Summary of previous lecture

- Schmidt decomposition
  - If $|\psi_{AB}\rangle$ is a pure state of a composite system AB, it can be always represented as $|\psi_{AB}\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$ where $|i_A\rangle$ and $|i_B\rangle$ are set of orthonormal states in corresponding vector space
  - Singular value decomposition
  - Limitation: applicable only to a pure state & bipartite system
- Determination of entangled state vs. separable state
  - Partial trace
  - Schmidt number ⇔ equivalent to a rank of the coefficient (amplitude) matrix
- Purification
  - Through an introduction of fictitious system R, a mixed state ρ^A of system A can be written in a form of pure state $|AR\rangle \equiv \sum_i \sqrt{p_i}|i^A\rangle|i^R\rangle$
  - Note that this is completely different from entanglement purification or entanglement distillation

Fidelity and Noise

- Fidelity (양자 충실도 또는 양자 신뢰도)
  - Measure of the similarity between two quantum state
  - Fidelity between two pure states $|\psi\rangle$ and $|\phi\rangle$
    - If two quantum states $|\psi\rangle$ and $|\phi\rangle$ are pure, then $F=\sqrt{|\langle\psi|\phi\rangle|^2}$ will tell us the overlap between the two states.
    - $F=\sqrt{|\langle\psi|\phi\rangle|^2}=\sqrt{\langle\psi|\phi\rangle\langle\phi|\psi\rangle}=\sqrt{\mathrm{tr}\big((|\psi\rangle\langle\psi|)(|\phi\rangle\langle\phi|)\big)}=\sqrt{\mathrm{tr}\big(\rho_{|\psi\rangle}\cdot\rho_{|\rho\rangle}\big)}$
  - Fidelity between a pure state $|\psi\rangle$ and a mixed state ρ
    - $F=\sqrt{\mathrm{tr}(|\psi\rangle\langle\psi|\rho)}=\sqrt{\langle\psi|\rho|\psi\rangle}$
  - Fidelity between two density matrices σ,ρ
    - $F\equiv\mathrm{tr}\sqrt{\sigma^{1/2}\rho\sigma^{1/2}}$
    - When applied to pure state $|\psi\rangle$ and mixed state ρ: $\sigma^{1/2}=|\psi\rangle\langle\psi|$

$$\Rightarrow\ F=\mathrm{tr}\sqrt{|\psi\rangle\langle\psi|\rho|\psi\rangle\langle\psi|}=\sqrt{\langle\psi|\rho|\psi\rangle}\mathrm{tr}\sqrt{|\psi\rangle\langle\psi|}=\sqrt{\langle\psi|\rho|\psi\rangle}$$

Fidelity and Noise

- Review of classical error analysis
  - Error probability of single bit: p
  - For a simple encoding scheme: 0➔ 000, 1➔ 111
  - The probability that two or more bits are flipped is
    $$p_e = 3p^2(1-p) + p^3 = 3p^2 - 2p^3$$

- Assume the original state is $|\psi\rangle$. After this quantum state go through a noisy channel, its quantum state can be described as
  $$\rho = (1-p)|\psi\rangle\langle\psi| + p(X|\psi\rangle)(\langle\psi|X)$$
- Fidelity of the quantum state after the noisy channel:
  - $F_1 = \sqrt{\langle\psi|\rho|\psi\rangle} = \sqrt{(1-p) + p\langle\psi|X|\psi\rangle\langle\psi|X|\psi\rangle} \geq \sqrt{(1-p)}$
- After the quantum error correction:
  - $\rho = [(1-p)^3 + 3p(1-p)^2]|\psi\rangle\langle\psi| + \cdots$
  - $F_2 = \sqrt{\langle\psi|\rho|\psi\rangle} \geq \sqrt{(1-p)^3 + 3p(1-p)^2} = \sqrt{1 - 3p^2 + 2p^3} > F_1$

