Optimal Liquidation Problems in a Randomly-Terminated Horizon

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Abstract

In this paper, we study optimal liquidation problems in a randomly-terminated horizon. We consider the liquidation of a large single-asset portfolio with the aim of minimizing a combination of volatility risk and transaction costs arising from permanent and temporary market impact. Three different scenarios are analyzed under Almgren-Chriss's market impact model to explore the relation between optimal liquidation strategies and potential inventory risk arising from the uncertainty of the liquidation horizon. For cases where no closed-form solutions can be obtained, we verify comparison principles for viscosity solutions and characterize the value function as the unique viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation.

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1 Introduction

Understanding trade execution strategies is a key issue for financial market practitioners and has attracted growing attention from the academic researchers. An important problem faced by equity traders is how to liquidate large orders. Different from small orders, an immediate execution of large orders is often impossible or at a very high cost

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due to insufficient liquidity. A slow liquidation process, however, is often costly, since it may involve undesirable inventory risk. Almgren and Chriss [2] provided one of the early studies on the optimal execution strategy of large trades, taking into account the volatility risk and liquidation costs. In order to produce tractable and analytical results, they set the market impact cost per share to be linear in the rate of trading. Schied and Schoneborn [21] considered the infinite-horizon optimal portfolio liquidation problem for a von Neumann-Morgenstern investor under the liquidity model of Almgren [1], in which a power law cost function was introduced to determine optimal trading strategies. However, most of the literature on optimal liquidation strategies mainly considered a known pre-determined time horizon or infinite horizon. The case of unknown (or more precisely, randomly-terminated) time horizon is not fully addressed. In some situation, it is more realistic to assume that the liquidation horizon depends on some stochastic factors of the model. For example, some financial markets adopt the circuit-breaking mechanism, which makes the horizon of the investor subject to the stock price movement. Once the stock price hits the daily limits, all transactions of the stock will be suspended.

In this paper, we consider a randomly-terminated time horizon under three different scenarios that an agent might encounter in a financial market. Almgren-Chriss's market impact model is employed to describe the underlying asset price:

$$\begin{cases} dS_t = f(\theta_t)dt + \sigma dW_t^S, \\ \widetilde{S}_t = g(\theta_t) + S_t, \end{cases}$$

where the constant $\sigma > 0$ is the absolute volatility of the asset price S_t, W_t^S is an one-dimensional standard Brownian motion, $\widetilde{S}_t$ is the actual transaction price, $\{\theta_t, t \geq 0\}$ is an admissible control process, $f(\theta_t)$ and $g(\theta_t)$ represent, respectively, the permanent and temporary components of the market impact. We consider the liquidation problem of a large single-asset portfolio with the aim of minimizing a combination of volatility risk and transaction costs arising from permanent and temporary market impact.

We first consider the case with a pre-determined time horizon T, which can be used as a benchmark for other cases with randomly-terminated time horizon. In general, it is required that a liquidation strategy θ_t should satisfy the *hands-clean* condition:

$$X_T = X_0 - \int_0^T \theta_t dt = 0,$$

where X_t is the number of shares held by the trader at the time t. We first work on a subclass of *deterministic* controls, which do not allow for inter-temporal updating, satisfying the *hands-clean* condition. Obviously, the deterministic strategy obtained in the subclass might be no longer optimal when taking into account the entire class of admissible controls. We then temporarily relax the *hands-clean* condition, and allow an immediate final liquidation (if necessary) so that the number of shares owned at the time $t = T$ is $X_T = 0$. We employ the dynamic programming (DP) approach to solve the stochastic control problems and prove that the optimal liquidation strategy actually converges to the deterministic strategy when the transaction cost involved by liquidating the outstanding position X_{T-} approaches to infinity.

We then move to analyze the randomly-terminated cases. Two different scenarios are analyzed to shed light on the relationship between liquidation strategies and potential position risk arising from the uncertainty of the time horizon. First, we consider the scenario where the liquidation process is terminated by an exogenous trigger event. We model the occurrence time of a trigger event to be random and its hazard rate process is given by $\{l(t), t \geq 0\}$. Once this event occurs, all liquidation processes will be forced to suspend. Compared with the case without trigger event, agents facing the scenario that an exogenous trigger event might occur during the trading horizon would like to accelerate the rate of liquidating to reduce their exposure to potential position risk and eventually in a smaller position when the trigger event occurs. Their strategy has a steeper gradient and is more "convex" when compared with those who are not threatened by this trigger event. Second, we consider the case when the liquidation process is subject to counterparty risk. Different from the exogenous trigger event setting, information set available to the counterparty risk modeler is more refined in terms of predictability. To model counterparty risk, we adopt the structural firm value approach, originated from Black and Scholes [3], and Merton [14], and let the firm's asset value follow a geometric Brownian motion:

$$\frac{dY_t}{Y_t} = \beta dt + \xi dW_t^Y.$$

The incorporation of counterparty risk into the study of optimal liquidation does not come without cost. In order to examine its impact on optimal trading strategies, we have to introduce and employ viscosity solutions. By verifying the comparison principles for viscosity solutions, we characterize the value function as the unique viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. This equation can be numerically solved. We further analyze the effectiveness of the numerical method and illustrate that the computational error is sufficiently small.

The remainder of this paper is structured as follows. The background and basic models of an agent's liquidation problem are introduced in Section 2. Section 3 discusses typical liquidating problems under the benchmark model. In Sections 4–5, we discuss two different scenarios with randomly-terminated time horizons. Viscosity solution approach is adopted in these sections to study in great generality stochastic control problems. By combining these results with comparison principles, we characterize the value function as the unique viscosity solution of the associated dynamic programming equation, and this can then be used to obtain further results. Finally, concluding remarks are given in Section 6. For the sake of self-containedness, we provide the technical proofs in the Appendix.

2 Problem Setup

In this section, we first describe the market environment of the agent. We then present a market impact model to discuss the optimal liquidating problem.

2.1 The Market Environment and Market Impact Model

The agent starts at $t = 0$ and has to liquidate a large position in a risky asset by time T. This terminal time can be either deterministic or random, depending on the scenario that the agent is facing. For simplicity, we assume that the agent withholds the liquidation proceeds. In other words, he/she does not deposit the liquidation proceeds in his/her money market account. At any time $t \in [0, T]$, we adopt the following notations for the agent's portfolio:

(i) $V_t = C_t + X_t S_t$, portfolio value;

(ii) C_t, balance of risk free bank account;

(iii) X_t, number of shares of underlying asset;

(iv) S_t, price of the underlying risky asset.

The initial conditions are $C_0 = 0$, $S_0 = s$, and $X_0 = Q$.

Suppose the risky asset can be continuously liquidated during the trading horizon, namely, there is always sufficient liquidity for their execution[1]. Let $\{\theta_t\}_{t\in[0,T]}$ denote the liquidation process. The shares held by the trader at any time $t \in [0, T]$ can be written as follows:

$$X_t = Q - \int_0^t \theta_u du.$$

We consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ endowed with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Definition 1 *A stochastic process $\theta(\cdot) = \{\theta_u, 0 \leq u \leq T\}$ is called an admissible control process if all of the following conditions hold:*

(i) (Adaptivity) *For each $t \in [0, T]$, θ_t is $\mathcal{F}_t$-adapted;*

(ii) (Non-negativity) *$\theta_t \in \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of nonnegative real values;*

(iii) (Consistency)

$$\int_0^T \theta_t dt \leq Q;$$

(iv) (Square-integrability)

$$\mathbb{E}\left[\int_0^T |\theta_t|^2 dt\right] < \infty;$$

(v) (L_∞-integrability)

$$\mathbb{E}\left[\max_{0\leq t\leq T} |\theta_t|\right] < \infty.$$

Furthermore, denote Θ_t as the collection of admissible controls with respect to the initial time $t \in [0, T)$ and $\widehat{\Theta}_t$ as the collection of controls only satisfying condition (i), (iv) and (v).

We assume that the risky asset exhibits a price impact due to the feedback effects of the agent's liquidation strategy. For any given admissible control $\theta(\cdot) \in \Theta_0$, the market mid-price of the stock is assumed to follow the dynamics:

$$dS_t = f(\theta_t)dt + \sigma dW_t^S, \tag{1}$$

where $\{W_t^S\}$ is a standard Brownian motion with filtration $\{\mathcal{F}_t\}$, the constant $\sigma > 0$ is the absolute volatility of the asset price, and $f(\cdot)$ is the permanent component of the

[1] For simplicity, the transaction fees will not be considered in this paper.

market impact. For simplicity, we further assume that f is time homogeneous, namely, $f(\cdot)$ is independent of t.

Generally speaking, the actual transaction price $\widetilde{S}_t$ is not always the same as the market mid-price S_t, since the market is not perfectly liquid, see, for example, Almgren and Chriss [2]. We assume $\widetilde{S}_t = S_t + g(\theta_t)$ and call $g(\theta)$ the temporary price impact. Intuitively, the function $g(\cdot)$ captures quantitatively how the limit order books available in the market are eaten up at different levels of trading speeds.

Assumption 0. The price dynamics follow a simple Almgren-Chriss linear market impact model (see, Almgren and Chriss [2]):

$$f(\theta) = -\eta \cdot \theta \quad \text{and} \quad g(\theta) = -\nu \cdot \theta,$$

where η and ν are positive constants.

An agent who holds the stock receives the capital gain or loss due to stock price movements. Thus, if the agent's position is marked to market using the book value, ignoring market impact that would be incurred in converting these shares into cash, at any time t, the agent's portfolio value $V_t = C_t + X_t S_t$ satisfies

$$\begin{cases} V_0 = Q \cdot s \\ dV_t = (\widetilde{S}_t - S_t)\theta_t dt + X_t dS_t. \end{cases} \tag{2}$$

At any time $t \in [0, T)$ before the end of trading,

$$\begin{aligned} V_t &= V_0 + \int_0^t (\widetilde{S}_u - S_u)\theta_u du + \int_0^t X_u dS_u \\ &= V_0 + \int_0^t \Big[(\widetilde{S}_u - S_u)\theta_u + X_u f(\theta_u) \Big] du + \int_0^t \sigma X_u dW_u^S. \end{aligned}$$

2.2 Hands-clean Condition

Let us recall that our task is to liquidate a large-size position by the time T. Generally speaking, it is required that the *hands-clean* condition should be satisfied:

$$X_T = X_0 - \int_0^T \theta_t dt = 0. \tag{3}$$

This technical condition, however, introduces some unexpected properties to the stochastic control problem. To tackle this problem, we temporary relax the *hands-clean* condition and allow an inmmediate final liquidation (if necessary) so that the number of shares owned at $t = T$ equals zero. That is, given the state variables (S_t, C_t, X_t) at the instant before the end of trading $t = T-$, if $X_{T-} \neq 0$, then we will have an immediate final liquidation so that $X_T = 0$. The liquidation proceeds C_T after this final trade is

$$C_T = C_{T-} + X_{T-} \left(S_{T-} - \mathcal{C}^o(X_{T-}) \right),$$

where $\mathcal{C}^o(X_{T-}) = \phi X_{T-}$, for some constant $\phi > 0$, is the cost involved from liquidating the outstanding position X_{T-}. Thus, we have

$$V_T = V_{T-} - \phi X_{T-}^2.$$

The gain/loss from liquidating the outstanding position, $R_T = V_T - V_0$, is given by

$$R_T = \int_{[0,T)} \Big[(\widetilde{S}_t - S_t)\theta_t + X_t f(\theta_t) \Big] dt + \int_{[0,T)} \sigma X_t dW_t^S - \phi X_{T-}^2. \tag{4}$$

2.3 Performance Criterion

Under the normal circumstance, investors are risk averse and demand a higher return for a riskier investment. The mean-variance criterion is popular for taking both return and risk into account. However, the mean-variance criterion may induce a potential problem of time-inconsistency, namely, planned and implemented policies are different. As mentioned in Rudloff et al. [19], a major reason for developing dynamic models instead of static ones is the fact that one can incorporate the flexibility of dynamic decisions to improve the objective function. Time-inconsistent criteria are generally not favorable to introduce in the study, since the associated policies are sub-optimal.

To take both return and risk into account, instead of adopting the mean-variance criterion, we are most interested in the mean-quadratic optimal agency execution strategies, as they are proved to be time-consistent in [2, 6, 22]. In this section, we will introduce the quadratic variation and the corresponding objective function as follows.

2.3.1 Quadratic Variation

Formally, the *quadratic variation* of the portfolio value V on $[0, T)$ is defined to be

$$[V, V]([0, T)) = \int_{[0,T)} \sigma^2 X_t^2 dt. \tag{5}$$

From the interpretation of Eq. (5), minimizing quadratic variation corresponds to minimizing volatility in the portfolio value process.

2.3.2 Objective Function

Let $\gamma > 0$ be a constant corresponding to the risk aversion. Then the agent's objective is to find the optimal control for

$$\begin{aligned} & \max_{\theta(\cdot)\in\Theta_0} \mathbb{E}[R_T - \gamma[V, V]([0, T))] \\ = \; & \max_{\theta(\cdot)\in\Theta_0} \mathbb{E}\left[\int_{[0,T)} \left[(\widetilde{S}_t - S_t)\theta_t + X_t f(\theta_t) - \gamma\sigma^2 X_t^2 \right] dt - \phi X_{T-}^2 \right]. \end{aligned} \tag{6}$$

3 The Benchmark Model for Optimal Liquidation (Model 1)

Assumption 1 *The liquidation horizon T is a finite-valued, pre-determined, and positive constant.*

In this section, we present our benchmark model under Assumption 0 for the optimal liquidation problem. We first work on a subclass of *deterministic* controls[2] satisfying the *hands-clean* condition (3), and then move to the dynamic programming (DP) approach considering over the entire class of admissible controls. We prove that when the transaction cost involved by liquidating the outstanding position X_{T-} approaches to infinity, the optimal liquidation strategy obtained from DP approach converges to the deterministic one.

[2] Controls that do not allow for inter-temporal updating.

3.1 Deterministic Control

Let us first consider the case in which $\theta(\cdot)$ ranges only over the sub-class Θ_0^{det} of *deterministic* strategies in Θ_0 satisfying the *hands-clean* condition

$$\int_0^T \theta_t dt = Q.$$

That is, $X_{T-} = 0$, and the agent's objective is to find the optimal strategy for

$$\begin{aligned}
&\max_{\theta(\cdot)\in\Theta_0^{det}} \mathbb{E}\int_{[0,T)} \left[(\widetilde{S}_t - S_t)\theta_t + X_t f(\theta_t) - \gamma\sigma^2 X_t^2\right] dt \\
=\ &\max_{\theta(\cdot)\in\Theta_0^{det}} \mathbb{E}\int_{[0,T)} \left[g(\theta_t)\theta_t + X_t f(\theta_t) - \gamma\sigma^2 X_t^2\right] dt.
\end{aligned} \tag{7}$$

The cost function of the deterministic control problem (7) is

$$\mathcal{H}(X_t, \theta_t, \Lambda_t, t) \equiv g(\theta_t)\theta_t + f(\theta_t)X_t - \gamma\sigma^2 X_t^2 - \Lambda_t\theta_t,$$

where Λ_t is the *Lagrange multiplier* (also called the *adjoint state*). The differential equation for the deterministic system is:

$$\frac{dX_t}{dt} = -\theta_t \quad \text{with} \quad X_0 = Q.$$

We assume the Hamiltonian $\mathcal{H}$ has continuous first-order derivatives in state, adjoint state, and control variables, namely, $\{X_t, \Lambda_t, \theta_t\}$. Then the necessary conditions (also called *Hamilton's equation*) for having an interior optimum of the Hamiltonian $\mathcal{H}$ at $\{X_t^{det,*}, \Lambda_t^{det,*}, \theta_t^{det,*}\}$, are given by

$$\begin{cases}
\dfrac{dX_t^{det,*}}{dt} = \left.\dfrac{\partial\mathcal{H}}{\partial\Lambda}\right|_{(X_t^{det,*},\theta_t^{det,*},\Lambda_t^{det,*},t)} \\
-\dfrac{d\Lambda_t^{det,*}}{dt} = \left.\dfrac{\partial\mathcal{H}}{\partial X}\right|_{(X_t^{det,*},\theta_t^{det,*},\Lambda_t^{det,*},t)} \\
0 = \left.\dfrac{\partial\mathcal{H}}{\partial\theta}\right|_{(X_t^{det,*},\theta_t^{det,*},\Lambda_t^{det,*},t)}.
\end{cases} \tag{8}$$

It follows from the critical conditions in Eq. (8) and Assumption 0 that

$$\begin{cases}
\ddot{X}_t^{det,*} = \dfrac{\gamma\sigma^2}{\nu} X_t^{det,*}, \\
X_0^{det,*} = Q, \\
X_T^{det,*} = 0.
\end{cases} \tag{9}$$

An explicit solution, which is unique according to Lasota and Opial [13], is given by

$$\begin{cases}
X_t^{det,*} = Q\dfrac{\sinh\left(\sqrt{\frac{\gamma\sigma^2}{\nu}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\gamma\sigma^2}{\nu}}T\right)}, \\
\theta_t^{det,*} = Q\sqrt{\dfrac{\gamma\sigma^2}{\nu}}\dfrac{\cosh\left(\sqrt{\frac{\gamma\sigma^2}{\nu}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\gamma\sigma^2}{\nu}}T\right)}.
\end{cases} \tag{10}$$

There is a very interesting phenomenon in the *deterministic* control problem: the solution (10) has nothing to do with the permanent price impact η. If a position of size Q units with initial market price s is fully liquidated by time T, i.e. $X_T = 0$, the expected value of the resulting cash becomes

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \widetilde{S}_t\theta_t dt\right] &= \mathbb{E}\left[\int_0^T S_t\theta_t dt - \nu\int_0^T \theta_t^2 dt\right] \\
&= Q\cdot s + \mathbb{E}\left[\int_0^T X_t dS_t - \nu\int_0^T \theta_t^2 dt\right] \\
&= Q\cdot s - \underbrace{\mathbb{E}\left[\nu\int_0^T \theta_t^2 dt\right]}_{\text{(temporary impact cost)}} - \underbrace{\frac{1}{2}\eta Q^2.}_{\text{(permanent impact cost)}}
\end{aligned}$$

Clearly, the permanent impact cost is independent of the time taken or strategy used to execute the liquidation.

