Summary of previous lecture

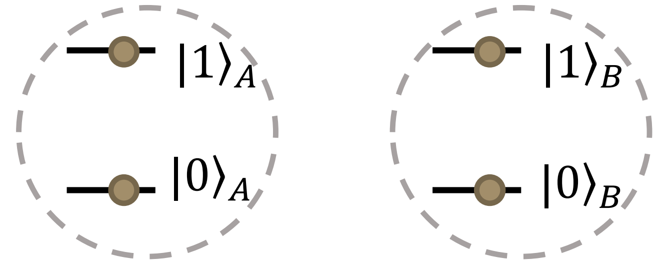
- Polarization measurement
  - Classical or quantum?
  - Consider the case when there is only single incoming photon to the measurement device
- Uncertainty principle
  - Measurements corresponding to two commuting Hermitian operators ➔ order of the measurements does not matter
- Quantum bits
  - $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$
  - Bloch sphere
- Single-qubit gates
  - X, Y, Z, Hadamard
  - Corresponds to a rotation on the Bloch sphere

Multiple qubits

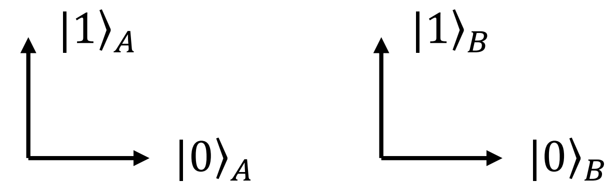
- Section 1.2.1
- Assume there are two qubits:
  - In principle, we can treat the quantum states of each qubit separately: $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$, $|\psi\rangle_B = \alpha|0\rangle_B + \beta|1\rangle_B$
  - However, if there is any interaction between two qubits, we cannot treat each qubit separately. For example, if there is an operation which flips $0_B \rightarrow 1_B$ and $1_B \rightarrow 0_B$ when qubit A is 1_A, we need to consider the situation for $0_A 0_B$, $0_A 1_B$, $1_A 0_B$, $1_A 1_B$.

➔ Overall quantum state should be represented as a single vector in a Hilbert space.

  - $|\psi\rangle_{AB} = \alpha|0\rangle_A|0\rangle_B + \beta|0\rangle_A|1\rangle_B + \gamma|1\rangle_A|0\rangle_B + \delta|1\rangle_A|1\rangle_B$



Qubit A   Qubit B



Controlled-NOT gate

Input		Output	
A	B	A	B
0_A	0_B	0_A	0_B
0_A	1_B	0_A	1_B
1_A	0_B	1_A	1_B
1_A	1_B	1_A	0_B

Tensor product

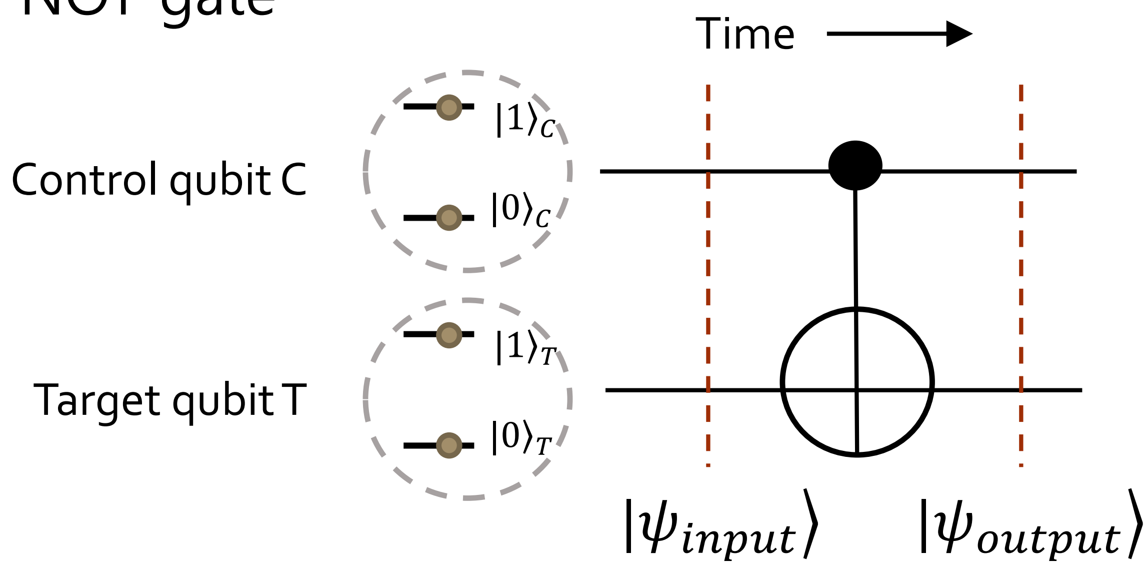
- Section 2.1.7
- Example: $|\psi\rangle_{AB} = \alpha|0\rangle_A \otimes |0\rangle_B + \beta|0\rangle_A \otimes |1\rangle_B + \gamma|1\rangle_A \otimes |0\rangle_B + \delta|1\rangle_A \otimes |1\rangle_B$
- Suppose $\mathbb{V}$ and $\mathbb{W}$ are Hilbert spaces of dimension m and n, respectively. Then $\mathbb{V} \otimes \mathbb{W}$ creates a new Hilbert space of dimension mn.
- The elements of $\mathbb{V} \otimes \mathbb{W}$ are linear combinations of 'tensor products' $|v\rangle \otimes |w\rangle$ of elements $|v\rangle$ of $\mathbb{V}$ and $|w\rangle$ of $\mathbb{W}$.
- Basis of $\mathbb{V} \otimes \mathbb{W}$: $|i\rangle \otimes |j\rangle$ when $|i\rangle$ and $|j\rangle$ are orthonormal basis of space $\mathbb{V}$ and $\mathbb{W}$, respectively.