Shannon entropy

- Section 11.1
- When a random variable X is given, the Shannon entropy of X quantifies how much information we gain, when we learn the value of X.
- Equivalently, the entropy of X measures the amount of *uncertainty* about X before we learn its value.
- When the probability of obtaining X is p_X, entropy is a function of a probability distribution, $p_1, \dots, p_n$. The Shannon entropy associated with this probability distribution is defined by

$$H(X) \equiv H(p_1, \dots, p_n) \equiv -\sum_x p_x \log p_x$$

where "log" are taken to base two, and when $p_x = 0$, $0 \cdot \log 0 \equiv 0$.

Shannon entropy

- $H(X) \equiv H(p_1, \ldots, p_n) \equiv -\sum_x p_x \log p_x$
- Quantifies the resources needed to store or transmit information
- Example
  - If Alice always sends 0100100101_2 to Bob, how much information is being transmitted?
  - If Alice sends either 0100100101_2 or 0111101011_2 to Bob, how much information is being transmitted?
  - An information source produces one of four symbols, 1, 2, 3, or 4.
    - What is the Shannon entropy when each symbol has the same probability?
    - $H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = -4 \times \left(\frac{1}{4} \log_2\left(\frac{1}{4}\right)\right) = 2$ bits

Shannon entropy

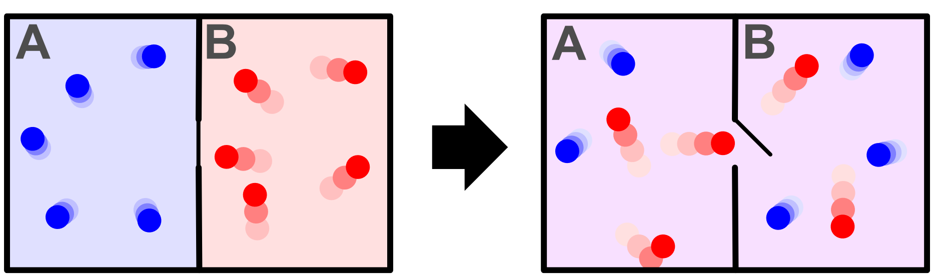
- $H(X) \equiv H(p_1, \ldots, p_n) \equiv -\sum_x p_x \log p_x$
- Quantifies the resources needed to store or transmit information
- Example
  - What is the Shannon entropy when symbol 1 is produced with probability of 1/2, symbol 2 with probability of 1/4, and the symbols 3 and 4 both with probability of 1/8 ?
    - $H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) = -\left(\frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{1}{4}\log_2\left(\frac{1}{4}\right) + 2 \cdot \frac{1}{8}\log_2\left(\frac{1}{8}\right)\right) = \frac{2+2+3}{4} = \frac{7}{4}$
    - Encoding: symbol 1 ➔ 1_2, symbol 2 ➔ 01_2, symbol 3 ➔ 001_2, symbol 4 ➔ 000_2
    - Average length of the compressed string: $\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 = \frac{7}{4}$
  - How many bits are necessary to transmit $2^{10} = 1{,}024$ bits of information when each bit has a probability of $p_1 = 1/2^{10}$ and $p_0 = 1 - 1/2^{10}$?
    - Approximation: among 1,024 bits, only 1 bit will be 1. ➔ can be represented by 10 bits
    - $H\left(1 - \frac{1}{2^{10}}, \frac{1}{2^{10}}\right) = -\left\{\left(1 - \frac{1}{2^{10}}\right)\log_2\left(1 - \frac{1}{2^{10}}\right) + \frac{1}{2^{10}}\log_2\left(\frac{1}{2^{10}}\right)\right\} \approx -1.44\left(-\frac{1}{2^{10}}\right) - \frac{-10}{2^{10}}$ where $\log_2(1+x) \approx \frac{x - x^2/2}{\ln 2}$
    - Average bits necessary: $2^{10} \times H\left(1 - \frac{1}{2^{10}}, \frac{1}{2^{10}}\right) \approx 1.4 + 10 = 11.4$