3.2 Dynamic Programming Approach

Obviously, if we are allowed to update dynamically, namely, replacing Θ_0^{det} by the entire class of admissible strategies Θ_0, then one will be able to further improve his/her performance. In this section, we consider a stochastic approach. We employ the DP method to solve the stochastic control problem (6). This approach yields a Hamilton-Jacobi-Bellman (HJB) equation. When this HJB equation can be solved by an explicit smooth solution, the verification theorem then validates the optimality of the candidate solution to the HJB equation. For more details about the verification theorem, we refer interested readers to Pham [18] (Chapter 3), Øksendal [15] (Chapter 11), and Øksendal and Sulem [16] (Chapter 3).

Let $U(t,q)$ be the optimal value function beginning at a time $t\in[0,T)$ with initial value $X_t = q$, namely[3],

$$U(t,q) = \max_{\theta(\cdot)\in\Theta_t} \mathbb{E}\left[\int_{[t,T)} \big[-\nu\theta_u^2 - \eta X_u\theta_u - \gamma\sigma^2 X_u^2\big]du - \phi X_{T-}^2 \Big| \mathcal{F}_t\right]. \tag{11}$$

Temporarily assuming that $U(t,q)\in\mathcal{C}^{1,2}([0,T)\times(0,+\infty))$.[4] From the DP principle, U must satisfy the following HJB equation:

$$\begin{cases} \partial_t U - \gamma\sigma^2 q^2 - \min\limits_{\theta_t\in\Theta_t}\left\{\nu\theta_t^2 + (\eta q + \partial_q U)\cdot\theta_t\right\} = 0 \\ U(T-,q) = -\phi q^2. \end{cases} \tag{12}$$

We remark that the optimization problem included in Eq. (12) is a constrained optimization problem with constraints: (a1) $\theta_t\in\mathbb{R}_+$; and (a2) $\int_{[0,T)}\theta_t dt \le Q$. Generally

[3]It is worth noting that the value function U does not depend explicitly on the stock price S_t.

[4]$\mathcal{C}^{1,2}([0,T)\times(0,+\infty))$ is the space of functions $f(t,q)$ which is continuously differentiable in t, and twice continuously differentiable in q.

speaking, there is no straightforward method to solve this kind of problems. One simple way to handle this problem is to consider the corresponding unconstrained optimization problem:

$$\begin{cases} \partial_t U - \gamma\sigma^2 q^2 - \min\limits_{\theta_t \in \widehat{\Theta}_t} \left\{ \nu\theta_t^2 + (\eta q + \partial_q U)\cdot\theta_t \right\} = 0 \\ U(T-, q) = -\phi q^2, \end{cases} \tag{13}$$

and then verify that the obtained result indeed satisfies all the constraints. From the HJB equation, Eq. (13), the optimal trading strategy without constraints is given by

$$\theta_t^{\phi,*} = -\frac{1}{2\nu}\big(\partial_q U + \eta q\big).$$

Thus the value function U solves the following Ordinary Differential Equation (ODE):

$$\begin{cases} \partial_t U - \gamma\sigma^2 q^2 + \dfrac{1}{4\nu}\big(\partial_q U + \eta q\big)^2 = 0 \\ U(T-, q) = -\phi q^2. \end{cases} \tag{14}$$

Theorem 1 *There is at most one $\mathcal{C}^{1,2}([0,T)\times(0,\infty))$ solution to Eq. (14).*

Proof: Let f_1 and f_2 be two $\mathcal{C}^{1,2}([0,T)\times(0,\infty))$ solutions to Eq. (14). Define $\widetilde{f} = f_1 - f_2$. Then the new function $\widetilde{f}$ satisfies the following Partial Differential Equation (PDE):

$$\begin{cases} \partial_t \widetilde{f} + \dfrac{1}{4\nu}\big[\partial_q(f_1+f_2) + 2\eta q\big]\partial_q \widetilde{f} = 0 \\ \widetilde{f}(T-, q) = 0. \end{cases}$$

Since the evolution equation for $\widetilde{f}$ is linear and first-order, one can solve the above problem explicitly by the method of characteristics, and find that $\widetilde{f} \equiv 0$ is the unique solution to this problem. As a result, $f_1 \equiv f_2$. □

To solve Eq. (14), we consider an ansatz that is quadratic in the variable q:

$$U(t,q) = a(t) + b(t)q + c(t)q^2.$$

According to Theorem 1, if the above ansatz is a solution of Eq. (14), then it must be the unique solution. Under this setting, the optimal liquidating strategy takes the following form:

$$\theta_t^{\phi,*} = -\frac{1}{2\nu}\left\{b(t) + [2c(t)+\eta]q\right\}.$$

A direct substitution yields that the coefficients $a(t)$, $b(t)$ and $c(t)$ must satisfy the following ODEs:

$$\begin{cases} \dot{c}(t) = \gamma\sigma^2 - \dfrac{1}{4\nu}[2c+\eta]^2 \\ \dot{b}(t) = -\dfrac{1}{2\nu}b(t)[2c+\eta] \\ \dot{a}(t) = -\dfrac{1}{4\nu}b^2 \end{cases} \tag{15}$$

with terminal conditions: $a(T) = 0$, $b(T) = 0$ and $c(T) = -\phi$. Since System (15) is partially decoupled, we can find the exact solution via direct integrations. As a result, they are given by

$$\begin{cases} c(t) = \dfrac{1}{2\xi}\left[\dfrac{\zeta e^{-4\gamma\xi\sigma^2(T-t)} - 1}{\zeta e^{-4\gamma\xi\sigma^2(T-t)} + 1}\right] - \dfrac{\eta}{2} \\ b(t) = 0 \\ a(t) = 0 \end{cases} \tag{16}$$

where the constants ζ and ξ are given by

$$\zeta = \frac{1 - \xi(2\phi - \eta)}{1 + \xi(2\phi - \eta)} \quad \text{and} \quad \xi = \frac{1}{2\sigma\sqrt{\gamma\nu}}.$$

It is worth noting that

$$\dot{X}_t^{\phi,*} = -\theta_t^{\phi,*} = \frac{1}{2\nu}[2c(t) + \eta]X_t^{\phi,*}, \tag{17}$$

and that $X_0^{\phi,*} = Q$. Therefore,

$$X_t^{\phi,*} = Q \cdot \exp\left(\frac{1}{2\nu}\int_0^t [2c(u) + \eta]du\right).$$

As to the results obtained in this section, we have the following proposition.

Proposition 1 *It is assumed that model parameters satisfy the condition:*

$$2\phi > \eta + 2\sigma\sqrt{\gamma\nu}. \tag{18}$$

That is, market liquidity risk dominates the potential arbitrage opportunity introduced by permanent impact and potential position risk involved by price fluctuations. Then, $c(t)$ is a strictly decreasing function in t and $c(t) + \eta \le 0$ for any $t \in [0, T)$. Furthermore, we have that

(b1) $\theta_t^{\phi,*} \ge 0$, *for any time $t \in [0, T)$; and that*

(b2) $\int_{[0,T)} \theta_t^{\phi,*} dt \le Q$.

The obtained optimal trading strategy (17) is also the optimal trading strategy for the constrained problem.

Proof: Notice that the graph of the function $c(t)$ depends on the coefficient $\zeta = (1 - x)/(1 + x)$, with $x = (2\phi - \eta)/2\sigma\sqrt{\gamma\nu}$. Under Assumption (18), $x > 1$, and hence $-1 < \zeta < 0$. Therefore,

$$\frac{\partial c(t)}{\partial t} < 0,$$

i.e., $c(t)$ is a strictly decreasing function in t, and $c(t) + \eta/2 \le 0$ always holds for any $t \in [0, T)$. Thus, we conclude that

$$\theta_t^{\phi,*} = -Q\frac{1}{2\nu}[2c(t) + \eta]e^{\frac{1}{2\nu}\int_0^t [2c(u)+\eta]du} \ge 0,$$

for any time $t \in [0,T)$, and that

$$\int_{[0,T)} \theta_t^{\phi,*} dt = Q\Big[1 - e^{\frac{1}{2\nu}\int_0^t [2c(u)+\eta]du}\Big] \le Q.$$

$\square$

Let $U_T(t,q)$ denote the value function of the optimization problem (11) with time horizon T, then for any $T_1 > T_2 > t$, we have

$$U_{T_1}(t,q) > U_{T_2}(t,q), \tag{19}$$

provided that the condition (18) holds. This is consistent with the fact that an investor's ability to bear risk relates to his/her time horizon for investment[5].

3.3 Relation between Deterministic and Stochastic Control

Theorem 2 *When the transaction fees involved by liquidating the outstanding position X_{T-} approaches to infinity, the limit of the optimal stochastic control process $(\theta_t^{\phi,*})_{t\in[0,T)}$ satisfies the hands-clean condition and it converges (point-wise) to the optimal deterministic control process $(\theta_t^{det,*})_{t\in[0,T)}$. Meanwhile, the optimal trajectory $X_t^{\phi,*}$ converges (point-wise) to the one determined in the deterministic system $X_t^{det,*}$. That is, as $\phi \to \infty$, we have*

1. $X_{T-}^{\phi,*} \to 0$;
2. $\lim_{\phi\to\infty} \theta_t^{\phi,*} = \theta_t^{det,*}$ *point-wise;*
3. $\lim_{\phi\to\infty} X_t^{\phi,*} = X_t^{det,*}$ *point-wise.*

Proof: We complete the proof by the following two steps:

Step 1 (Hands-clean condition) We first prove that, as $\phi \to \infty$, $X_{T-}^{\phi,*} \to 0$. We note that

$$X_t^{\phi,*} = Q \cdot \exp\left(\int_0^t \frac{1}{2\nu}[2c(u)+\eta]du\right).$$

A simple calculation yields

$$e^{\int_u^t \frac{1}{2\nu}[2c(r)+\eta]dr} = \frac{\zeta e^{-4\gamma\xi\sigma^2(T-t)}+1}{\zeta e^{-4\gamma\xi\sigma^2(T-u)}+1} e^{-2\gamma\xi\sigma^2(t-u)}. \tag{20}$$

As $\phi \to \infty$, $\zeta \to -1$, and hence

$$X_{T-}^{\phi,*} = \frac{Q(\zeta+1)}{\zeta e^{-2\gamma\xi\sigma^2 T} + e^{2\gamma\xi\sigma^2 T}} \to 0.$$

[5] The ability to bear risk is measured mainly in terms of objective factors, such as time horizon, expected income, and the level of wealth relative to liability.

Step 2 (Convergence) We then prove that as $\phi \to \infty$,

- $\lim\limits_{\phi\to\infty} \theta_t^{\phi,*} = \theta_t^{det,*}$ point-wise; and
- $\lim\limits_{\phi\to\infty} X_t^{\phi,*} = X_t^{det,*}$ point-wise.

First, we have

$$\lim_{\phi\to\infty} X_t^{\phi,*} = \lim_{\phi\to\infty} Qe^{\int_0^t \frac{1}{2\nu}[2c(u)+\eta]du} = Q\frac{e^{2\gamma\xi\sigma^2(T-t)} - e^{-2\gamma\xi\sigma^2(T-t)}}{e^{2\gamma\xi\sigma^2 T} - e^{-2\gamma\xi\sigma^2 T}} = X_t^{det,*}.$$

For any time $t \in [0, T)$,

$$\lim_{\phi\to\infty}[2c(t)+\eta] = \frac{1}{\xi}\frac{e^{-4\gamma\xi\sigma^2(T-t)}+1}{e^{-4\gamma\xi\sigma^2(T-t)}-1}.$$

Thus, we have

$$\lim_{\phi\to\infty} \theta_t^{\phi,*} = \lim_{\phi\to\infty} -\frac{1}{2\nu}[2c(t)+\eta]X_t^{\phi,*} = \frac{Q}{2\nu\xi}\frac{e^{-2\gamma\xi\sigma^2(T-t)} + e^{2\gamma\xi\sigma^2(T-t)}}{e^{2\gamma\xi\sigma^2 T} - e^{-2\gamma\xi\sigma^2 T}} = \theta_t^{det,*}.$$

$\square$

In Figure 1, we illustrate how the transaction fees involved by liquidating the outstanding position $X_{T-}^{\phi,*}$, $\phi|X_{T-}^{\phi,*}|^2$, affects the agent's liquidating speed. We chose the following values of the model parameters: $T = 1$ *day*, $Q = 100$ *units*, $\gamma = 0.1$, $\sigma = 0.2$, $\eta = 0.001$ and $\nu = 0.003$.

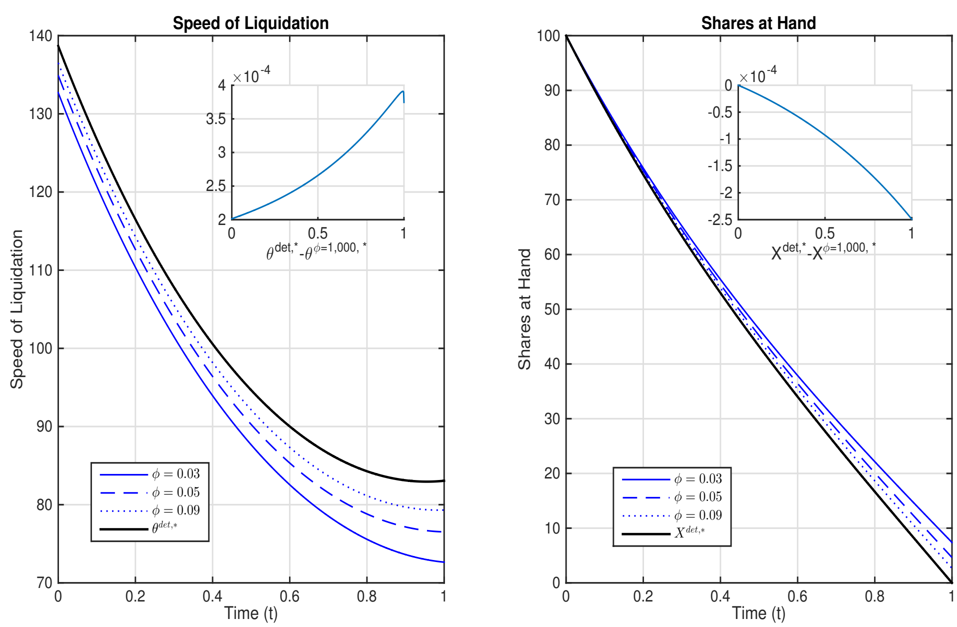

Figure 1: Optimal deterministic control vs. Optimal stochastic controls (model parameters: $T = 1$ *day*, $Q = 100$ *units*, $\gamma = 0.1, \sigma = 0.2, \eta = 0.001, \nu = 0.003$).

Figure 1 illustrates that the speed of liquidation which is free of *hands-clean* condition is always slower than that under the constraint of *hands-clean* condition. As the transaction fees involved by liquidating the outstanding position $X_{T-}^{\phi,*}$ increases (namely, as ϕ increases), the agent's liquidating speed increases, indicating that the optimal stochastic control moves closer to the optimal deterministic control. The embedded subfigures in Figure 1 show, respectively, the differences between the deterministic and stochastic liquidating strategies and the corresponding trajectories with $\phi = 1,000$. Both of them are of magnitude 10^{-4}.

4 Optimal Liquidation Strategy Subject to an Exogenous Trigger Event (Model 2)

In this section, we extend our results to models with an exogenous event, which does not depend on the information structure $\{\mathcal{F}_t\}_{t\geq 0}$.

Assumption 2 *The liquidation process will be suspended, if an exogenous trigger event occurs.*

We model the occurrence time of a trigger event, denoted by κ, to be random, and the hazard rate is given by $l(t)$. The survival probability at time t is

$$P(t) = \exp\Big(- \int_0^t l(u)du \Big). \tag{21}$$

The liquidation horizon is then defined by

$$\tau = \min\{T, \kappa\}, \tag{22}$$

where the constant $T \in (0, \infty)$ is a pre-determined time horizon. A direct computation yields the following proposition:

Proposition 2 *For $t < T$, the density function of τ is*

$$f_\tau(t) = l(t) \exp\Big(- \int_0^t l(u)du \Big).$$

The probability that τ takes the value of T is $\mathbb{P}(\tau = T) = P(T)$.

