Revisit of Postulate 2

- Recall Postulate 2: the evolution of a closed quantum system is described by a unitary transformation
  - $|\psi\rangle\ at\ t_1 \xrightarrow{\text{unitary transformation}} |\psi'\rangle\ at\ t_2$
- Postulates of quantum mechanics does not tell us how the open system will evolve ➔ We need to guess from the given postulates. ➔ Density Matrix

Density matrix

- Section 2.4 The density operator
- When two particles are entangled, if we measure one of the particles but don't know the measurement result, how can we represent the quantum state of the other particle?
  - $|\psi^-\rangle = [|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B]/\sqrt{2}$
  - If A measures $|0\rangle_A$, B remains in $|1\rangle_B$ state.
  - If A measures $|1\rangle_A$, B remains in $|0\rangle_B$ state.
  - From the above state $|\psi^-\rangle$, we know that $|0\rangle_B$ or $|1\rangle_B$ will remain with 50% of probability.
  - $\rho_B = \frac{1}{2}|0\rangle_{BB}\langle 0| + \frac{1}{2}|1\rangle_{BB}\langle 1| = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$
  - What happens if A was measured in $|D\rangle_A$ & $|A\rangle_A$ basis?
  - $\rho_B = \frac{1}{2}|A\rangle_{BB}\langle A| + \frac{1}{2}|D\rangle_{BB}\langle D| = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$
  - The same result will be obtained with measurements in other basis or even without any measurements.

Ensembles of quantum states

- Section 2.4.1
- Suppose a quantum system is in one of a number of states $|\psi_i\rangle$, where i is an index, with respective probabilities p_i.
- $\{p_i, |\psi_i\rangle\}$ is called an *ensemble* of *pure states.*
- The density operator (or density matrix) is defined as

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i|.$$

- Example
  - 90% of $|0\rangle$ and 10% of $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$
  - $\rho = \frac{9}{10}|0\rangle\langle 0| + \frac{1}{10}|+\rangle\langle +| = \frac{19|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|}{20} = \begin{bmatrix} \frac{19}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} \end{bmatrix}$

Reformulation of postulate 2

- Reformulation of all the postulates in terms of density matrix
- Postulate 2
  - Example: 90% of $|0\rangle$ and 10% of $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$
    $|0\rangle \xrightarrow{U} U|0\rangle$, $|+\rangle \xrightarrow{U} U|+\rangle$ ➔ $\rho_{final} = \frac{9}{10}(U|0\rangle)(\langle 0|U^\dagger) + \frac{1}{10}(U|+\rangle)(\langle +|U^\dagger)$
  - Initial state $|\psi_i\rangle$ with probability of p_i ➔ $U|\psi_i\rangle$ with the same probability of p_i

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \xrightarrow{U} \sum_i p_i (U|\psi_i\rangle\langle\psi_i|U^\dagger) = U\rho U^\dagger$$

Reformulation of postulate 3

- Postulate 3
  - Example: 90% of $|0\rangle$ and 10% of $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$. What is the probability of measuring 0?
    - $p(0|0) = \langle 0|M_0^\dagger M_0|0\rangle = 1$, $p(0|+) = \langle +|M_0^\dagger M_0|+\rangle = \frac{1}{2}$.
    - Total probability is $0.9 \times 1 + 0.1 \times \frac{1}{2} = 0.95$
  - If initial state was $|\psi_i\rangle$, the probability of getting result m is $p(m|i) = \langle \psi_i|M_m^\dagger M_m|\psi_i\rangle = \mathrm{tr}\left(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|\right)$
    - Note $\langle\beta|\alpha\rangle = \langle\beta|\left(\sum_{j=1}^n |j\rangle\langle j|\right)|\alpha\rangle = \sum_{j=1}^n \langle j|\alpha\rangle\langle\beta|j\rangle = \mathrm{tr}(|\alpha\rangle\langle\beta|)$

$$p(m) = \sum_i p_i \cdot p(m|i) = \sum_i p_i \mathrm{tr}\left(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|\right) = \mathrm{tr}\left(M_m^\dagger M_m \rho\right)$$

  - Example: from previous page, $\rho = \begin{bmatrix} \frac{19}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} \end{bmatrix}$.

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow p(m) = \mathrm{tr}\left(M_m^\dagger M_m \rho\right) = \mathrm{tr}\left(\begin{bmatrix} \frac{19}{20} & 0 \\ 0 & 0 \end{bmatrix}\right) = 0.95$$

Reformulation of postulate 3

- Postulate 3
  - From the previous page,
    - if initial state was $|\psi_i\rangle$, the probability of getting result m is $p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle$.
    - $p(m) = \mathrm{tr}\left(M_m^\dagger M_m \rho\right)$.
  - After measurement of $|\psi_i\rangle$ with result m,

➔ $|\psi_i^m\rangle = \dfrac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$ with probability of $p(i|m)$

➔ $\rho_m = \sum_i p(i|m)|\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(i|m)\dfrac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}$

➔ By elementary probability theory, $p(i|m) = \dfrac{p(m,i)}{p(m)} = p(m|i)p_i/p(m)$

➔ $\rho_m = \sum_i p_i \dfrac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\mathrm{tr}\left(M_m^\dagger M_m\rho\right)} = \dfrac{M_m\rho M_m^\dagger}{\mathrm{tr}\left(M_m^\dagger M_m\rho\right)}$

Ensembles of quantum states

- A quantum system whose state $|\psi\rangle$ is known exactly is said to be in a pure state ➔ $\rho = |\psi\rangle\langle\psi|$
- Otherwise, ρ is in a mixed state. Or mixture of the different pure states in the ensemble.
- $\mathrm{tr}(\rho^2) = \begin{cases} 1 & \Rightarrow \text{Pure state} \\ < 1 & \Rightarrow \text{Mixed state} \end{cases}$
- Mixture of mixed states
  - A quantum state is prepared in the state ρ_i with probability p_i for $i = 1 \cdots n \overset{?}{\rightarrow} \rho = \sum_{i=1}^{n} p_i \rho_i$
  - Proof
    - ρ_i will arise from some ensemble $\{p_{ij}, |\psi_{ij}\rangle\}$ of pure states ➔ $\{p_1 \cdot p_{1j}, |\psi_{1j}\rangle\}$, $\{p_2 \cdot p_{2j}, |\psi_{2j}\rangle\}$, …, $\{p_n \cdot p_{nj}, |\psi_{nj}\rangle\}$
    - $\rho = \sum_{i=1}^{n} \sum_{j=1}^{m_i} p_i \cdot p_{ij} |\psi_{ij}\rangle\langle\psi_{ij}| = \sum_{i=1}^{n} p_i \sum_{j=1}^{m_i} p_{ij} |\psi_{ij}\rangle\langle\psi_{ij}| = \sum_{i=1}^{n} p_i \rho_i$