- Linear operator
  - Assume A and B are linear operators on $\mathbb{V}$ and $\mathbb{W}$, respectively, then we can define a linear operator $A \otimes B$ on $\mathbb{V} \otimes \mathbb{W}$ by the equation, $A \otimes B(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle$
  - Linearity of $A \otimes B$: $A \otimes B(\sum_i a_i|v_i\rangle \otimes |w_i\rangle) \equiv \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$
- Simplified notations
  - $|v\rangle \otimes |w\rangle$ ➔ $|v\rangle|w\rangle$ or $|v, w\rangle$ or $|vw\rangle$
  - Example: $|0\rangle_A \otimes |1\rangle_B$ ➔ $|0\rangle_A|1\rangle_B$ or $|0_A, 1_B\rangle$ or $|0,1\rangle_{AB}$ or $|01\rangle$ etc.

Tensor product

- Example: Controlled-NOT gate

Input		Output	
C	T	C	T
0_C	0_T	0_C	0_T
0_C	1_T	0_C	1_T
1_C	0_T	1_C	1_T
1_C	1_T	1_C	0_T



- When input qubits are $|\psi_{input}\rangle = |0\rangle_C|0\rangle_T$ ➔ $|\psi_{output}\rangle = |0\rangle_C|0\rangle_T$
- When input qubits are $|\psi_{input}\rangle = |1\rangle_C|0\rangle_T$ ➔ $|\psi_{output}\rangle = |1\rangle_C|1\rangle_T$
- When input qubits are $|\psi_{input}\rangle = (|0\rangle_C + |1\rangle_C)|0\rangle_T/\sqrt{2} = (|0\rangle_C|0\rangle_T + |1\rangle_C|0\rangle_T)/\sqrt{2}$

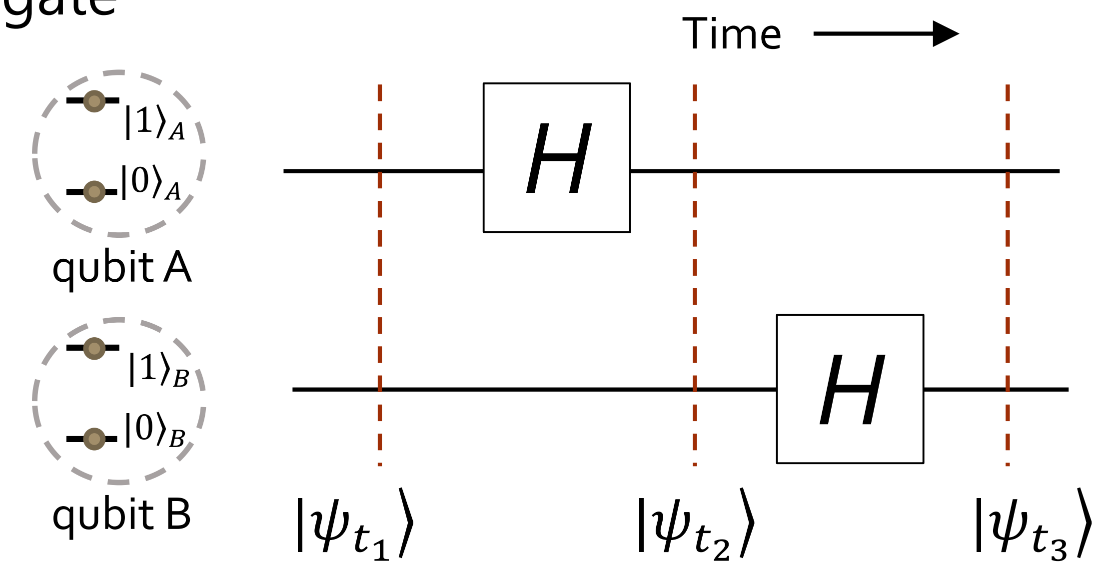
➔ $|\psi_{output}\rangle = (|0\rangle_C|0\rangle_T + |1\rangle_C|1\rangle_T)/\sqrt{2}$

⇔ Gate operation will be linear on different states. Why?