Denote by $\mathcal{G}_t$ the event $\{\tau > t\} = \{$the trigger event has *not* occurred by the time $t\}$. At any time $t < \tau$, i.e., the trigger event has not occurred prior to time t, the agent's objective is to find the optimal control for

$$\max_{\theta(\cdot)\in\Theta_t} \mathbb{E}\left[\int_t^{\tau-} \Pi(\theta_u, X_u)du - \phi X_{\tau-}^2 \big| \mathcal{F}_t \vee \mathcal{G}_t \right] \tag{23}$$

where

$$\Pi(\theta_t, X_t) = g(\theta_t)\theta_t + f(\theta_t)X_t - \gamma\sigma^2 X_t^2,$$

and $\mathcal{F}_t$ is the information structure available to the agent up to and including time t. If the trigger event occurs at time t, all market transactions will be suspended at that time. The agent will end up with an outstanding position X_t.

It is worth noting that

$$
\begin{aligned}
E\Big[\int_t^{\tau-}\Pi(\theta_u,X_u)du\Big|\mathcal{F}_t\vee\mathcal{G}_t\Big] &= E\Big[\int_{[t,T)}\mathbb{I}_{\{u<\tau\}}\Pi(\theta_u,X_u)du\Big|\mathcal{F}_t\vee\mathcal{G}_t\Big] \\
&= E\Big[\int_{[t,T)}\mathbb{P}(\tau>u|\mathcal{G}_t)\Pi(\theta_u,X_u)du\Big|\mathcal{F}_t\Big],
\end{aligned} \tag{24}
$$

and that

$$
\mathbb{P}(\tau>u|\mathcal{G}_t)=\mathbb{P}(\tau>u|\tau>t)=e^{-\int_t^u l(r)dr}.
$$

Here, the indicator function $\mathbb{I}_{\{\cdot\}}$ takes the value 1 when its argument is true and the value 0, otherwise. The last equality in Eq. (24) follows from the assumption that the trigger event is exogenous and does not depend on information structure $\mathcal{F}_t$.

Therefore, we have

$$
\begin{aligned}
&\max_{\theta(\cdot)\in\Theta_t}\mathbb{E}\left[\int_t^{\tau-}\Pi(\theta_u,X_u)du-\phi X_{\tau-}^2\Big|\mathcal{F}_t\vee\mathcal{G}_t\right] \\
=\;&\max_{\theta(\cdot)\in\Theta_t}\mathbb{E}\Big[\int_{[t,T)}e^{-\int_t^u l(r)dr}\left[\Pi(\theta_u,X_u)-\phi\cdot l(u)X_u^2\right]du-\phi e^{-\int_{[t,T)}l(r)dr}X_{T-}^2\Big|\mathcal{F}_t\Big].
\end{aligned} \tag{25}
$$

That is, the optimal liquidating problem with a random horizon τ defined in Eq. (22) is equivalent to an optimal liquidating problem with a finite horizon T, a consumption process $\{\Pi(\theta_t,X_t)-\phi\cdot l(t)X_t^2\}_{t\ge 0}$, a discount process $\{l(t),t\ge 0\}$, and a terminal condition $-\phi X_{T-}^2$.

4.1 Deterministic Control

Let us first consider the case in which $\theta(\cdot)$ ranges only over the subclass Θ_0^{det} of *deterministic* strategies in Θ_0 satisfying the *hands-clean* condition (3)[6], namely, $X_{T-}=0$. Thus, the agent's objective, before the trigger event occurs, is to find the optimal control for

$$
\max_{\theta(\cdot)\in\Theta_0^{det}}\mathbb{E}\int_{[0,T)}e^{-\int_0^t l(u)du}\left[\Pi(\theta_t,X_t)-\phi\cdot l(t)X_t^2\right]dt.
$$

The cost function of the deterministic control problem is

$$
\mathcal{H}(X_t,\theta_t,\Lambda_t,t)\equiv P(t)\left[\Pi(\theta_t,X_t)-\phi\cdot l(t)X_t^2\right]-\Lambda_t\theta_t,
$$

where Λ_t is the *Lagrange multiplier*, and $P(t)$ is the survival probability defined in Eq. (21). The differential equation for the deterministic system dynamics is

$$
\frac{dX_t}{dt}=-\theta_t \quad\text{and}\quad X_0=Q.
$$

We assume that the Hamiltonian $\mathcal{H}$ has continuous first-order derivatives in the state, adjoint state, and the control variable, namely, $\{X_t,\Lambda_t,\theta_t\}$. Then the necessary conditions

[6]The *hands-clean* condition only makes sense for the equivalent problem (25). While considering the original optimization problem (23), where the terminal time is a stopping time, the *hands-clean* condition is no longer valid.

for having an interior point optimum of the Hamiltonian $\mathcal{H}$ at $\{X_t^{det,*}, \Lambda_t^{det,*}, \theta_t^{det,*}\}$ are given by

$$\begin{cases} \dfrac{dX_t^{det,*}}{dt} &= \left.\dfrac{\partial \mathcal{H}}{\partial \Lambda}\right|_{(X_t^{det,*},\theta_t^{det,*},\Lambda_t^{det,*},t)} \\ -\dfrac{d\Lambda_t^{det,*}}{dt} &= \left.\dfrac{\partial \mathcal{H}}{\partial X}\right|_{(X_t^{det,*},\theta_t^{det,*},\Lambda_t^{det,*},t)} \\ 0 &= \left.\dfrac{\partial \mathcal{H}}{\partial \theta}\right|_{(X_t^{det,*},\theta_t^{det,*},\Lambda_t^{det,*},t)}. \end{cases} \tag{26}$$

It follows from Eq. (26) that

$$\begin{cases} \ddot{X}_t^{det,*} = l(t)\dot{X}_t^{det,*} + \dfrac{\gamma\sigma^2 + (\phi - \frac{\eta}{2}) \cdot l(t)}{\nu} X_t^{det,*} \\ X_0^{det,*} = Q \\ X_T^{det,*} = 0. \end{cases} \tag{27}$$

Regarding this linear second-order boundary value problem (BVP), its existence and uniqueness are standard. Interested readers can refer to, for example Hwang [11], for more details.

Consider the case when $l(t) \equiv \lambda \neq 0$, which corresponds to the case of constant hazard rate, an explicit solution is given by

$$\begin{cases} X_t^{det,*} = Qe^{\frac{\lambda}{2}t}\dfrac{\sinh(\alpha(T-t))}{\sinh(\alpha T)}, \\ \theta_t^{det,*} = -Qe^{\frac{\lambda}{2}t}\dfrac{\left[\frac{\lambda}{2}\sinh(\alpha(T-t)) - \alpha\cosh(\alpha(T-t))\right]}{\sinh(\alpha T)}, \end{cases}$$

where

$$\alpha = \sqrt{\frac{\lambda^2}{4} + \frac{\gamma\sigma^2 + (\phi - \frac{\eta}{2})\lambda}{\nu}}.$$

It is worth noting that (i) when $\lambda = 0$, the model degenerates to Model 1; (ii) as $\phi \to \infty$, $\lim_{\phi\to\infty} \theta_t^{det,*} = 0$ and $\lim_{\phi\to\infty} X_t^{det,*} = 0$, for all $t \in (0, T]$; and $\lim_{\phi\to\infty} \theta_0^{det,*} = \infty$. That is, as the final liquidation fee, ϕ per share, approaches infinity, the trader would immediately complete the transaction at the beginning of the trading horizon.

4.2 Dynamic Programming Approach

Let us consider the case of allowing dynamic updating, i.e., replacing Θ_0^{det} by the entire class of admissible strategies Θ_0. Let $F(t, q)$ denote the optimal value function of Eq. (25) at any time prior to the occurrence of the trigger event. Under appropriate regularity assumptions, F satisfies the following HJB equation:

$$l(t)F = \partial_t F - [\gamma\sigma^2 + \phi \cdot l(t)]q^2 - \min_{\theta_t \in \Theta_t} \left\{\nu\theta_t^2 + (\partial_q F + \eta q) \cdot \theta_t\right\} \tag{28}$$

subject to the terminal condition: $F(T, q) = -\phi q^2$. Here, $l(t)$ is the given hazard rate. Similarly, we consider relaxing the constraints associated with the HJB equation and solve the unconstrained optimization problem. We then prove that the obtained optimal control does satisfy all the constraints. The associated optimal trading strategy is

$$\theta_t^{\phi,*} = -\frac{1}{2\nu}\big(\partial_q F + \eta q\big),$$

and hence the value function satisfies

$$\begin{cases} \partial_t F - [\gamma\sigma^2 + \phi \cdot l(t)]q^2 + \dfrac{1}{4\nu}\big(\partial_q F + \eta q\big)^2 - l(t)F = 0 \\ F(T-, q) = -\phi q^2. \end{cases} \tag{29}$$

Regarding Eq. (29), we have the following theorem for the uniqueness of classical solutions.

Theorem 3 *There is at most one $\mathcal{C}^{1,2}([0,T) \times (0,\infty))$ solution to Eq. (29).*

Proof: Suppose f_1 and f_2 are two $\mathcal{C}^{1,2}([0,T) \times (0,\infty))$ solutions to Eq. (29). Define $\widetilde{f} = f_1 - f_2$. Then the new function $\widetilde{f}$ satisfies the following problem:

$$\begin{cases} \partial_t \widetilde{f} + \dfrac{1}{4\nu}\big[\partial_q(f_1 + f_2) + 2\eta q\big]\partial_q \widetilde{f} - l(t)\widetilde{f} = 0 \\ \widetilde{f}(T-, q) = 0. \end{cases}$$

Since the evolution equation for $\widetilde{f}$ is linear and first-order, one can solve the above problem explicitly by the method of characteristics, and find that $\widetilde{f} \equiv 0$ is the unique solution to this problem. As a result, $f_1 \equiv f_2$. □

Similar to Section 3.2, we consider an ansatz that is quadratic in the variable q:

$$F(t,q) = \widetilde{a}(t) + \widetilde{b}(t)q + \widetilde{c}(t)q^2.$$

Substituting the ansatz into Eq. (29), we know that the coefficients $\widetilde{a}(t)$, $\widetilde{b}(t)$ and $\widetilde{c}(t)$ must satisfy the following partially decoupled system:

$$\begin{cases} \dot{\widetilde{c}}(t) = l(t)\widetilde{c}(t) + \gamma\sigma^2 + \phi \cdot l(t) - \dfrac{1}{4\nu}[2\widetilde{c}(t) + \eta]^2 \\ \dot{\widetilde{b}}(t) = l(t)\widetilde{b}(t) - \dfrac{1}{2\nu}\widetilde{b}(t)[2\widetilde{c}(t) + \eta] \\ \dot{\widetilde{a}}(t) = l(t)\widetilde{a}(t) - \dfrac{1}{4\nu}\widetilde{b}^2(t) \end{cases} \tag{30}$$

with terminal conditions: $\widetilde{a}(T-) = 0$, $\widetilde{b}(T-) = 0$ and $\widetilde{c}(T-) = -\phi$.

It is straightforward to verify that

$$\widetilde{b}(t) \equiv 0 \quad \text{and} \quad \widetilde{a}(t) \equiv 0.$$

However, the equation satisfied by $\widetilde{c}(t)$ is a Riccati equation, which can be reduced to a second-order linear ODE:

$$u'' - l(t)u' - \frac{\gamma\sigma^2 + (\phi - \frac{\eta}{2})l(t)}{\nu}u = 0, \tag{31}$$

where u is defined implicitly via $\widetilde{c}(t) = \frac{\nu u'}{u} - \frac{\eta}{2}$. For this second-order linear ODE, its existence and uniqueness are standard. Even though we know the existence and uniqueness of the solution, it is still difficult to solve it in a closed-form for a general hazard rate $l(t)$. The above second-order linear ODE can be easily solved in two cases: (i) its coefficients are constant; or (ii) its coefficients adopt particular forms.

If closed-form solutions cannot be obtained, finite difference method can be applied to solving the BVP numerically. For more details, see, for example, Hwang [11].

Theorem 4 (Constant hazard rate). *When the hazard rate is a constant, i.e., $l(t) \equiv \lambda$, the unknown function $\widetilde{c}(t)$ can be explicitly solved. It is given by*

$$\widetilde{c}(t) = \frac{1}{2\hat{\xi}} \left[\frac{\hat{\zeta} e^{-2\alpha(T-t)} - 1}{\hat{\zeta} e^{-2\alpha(T-t)} + 1} \right] + \frac{\lambda\nu - \eta}{2}$$

with

$$\alpha = \sqrt{\frac{\lambda^2}{4} + \frac{\gamma\sigma^2 + (\phi - \frac{\eta}{2})\lambda}{\nu}}, \quad \hat{\zeta} = \frac{1 - \hat{\xi}(2\phi + \lambda\nu - \eta)}{1 + \hat{\xi}(2\phi + \lambda\nu - \eta)} \quad \text{and} \quad \hat{\xi} = \frac{1}{2\alpha\nu}.$$

A direct verification yields that when $\lambda = 0$, $\widetilde{c}(t) = c(t)$. The results derived under Model 2 coincide with those derived under Model 1. The optimal liquidating strategy for the unconstrained problem can then be derived through the following relation:

$$\theta_t^{\phi,*} = -\frac{1}{2\nu}[2\widetilde{c}(t) + \eta]q, \tag{32}$$

and hence,

$$X_t^{\phi,*} = Q \cdot \exp\left(\int_0^t \frac{1}{2\nu}[2\widetilde{c}(u) + \eta]du \right).$$

The following theorem provides us the relation between the optimal deterministic control and the optimal stochastic control.

Theorem 5 *When the transaction fees involved by liquidating the outstanding position X_{T-} approaches to infinity, the optimal stochastic control process $(\theta_t^{\phi,*})_{t\in[0,T)}$ before the trigger event occurs converges (point-wise) to the optimal deterministic control process $(\theta_t^{det,*})_{t\in[0,T)}$. Meanwhile, the optimal trajectory $X_t^{\phi,*}$ converges (point-wise) to the one determined in the deterministic system $X_t^{det,*}$: as $\phi \to \infty$, for any time $t \in [0,T)$,*

1. $\lim_{\phi\to\infty} \theta_t^{\phi,*} = \lim_{\phi\to\infty} \theta_t^{det,*}$;

2. $\lim_{\phi\to\infty} X_t^{\phi,*} = \lim_{\phi\to\infty} X_t^{det,*}$.

Proof: The proof of this theorem is very similar to that of Theorem 2. Therefore we will not provide all the details; instead we will just outline the proof as follows. First, a simple calculation yields that

$$\begin{aligned}
\lim_{\phi\to\infty} X_t^{\phi,*} &= \lim_{\phi\to\infty} Q e^{-(\alpha - \frac{\lambda}{2})t} \frac{\hat{\zeta} e^{-2\alpha(T-t)} + 1}{\hat{\zeta} e^{-2\alpha T} + 1} \\
&= \lim_{\phi\to\infty} Q e^{-(\alpha - \frac{\lambda}{2})t} \frac{e^{-2\alpha(T-t)} - 1}{e^{-2\alpha T} - 1} \\
&= \lim_{\phi\to\infty} X_t^{det,*}.
\end{aligned}$$

Following the relations

$$\begin{cases} \theta_t^{\phi,*} = -\dfrac{1}{2\nu}[2\widetilde{c}(t)+\eta]X_t^{\phi,*}, \\ \lim\limits_{\phi\to\infty}\dfrac{1}{2\nu}[2\widetilde{c}(t)+\eta] = \lim\limits_{\phi\to\infty}\Big[\alpha\dfrac{e^{-2\alpha(T-t)}+1}{e^{-2\alpha(T-t)}-1}+\dfrac{\lambda}{2}\Big], \end{cases}$$

we can further verify that $\lim_{\phi\to\infty}\theta_t^{\phi,*} = \lim_{\phi\to\infty}\theta_t^{det,*}$. $\square$

We remark that if condition (18) is satisfied, then

$$\frac{d\widetilde{c}(t)}{dt} < 0 \quad \text{and} \quad 2\widetilde{c}(0)+\eta < \lambda\nu - \frac{1}{\hat{\xi}} < 0.$$

Therefore, $2\widetilde{c}(t)+\eta < 0$ always holds for any time $t \in [0,T)$. We can further verify that (a1) $\theta_t^{\phi,*} \geq 0$ holds for any time $t \in [0,T)$; and (a2) $\int_{[0,T)}\theta_t^{\phi,*}dt \leq Q$. That is, the obtained optimal trading strategy in Eq. (32) is also the optimal trading strategy for the constrained problem.

