{B} = \sum_j \lambda_j^B |e_j\rangle\langle e_j|$$

$$\Rightarrow \hat{A}\hat{B} - \hat{B}\hat{A} = \sum_{ij} \left(\lambda_i^A \lambda_j^B |e_i\rangle\langle e_i|e_j\rangle\langle e_j| - \lambda_i^B \lambda_j^A |e_i\rangle\langle e_i|e_j\rangle\langle e_j| \right)$$

$$= \sum_{ij} \lambda_i^A \lambda_j^B \delta_{ij} |e_i\rangle\langle e_j| - \lambda_i^B \lambda_j^A \delta_{ij} |e_i\rangle\langle e_j|$$

$$= \sum_i (\lambda_i^A \lambda_i^B - \lambda_i^B \lambda_i^A) |e_i\rangle\langle e_i| = 0. \Rightarrow \text{They commute.}$$

(B) Define eigenvector set of $\begin{cases} A \\ B \end{cases}$ as $\begin{matrix} \{|e_j\rangle_A\}_j = \beta_A \\ \{|e_j\rangle_B\}_j = \beta_B \end{matrix}$

& corresponding eigenvalue set as $\begin{cases} \{\lambda_j^A\}_j \\ \{\lambda_j^B\}_j \end{cases}$

$$\Rightarrow AB|e_j\rangle_A = BA|e_j\rangle_A = B\lambda_j^A|e_j\rangle_A = \lambda_j^A (B|e_j\rangle_A)$$

$\Rightarrow B|e_j\rangle_A \in \beta_A$ with λ_j^A $\Rightarrow$ Only when $B|e_j\rangle_A = \alpha|e_j\rangle_A$ as there is no degeneracy.

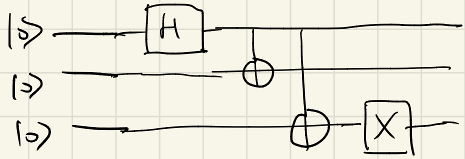
$\Rightarrow \underline{|e_j\rangle_A \in \beta_B}$ $\Rightarrow \beta_A \subset \beta_B$

In the same way, $\beta_B \subset \beta_A \Rightarrow \beta_A = \beta_B$ ☒

If you assumed the same dimension, -1.

Not showing reverse case : -1.

Prob. 5.

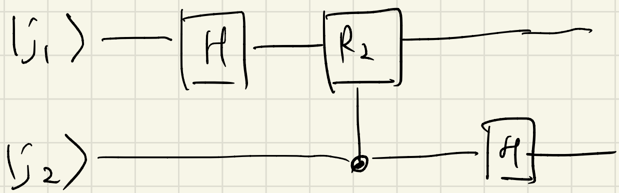


Prob 6.

$$y[n] = \frac{1}{\sqrt{4}}\left(1 \cdot e^{0} + 2 \cdot e^{i\frac{2\pi \cdot 2 \cdot n}{4}}\right) = \frac{1}{2}\left(1 + 2 \cdot (-1)^{n}\right)$$

$\Rightarrow$ $y[0] = \frac{3}{2}$, $y[1] = -\frac{1}{2}$, $y[2] = \frac{3}{2}$, $y[3] = -\frac{1}{2}$.

Prob. 7. (1) SWAP X.



$$\text{Matrix} = (I_1 \otimes H_2)\cdot(\text{Controlled} - R_2)\,(H_1 \otimes I_2)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2\pi i/4} \end{pmatrix}\begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & i \\ & & 1 & -i \end{pmatrix}\begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \\ 1 & -i & -1 & i \end{pmatrix}$$

(2) SWAP O.

$$\begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \\ 1 & -i & -1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Qubit 1,2 위치를 다르게 하면 transpose.

Circuit 맞다: 2점.
Controlled - R2 를 안다: +2점
Approach 좋았다: +2점.
계산 실수: -2점.