Von Neumann entropy

- Section 11.3
- When a quantum state is given in terms of density operator ρ, von Neumann entropy is defined as
$$S(\rho) \equiv -\mathrm{tr}(\rho \log \rho)$$
where "$\log$" are taken to base two.
- If λ_x are the eigenvalues of ρ, then $S(\rho) = -\sum_x \lambda_x \log \lambda_x$
- Example
  - The completely mixed density operator in a d-dimensional space, I/d, has entropy $\log d$
  - What is the entropy of pure state?
  - What is the entropy of Bell state?

Entropy in Physics

- Statistical mechanics
  - $S = k_B \ln \Omega$ where k_B is Boltzmann's constant and Ω is the number of microstates and the probability of certain microstate to occur is equal.
- Thermodynamics
  - Second law of thermodynamics
    - The total entropy of an isolated system can never decrease over time.
    - The total entropy of a system and its surroundings can remain constant in ideal cases where the system is in thermodynamic equilibrium, or is undergoing a (fictive) reversible process.
    - In all processes that occur, including spontaneous processes, the total entropy of the system and its surroundings increases and the process is irreversible in the thermodynamic sense.
    - For example, heat always flows spontaneously from hotter to colder bodies, and never the reverse.
  - $dS \geq dQ/T$ when a small amount of energy dQ is introduced into the system at a certain temperature T.



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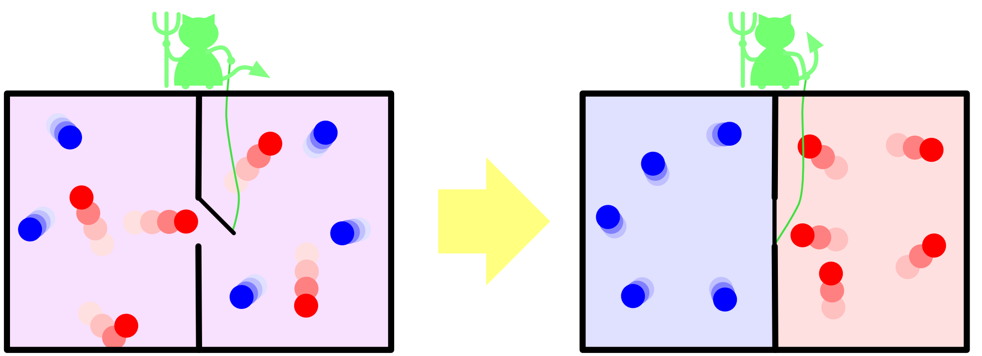
Energy and computation

- Section 3.2.5
- What is the fundamental energy requirement for computation?
- Landauer's principle
  - Suppose a computer erases a single bit of information. The amount of energy dissipated into the environment is at least $k_B T \ln 2$, where k_B is Boltzmann's constant, and T is the temperature of the environment of the computer.
- Computers these days dissipate roughly $500 k_B T \ln 2$ (Joule) in energy for each elementary logical operations.
- Can we develop a circuit which does not consume any energy in principle?

Maxwell's demon

- In 1871, James Clark Maxwell proposed the existence of a machine that apparently violated the second law of thermodynamics.
  - He envisioned a demon which can reduce the entropy of a gas cylinder initially at equilibrium by individually separating the fast and slow molecules into the two halves of the cylinder.
  - This demon sits at a little door at the middle partition, and when a fast molecule approaches from the left side the demon opens a door between the partitions, allowing the molecule through, and then closes the door.
  - By doing this many times, the total entropy of the cylinder can be decreased.
- The result of velocity measurement should be stored in some memory, but because any memory is finite, the demon must eventually begin erasing information from the memory. This act of erasing information increases entropy of the combined system – demon, gas cylinder, and their environment.



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