4.3 Numerical Results

In this section, we provide some numerical results to illustrate the effects of exogenous trigger event on the agent's liquidating strategy. Suppose the size of the target order to be liquidated is $Q = 100$ *units*, the liquidation time $T = 1$ *day*, and the hazard rate at which the trigger event occurs is $\lambda = 1$. The model parameters' values are set as follows:

$$\gamma = 0.1, \quad \sigma = 0.2, \quad \eta = 0.001, \quad \nu = 0.003, \quad \phi = 0.1.$$

Figure 2 provides a comparison of liquidation strategies under two different settings: one without trigger event (Model 1), and the other with trigger event (Model 2). In the upper-panel and middle-panel plots given in Figure 2, we can observe that an exogenous trigger event occurs at time $t = 0.46$. At that time, all trades are suspended in Model 2:

$$\theta_t|_{t\in(0.46,1]} = 0 \quad \text{and} \quad X_t|_{t\in(0.46,1]} = X_{t=0.46}.$$

Since our objective is to liquidate a large position before time $T = 1$ (Model 1) or time $\tau = \min\{0.46, 1\}$ (Model 2), agents facing the scenario that an exogenous trigger event might occur during the trading horizon (Model 2) would like to accelerate the rate of liquidating to reduce their exposure to potential position risk and eventually in a smaller position when the trigger event occurs. Their strategy is more "convex" compared with those who are not threatened by this trigger event, as can be seen from the upper-panel plot given in Figure 2.

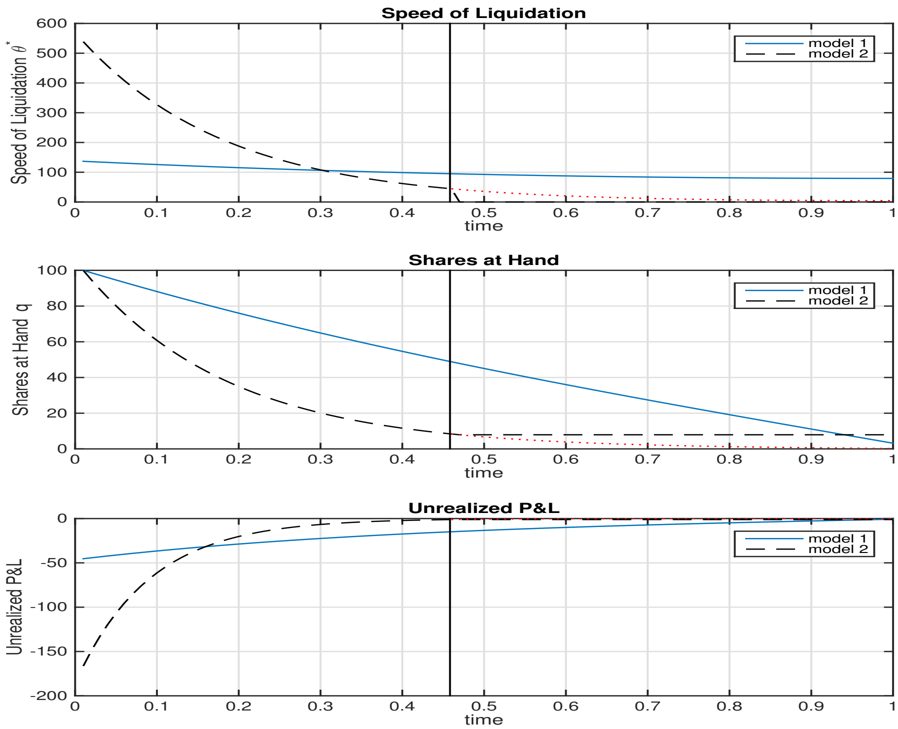

Figure 2: A comparison of liquidation strategies under two different settings: one without trigger event (Model 1), and the other with trigger event (Model 2).

The lower-panel plot given in Figure 2 depicts the updated unrealized Profit and Loss (P&L) profile of the DP problems as a function of time t:

$$U(t,q) = c(t)q^2 \quad \text{and} \quad F(t,q) = \widetilde{c}(t)q^2.$$

Notice that at any time $t \in [0,T]$, according to the DP principle, the value function at time $t = 0$ can be written as follows:

$$U(0,Q) = \underbrace{R_t^* - \gamma[V^*,V^*]([0,t))}_{\text{realized P\&L}} + \underbrace{U(t,X_t)}_{\text{unrealized P\&L}}.$$

Here $R_t^* - \gamma[V^*,V^*]([0,t))$ can be regarded as the realized P&L, and $U(t,X_t)$ can be regarded as the unrealized P&L. As we can see from Figure 2, at the very beginning, due to the potential position risk incurred by exogenous trigger events, the unrealized P&L under the second setting is significantly smaller than that under the first setting. This gap would eventually be narrowed through the adjustment of the trading strategy, and at time $t = 0.15$, before the occurrence of the trigger event, this situation is completely reversed.

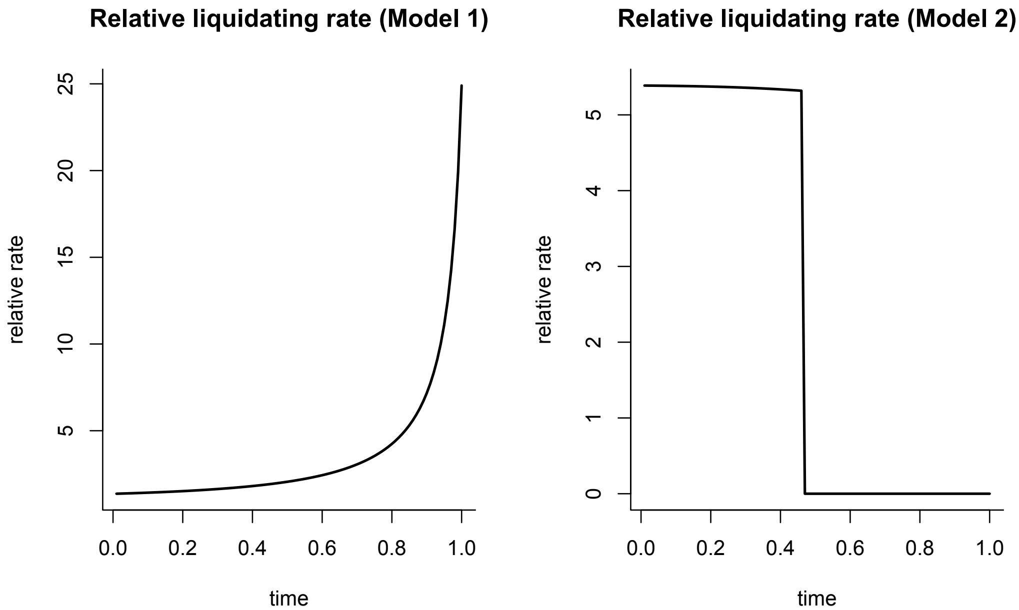

Figure 3: A comparison of relative liquidating speed θ/q under two different settings: one without trigger event (Model 1), and the other with trigger event (Model 2).

Figure 3 displays the relative liquidating speeds θ_t/X_t under the two different settings. We clearly see that the relative liquidating rate under the first setting (Model 1) depends on time-to-maturity in a monotonic way. Indeed, as t approaches to the time horizon T, there is a real need for an agent to liquidate because the liquidation cost ϕX_{T-}^2 at time T is high. However, if the agent faces an additional risk that an exogenous trigger event might occur during the liquidating horizon, when the related risk is high, he/she needs to trade faster to reduce this risk. As in the above figure, the relative liquidating rate under the second setting shows a nearly constant during the period $[0, \tau)$.

5 Optimal Liquidation Strategy Subject to Counterparty Risk (Model 3)

In this section, we assume that the trigger event is not exogenous. It is incurred by the evolvement of the market value of the stock issuer.

5.1 The Hitting Time

Let the stock issuer's market value Y_t evolves over time according to

$$\begin{cases} dY_t = Y_t\big(\beta dt + \xi dW_t^Y\big), \quad Y_0 = y_0 \\ dW_t^Y dW_t^S = \rho, \end{cases}$$

where the constant β is the mean rate of return of the company, the constant ξ is the volatility, $\{W_t^S\}$ and $\{W_t^Y\}$ are two correlated Brownian motions, and the constant ρ is the correlation coefficient, with $|\rho| < 1$. Thus, we have

$$Y_t = y_0 e^{(\beta - \frac{\xi^2}{2})t + \xi W_t^Y}.$$

We assume that once the company's market value falls down to a pre-determined limit $\alpha^* > 0$ (it is pre-assumed that $y_0 > \alpha^*$), a great switch will be involved in this company and the liquidation process will be forced to suspend. Let

$$m(y_0) = \frac{1}{\xi} \ln\left(\frac{y_0}{\alpha^*}\right) > 0 \quad \text{and} \quad \alpha = \frac{1}{\xi}\left(\frac{\xi^2}{2} - \beta\right),$$

and define $\widehat{W}_t^Y = W_t^Y + \alpha t$. According to the information, the time at which this switch occurs is defined by[7]

$$\kappa_{m(y_0)} = \inf\{t \geq 0 : \widehat{W}_t^Y = m(y_0)\}.$$

The liquidation horizon is then given by $\tau_{m(y_0)} = \min\{\kappa_{m(y_0)}, T\}$, which is a stopping time. Due to the Markov property of the Brownian motion $\{\widehat{W}_t^Y\}$, given $\kappa_{m(y_0)} > t$ and $Y_t = y$, the conditional distribution of $\kappa_{m(y_0)}$ is given by

$$\kappa_{m(y_0)} | \{\kappa_{m(y_0)} > t \vee Y_t = y\} = t + \kappa_{m(y)}.$$

Therefore, we obtain

$$\tau_{m(y_0)} | \{\tau_{m(y_0)} > t \vee Y_t = y\} = t + \min\{\kappa_{m(y)}, T - t\} =: \tau_{t,m(y)}. \tag{33}$$

Proposition 3 *For the Brownian motion, $\widehat{W}_t^Y = W_t^Y + \alpha t$, $\alpha \neq 0$, define*

$$\kappa_m = \inf\{t \geq 0 : \widehat{W}_t^Y = m\}, \quad m > 0.$$

The Laplace transform of κ_m is

$$\mathbb{E}[e^{-u\kappa_m}] = e^{\alpha m - m\sqrt{2u + \alpha^2}}, \quad \textit{for all } u > 0.$$

Recall that, at any time t prior to the time horizon $\tau_{t,m(y)}$ (which is defined in Eq. (33)) with initial value $Y_t = y$ and $X_t = q$, the agent's objective is to find the optimal control for

$$\begin{aligned} H(t, y, q) &= \max_{\theta(\cdot) \in \Theta_t} \mathbb{E}\left[\int_t^{\tau_{t,m(y)}-} \Pi(\theta_r, X_r) dr - \phi X^2_{\tau_{t,m(y)}-} \Big| \mathcal{F}_t\right] \\ &= \max_{\theta(\cdot) \in \Theta_t} \mathbb{E}\left[\int_{[t,T)} \mathbb{I}_{\{r < \tau_{t,m(y)}\}} \Pi(\theta_r, X_r) dr - \phi X^2_{\tau_{t,m(y)}-} \Big| \mathcal{F}_t\right], \end{aligned} \tag{34}$$

where

$$\Pi(\theta_t, X_t) = g(\theta_t)\theta_t + f(\theta_t) X_t - \gamma \sigma^2 X_t^2.$$

We verify comparison principles for viscosity solutions and characterize the value function as the unique viscosity solution of the associated HJB equation.

[7] We use the fact that $-W_t^Y$ has the same distribution as W_t^Y.

Remark 1 *Assume that the company's market value has not hit the pre-determined level* α^* *by time* t. *By the definition of* $m(y)$, *we have* $m(y) \to \infty$ *as* $y \to \infty$. *According to Proposition 2,* $\lim_{y\to\infty} \mathbb{E}[e^{-u\kappa_{m(y)}}] = 0$, *for all* $u > 0$. *This implies that,*

$$0 \le e^{-uN}\mathbb{P}(\kappa_{m(y)} \le N) \le \mathbb{E}[e^{-u\kappa_{m(y)}}],$$

for any positive integer N. *Hence,* $\lim_{y\to\infty} \mathbb{P}(\kappa_{m(y)} \le N) = 0$. *Passing to the limit* $N \to \infty$, *we conclude that*

$$\lim_{y\to\infty} \mathbb{P}(\kappa_{m(y)} < \infty) = 0 \quad \textit{and} \quad \lim_{y\to\infty} \mathbb{P}(\kappa_{m(y)} = \infty) = 1.$$

Therefore, $\lim_{y\to\infty} \mathbb{P}(\tau_{t,m(y)} = T) = 1$.

5.2 Dynamic Programming Approach

In this section, we discuss some analytical properties of the value function H without proofs. Some technical proofs will be provided in Sections 5.3 and 5.4.

Theorem 6 *Let* $H(t, y, q)$ *denote the value function in Eq. (34) at any time* t *before the process* $\{Y_t, t \ge 0\}$ *touches the pre-determined limit* α^* *and* $(Y_t, X_t) = (y, q)$. *Suppose the value function* $H(t, y, q)$ *is sufficiently smooth*[8], *i.e.,* $H \in \mathcal{C}^{1,2,2}([0, T) \times (\alpha^*, +\infty) \times (0, +\infty))$, *then* $H(t, y, q)$ *satisfies the HJB equation*

$$-(\partial_t + \beta y \partial_y + \frac{1}{2}\xi^2 y^2 \partial_{yy})H + \gamma\sigma^2 q^2 - \max_{\theta_t \in \Theta_t} \Big\{ g(\theta_t)\theta_t + f(\theta_t)q - \partial_q H \cdot \theta_t \Big\} = 0, \tag{35}$$

in the region

$$\{(t, y, q) : 0 \le t < T, y > \alpha^*, q > 0\}$$

and satisfies the boundary conditions

$$(a)\ \ H(T, y, q) = -\phi q^2, \qquad y > \alpha^*, \tag{35.a}$$

$$(b)\ \ H(t, \alpha^*, q) = -\phi q^2, \qquad 0 \le t < T, \tag{35.b}$$

$$(c)\ \ \lim_{y\to\infty} H(t, y, q) = U(t, q), \tag{35.c}$$

where $U(t, q)$ *is the value function of the optimization problem that we considered in Model 1.*

It is worth noting that for any $y \ge \alpha^*$, $H(t, y, q) \le U(t, q)$, and for any $y_1 \ge y_2 \ge \alpha^*$, $H(t, y_1, q) \ge H(t, y_2, q)$. The rationale behind this is intuitive. Compared with "default-free" model (Model 1), greater counterparty risk gives a smaller value function.

[8] $\mathcal{C}^{1,2,2}([0, T) \times (\alpha^*, \infty) \times (0, \infty))$ is the space of functions $f(t, y, q)$ which is continuously differentiable in t, and twice continuously differentiable in y and q.

5.3 Monotonicity and Continuity of $H(t,y,q)$

In Section 5.2, we presented without proof the primary analytical properties of the value function $H(t,y,q)$. In this section, we prove the monotonicity, growth rate control and continuity of $H(t,y,q)$ as follows.

Theorem 7 *Assume that condition (18) is satisfied: $2\phi > \eta + 2\sigma\sqrt{\gamma\nu}$. Then, we have*

(i) **(Monotonicity)** *$H(t,y,q)$ is an increasing function in y, and a decreasing function in t;*

(ii) **(Continuity)** *$H(t,y,q)$ is locally Hölder continuous in t with exponent $1/2$, and locally Lipschitz continuous in both y and q in $[0,T)\times(\alpha^*,\infty)\times(0,\infty)$;*

(iii) **(Growth Rate Control)** *$H(t,y,q)$ satisfies a quadratic growth condition with respect to the inventory variable q: for any $(t,y,q)\in[0,T)\times(\alpha^*,\infty)\times(0,\infty)$,*

$$H(t,y,q) \le \left(\phi + \big[\frac{(2\phi-\eta)^2}{4\nu} - \gamma\sigma^2\big](T-t)\right)q^2.$$

To avoid confusion, in the sections to follow, we denote $X(t)$, the number of shares at time t, and $\{X^{\theta}_{t,q}(u)\}_{u\ge t}$, a trajectory of $X(\cdot)$ given $X(t)=q$ and a trading strategy θ. To prove Theorem 7, we first convert the original control problem into a problem without terminal bequest function. Since $g(x) = -\phi x^2$ is continuously differentiable, and $\mathbb{E}[\tau_{t,m(y)}|\mathcal{F}_t] < T < \infty$, we can apply Dynkin's formula to $-\phi X^2(t)$ and rewrite the value function H as

$$H(t,y,q) = -\phi q^2 + \max_{\theta(\cdot)\in\Theta_t}\mathbb{E}\left[\int_t^{\tau_{t,m(y)}-} L(\theta_r, X^{\theta}_{t,q}(r))dr\Big|\mathcal{F}_t\right]$$

where

$$L(\theta,q) = \Pi(\theta,q) + 2\phi q\theta. \tag{36}$$

Define a new value function as

$$\widehat{H}(t,y,q) = \max_{\theta(\cdot)\in\Theta_t}\mathbb{E}\left[\int_t^{\tau_{t,m(y)}-} L(\theta_r, X^{\theta}_{t,q}(r))dr\Big|\mathcal{F}_t\right].$$

Proof of Theorem 7.