Tensor product

- Example: Hadamard gate

Input	Output
$\|0\rangle$	$(\|0\rangle + \|1\rangle)/\sqrt{2}$
$\|1\rangle$	$(\|0\rangle - \|1\rangle)/\sqrt{2}$



  - Assume that at t_1, $|\psi_{t_1}\rangle = |0\rangle_A |1\rangle_B$.
  - At t_2, only qubit A should be changed by Hadamard gate. How can we write such kind of situation in equation?
  - $|\psi_{t_2}\rangle = (H_A \otimes I_B)(|0\rangle_A \otimes |1\rangle_B) = (H_A|0\rangle_A) \otimes (I_B|1\rangle_B) = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \otimes |1\rangle_B$
  - $|\psi_{t_3}\rangle = (I_A \otimes H_B)|\psi_{t_2}\rangle = \left(I_A \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right) \otimes (H_B|1\rangle_B) = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \otimes \frac{|0\rangle_B - |1\rangle_B}{\sqrt{2}}$
  - Of course, this operation can be written as $H_A \otimes H_B$:

$$|\psi_{t_3}\rangle = (H_A \otimes H_B)(|0\rangle_A \otimes |1\rangle_B) = (H|0\rangle_A) \otimes (H|1\rangle_B) = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \otimes \frac{|0\rangle_B - |1\rangle_B}{\sqrt{2}}$$

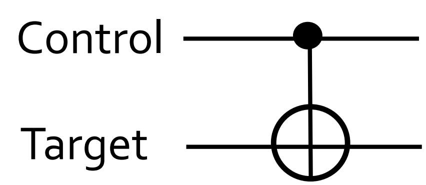
Tensor product

- Matrix representation (also known as Kronecker product)
  - When A is an $m \times n$ matrix and B is an $p \times q$ matrix,
  - $$A \otimes B \equiv \overbrace{\begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix}}^{nq \text{ columns}} \left.\vphantom{\begin{bmatrix} A \\ A \\ \vdots \\ A \end{bmatrix}}\right\} mp \text{ rows}$$
  - Examples
    - $$\begin{bmatrix}1\\2\end{bmatrix} \otimes \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix} 1 \times \begin{bmatrix}2\\3\end{bmatrix} \\ 2 \times \begin{bmatrix}2\\3\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \times 2 \\ 1 \times 3 \\ 2 \times 2 \\ 2 \times 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}$$
    - $X \otimes Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes Y = \begin{bmatrix} 0 \cdot Y & 1 \cdot Y \\ 1 \cdot Y & 0 \cdot Y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$ where $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Multiple qubit gates

- Classical NAND gate: universal gate in classical digital logic
  - Is classical XOR gate universal?
- Controlled-NOT gate
  - $|C,T\rangle \rightarrow |C,T\oplus C\rangle$ ➔ Unitary gate ⇔ invertible gate
  - $|0\rangle_C\langle 0| \otimes I_T + |1\rangle_C\langle 1| \otimes X_T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
  - Is $|0\rangle_C\langle 0| \otimes I_T + |1\rangle_C\langle 1| \otimes X_T$ unitary?

Input		Output	
Control	Target	Control	Target
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$



Tensor product

- Linear operator
  - Assume A and B are linear operators on $\mathbb{V}$ and $\mathbb{W}$, respectively, then we can define a linear operator $A \otimes B$ on $\mathbb{V} \otimes \mathbb{W}$ by the equation, $A \otimes B(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle$
  - Linearity of $A \otimes B$: $A \otimes B(\sum_i a_i |v_i\rangle \otimes |w_i\rangle) \equiv \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$
  - Linear combination of tensor products of operators: $C = \sum_j c_j A_j \otimes B_j$

  ➔ $\left(\sum_j c_j A_j \otimes B_j\right)|v\rangle \otimes |w\rangle \equiv \sum_j c_j A_j|v\rangle \otimes B_j|w\rangle$
  - Example: Controlled-NOT gate: $|0\rangle_A\langle 0| \otimes I_B + |1\rangle_A\langle 1| \otimes X_B$

Tensor product

- Definition of inner product in the tensor product space
  - $\left(\sum_i a_i^* \langle v_i| \otimes \langle w_i|\right)\left(\sum_j b_j |v'_j\rangle \otimes |w'_j\rangle\right) \equiv \sum_{i,j} a_i^* b_j \langle v_i|v'_j\rangle\langle w_i|w'_j\rangle$
  - $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|1\rangle \otimes |1\rangle$ are all orthonormal to each other.

- Notation
  - $|\psi\rangle^{\otimes 2} = |\psi\rangle \otimes |\psi\rangle$, $|\psi\rangle^{\otimes k} = |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle$
  - $H^{\otimes 2} = H \otimes H$, $H^{\otimes k} = H \otimes H \otimes \cdots \otimes H$
  - Frequently tensor product symbol $\otimes$ is omitted, and the tensor product state is written as $|v\rangle|w\rangle$, $|v, w\rangle$, or $|vw\rangle$ etc.
  - For example, $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ are computational basis