양자 컴퓨팅 및 정보의 기초 Homework #1

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1.a) 2개의 영벡터 $|0\rangle$, $|0'\rangle$이 있다고 가정하면

$|0\rangle = |0\rangle + |0'\rangle = |0'\rangle + |0\rangle = |0'\rangle$,

$\therefore |0\rangle = |0'\rangle$ 이므로 $|0\rangle$은 유일하다.

1.b) $0|v\rangle = (0+0)|v\rangle = 0|v\rangle + 0|v\rangle$

$\Rightarrow |0\rangle = 0|v\rangle - 0|v\rangle = 0|v\rangle + 0|v\rangle - 0|v\rangle = 0|v\rangle$,

$\therefore |0\rangle = 0|v\rangle$

1.c) 1.d)에 의해 $|v\rangle$의 역원은 $|-v\rangle$이고 유일하다.

$|v\rangle + (-|v\rangle) = (1-1)|v\rangle = |0\rangle$ 이므로

$-|v\rangle$는 $|v\rangle$의 역원이다.

$\therefore -|v\rangle = |-v\rangle$

1.d) 2개의 역원 $|-v\rangle$, $|-v'\rangle$이 있다고 가정하면

$|-v\rangle = |-v\rangle + (|v\rangle + |-v'\rangle) = (|-v\rangle + |v\rangle) + |-v'\rangle) = |-v'\rangle$,

$\therefore |-v\rangle = |-v'\rangle$ 이므로 $|-v\rangle$는 역원이고 유일하다.

2.a) (X) $\Rightarrow$ 덧셈에 대한 항등원 $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$이 존재하지 않는다.

2.b) (O) $\Rightarrow$ 덧셈과 상수배에 대해 닫혀있고,
덧셈의 항등원과 역원이 존재한다.

2.c) (X) $\Rightarrow$ 덧셈을 하면 (2,2)성분이 3이 아니게 되므로
덧셈에 대해 닫혀있지 않다.

2.d) (O) $\Rightarrow$ 덧셈과 상수배에 대해 닫혀있고
덧셈의 항등원과 역원이 존재한다.

2.e) (X) $\Rightarrow$ XOR 연산은 덧셈에 대한 항등원이 존재하지 않는다.

3. 2개의 representation $\sum_{i=1}^{n} v_i|i\rangle$, $\sum_{i=1}^{n} u_i|i\rangle$가 있다고 가정하면

$|0\rangle = v - v = \sum_{i=1}^{n}(v_i - u_i)|i\rangle$ 이므로

$1 \le i \le n$ 일 때 $v_i = u_i$ 이다.

따라서 v의 representation $\sum_{i=1}^{n} v_i|i\rangle$ 은 유일하다.

4. $A \circ B = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\cdot B_{ij}$, $Tr(A^TB) = \sum_{i=1}^{3} A^TB_{ii}$

+ $A^TB_{11} = \sum_{i=1}^{3} A_{i1}B_{i1}$, $A^TB_{22} = \sum_{i=1}^{3} A_{i2}B_{i2}$, $A^TB_{33} = \sum_{i=1}^{3} A_{i3}B_{i3}$,

$\therefore Tr(A^TB) = \sum_{i=1}^{3} A^TB_{ii} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}B_{ij} = A \circ B$

5. orthogonal vectors $\{|t_1\rangle, \cdots, |t_n\rangle\}$를 구하면

orthonormal vectors $\{|u_1\rangle, \cdots, |u_n\rangle\}$은 $|u_i\rangle = \dfrac{|t_i\rangle}{|t_i|}$ 이다.