(i) One approach to verify the monotonicity in y is directly applying the definition. Let $\theta^{y_2,*}$ denote the optimal control process with respect to the stopping time $\tau_{t,m(y_2)}$. For any positive numbers $y_1 \ge y_2 > \alpha^*$, we have $\tau_{t,m(y_1)} \ge \tau_{t,m(y_2)}$. From this observation, we have

$$\begin{aligned}
\widehat{H}(t,y_1,q) &= \max_{\theta(\cdot)\in\Theta_t}\mathbb{E}\left[\int_t^{\tau_{t,m(y_1)}-} L(\theta_r, X^{\theta}_{t,q}(r))dr\Big|\mathcal{F}_t\right]\\
&\ge \mathbb{E}\left[\int_t^{\tau_{t,m(y_2)}-} L(\theta^{y_2,*}_r, X^{\theta^{y_2,*}}_{t,q})dr\Big|\mathcal{F}_t\right]\\
&\quad + \max_{\theta(\cdot)\in\Theta_{\tau_{t,m(y_2)}}}\underbrace{\mathbb{E}\left[\int_{\tau_{t,m(y_2)}}^{\tau_{t,m(y_1)}-} L(\theta_r, X^{\theta}_{\tau_{t,m(y_2)},X^{\theta^{y_2,*}}_{t,q}(\tau_{t,m(y_2)}-)}(r))dr\Big|\mathcal{F}_t\right]}_{\text{(II)}}.
\end{aligned}$$

Since

$$L(\theta,q)=\Pi(\theta,q)+2\phi q\theta=-\nu\theta^2+(2\phi-\eta)q\theta-\gamma\sigma^2q^2.$$

Under the assumption that $2\phi>\eta+2\sigma\sqrt{\gamma\nu}$, for any $q\in(0,+\infty)$, we can always choose $\theta:=\frac{2\phi-\eta}{2\nu}q\geq 0$ such that $L(\theta,q)\geq 0$, and hence, (II) ≥ 0. Therefore, $\widehat{H}(t,y_1,q)\geq\widehat{H}(t,y_2,q)$, and hence $H(t,y_1,q)\geq H(t,y_2,q)$.

Another approach to this question is to apply the result in Section 3.2. Let $U_T(t,q)$ denote the value function of the optimization problem (11)[9] with time horizon T. Under condition (18), for any $T_1>T_2>t$, we have Inequality (19):

$$U_{T_1}(t,q)>U_{T_2}(t,q).$$

If we set $T_1=\tau_{t,m(y_1)}$ and $T_2=\tau_{t,m(y_2)}$, then

$$H(t,y_1,q)=\mathbb{E}[U_{\tau_{t,m(y_1)}}(t,q)|\mathcal{F}_t]\geq\mathbb{E}[U_{\tau_{t,m(y_2)}}(t,q)|\mathcal{F}_t]=H(t,y_2,q).$$

Similarly, we can verify the monotonicity of $H(t,y,q)$ in t. If we set $\bar{U}(\iota,q)=U_T(t,q)$, where $\iota=T-t$ is the time to maturity, then, according to Proposition 1, for any $0\leq\iota_2<\iota_1<T$,

$$\bar{U}(\iota_1,q)>\bar{U}(\iota_2,q).$$

For any $0\leq t_1<t_2<T$. Let $\iota_1=\tau_{t_1,m(y)}-t_1$ and $\iota_2=\tau_{t_2,m(y)}-t_2$. By the definition of $\tau_{t,m(y)}$, we have $\iota_1\geq\iota_2$, and hence

$$H(t_1,y,q)=\mathbb{E}[\bar{U}(\iota_1,q)|\mathcal{F}_t]\geq\mathbb{E}[\bar{U}(\iota_2,q)|\mathcal{F}_t]=H(t_2,y,q).$$

(ii) In order to prove the continuity of H, it suffices to show that for any two points (t_1,y_1,q_1) and (t_2,y_2,q_2) in the region

$$\{(t,y,q):0\leq t<T,\alpha^*<y,0<q\},$$

there exist three (t,y)-independent, polynomial-growth (with respect to (q_1,q_2)) coefficients $K_1(q_1,q_2)$, $K_2(q_1,q_2)$, and $K_3(q_1,q_2)$, such that

$$\begin{aligned}|\widehat{H}(t_1,y_1,q_1)-\widehat{H}(t_2,y_2,q_2)| &\leq |\widehat{H}(t_1,y_1,q_1)-\widehat{H}(t_1,y_2,q_1)|\\ &\quad+|\widehat{H}(t_1,y_2,q_1)-\widehat{H}(t_1,y_2,q_2)|\\ &\quad+|\widehat{H}(t_1,y_2,q_2)-\widehat{H}(t_2,y_2,q_2)|\\ &\leq K_1|y_1-y_2|+K_2|q_1-q_2|+K_3(|t_2-t_1|^{\frac{1}{2}}+|t_2-t_1|).\end{aligned}$$

We divide the proof into three parts: one is for the variable y, another one is for the variable q, and the rest is for the variable t.

Step 1 (Variable y). For any positive numbers $y_1\geq y_2>\alpha^*$, we have $\tau_{t,m(y_1)}\geq\tau_{t,m(y_2)}$, and hence

$$\begin{aligned}&\widehat{H}(t,y_1,q)\\ =\;&\mathbb{E}\left[\int_t^{\tau_{t,m(y_2)}-}L(\theta_r^{y_1,*},X_{t,q}^{\theta^{y_1,*}})dr+\int_{\tau_{t,m(y_2)}}^{\tau_{t,m(y_1)}-}L(\theta_r^{y_1,*},X^{\theta^{y_1,*}}_{\tau_{t,m(y_2)},X_{t,q}^{\theta^{y_1,*}}(\tau_{t,m(y_2)}-)}(r))dr\Big|\mathcal{F}_t\right]\\ \leq\;&\widehat{H}(t,y_2,q)+\max_{\theta(\cdot)\in\Theta_{\tau_{t,m(y_2)}}}\mathbb{E}\left[\int_{\tau_{t,m(y_2)}}^{\tau_{t,m(y_1)}-}L(\theta_r,X^{\theta}_{\tau_{t,m(y_2)},X_{t,q}^{\theta^{y_1,*}}(\tau_{t,m(y_2)}-)}(r))dr\Big|\mathcal{F}_t\right],\end{aligned}\tag{37}$$

[9] The associated liquidation problem without countparty risk.

where $\theta^{y_1,*}$ is the optimal control process with respect to the stopping time $\tau_{t,m(y_1)}$. Thus, using part (i), we have

$$\begin{aligned}
&|\widehat{H}(t,y_1,q)-\widehat{H}(t,y_2,q)| = \widehat{H}(t,y_1,q)-\widehat{H}(t,y_2,q)\\
\leq & \max_{\theta(\cdot)\in\Theta_{\tau_{t,m(y_2)}}} \mathbb{E}\left[\int_{\tau_{t,m(y_2)}}^{\tau_{t,m(y_1)}-} L(\theta_r, X^{\theta}_{\tau_{t,m(y_2)},X^{\theta^{y_1,*}}_{t,q}(\tau_{t,m(y_2)}-)}(r))dr\Big|\mathcal{F}_t\right].
\end{aligned}\tag{38}$$

A completing square yields

$$L(\theta,X)=-\nu\big[\theta-\frac{2\phi-\eta}{2\nu}X\big]^2+\big[\frac{(2\phi-\eta)^2}{4\nu}-\gamma\sigma^2\big]|X|^2,$$

so[10]

$$\begin{aligned}
&\max_{\theta(\cdot)\in\Theta_{\tau_{t,m(y_2)}}} \mathbb{E}\left[\int_{\tau_{t,m(y_2)}}^{\tau_{t,m(y_1)}-} L(\theta_r, X^{\theta}_{\tau_{t,m(y_2)},X^{\theta^{y_1,*}}_{t,q}(\tau_{t,m(y_2)}-)}(r))dr\Big|\mathcal{F}_t\right]\\
\leq & \big[\frac{(2\phi-\eta)^2}{4\nu}-\gamma\sigma^2\big]q^2\mathbb{E}\left[\int_{\tau_{t,m(y_2)}}^{\tau_{t,m(y_1)}-} dr\Big|\mathcal{F}_t\right]\\
= & \big[\frac{(2\phi-\eta)^2}{4\nu}-\gamma\sigma^2\big]q^2\mathbb{E}\left[\tau_{t,m(y_1)}-\tau_{t,m(y_2)}|\mathcal{F}_t\right]
\end{aligned}$$

because

$$\left|X^{\theta}_{\tau_{t,m(y_2)},X^{\theta^{y_1,*}}_{t,q}(\tau_{t,m(y_2)}-)}(r)\right|^2 \leq \left|X^{\theta^{y_1,*}}_{t,q}(\tau_{t,m(y_2)}-)\right|^2 \leq q^2.$$

By the definition of $\tau_{t,m(y)}$, we have $\tau_{t,m(y_1)}-\tau_{t,m(y_2)} \leq \kappa_{m(y_1)}-\kappa_{m(y_2)}$. According to Proposition 3 and the definition of $m(y)$, there exists a constant $c_0>0$ such that

$$\mathbb{E}[\kappa_{m(y_1)}-\kappa_{m(y_2)}|\mathcal{F}_t] \leq c_0|\ln(y_1)-\ln(y_2)|,$$

and hence,

$$|\widehat{H}(t,y_1,q)-\widehat{H}(t,y_2,q)| \leq c_0\big[\frac{(2\phi-\eta)^2}{4\nu}-\gamma\sigma^2\big]q^2|\ln(y_1)-\ln(y_2)|. \tag{39}$$

Since $|\ln(y_1)-\ln(y_2)| \leq \frac{1}{\alpha^*}|y_1-y_2|$, for any $y_1,y_2\in(\alpha^*,+\infty)$. There exists a (t,y)-independent, quadratic-growth coefficient $K_1(q)$ so that

$$|\widehat{H}(t,y_1,q)-\widehat{H}(t,y_2,q)| \leq K_1(q)|y_1-y_2|.$$

Step 2 (Variable q). Let $q_1,q_2\in(0,+\infty)$ satisfying $|q_1-q_2|\leq 1$. Consider the value functions $\widehat{H}(t,y,q_1)$ and $\widehat{H}(t,y,q_2)$. By the definition and the relation $|\max f-\max g|\leq \max|f-g|$, we have

$$\begin{aligned}
&|\widehat{H}(t,y,q_1)-\widehat{H}(t,y,q_2)|\\
\leq & \max_{\theta(\cdot)\in\Theta_t}\mathbb{E}\left[\int_t^T \mathbb{I}_{t\leq r<\tau_{t,m(y)}}\cdot\big|L(\theta_r,X^{\theta}_{t,q_1}(r))-L(\theta_r,X^{\theta}_{t,q_2}(r))\big|dr\Big|\mathcal{F}_t\right]\\
\leq & \max_{\theta(\cdot)\in\Theta_t}\mathbb{E}\left[\int_t^T \big|L(\theta_r,X^{\theta}_{t,q_1}(r))-L(\theta_r,X^{\theta}_{t,q_2}(r))\big|dr\Big|\mathcal{F}_t\right]\\
\leq & K_2(q_1,q_2)|q_1-q_2|,
\end{aligned}\tag{40}$$

[10] It is worth noting that condition (18) implies that $\frac{(2\phi-\eta)^2}{4\nu}-\gamma\sigma^2>0$.

where $K_2(q_1, q_2)$ is a polynomial-growth coefficient. The last inequality follows from Definition 1 and the fact that

$$X^{\theta}_{t,q_1}(r) - X^{\theta}_{t,q_2}(r) = q_1 - q_2,$$

for any trading strategy $\theta \in \Theta_t$.

Step 3 (Variable t). Let $0 \le t_1 < t_2 < T$, and $(y, q) \in (\alpha^*, \infty) \times (0, +\infty)$. By the DP principle,

$$\widehat{H}(t_1, y, q) = \mathbb{E}\left[\int_{t_1}^{t_2} L(\theta_r, X^{\theta^*}_{t_1,q}(r))dr + \widehat{H}(t_2, Y_{t_1,y}(t_2), X^{\theta^*}_{t_1,q}(t_2))\Big|\mathcal{F}_{t_1}\right],$$

where θ^* is the optimal control process. Therefore, by part (i),

$$\begin{aligned} & |\widehat{H}(t_1, y, q) - \widehat{H}(t_2, y, q)| = \widehat{H}(t_1, y, q) - \widehat{H}(t_2, y, q) \\ = \;& \mathbb{E}\left[\int_{t_1}^{t_2} L(\theta^*_r, X^{\theta^*}_{t_1,q}(r))dr\Big|\mathcal{F}_{t_1}\right] + \mathbb{E}\big[\widehat{H}(t_2, Y_{t_1,y}(t_2), X^{\theta^*}_{t_1,q}(t_2)) - \widehat{H}(t_2, y, q)\big|\mathcal{F}_{t_1}\big] \\ =: \;& I_1 + I_2. \end{aligned}$$

For the second term I_2, we have

$$\begin{aligned} I_2 \;\le\;& \mathbb{E}\left[\big|\widehat{H}(t_2, Y_{t_1,y}(t_2), X^{\theta^*}_{t_1,q}(t_2)) - \widehat{H}(t_2, Y_{t_1,y}(t_2), q)\big|\Big|\mathcal{F}_{t_1}\right] \\ & + \mathbb{E}\left[\big|\widehat{H}(t_2, Y_{t_1,y}(t_2), q) - \widehat{H}(t_2, y, q)\big|\Big|\mathcal{F}_{t_1}\right] \\ \le\;& \mathbb{E}[K_2(X^{\theta^*}_{t_1,q}(t_2), q)|X^{\theta^*}_{t_1,q}(t_2) - q||\mathcal{F}_{t_1}] + C_2(q)\mathbb{E}[|\ln(Y_{t_1,y}(t_2)) - \ln(y)||\mathcal{F}_{t_1}] \\ \le\;& C_1(q)\mathbb{E}[|X^{\theta^*}_{t_1,q}(t_2) - q||\mathcal{F}_{t_1}] + C_2(q)\mathbb{E}[|\ln(Y_{t_1,y}(t_2)) - \ln(y)||\mathcal{F}_{t_1}], \end{aligned} \tag{41}$$

where $C_1(q)$ and $C_2(q)$ are two (t, y)-independent, polynomial-growth coefficients. The second-to-last inequality follows form the results in Eq. (39) and Eq. (40). The last inequality follows from the fact that $|X^{\theta^*}_{t_1,q}(t_2)| \le q$. Noticing that

1. by the completing square trick as used in Step 1,

$$\begin{aligned} I_1 + C_1(q)\mathbb{E}[|X^{\theta^*}_{t_1,q}(t_2) - q||\mathcal{F}_{t_1}] \;=\;& \mathbb{E}\left[\int_{t_1}^{t_2}\left[L(\theta^*_r, X^{\theta^*}_{t_1,q}(r)) + C_1(q)\theta^*_r\right]dr\Big|\mathcal{F}_{t_1}\right] \\ \le\;& \Big(\frac{[(2\phi - \eta) + C_1(q)/q]^2}{4\nu} - \gamma\sigma^2\Big)q^2|t_2 - t_1|. \end{aligned}$$

   and that

2. $\mathbb{E}[|\ln(Y_{t_1,y}(t_2) - \ln(y)||\mathcal{F}_{t_1}] = \mathbb{E}\left[|Z_{t_2-t_1}|\right]$, where

$$Z_{t_2-t_1} = (\beta - \frac{\xi^2}{2})(t_2 - t_1) + \xi W^Y_{t_2-t_1}$$

   is a normally distributed random variable with mean $(\beta - \frac{\xi^2}{2})(t_2 - t_1)$ and variance $\xi^2(t_2 - t_1)$. Let $f_z(x)$ be the probability density function of $Z_{t_2-t_1}$, then

$$\begin{aligned}
\mathbb{E}\left[|Z_{t_2-t_1}|\right] &= \int_{-\infty}^{\infty} |x| f_z(x)dx \\
&\leq \sqrt{\int_{-\infty}^{\infty} x^2 f_z(x)dx} \cdot \sqrt{\int_{-\infty}^{\infty} f_z(x)dx} \\
&= \sqrt{\mathbb{E}[Z^2_{t_2-t_1}]} \\
&= \sqrt{Var(Z_{t_2-t_1}) + \mathbb{E}[Z_{t_2-t_1}]^2} \\
&= \sqrt{\xi^2(t_2-t_1) + (\beta - \frac{\xi^2}{2})^2(t_2-t_1)^2}.
\end{aligned}$$

Therefore, by Inequality (41), there exists some polynomial-growth coefficient $K_3(q)$ such that

$$|\widehat{H}(t_1, y, q) - \widehat{H}(t_2, y, q)| = I_1 + I_2 \leq K_3(q)(\sqrt{|t_2 - t_1|} + |t_2 - t_1|).$$

Combining the results in Steps 1, 2 and 3, we conclude that $\widehat{H}$ is locally Hölder continuous in t with exponential $1/2$, and locally Lipschitz continuous in both y and q. Since $H(t, y, q) = -\phi q^2 + \widehat{H}(t, y, q)$, we conclude that H has the same continuity property in $[0, T) \times (\alpha^*, \infty) \times (0, +\infty)$.

