Summary of the Previous Lecture

- Linear operator
  - Matrix representation
  - Completeness relation & projection operator
  - Outer product and construction of a new mapping
- Adjoint of an operator
  - Adjoint of an operator Ω is defined as a new operator $\Omega^\dagger$ that can transform $\langle V|$ to $\langle \Omega V|$ such that $\langle V|\Omega^\dagger = \langle \Omega V|$ when $\Omega|V\rangle = |\Omega V\rangle$.
  - In terms of matrix elements, corresponds to transpose & complex conjugation ➔ $\left(\Omega^\dagger\right)_{ij} = \Omega_{ji}^*$
- Hermitian ⇔ $\Omega^\dagger = \Omega$ ➔ similar to symmetric matrix
- Unitary ⇔ $UU^\dagger = I$ ➔ similar to orthogonal matrix
- Trace of a matrix ⇔ $\mathrm{Tr}(\Omega) = \sum_{i=1}^{n} \Omega_{ii}$
- Eigenvalue Problem
  - $\Omega|V\rangle = \omega|V\rangle \Leftrightarrow \begin{bmatrix} \Omega_{11}-\omega & \Omega_{12} & \cdots \\ \Omega_{21} & \Omega_{22}-\omega & \\ \vdots & & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} \Leftrightarrow \det(\Omega - \omega I) = 0$

Degeneracy

- Example: find out the eigenvalues & eigenvectors for $\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
  - $\det(\Omega - \omega I) = \begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & 2-\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = -(\omega-2)^2\omega = 0$
  - For $\omega = 0$, corresponding eigenvector is $|\omega = 0\rangle = \frac{1}{2^{1/2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
  - For $\omega = 2$,

$$\begin{bmatrix} 1-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x_1 + x_3 = 0 \\ 0 = 0 \\ x_1 - x_3 = 0 \end{matrix} \Rightarrow x_1 = x_3$$

We have freedom in choosing x_2 as long as $x_1 = x_3$ is satisfied.

➔ $\begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

➔ For example, $|\omega = 2, \text{first}\rangle = \frac{1}{3^{1/2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $|\omega = 2, \text{second}\rangle = \frac{1}{6^{1/2}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ are another possibility.

- For eigenvectors with the same eigenvalue, we need an additional label such as α for the eigenvector like $|\omega, \alpha\rangle$

Properties of Hermitian Operators

- **Theorem 9**: the eigenvalues of a Hermitian operator are **real**.
  - Let $\Omega|\omega\rangle = \omega|\omega\rangle$
  - Apply $\langle\omega|$ on both sides: $\langle\omega|\Omega|\omega\rangle = \omega\langle\omega|\omega\rangle$
  - Take complex conjugation of both sides: $\langle\omega|\Omega^{\dagger}|\omega\rangle = \omega^{*}\langle\omega|\omega\rangle = \langle\omega|\Omega|\omega\rangle = \omega\langle\omega|\omega\rangle$
  - Because $\langle\omega|\omega\rangle \neq 0$, ω should be real.
- When there is no degeneracy, two eigenvectors of a Hermitian operator with different eigenvalues are orthogonal.
  - Let $\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle$ and $\Omega|\omega_j\rangle = \omega_j|\omega_j\rangle$
  - $\langle\omega_j|\Omega|\omega_i\rangle = \omega_i\langle\omega_j|\omega_i\rangle$, $\langle\omega_i|\Omega|\omega_j\rangle = \omega_j\langle\omega_i|\omega_j\rangle$
  - Take complex conjugation of $\langle\omega_j|\Omega|\omega_i\rangle$, then $\omega_i\langle\omega_i|\omega_j\rangle = \omega_j\langle\omega_i|\omega_j\rangle$.
  - $(\omega_i - \omega_j)\langle\omega_i|\omega_j\rangle = 0$ ➔ Because $\omega_i \neq \omega_j$, $\langle\omega_i|\omega_j\rangle = 0$.
- **Theorem 10**: To every Hermitian operator Ω, there exists (at least) a basis consisting of its orthonormal eigenvectors. It is diagonal in this eigenbasis and has its eigenvalues as its diagonal entries.
  - $\begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & \vdots \\ 0 & \dots & \ddots \end{bmatrix}$
  - Full proof is in page 36 of the reference.

Properties of Unitary Operators

- **Theorem 11**: the eigenvalues of a unitary operator are complex numbers of unit modulus.
- **Theorem 12**: the eigenvectors of a unitary operator are mutually orthogonal (assuming there is no degeneracy).
  - Let $U|u_i\rangle = u_i|u_i\rangle$ and $U|u_j\rangle = u_j|u_j\rangle$
  - $\langle u_j|U^\dagger U|u_i\rangle = u_j^* u_i\langle u_j|u_i\rangle = \langle u_j|u_i\rangle$
  - $(u_j^* u_i - 1)\langle u_j|u_i\rangle = 0$
  - If $i = j$, $u_i^* u_i = 1$. ➔ Theorem 11
  - If $i \neq j$, $\langle u_j|u_i\rangle = 0$. ➔ Theorem 12
- Orthogonality of unitary and Hermitian operator
  - Normal operator ⇔ $AA^\dagger = A^\dagger A$ (p. 70 of the textbook)

From Chap. 1 of "Principles of Quantum Mechanics", Ramamurti Shankar, 2nd ed. Plenum Press (1994)

Basis Transformation of an Operator

- Assume that $\mathbb{O}$ is a matrix representation of an operator Ω in some orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$.
- If we switch the basis from $|1\rangle, |2\rangle, \dots, |n\rangle$ to a new orthonormal basis $|I\rangle, |II\rangle, \dots, |N\rangle$, the new matrix representation $\mathbb{O}'$ of an operator Ω in the new orthonormal basis can be obtained by $\mathbb{O}' = \mathbb{U}^{\dagger}\mathbb{O}\mathbb{U}$.
- $\mathbb{U}$ is a matrix representation of $U = \sum_{m=1}^{n} |M\rangle\langle m|$ in the original basis $|1\rangle, |2\rangle, \dots, |n\rangle$.

From Chap. 1 of "Principles of Quantum Mechanics", Ramamurti Shankar, 2nd ed. Plenum Press (1994)

Euler Relation

- $e^{ix} = \cos x + i \sin x$
- Proof
  - From Taylor expansion:
    - $f(x) = f(0) + \left.\frac{df}{dx}\right|_{x=0} x + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=0} x^2 + \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=0} x^3 + \cdots$
  - $\sin x = 0 + \frac{1}{1!}x + \frac{-0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
  - $\cos x = 1 + \frac{-0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$
  - $e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots = \sum_{n=0}^{\infty} \frac{z}{n!}$
  - If $z = ix$,
    - $e^{ix} = 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \ldots\right)$$

$$= \cos x + i \sin x$$