+ $|t_1\rangle = |v_1\rangle$

$$|t_2\rangle = |v_2\rangle - \frac{\langle t_1|v_2\rangle}{\langle t_1|t_1\rangle}|t_1\rangle$$

$\vdots$

$$|t_n\rangle = |v_n\rangle - \sum_{i=1}^{n-1}\frac{\langle t_i|v_n\rangle}{\langle t_i|t_i\rangle}|t_i\rangle$$

orthogonal 증명)

i) $n=1$ 일 때 $\{|t_1\rangle\}$은 자명히 orthogonal 하다.

ii) $n=k$일 때 $\{|t_1\rangle, \cdots, |t_k\rangle\}$가 orthogonal 하다고 가정하고

iii) $n=k+1$일 때 $1 \le j \le k$인 임의의 j에 대해

$$\langle t_j|t_{k+1}\rangle = \langle t_j|v_{k+1}\rangle - \sum_{i=1}^{k}\frac{\langle t_i|v_{k+1}\rangle}{\langle t_i|t_i\rangle}\langle t_j|t_i\rangle$$

$$= \langle t_j|v_{k+1}\rangle - \frac{\langle t_j|v_{k+1}\rangle}{\cancel{\langle t_j|t_j\rangle}}\cancel{\langle t_j|t_j\rangle} = 0,$$

$\therefore \{|t_1\rangle, \cdots, |t_n\rangle\}$ = orthogonal → $\{|u_1\rangle, \cdots, |u_n\rangle\}$ = orthonormal

5. $|v_1\rangle = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $|v_2\rangle = \begin{bmatrix}2\\3\\0\end{bmatrix}$, $|v_3\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix}$ 일 때

$|t_1\rangle = |v_1\rangle = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $|t_2\rangle = |v_2\rangle - \frac{\langle t_1|v_2\rangle}{\langle t_1|t_1\rangle}|t_1\rangle = \begin{bmatrix}2\\3\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\3\\-1\end{bmatrix}$,

$|t_3\rangle = |v_3\rangle - \frac{\langle t_1|v_3\rangle}{\langle t_1|t_1\rangle}|t_1\rangle - \frac{\langle t_2|v_3\rangle}{\langle t_2|t_2\rangle}|t_2\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{11}\begin{bmatrix}1\\3\\-1\end{bmatrix} = \frac{4}{11}\begin{bmatrix}-3\\2\\3\end{bmatrix}$,

$\therefore |u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$, $|u_2\rangle = \frac{1}{\sqrt{11}}\begin{bmatrix}1\\3\\-1\end{bmatrix}$, $|u_3\rangle = \frac{1}{\sqrt{22}}\begin{bmatrix}-3\\2\\3\end{bmatrix}$

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6. $$\begin{aligned}|V+W|^2 &= \langle V+W|V+W\rangle \\ &= \langle V|V\rangle + \langle V|W\rangle + \langle W|V\rangle + \langle W|W\rangle \\ &\le |V|^2 + |V||W| + |W||V| + |W|^2 = (|V|+|W|)^2,\end{aligned}$$

$|V+W| \ge 0$ 이고 $|V|+|W| \ge 0$ 이므로 $|V+W| \le |V|+|W|$ 이다.

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7. (a) $$\begin{aligned}(\Omega^+)_{ij} &= \langle i|\Omega^+|j\rangle = \langle \Omega i|j\rangle \\ &= \langle j|\Omega i\rangle^* = \langle j|\Omega|i\rangle^* = \Omega^*_{ji},\end{aligned}$$

$\therefore ((\Omega^+)^+)_{ij} = (\Omega^+)^*_{ji} = \Omega_{ij} \to (\Omega^+)^+ = \Omega$

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7. (b) $$\begin{aligned}(\Omega\Lambda)^+_{ij} &= \langle i|(\Omega\Lambda)^+|j\rangle = \langle \Omega\Lambda i|j\rangle \\ &= \langle \Lambda i|\Omega^+|j\rangle = \langle i|\Lambda^+\Omega^+|j\rangle = \Lambda^+\Omega^+_{ij},\end{aligned}$$

$\therefore (\Omega\Lambda)^+ = \Lambda^+\Omega^+$

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7. (c) $\langle V|\Omega|W\rangle^* = \langle V|\Omega W\rangle^* = \langle \Omega W|V\rangle = \langle W|\Omega^+|V\rangle$