(iii) Since

$$H(t, y, q) = -\phi q^2 + \max_{\theta(\cdot)\in\Theta_t} \mathbb{E}\left[\int_t^{\tau_{t,m(y)}-} L(\theta_r, X^\theta_{t,q}(r))dr \Big| \mathcal{F}_t\right],$$

by the completing square trick as used in part (ii), Step 1, we have

$$\begin{aligned}
|H(t, y, q)| &\leq \phi q^2 + \left[\frac{(2\phi - \eta)^2}{4\nu} - \gamma\sigma^2\right] q^2 \mathbb{E}\left[\tau_{t,m(y)} - t|\mathcal{F}_t\right] \\
&\leq \left(\phi + \left[\frac{(2\phi - \eta)^2}{4\nu} - \gamma\sigma^2\right](T - t)\right) q^2.
\end{aligned}$$

That is, $H(t, y, q)$ satisfies a quadratic growth condition with respect to the inventory variable q, and is bounded in any compact subset of $[0, T) \times (\alpha^*, \infty) \times (0, \infty)$.

5.4 Viscosity Solutions

In Section 5.3, we discussed in detail the continuity of $H(t, y, q)$. Since we do not expect the value function H to be continuously differentiable, we cannot discuss the solution to the HJB equation (35) in the classical sense. Therefore we would like to introduce the concept of a viscosity solution.

Definition 2 *A continuous function $H(\cdot, \cdot, \cdot)$, on $[0, T) \times (\alpha^*, \infty) \times (0, +\infty)$ is a viscosity sub-solution (resp. super-solution) of the HJB equation (35), if for any $\mathcal{C}^{1,2,2}([0, T) \times (\alpha^*, \infty) \times (0, +\infty))$ function ψ and $(\bar{t}, \bar{y}, \bar{q}) \in [0, T) \times (\alpha^*, \infty) \times (0, +\infty)$ such that $H(t, y, q) - \psi(t, y, q)$ attaints its local maximum (resp. minimum) at $(\bar{t}, \bar{y}, \bar{q})$, we have*

$$\begin{aligned}
&-\Big(\partial_t + \beta y \partial_y + \frac{1}{2}\xi^2 y^2 \partial_{yy}\Big)\psi(\bar{t}, \bar{y}, \bar{q}) + \gamma\sigma^2\bar{q}^2 \\
&\quad - \max_{\theta_t\in\Theta_t}\Big\{g(\theta_t)\theta_t + f(\theta_t)\bar{q} - \partial_q\psi(\bar{t}, \bar{y}, \bar{q}) \cdot \theta_t\Big\} \leq 0; \quad \textit{(resp. } \geq 0\textit{)}.
\end{aligned}$$

The continuous function H is a viscosity solution if it is both a viscosity sub-solution and a viscosity super-solution.

For the value function $H(t, y, q)$, we have the following results:

Theorem 8 *The value function H is a viscosity solution of the HJB equation (35).*

Proof: We will prove that H is a viscosity super-solution and sub-solution of Eq. (35) in Steps 1 and 2, respectively.

Step 1: H is a viscosity super-solution of the HJB equation (35).

Without loss of generality, let

$$\min_{(t,y,q)\in[0,T)\times(\alpha^*,\infty)\times(0,+\infty)} (H-\psi)(t,y,q) = (H-\psi)(\bar{t},\bar{y},\bar{q}) = 0. \tag{42}$$

Assume that δ is sufficiently small such that

$$B_\delta(\bar{y},\bar{q}) := \{(y,q) : \sqrt{(y-\bar{y})^2 + (q-\bar{q})^2} < \delta\} \subseteq (\alpha^*, +\infty) \times (0, +\infty).$$

For any arbitrary constant control $\bar{\theta} \in \Theta_{\bar{t}}$, define

$$\widehat{\tau}(\bar{\theta}) = \inf\{t \geq \bar{t} : (Y_{\bar{t},\bar{y}}(t), X^{\bar{\theta}}_{\bar{t},\bar{q}}(t)) \notin B_\delta(\bar{y},\bar{q})\}.$$

For any $0 < \Delta t < T - \bar{t}$, define the stopping time

$$\tau(\bar{\theta}, \Delta t) = (\bar{t} + \Delta t) \wedge \widehat{\tau}(\bar{\theta}) \wedge \tau_{\bar{t},m(\bar{y})}.$$

By DP principle,

$$\begin{aligned} H(\bar{t},\bar{y},\bar{q}) \;\; \geq \;\; & \mathbb{E}\left[\int_{\bar{t}}^{\tau(\bar{\theta},\Delta t)-} \left[g(\bar{\theta})\bar{\theta} + f(\bar{\theta})X^{\bar{\theta}}_{\bar{t},\bar{q}}(r) - \gamma\sigma^2 (X^{\bar{\theta}}_{\bar{t},\bar{q}}(r))^2\right] dr\right] \\ & + \mathbb{E}\left[H(\tau(\bar{\theta},\Delta t), Y_{\bar{t},\bar{y}}(\tau(\bar{\theta},\Delta t)), X^{\bar{\theta}}_{\bar{t},\bar{q}}(\tau(\bar{\theta},\Delta t)))\right]. \end{aligned}$$

Eq. (42) implies that $H(t, y, q) \geq \psi(t, y, q)$ and $H(\bar{t},\bar{y},\bar{q}) = \psi(\bar{t},\bar{y},\bar{q})$, thus

$$\begin{aligned} \psi(\bar{t},\bar{y},\bar{q}) \;\; \geq \;\; & \mathbb{E}\left[\int_{\bar{t}}^{\tau(\bar{\theta},\Delta t)-} \left[g(\bar{\theta})\bar{\theta} + f(\bar{\theta})X^{\bar{\theta}}_{\bar{t},\bar{q}}(r) - \gamma\sigma^2 (X^{\bar{\theta}}_{\bar{t},\bar{q}}(r))^2\right] dr\right] \\ & + \mathbb{E}\left[\psi(\tau(\bar{\theta},\Delta t), Y_{\bar{t},\bar{y}}(\tau(\bar{\theta},\Delta t)), X^{\bar{\theta}}_{\bar{t},\bar{q}}(\tau(\bar{\theta},\Delta t)))\right]. \end{aligned}$$

Applying Itô's formula to $\psi(t, Y_{\bar{t},\bar{y}}(t), X^{\bar{\theta}}_{\bar{t},\bar{q}}(t))$ between $\bar{t}$ and $\tau(\bar{\theta}, \Delta t)$, we obtain

$$\begin{aligned} \mathbb{E}\Big[\frac{1}{\tau(\bar{\theta},\Delta t) - \bar{t}} \int_{\bar{t}}^{\tau(\bar{\theta},\Delta t)} & \big[g(\bar{\theta})\bar{\theta} + f(\bar{\theta})X^{\bar{\theta}}_{\bar{t},\bar{q}}(r) - \gamma\sigma^2 (X^{\bar{\theta}}_{\bar{t},\bar{q}}(r))^2 \\ & + \mathcal{L}\psi(r, Y_{\bar{t},\bar{y}}(r), X^{\bar{\theta}}_{\bar{t},\bar{q}}(r)) - \partial_q \psi(r, Y_{\bar{t},\bar{y}}(r), X^{\bar{\theta}}_{\bar{t},\bar{q}}(r)) \cdot \bar{\theta}\,\big] dr\Big] \leq 0, \end{aligned} \tag{43}$$

where

$$\mathcal{L} = \partial_t + \beta y \partial_y + \frac{1}{2}\xi^2 y^2 \partial_{yy}.$$

By the mean-value theorem, the random variable in the expectation (43) converges a.s. to

$$\mathcal{L}\psi(\bar{t},\bar{y},\bar{q}) - \gamma\sigma^2\bar{q}^2 + g(\bar{\theta})\bar{\theta} + f(\bar{\theta})\bar{q} - \partial_q\psi(\bar{t},\bar{y},\bar{q})\cdot\bar{\theta}$$

as[11] $\Delta t \to 0^+$. We then obtain

$$\mathcal{L}\psi(\bar{t},\bar{y},\bar{q}) - \gamma\sigma^2\bar{q}^2 + g(\bar{\theta})\bar{\theta} + f(\bar{\theta})\bar{q} - \partial_q\psi(\bar{t},\bar{y},\bar{q})\cdot\bar{\theta} \le 0.$$

We conclude the proof from the arbitrariness of $\bar{\theta} \in \Theta_{\bar{t}}$.

Step 2: H is a viscosity sub-solution of the HJB equation (35).

Without loss of generality, let

$$\max_{(t,y,q)\in[0,T)\times(\alpha^*,\infty)\times(0,+\infty)} (H-\psi)(t,y,q) = (H-\psi)(\bar{t},\bar{y},\bar{q}) = 0. \tag{44}$$

We will show the result by contradiction. Assume on the contrary that

$$\mathcal{L}\psi(\bar{t},\bar{y},\bar{q}) - \gamma\sigma^2\bar{q}^2 + \max_{\theta_t}\left\{g(\theta_t)\theta_t + f(\theta_t)\bar{q} - \partial_q\psi(\bar{t},\bar{y},\bar{q})\cdot\theta_t\right\} < 0.$$

Since $\psi \in \mathcal{C}^{1,2,2}([0,T)\times(\alpha^*,+\infty)\times(0,+\infty))$, there exist $\delta > 0$ and $\xi > 0$ such that

$$\mathcal{L}\psi(t,y,q) - \gamma\sigma^2 q^2 + \max_{\theta_t}\left\{g(\theta_t)\theta_t + f(\theta_t)q - \partial_q\psi(t,y,q)\cdot\theta_t\right\} < -\xi, \tag{45}$$

for any $(t,y,q) \in B_\delta(\bar{t},\bar{y},\bar{q})$. Here

$$B_\delta(\bar{t},\bar{y},\bar{q}) \;:=\; \{(t,y,q) : \sqrt{(t-\bar{t})^2 + (y-\bar{y})^2 + (q-\bar{q})^2} < \delta\}$$

is a 3-dimensional ball of radius δ. Without loss of generality, we can always choose δ to be sufficiently small so that $B_\delta(\bar{t},\bar{y},\bar{q}) \subseteq [0,T)\times(\alpha^*,\infty)\times(0,+\infty)$. For any arbitrary control process $\theta \in \Theta_{\bar{t}}$, we define

$$\widetilde{\tau}(\theta) = \inf\{t \ge \bar{t} : (t, Y_{\bar{t},\bar{y}}(t), X^{\theta}_{\bar{t},\bar{q}}(t)) \notin B_\delta(\bar{t},\bar{y},\bar{q})\}.$$

For any $0 < \Delta t < T - \bar{t}$, define

$$\tau'(\theta,\Delta t) = (\bar{t} + \Delta t) \wedge \widetilde{\tau}(\theta) \wedge \tau_{\bar{t},m(\bar{y})}.$$

By the DP principle, there exists a control process $\theta' \in \Theta_{\bar{t}}$ such that

$$\begin{aligned} H(\bar{t},\bar{y},\bar{q}) - \frac{\xi}{2}\Delta t \;\le\; & \mathbb{E}\left[\int_{\bar{t}}^{\tau'(\theta',\Delta t)-} \left[g(\theta'_r)\theta'_r + f(\theta'_r)X^{\theta'}_{\bar{t},\bar{q}}(r) - \gamma\sigma^2(X^{\theta'}_{\bar{t},\bar{q}}(r))^2\right]dr\right] \\ & + \mathbb{E}\left[H(\tau'(\theta',\Delta t), Y_{\bar{t},\bar{y}}(\tau'(\theta',\Delta t)), X^{\theta'}_{\bar{t},\bar{q}}(\tau'(\theta',\Delta t)))\right]. \end{aligned}$$

Eq. (44) implies that $H(t,y,q) \le \psi(t,y,q)$ and $H(\bar{t},\bar{y},\bar{q}) = \psi(\bar{t},\bar{y},\bar{q})$, thus

$$\begin{aligned} \psi(\bar{t},\bar{y},\bar{q}) - \frac{\xi}{2}\Delta t \;\le\; & \mathbb{E}\left[\int_{\bar{t}}^{\tau'(\theta',\Delta t)-} \left[g(\theta'_r)\theta'_r + f(\theta'_r)X^{\theta'}_{\bar{t},\bar{q}}(r) - \gamma\sigma^2(X^{\theta'}_{\bar{t},\bar{q}}(r))^2\right]dr\right] \\ & + \mathbb{E}\left[\psi(\tau'(\theta',\Delta t), Y_{\bar{t},\bar{y}}(\tau'(\theta',\Delta t)), X^{\theta'}_{\bar{t},\bar{q}}(\tau'(\theta',\Delta t)))\right]. \end{aligned}$$

[11] For any arbitrary constant control $\bar{\theta} \in \Theta_{\bar{t}}$, $\lim_{\Delta t\to 0}\tau(\bar{\theta},\Delta t) = \bar{t}$.

Applying Itô's formula to $\psi(t, Y_{\bar{t},\bar{y}}(t), X^{\theta'}_{\bar{t},\bar{q}}(t))$ between $\bar{t}$ and $\tau'(\theta', \Delta t)$, we obtain

$$-\frac{\xi}{2} \leq \mathbb{E}\Big[\frac{1}{\Delta t}\int_{\bar{t}}^{\tau'(\theta',\Delta t)} \big[g(\theta'_r)\theta'_r + f(\theta'_r)X^{\theta'}_{\bar{t},\bar{q}}(r) - \gamma\sigma^2(X^{\theta'}_{\bar{t},\bar{q}}(r))^2 \\ +\mathcal{L}\psi(r, Y_{\bar{t},\bar{y}}(r), X^{\theta'}_{\bar{t},\bar{q}}(r)) - \partial_q\psi(r, Y_{\bar{t},\bar{y}}(r), X^{\theta'}_{\bar{t},\bar{q}}(r)) \cdot \theta'_r\big]dr\Big]. \tag{46}$$

Letting $\Delta t \to 0$, Eq. (45) and Eq. (46) imply that

$$-\frac{\xi}{2} \leq -\lim_{\Delta t\to 0}\frac{\mathbb{E}[\tau'(\theta', \Delta t) - \bar{t}\,]}{\Delta t}\xi. \tag{47}$$

By the definition, we have

$$\begin{aligned} 1 \geq \frac{\mathbb{E}[\tau'(\theta', \Delta t) - \bar{t}\,]}{\Delta t} &\geq \frac{\Delta t \times \mathbb{P}(\widetilde{\tau}(\theta') \wedge \tau_{\bar{t},m(\bar{y})} > \bar{t} + \Delta t)}{\Delta t} \\ &= \mathbb{P}(\widetilde{\tau}(\theta') \wedge \tau_{\bar{t},m(\bar{y})} > \bar{t} + \Delta t) \to 1, \end{aligned}$$

as $\Delta t \to 0^+$, which implies that

$$\lim_{\Delta t\to 0}\frac{\mathbb{E}[\tau'(\theta', \Delta t) - \bar{t}]}{\Delta t} = 1.$$

Eq. (47) then yields $-\frac{\xi}{2} \leq -\xi$, which is a contradiction. Therefore

$$\mathcal{L}\psi(\bar{t}, \bar{y}, \bar{q}) - \gamma\sigma^2\bar{q}^2 + \max_{\theta_t\in\Theta_t}\Big\{g(\theta_t)\theta_t + f(\theta_t)\bar{q} - \partial_q\psi(\bar{t}, \bar{y}, \bar{q}) \cdot \theta_t\Big\} \geq 0.$$

Since the value function H is both a viscosity sub-solution and a viscosity super-solution, we conclude that it is a viscosity solution of the HJB equation (35).

□

5.5 Comparison Principle and Uniqueness

The dynamic programming (DP) method is a powerful tool to study stochastic control problems by means of the HJB equation. However, in the classical approach, the method is used when it is assumed a priori that the value function is sufficiently smooth. This, however, is not necessarily true even in some very simple cases.

To circumvent this difficulty, we adopt the viscosity solutions approach in Section 5.4. In this section, we combine the results obtained in the previous sections with comparison principles for viscosity solutions. We characterize the value function as the unique viscosity solution of the associated dynamic programming equation, Eq. (35), and this can then be used to obtain further results.