최대 점수: 6점 (구하지 못했을 때)

Prob. 8.

$$|(2)\rangle_{012345} = \alpha|00000z\rangle + \beta|01010z\rangle + \gamma|10100z\rangle + \delta|11101z\rangle$$

$$|(3)\rangle_{012345} = \alpha|00000z\rangle + \beta|01010z\rangle + \gamma|10100z\rangle + \delta|11101\bar{z}\rangle$$

$$|(4)\rangle_{012345} = \alpha|00000z\rangle + \beta|01010z\rangle + \gamma|10100z\rangle + \delta|11110\bar{z}\rangle$$

$$|(5)\rangle_{012345} = \alpha|00000z\rangle + \beta|01000z\rangle + \gamma|10000z\rangle + \delta|11000\bar{z}\rangle$$

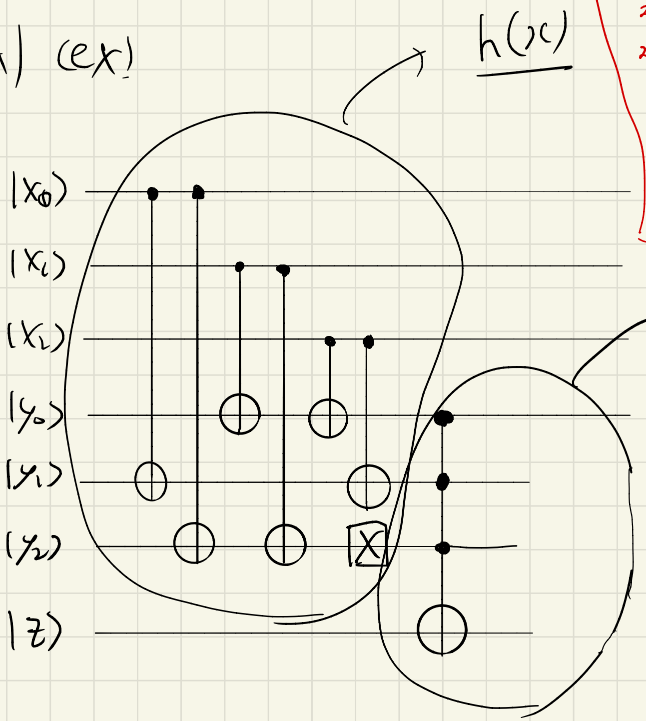
$$= \alpha|00\rangle|000\rangle|z\rangle + \beta|01\rangle|000\rangle|z\rangle + \gamma|10\rangle|000\rangle|z\rangle + \delta|11\rangle|000\rangle|\bar{z}\rangle$$

$$= \left(\alpha|00z\rangle + \beta|01z\rangle + \gamma|10z\rangle + \delta|11\bar{z}\rangle\right)_{015}|000\rangle_{234}.$$

번호랑 3점,

Prob. 9.

(A) ex)



h는 잘 그렸으나 f 실패 → 7점
진리표가 해 하나만 포함 + Circuit 잘그림 → 7점
진리표가 해 포함 X + Circuit 잘그림 → 5점
진리표가 있는데 Circuit X → 5점
진리표 없이 Circuit 잘못그림 → 0점
h를 안 만듦 → 5점

$|z\rangle$ —— $|z \oplus f(x)\rangle$ 이걸 빠뜨려 -1.

1 only when

$\Rightarrow y_1 y_2 \bar{y}_3$

$(\Rightarrow y_1 y_2 y_3 = 110)$

(진리표 없으면서 하면 4점)
M=1로 풀면 3점

(B) $x = 110, 001 \Rightarrow y = 110$

$\Rightarrow N = 8 \;\&\; M = 2.$

$\Rightarrow$ # of operation required

$$= \frac{\arccos\left(\sqrt{\frac{2}{8}}\right)}{\frac{\pi}{3}} = \frac{3}{\pi}\cdot\frac{\pi}{3} = 1.$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{8-2}{8}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$