7.d) $\mathrm{Tr}(\Omega\Lambda) = \sum_{i=1}^{n} \langle i|\Omega\Lambda|i\rangle$

$$= \sum_{i=1}^{n} \langle i|\Omega\left(\sum_{k=1}^{n}|k\rangle\langle k|\right)\Lambda|i\rangle$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \langle i|\Omega|k\rangle\langle k|\Lambda|i\rangle$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \langle k|\Lambda|i\rangle\langle i|\Omega|k\rangle$$

$$= \sum_{k=1}^{n} \langle k|\Lambda\left(\sum_{i=1}^{n}|i\rangle\langle i|\right)\Omega|k\rangle$$

$$= \sum_{k=1}^{n} \langle k|\Lambda\Omega|k\rangle = \mathrm{Tr}(\Lambda\Omega)$$

7.e) $\mathrm{Tr}(\Omega\Lambda\theta) = \mathrm{Tr}(\Omega(\Lambda\theta))$

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8. $\det(A-\omega I) = \begin{vmatrix} 3-\omega & 1 & -1 \\ 1 & 3-\omega & -1 \\ -1 & -1 & 5-\omega \end{vmatrix} = (2-\omega)(3-\omega)(6-\omega)$

i) $\omega = 2 \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} |\omega=2\rangle = |0\rangle \rightarrow |\omega=2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

ii) $\omega = 3 \rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} |\omega=3\rangle = |0\rangle \rightarrow |\omega=3\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

iii) $\omega = 6 \rightarrow \begin{bmatrix} -3 & 1 & -1 \\ 1 & -3 & -1 \\ -1 & -1 & -1 \end{bmatrix} |\omega=6\rangle = |0\rangle \rightarrow |\omega=6\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

8. $P = \begin{bmatrix} |w=2\rangle & |w=3\rangle & |w=6\rangle \end{bmatrix} \rightarrow A_D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

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9. $\det(A - wI) = \begin{vmatrix} 1-w & 3 & 0 \\ 3 & 1-w & 0 \\ 0 & 0 & -2-w \end{vmatrix} = (w+2)^2(4-w)$

i) $w = -2 \rightarrow \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} |w=-2\rangle = |0\rangle \rightarrow |w=-2,1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$|w=-2,2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

ii) $w = 4 \rightarrow \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} |w=4\rangle = |0\rangle \rightarrow |w=4\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$

A is diagonalizable ($\because$ $|w=-2,1\rangle$, $|w=-2,2\rangle$, $|w=4\rangle$ are linearly independent)

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10. $\det(A - wI) = \begin{vmatrix} -a-w & 0 & 0 \\ 0 & -a-w & 0 \\ 0 & 0 & a-w \end{vmatrix} = (a-w)(a+w)^2$

i) $w = a \rightarrow \begin{bmatrix} -2a & 0 & 0 \\ 0 & -2a & 0 \\ 0 & 0 & 0 \end{bmatrix} |w=a\rangle = |0\rangle \rightarrow |w=a\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

ii) $w = -a \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2a \end{bmatrix} |w=-a\rangle = |0\rangle \rightarrow |w=-a,1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$|w=-a,2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\det(B - \lambda I) = \begin{vmatrix} -\lambda & -ib & 0 \\ ib & -\lambda & 0 \\ 0 & 0 & b-\lambda \end{vmatrix} = (\lambda - b)^2(\lambda + b)$

i) $\lambda = b \rightarrow \begin{bmatrix} -b & -ib & 0 \\ ib & -b & 0 \\ 0 & 0 & 0 \end{bmatrix} |\lambda=b\rangle = |0\rangle \rightarrow |\lambda=b,1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$|\lambda=b,2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$

ii) $\lambda = -b \rightarrow \begin{bmatrix} b & -ib & 0 \\ ib & b & 0 \\ 0 & 0 & 2b \end{bmatrix} |\lambda=-b\rangle = |0\rangle \rightarrow |\lambda=-b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$

10. basis $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \quad U^{+} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix},$

$A' = U^{+}AU = \begin{bmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{bmatrix}, \quad B' = U^{+}BU = \begin{bmatrix} b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & b \end{bmatrix}$