Theorem 9 (Comparison Principle). *Let $J^{sub}(t, y, q)$ (resp. $J^{sup}(t, y, q)$) be an upper semi-continuous viscosity sub-solution (resp. lower semi-continuous viscosity super-solution) to Eq. (35), satisfying a polynomial growth condition with respect to q. Suppose there exists a positive constant $r < 1$ such that the growth rate of the solution with respect to y can be controlled by $[\ln(y)]^r$. If*

$$J^{sub} \leq J^{sup} \quad \text{on} \quad \{t = T\} \cup \{y = \alpha^*\} \cup \{q = 0\},$$

then

$$J^{sub} \leq J^{sup} \quad \text{in} \quad [0, T) \times (\alpha^*, +\infty) \times (0, +\infty).$$

Proof: We complete the proof in the following steps:

Step 1. Let $\varrho > 0$. Define $\widetilde{J}^{sub} = e^{\varrho t}J^{sub}$ and $\widetilde{J}^{sup} = e^{\varrho t}J^{sup}$. A straightforward calculation shows that $\widetilde{J}^{sub}$ (resp. $\widetilde{J}^{sup}$) is a sub-solution (resp. super-solution) to

$$\begin{aligned} &-\Big(\partial_t + \beta y\partial_y + \frac{1}{2}\xi^2 y^2\partial_{yy}\Big)J(t,y,q) + e^{\varrho t}\gamma\sigma^2 q^2 + \varrho J(t,y,q) \\ &- \max_{\theta_t\in\Theta_t}\Big\{e^{\varrho t}[g(\theta_t)\theta_t + f(\theta_t)q] - \partial_q J(t,y,q)\cdot\theta_t\Big\} = 0. \end{aligned} \tag{48}$$

Define

$$\mathcal{H}(t,y,q,p,N) = \max_{\theta\geq 0}\Big\{b(y,q,\theta)\cdot p + \frac{1}{2}tr(\Sigma\Sigma'(y)N) + f(t,y,q,\theta,p,N)\Big\}$$

where

$$\begin{cases} b(y,q,\theta) = \begin{pmatrix} \beta y \\ -\theta \end{pmatrix} \\ \Sigma(y) = \begin{pmatrix} \xi y & 0 \\ 0 & 0 \end{pmatrix} \\ f(t,y,q,\theta,p,N) = -e^{\varrho t}\gamma\sigma^2 q^2 + e^{\varrho t}[g(\theta)\theta + f(\theta)q], \end{cases}$$

Σ' is the transpose of Σ, and $(p,\Sigma) = \big((p_1,p_2),\Sigma\big) \in \mathbb{R}^2 \times \mathcal{S}_2$. Here $\mathcal{S}_2$ is the set of symmetric 2×2 matrices. Eq. (48) can then be rewritten as[12]

$$-\partial_t J(t,y,q) + \varrho J(t,y,q) - \mathcal{H}(t,y,q,D_{(y,q)}J, D^2_{(y,q)}J) = 0. \tag{49}$$

Step 2. (Penalization and perturbation of super-solution) From the boundary and polynomial growth conditions on J^{sub} and J^{sup}, we may choose an integer $r_2 > 1$, a positive constant $r_1 < 1$ and $a > 1/\alpha^*$ so that

$$\sup_{(t,y,q)\in[0,T)\times(\alpha^*,+\infty)\times(0,+\infty)} \frac{|J^{sub}(t,y,q)| + |J^{sup}(t,y,q)|}{1 + [\ln(ay)]^{r_1} + q^{r_2}} < \infty.$$

We then consider the function

$$\Upsilon(t,y,q) = e^{-\varrho t}(1 + [\ln(ay)]^{\frac{r_1+1}{2}} + q^{2r_2}).$$

A direct calculation shows that

$$\begin{aligned} & (\partial_t + \beta y\partial_y + \frac{1}{2}\xi^2 y^2\partial_{yy})\Upsilon \\ = \; & -e^{-\varrho t}\Big\{\varrho - \frac{\beta(1+r_1)}{2[\ln(ay)]^{\frac{1-r_1}{2}}} + \frac{\xi^2}{4}\left[\frac{1-r_1^2}{2}[\ln(ay)]^{\frac{r_1-3}{2}} + (1+r_1)[\ln(ay)]^{\frac{r_1-1}{2}}\right] \\ & +\varrho[\ln(ay)]^{\frac{r_1+1}{2}} + \varrho q^{2r}\Big\} \leq 0, \end{aligned}$$

[12]We denote the gradient vector and matrix of second-order partial derivatives of J, respectively, by

$$D_{(y,q)}J = \begin{pmatrix} J_y \\ J_q \end{pmatrix} \quad \text{and} \quad D^2_{(y,q)}J = \begin{pmatrix} J_{yy} & J_{yq} \\ J_{qy} & J_{qq} \end{pmatrix}.$$

as long as $\varrho > \max\left\{0, \frac{\beta(1+r_1)}{2[\ln(a\alpha^*)]^{\frac{1-r_1}{2}}}\right\}$. This implies that for all $\omega > 0$, the function $J^{sup}_\omega = J^{sup} + \omega\Upsilon$ is, as J^{sup}, a super-solution to Eq. (35). Furthermore, according to the growth conditions on J^{sub}, J^{sup} and Υ, we have, for all $\omega > 0$,

$$\lim_{\substack{(y,\,q)\,\in\,(\alpha^*,\,+\infty)\,\times\,(0,\,+\infty)\\ y\,+\,p\,\to\,+\infty}} \sup_{[0,T]}(J^{sub} - J^{sup}_\omega)(t, y, q) = -\infty.$$

Step 3. Let $\mathcal{D} = (\alpha^*, +\infty) \times (0, +\infty)$. Without loss of generality, we may assume that the supremum of the upper semi-continuous function $\widetilde{J}^{sub} - \widetilde{J}^{sup}$ on $[0, T] \times \bar{\mathcal{D}}$ is attained on $[0, T] \times (\mathcal{O} \cup \{y = \alpha^*\} \cup \{q = 0\})$ for some bounded open set $\mathcal{O}$ of $(\alpha^*, +\infty) \times (0, +\infty)$. Here $\bar{\mathcal{D}}$ denotes the closure of $\mathcal{D}$. Otherwise, we can consider $\widetilde{J}^{sub} - \widetilde{J}^{sup}_\omega$ (that we described in Step 2) instead of the original $\widetilde{J}^{sub} - \widetilde{J}^{sup}$, and then passing to the limit $\omega \to 0^+$ at the end of the argument below to obtain the same result.

Given that $\widetilde{J}^{sub} \le \widetilde{J}^{sup}$ on $\{t = T\} \cup \{y = \alpha^*\} \cup \{q = 0\}$, we need to prove that $\widetilde{J}^{sub} \le \widetilde{J}^{sup}$ on $[0, T] \times [\alpha^*, \infty) \times [0, +\infty)$. A rough framework for the proof is through proof by contradiction. We assume[13]

$$M := \sup_{[0,T]\times\bar{\mathcal{D}}}(\widetilde{J}^{sub} - \widetilde{J}^{sup}) = \max_{[0,T)\times\mathcal{O}}(\widetilde{J}^{sub} - \widetilde{J}^{sup}) > 0. \tag{50}$$

We now use the doubling of variables technique by considering, for any $\epsilon > 0$, the function (for more details, please refer to Pham [18] for instance)

$$\Psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = \widetilde{J}^{sub}(t_1, y_1, q_1) - \widetilde{J}^{sup}(t_2, y_2, q_2) - \psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2)$$

where

$$\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = \frac{1}{\epsilon^2}(t_1 - t_2)^2 + \frac{1}{\epsilon}\left[(y_1 - y_2)^2 + (q_1 - q_2)^2\right].$$

The upper semi-continuous function Ψ_ϵ attains its maximum, denoted by M_ϵ, on the compact set $[0, T]^2 \times \bar{\mathcal{O}}^2$ at $(t_1^\epsilon, t_2^\epsilon, (y_1^\epsilon, q_1^\epsilon), (y_2^\epsilon, q_2^\epsilon))$. Notice that

$$\begin{aligned} M \le M_\epsilon &= \Psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon) \\ &= \widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) - \widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon) - \psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon) \\ &\le \widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) - \widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon), \end{aligned} \tag{51}$$

for any $\epsilon > 0$. It follows from the Bolzano-Weierstrass theorem that there exists a subsequence of $(t_1^\epsilon, t_2^\epsilon, (y_1^\epsilon, q_1^\epsilon), (y_2^\epsilon, q_2^\epsilon))_\epsilon$ converging to some point $(\bar{t}_1, \bar{t}_2, (\bar{y}_1, \bar{q}_1), (\bar{y}_2, \bar{q}_2))$ $\in [0, T]^2 \times \bar{\mathcal{O}}^2$. From now on, we will consider such a convergent subsequence when necessary. Moreover, since the sequence $(\widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) - \widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon))_\epsilon$ is bounded, we see from Eq. (51) that the sequence $(\psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon))_\epsilon$ is also bounded, and hence, $\bar{t}_1 = \bar{t}_2 = \bar{t}$, $\bar{y}_1 = \bar{y}_2 = \bar{y}$, $\bar{q}_1 = \bar{q}_2 = \bar{q}$. Passing to the limit $\epsilon \to 0^+$ in Eq. (51), we obtain

[13] It is worth noting that under the hypothesis $\widetilde{J}^{sub} \le \widetilde{J}^{sup}$ on $\{t = T\} \cup \{y = \alpha^*\} \cup \{q = 0\}$, the positive maximum of $\widetilde{J}^{sub} - \widetilde{J}^{sup}$ cannot be attained on the boundary $\{t = T\} \cup \{y = \alpha^*\} \cup \{q = 0\}$.

(i) $M \leq (\widetilde{J}^{sub} - \widetilde{J}^{sup})(\bar{t}_1, \bar{y}_1, \bar{q}_1) \leq M$, and hence $M = (\widetilde{J}^{sub} - \widetilde{J}^{sup})(\bar{t}_1, \bar{y}_1, \bar{q}_1)$ with $(\bar{t}_1, \bar{y}_1, \bar{q}_1) \in [0,T) \times \mathcal{O}$ by Eq. (50);

(ii) As $\epsilon \to 0^+$, $M_\epsilon \to M$ and $\psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon) \to 0$.

Hence, by definition of $(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon)$,

(a) $(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon)$ is a local maximum of

$$(t, y, q) \to \widetilde{J}^{sub}(t, y, q) - \psi_\epsilon(t, t_2^\epsilon, y, y_2^\epsilon, q, q_2^\epsilon)$$

on $[0,T) \times \mathcal{D}$; and

(b) $(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon)$ is a local minimum of

$$(t, y, q) \to \widetilde{J}^{sup}(t, y, q) + \psi_\epsilon(t_1^\epsilon, t, y_1^\epsilon, y, q_1^\epsilon, q)$$

on $[0,T) \times \mathcal{D}$.

We define the second-order *superjets* $\mathcal{P}^{+,(1,2)}\widetilde{J}^{sub}(t,y,q)$ of $\widetilde{J}^{sub}$ at point $(t,y,q) \in [0,T) \times \mathcal{D}$, and the second-order *subjets* $\mathcal{P}^{-,(1,2)}\widetilde{J}^{sup}(t,y,q)$ of $\widetilde{J}^{sup}$ as follows:

$$\begin{aligned}
&\mathcal{P}^{+,(1,2)}\widetilde{J}^{sub}(t,y,q) \\
&= \Big\{(p_1, p_2, N) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2 : \\
&\limsup_{\substack{(\delta_t, \delta_y, \delta_q) \to 0 \\ (t+\delta_t, y+\delta_y, q+\delta_q) \in [0,T) \times \mathcal{D}}} \frac{\widetilde{J}^{sub}(t+\delta_t, y+\delta_y, q+\delta_q) - \widetilde{J}^{sub}(t,y,q) - p_1\delta_t - p_2 \cdot (\delta_y, \delta_q) - \frac{1}{2}N(\delta_y, \delta_q)' \cdot (\delta_y, \delta_q)}{|\delta_t| + |\delta_y|^2 + |\delta_q|^2} \leq 0\Big\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{P}^{-,(1,2)}\widetilde{J}^{sup}(t,y,q) \\
&= \Big\{(p_1, p_2, N) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2 : \\
&\limsup_{\substack{(\delta_t, \delta_y, \delta_q) \to 0 \\ (t+\delta_t, y+\delta_y, q+\delta_q) \in [0,T) \times \mathcal{D}}} \frac{\widetilde{J}^{sup}(t+\delta_t, y+\delta_y, q+\delta_q) - \widetilde{J}^{sup}(t,y,q) - p_1\delta_t - p_2 \cdot (\delta_y, \delta_q) - \frac{1}{2}N(\delta_y, \delta_q)' \cdot (\delta_y, \delta_q)}{|\delta_t| + |\delta_y|^2 + |\delta_q|^2} \geq 0\Big\},
\end{aligned}$$

where $\mathcal{S}_2$ is the set of symmetric 2×2 matrices. From the definitions, we obtain

$$\begin{aligned}
&\left(\partial_{t_1}\psi_\epsilon, D_{(y_1,q_1)}\psi_\epsilon, D^2_{(y_1,q_1)}\psi_\epsilon\right)(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon) \in \mathcal{P}^{+,(1,2)}\widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) \\
&\left(-\partial_{t_2}\psi_\epsilon, -D_{(y_2,q_2)}\psi_\epsilon, -D^2_{(y_2,q_2)}\psi_\epsilon\right)(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon) \in \mathcal{P}^{-,(1,2)}\widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon).
\end{aligned} \tag{52}$$

It is because (similar analysis can be applied to $\widetilde{J}^{sup}(t,y,q)$),

$$\begin{aligned}
\widetilde{J}^{sub}(t,y,q) &\leq \widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) + \psi_\epsilon(t, t_2^\epsilon, y, y_2^\epsilon, q, q_2^\epsilon) - \psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_2^\epsilon, q_2^\epsilon) \\
&= \widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) + \partial_{t_1}\psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_2^\epsilon, q_2^\epsilon)(t - t_1^\epsilon) \\
&\quad + D_{(y_1,q_1)}\psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_2^\epsilon, q_2^\epsilon) \cdot (y - y_1^\epsilon, q - q_1^\epsilon) \\
&\quad + \frac{1}{2}D^2_{(y_1,q_1)}\psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_2^\epsilon, q_2^\epsilon)(y - y_1^\epsilon, q - q_1^\epsilon) \cdot (y - y_1^\epsilon, q - q_1^\epsilon) \\
&\quad + o(|t - t_1^\epsilon| + |y - y_1^\epsilon|^2 + |q - q_1^\epsilon|^2).
\end{aligned}$$

Actually, Eq. (52) holds true for any test function $\psi \in \mathcal{C}^{1,2,2}([0,T) \times (\alpha^*, +\infty) \times (0, +\infty))$ of $\widetilde{J}^{sub}$ and $\widetilde{J}^{sup}$, and the converse property also holds true: for any

$(p_1, p_2, N) \in \mathcal{P}^{+,(1,2)}\widetilde{J}^{sub}(t, y, q)$, there exists $\psi \in \mathcal{C}^{1,2,2}([0,T) \times (\alpha^*, +\infty) \times (0, +\infty))$ satisfying

$$(p_1, p_2, N) = \left(\partial_t\psi, D_{(y,q)}\psi, D^2_{(y,q)}\psi\right)(t, y, q) \in \mathcal{P}^{+,(1,2)}\widetilde{J}^{sub}(t, y, q).$$

We refer to Lemma 4.1 in Chapter V of [7] for more details. An equivalent definition of viscosity solutions in terms of *superjets* and *subjets* are given in the following proposition.

Proposition 4 *An upper semi-continuous (resp. lower semi-continuous) function ω on $[0,T) \times \mathcal{D}$ is a viscosity sub-solution (resp. super-solution) of Eq. (49) on $[0,T) \times \mathcal{D}$ if and only if for all $(t, y, q) \in [0,T) \times \mathcal{D}$ and all $(p_1, p_2, N) \in \mathcal{P}^{+,(1,2)}\omega(t, y, q)$ (resp. $\mathcal{P}^{-,(1,2)}\omega(t, y, q)$),*

$$-p_1 + \varrho\omega(t, y, q) - \mathcal{H}(t, y, q, p_2, N) \leq \textit{(resp. } \geq\textit{)} \quad 0.$$

The key tool in the comparison proof for second-order equations in the theory of viscosity solutions is a lemma in analysis due to Ishii. We state this lemma without proof, and refer the reader to Lemma 4.4.6 in [18, P. 80] and Lemma 3.6 in [12, P. 32] for more details.

Lemma 1 (Ishii's lemma) *Let U (resp. V) be a upper semi-continuous (resp. lower semi-continuous) function on $[0,T) \times \mathbb{R}^n$, $\psi \in \mathcal{C}^{1,1,2,2}([0,T)^2 \times \mathbb{R}^n \times \mathbb{R}^n)$, and $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in [0,T)^2 \times \mathbb{R}^n \times \mathbb{R}^n$ a local maximum of $U(t,x) - V(s,y) - \psi(t,s,x,y)$. Then, for all $\eta > 0$, there exist $N_1, N_2 \in \mathcal{S}_n$ satisfying*

$$\begin{aligned} &(\partial_t\psi, D_x\psi, N_1)(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in \mathcal{P}^{+,(1,2)}U(\bar{t}, \bar{x}), \\ &(-\partial_s\psi, -D_y\psi, N_2)(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in \mathcal{P}^{-,(1,2)}V(\bar{s}, \bar{y}), \end{aligned}$$

and

$$\begin{pmatrix} N_1 & 0 \\ 0 & -N_2 \end{pmatrix} \leq D^2_{(x,y)}\phi(\bar{t}, \bar{s}, \bar{x}, \bar{y}) + \eta\left(D^2_{(x,y)}\phi(\bar{t}, \bar{s}, \bar{x}, \bar{y})\right)^2.$$

Here, $\mathcal{S}_n$ is the set of symmetric $n \times n$ matrices.

