1a)

Suppose there are two null vectors. . By definition,

(There exists a null vector obeying )

(commutativity in addition)

(There exists a null vector obeying )

. Therefore, the null vector is unique.

1b)

(There exists a null vector obeying )

(For every vector there exists an inverse under addition, such that )

(associativity in addition)

()

(distributivity in scalar multiplication)

(For every vector there exists an inverse under addition, such that )

1c)

(For every vector there exists an inverse under addition, such that )

(There exists a null vector obeying )

1d)

Suppose there are two additive inverses for, . Then,

(There exists a null vector obeying )

(For every vector there exists an inverse under addition, such that )

(associativity in addition)

(For every vector there exists an inverse under addition, such that )

(There exists a null vector obeying )

**2a)** False. Counter example:

Not closed, no zero vector, etc => all valid counter examples

Wrong answer: 0 points

2b) True. It should be shown that the space composed of only the null vector satisfies all the axioms. The minimum element that a vector space should contain is the null vector.

**2c)** False. Counter example:

The null vector is not defined, not closed => all valid counter examples

Wrong answer: 0 points

2d) True. It should be shown that the given set satisfies all the axioms.

**2e)** True. All the axioms are satisfied.

Wrong answer: 0 points

3. Suppose there are two representations of an identical vector 𝑣.

Then

But since is a basis, that is, it forms a linearly independent set, for all i in order to satisfy the above equation. implies that the representation is unique.

4)

i. From ,

.

ii.

iii. From and

**5a)**

is a linearly independent set.

For

For let

.

Such that is orthogonal to all vectors . ( )

Now then

.

is orthogonal to all vectors . ( )

Therefore, by induction, the Gram-Schmidt procedure generates a set of orthogonal vectors.

If you provided the construction procedure only, without proof through induction, -2 points

**5b)**

You receive half out of the full score if you provided the correct answers to half of the problem (i.e. 5a) no proof, but 5b) correct)

6)

()

(Schwarz inequality)

7a)

7b)

7c)

7d)

7e)

Let , or ,

Use 7d)

**8)**

The characteristic equation is

Solving the eigenvalue problem, we obtain the pairs of eigenvalues and eigenvectors:

i) , ii) , iii)

If you got the wrong eigenvector/eigenvalue, -1 points.

**9)**

The characteristic equation is

Solving the eigenvalue problem, we obtain the pairs of eigenvalues and eigenvectors:

i) , ii) ,

Even though the eigenvalue spectrum is degenerate in , the eigenspace is diagonal in the eigenbasis . Therefore, the total matrix is diagonalizable in the basis

If you got the wrong eigenvector/eigenvalue, -1 points.

If you answered that the matrix is not diagonalizable, 0 points.

**10)**

1. Matrix has the eigenvectors and eigenvalues

i) , ii) ,

ii) Block diagonalize matrix with respect to the eigenspaces of matrix .

The eigenvalue and eigenvector of matrix obtained from is and . (simply append with .)

The eigenvalue and eigenvector of matrix obtained from is b and . (simply append with .))

Then the simultaneously diagonalizing eigenbasis is , .

The eigenvalues of and are with respect to

That is,

If you got the wrong eigenvector/eigenvalue, -1 points.

If you did not show the eigenvalues of A in the common diagonal basis, -1 points.

You receive half out of the full score if you provided the correct answers to half of the problem (i.e. Correct diagonalization of A, but wrong answer for B)