We shall use Ishii's lemma with

$$\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = \frac{1}{\epsilon^2}(t_1 - t_2)^2 + \frac{1}{\epsilon}\left[(y_1 - y_2)^2 + (q_1 - q_2)^2\right].$$

Then, direct differentiations yield

$$\begin{cases} \partial_{t_1}\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = -\partial_{t_2}\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = \dfrac{2}{\epsilon^2}(t_1 - t_2) \\ \partial_{y_1}\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = -\partial_{y_2}\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = \dfrac{2}{\epsilon}(y_1 - y_2) \\ \partial_{q_1}\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = -\partial_{q_2}\psi_\epsilon(t_1, t_2, y_1, y_2, q_1, q_2) = \dfrac{2}{\epsilon}(q_1 - q_2), \end{cases} \tag{53}$$

$$D^2_{((y_1,q_1),(y_2,q_2))}\psi(t_1, t_2, y_1, y_2, q_1, q_2) = \frac{2}{\epsilon}\begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix},$$

and

$$\left(D^2_{(y_1,q_1),(y_2,q_2)}\psi(t_1,t_2,y_1,y_2,q_1,q_2)\right)^2 = \frac{8}{\epsilon^2}\begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}.$$

Furthermore, by choosing $\eta = \epsilon$ in Lemma 1, there exist matrices N_1 and $N_2 \in S_2$ such that

$$\begin{pmatrix} N_1 & 0 \\ 0 & -N_2 \end{pmatrix} \leq \frac{10}{\epsilon}\begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}. \tag{54}$$

In view of Ishii's lemma and Eq. (53),

$$\begin{aligned}
&\left(\frac{2}{\epsilon^2}(t_1^\epsilon - t_2^\epsilon),\ \left(\frac{2}{\epsilon}(y_1^\epsilon - y_2^\epsilon)\ ,\frac{2}{\epsilon}(q_1^\epsilon - q_2^\epsilon)\right),\ N_1\right) \in \mathcal{P}^{+,(1,2)}\widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon),\\
&\left(\frac{2}{\epsilon^2}(t_1^\epsilon - t_2^\epsilon),\ \left(\frac{2}{\epsilon}(y_1^\epsilon - y_2^\epsilon)\ ,\frac{2}{\epsilon}(q_1^\epsilon - q_2^\epsilon)\right),\ N_2\right) \in \mathcal{P}^{-,(1,2)}\widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon).
\end{aligned}$$

From viscosity sub-solution and super-solution characterization (i.e., Proposition 4) of $\widetilde{J}^{sub}$ and $\widetilde{J}^{sup}$ in terms of *superjets* and *subjets*, we then have[14]

$$\begin{aligned}
-\frac{2}{\epsilon^2}(t_1^\epsilon - t_2^\epsilon) + \varrho\widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) - \mathcal{H}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon, \left(\frac{2}{\epsilon}(y_1^\epsilon - y_2^\epsilon), \frac{2}{\epsilon}(q_1^\epsilon - q_2^\epsilon)\right), N_1) \leq 0,\\
-\frac{2}{\epsilon^2}(t_1^\epsilon - t_2^\epsilon) + \varrho\widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon) - \mathcal{H}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon, \left(\frac{2}{\epsilon}(y_1^\epsilon - y_2^\epsilon), \frac{2}{\epsilon}(q_1^\epsilon - q_2^\epsilon)\right), N_2) \geq 0.
\end{aligned}$$

By subtracting the above two inequalities, we get

$$\begin{aligned}
\varrho[\widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) - \widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon)] \;\leq\; & \mathcal{H}\left(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon, \left(\frac{2}{\epsilon}(y_1^\epsilon - y_2^\epsilon), \frac{2}{\epsilon}(q_1^\epsilon - q_2^\epsilon)\right), N_1\right)\\
& -\mathcal{H}\left(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon, \left(\frac{2}{\epsilon}(y_1^\epsilon - y_2^\epsilon), \frac{2}{\epsilon}(q_1^\epsilon - q_2^\epsilon)\right), N_2\right).
\end{aligned} \tag{55}$$

Let us recall that

$$\mathcal{H}(t,y,q,p,N) = \frac{1}{2}tr(\Sigma\Sigma'(y)N) + \beta y p_1 - e^{\varrho t}\gamma\sigma^2 q^2 + \widehat{\mathcal{H}}(t,q,p_2),$$

where $g(\theta) = -\nu\theta$, $f(\theta) = -\eta\theta$ and

$$\begin{aligned}
\widehat{\mathcal{H}}(t,q,p_2) &= \max_{\theta\geq 0}\left\{e^{\varrho t}[g(\theta)\theta + f(\theta)q] - p_2\theta\right\}\\
&= \begin{cases} 0, & \text{if } p_2 \geq -\eta q e^{\varrho t}\\ \frac{1}{4\nu}e^{\varrho t}\left[\eta q + e^{-\varrho t}p_2\right]^2, & \text{otherwise.}\end{cases}
\end{aligned}$$

The consequence on $\widehat{\mathcal{H}}$ is the crucial inequality

$$\left|\widehat{\mathcal{H}}(t,q,p_2) - \widehat{\mathcal{H}}(t',q',p_2)\right| \leq C((1+|p_2|^2)|t-t'| + (1+|p_2|)|q-q'|), \tag{56}$$

[14] We use the facts that $\psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon) = \widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon)$ and $-\psi_\epsilon(t_1^\epsilon, t_2^\epsilon, y_1^\epsilon, y_2^\epsilon, q_1^\epsilon, q_2^\epsilon) = \widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon)$.

for all $(t, t', q, q', p_2) \in [0,T]^2 \times (0,+\infty)^2 \times \mathbb{R}$, where C is a constant depending on η, ν, ρ, T and $\mathcal{O}$. We refer interested readers to Appendix A for the proof of Inequality (56). Therefore,

$$\begin{aligned}
& \varrho[\widetilde{J}^{sub}(t_1^\epsilon, y_1^\epsilon, q_1^\epsilon) - \widetilde{J}^{sup}(t_2^\epsilon, y_2^\epsilon, q_2^\epsilon)] \\
\leq\ & \frac{1}{2}tr(\Sigma\Sigma'(y_1^\epsilon)N_1 - \Sigma\Sigma'(y_2^\epsilon)N_2) + \frac{2}{\epsilon}\beta|y_1^\epsilon - y_2^\epsilon|^2 - e^{\varrho t_1^\epsilon}\gamma\sigma^2 q_1^\epsilon + e^{\varrho t_2^\epsilon}\gamma\sigma^2 q_2^\epsilon \\
& + C\left(\frac{4}{\epsilon^2}|t_1^\epsilon - t_2^\epsilon|\cdot|q_1^\epsilon - q_2^\epsilon|^2 + |t_1^\epsilon - t_2^\epsilon| + |q_1^\epsilon - q_2^\epsilon| + \frac{2}{\epsilon}|q_1^\epsilon - q_2^\epsilon|^2\right).
\end{aligned} \tag{57}$$

Due to the fact that $\psi_\epsilon \to 0$, as $\epsilon \to 0^+$, we have

$$|t_1^\epsilon - t_2^\epsilon| = o(\epsilon), \quad |y_1^\epsilon - y_2^\epsilon| = o(\epsilon^{1/2}) \quad \text{and} \quad |q_1^\epsilon - q_2^\epsilon| = o(\epsilon^{1/2}).$$

Hence, as $\epsilon \to 0^+$,

$$\frac{4}{\epsilon^2}|t_1^\epsilon - t_2^\epsilon|\cdot|q_1^\epsilon - q_2^\epsilon|^2 + |t_1^\epsilon - t_2^\epsilon| + |q_1^\epsilon - q_2^\epsilon| + \frac{2}{\epsilon}|q_1^\epsilon - q_2^\epsilon|^2 \to 0.$$

We now use Eq. (54) to obtain

$$\begin{aligned}
& \frac{1}{2}tr(\Sigma\Sigma'(y_1^\epsilon)N_1 - \Sigma\Sigma'(y_2^\epsilon)N_2) \\
\leq\ & \frac{1}{2}tr\left(\begin{bmatrix} \Sigma\Sigma'(y_1^\epsilon) & \Sigma(y_1^\epsilon)\Sigma'(y_2^\epsilon) \\ \Sigma(y_2^\epsilon)\Sigma'(y_1^\epsilon) & \Sigma\Sigma'(y_2^\epsilon) \end{bmatrix}\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}\right) \\
\leq\ & \frac{5}{\epsilon}tr\left(\begin{bmatrix} \Sigma\Sigma'(y_1^\epsilon) & \Sigma(y_1^\epsilon)\Sigma'(y_2^\epsilon) \\ \Sigma(y_2^\epsilon)\Sigma'(y_1^\epsilon) & \Sigma\Sigma'(y_2^\epsilon) \end{bmatrix}\begin{bmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{bmatrix}\right) \\
=\ & \frac{5}{\epsilon}tr\left(\Sigma\Sigma'(y_1^\epsilon) - \Sigma(y_1^\epsilon)\Sigma'(y_2^\epsilon) - \Sigma(y_2^\epsilon)\Sigma'(y_1^\epsilon) + \Sigma\Sigma'(y_2^\epsilon)\right) \\
=\ & \frac{5}{\epsilon}tr\left([\Sigma(y_1^\epsilon) - \Sigma(y_2^\epsilon)][\Sigma'(y_1^\epsilon) - \Sigma'(y_2^\epsilon)]\right) \\
=\ & \frac{5}{\epsilon}||\Sigma(y_1^\epsilon) - \Sigma(y_2^\epsilon)||^2 \\
=\ & \frac{5}{\epsilon}\xi^2|y_1^\epsilon - y_2^\epsilon|^2.
\end{aligned} \tag{58}$$

Combining the results in Eq. (57) and (58), we conclude via passing to the limit $\epsilon \to 0^+$ that $\varrho M \leq 0$, which contradicts to Eq. (50). As a result, the assumption Eq. (50) is false, and hence, the comparison principle $J^{sub} \leq J^{sup}$ on $[0,T) \times \mathcal{D}$ holds.

□

In Theorem 8, we prove that $H(t,y,q)$ is a viscosity solution of the HJB equation (35). In the proof of Theorem 7, we verify that

(g1) $H(t,y,q)$ satisfies a polynomial growth condition with respect to the inventory variable q; and that

(g2) $H(t,y,q)$ can be controlled by a y-independent term:

$$\left(\phi + [\frac{(2\phi-\eta)^2}{4\nu} - \gamma\sigma^2](T-t)\right)q^2.$$

Combining these results with the comparison principle, we can prove that the value function is the unique viscosity solution of Eq. (35). We provide this result in the following corollary:

Corollary 1 (Uniqueness). *The value function H is the unique viscosity solution of the HJB equation (35), satisfying the boundary and terminal conditions Eq. (35.a), (35.b) and (35.c), as well as the growth conditions (g1) and (g2).*

Proof: Suppose H_1 and H_2 are two viscosity solutions of the HJB equation, Eq. (35), then $H_1(T,y,q) = H_2(T,y,q) = -\phi q^2$, $H_1(t,\alpha^*,q) = H_2(t,\alpha^*,q) = -\phi q^2$ and $H_1(t,y,0) = H_2(t,y,0) = 0$. According to Theorem 9, if we view H_1 as the sub-solution and H_2 as the super-solution, then

$$H_1(\cdot,\cdot,\cdot) \leq H_2(\cdot,\cdot,\cdot)$$

holds over $[0,T) \times \mathcal{D}$; on the other hand, if we view H_2 as the sub-solution and H_1 as the super-solution, then

$$H_2(\cdot,\cdot,\cdot) \leq H_1(\cdot,\cdot,\cdot)$$

holds over $[0,T) \times \mathcal{D}$. Hence $H_1(\cdot,\cdot,\cdot) \equiv H_2(\cdot,\cdot,\cdot)$. According to Theorem 8, the value function $H(t,y,q)$ is a viscosity solution of Eq. (35), so it is the unique one. □

5.6 Numerical Scheme

In this section, we mainly discuss the application of finite difference methods on optimal liquidation problems.

5.6.1 The Value Function

Similar to Section 3.2, we consider an ansatz of $H(t,y,q)$ that is quadratic in the variable q:

$$H = h^H(t,y)q^2. \tag{59}$$

With Assumption (59), our optimal liquidation strategy for the associated unconstrained problem in Eq. (35) can be written in the following feedback form:

$$\theta_t^{\phi,*} = -\frac{1}{2\nu}[2h^H(t,y) + \eta]q. \tag{60}$$

According to the results in Theorem 6 and Theorem 7, we have

$$H(t,\alpha^*,q) \leq H(t,y,q) \leq U(t,q), \quad \text{for any } y > \alpha^*.$$

Therefore,

$$\theta_t^{\phi,*} = -\frac{1}{2\nu}[2h^H(t,y) + \eta]q \geq -\frac{1}{2\nu}[2c(t) + \eta]q \geq 0.$$

Similar arguments can then be applied here to prove that,

$$\int_{[0,T)} \theta_t^{\phi,*} dt = Q - X_{T-}^{\phi,*} \leq Q.$$

That is, the unconstrained optimal liquidation strategy (60) is the optimal one for the original constrained problem. The unknown function $h^H(t,y)$ then solves the following PDE:

$$
\text{(I)} \qquad \begin{cases} \Big(\partial_t + \beta y \partial_y + \dfrac{1}{2}\xi^2 y^2 \partial_{yy}\Big) h^H - \gamma\sigma^2 + \dfrac{1}{4\nu}[2h^H(t,y)+\eta]^2 = 0 \\ h^H(T,y) = -\phi, \qquad\qquad y > \alpha^* \\ h^H(t,\alpha^*) = -\phi, \qquad\qquad 0 \le t \le T \\ \lim\limits_{y\to\infty} h^H(t,y) = c(t) \end{cases}
$$

where $c(t)$ is given in Eq. (16).

5.6.2 Numerical Scheme for h^H

Define variables $x = \log \frac{y}{\alpha^*}$ and $\tau = T - t$. Set $\widetilde{h}^H = 2h^H + \eta$. The PDE satisfied by the unknown function $\widetilde{h}^H$ (similar methods can be used to the function f^H) can then be written as

$$
\text{(I')} \qquad \begin{cases} \partial_\tau \widetilde{h}^H = (\beta - \dfrac{\xi^2}{2})\partial_x \widetilde{h}^H + \dfrac{\xi^2}{2}\partial_{xx}\widetilde{h}^H - 2\gamma\sigma^2 + \dfrac{1}{2\nu}(\widetilde{h}^H)^2 \\ \widetilde{h}^H(0,x) = -2\phi + \eta \\ \widetilde{h}^H(\tau,0) = -2\phi + \eta \\ \lim\limits_{x\to\infty} \widetilde{h}^H(t,x) = 2c(t) + \eta. \end{cases}
$$

To approximate the solution of the PDE, we discretize variables τ and x with step sizes $\Delta\tau$ and Δx, respectively. The value of $\widetilde{h}^H$ at a grid point $\tau_i = i\Delta\tau$ ($i = 0, 1, \cdots, N$ with $\tau_N = T$) and $x_j = j\Delta x$ is denoted by $\widetilde{h}_j^{H,i}$. To approximate the infinite boundary condition

$$
\lim_{x\to\infty} \widetilde{h}^H(t,x) = 2c(t) + \eta,
$$

we choose a sufficiently large integer M, and set $\widetilde{h}_M^{H,i} = 2c_M^i + \eta$. In this section, we use the explicit difference method to perform these numerical simulations:

$$
\begin{cases} \partial_\tau \widetilde{h}^H = (\widetilde{h}_j^{H,i+1} - \widetilde{h}_j^{H,i})/\Delta\tau \\ \partial_x \widetilde{h}^H = (\widetilde{h}_{j+1}^{H,i} - \widetilde{h}_{j-1}^{H,i})/(2\Delta x) \\ \partial_{xx}\widetilde{h}^H = (\widetilde{h}_{j+1}^{H,i} + \widetilde{h}_{j-1}^{H,i} - 2\widetilde{h}_j^{H,i})/\Delta x^2. \end{cases}
$$

To numerically solve the nonlinear PDE, we use a single Picard iteration, i.e., approximating a nonlinear term like $(\widetilde{h}_j^{H,i})^2$ by $\widetilde{h}_j^{H,i}\widetilde{h}_j^{H,i+1}$. We define

$$
r = \Delta\tau\xi^2/\Delta x^2, \quad v = r/2 - \Delta\tau\big(\beta - \frac{\xi^2}{2}\big)/(2\Delta x), \quad u = r/2 + \Delta\tau\big(\beta - \frac{\xi^2}{2}\big)/(2\Delta x).
$$

After implementing the scheme, we end up with a problem of solving linear system of equations:

$$
B_i \widetilde{\mathbf{h}}^{H,i+1} = A\widetilde{\mathbf{h}}^{H,i} - 2\Delta\tau\gamma\sigma^2\mathbf{e} \tag{61}
$$

where $\mathbf{e} = (1, 1, \cdots, 1)^T$ is a $(M+1)$-dimensional vector, B_i and A are square matrices of size $(M+1)$ given by

$$B_i = \begin{pmatrix} 1 - \frac{\Delta\tau \widetilde{h}_0^{H,i}}{2\nu} & & \\ & \ddots & \\ & & 1 - \frac{\Delta\tau \widetilde{h}_M^{H,i}}{2\nu} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1-r & u & & \\ v & 1-r & u & \\ & \ddots & \ddots & \\ & & v & 1-r \end{pmatrix}.$$

The above system is solved for every time step moving forward in time given an initial conditions $\widetilde{\mathbf{h}}^{H,0} = (h_0^{H,0}, \cdots, h_M^{H,0})^T$ and B_0. Since the associated problem has defined boundary conditions, for the solution vector $\widetilde{\mathbf{h}}^{H,i}$ of size $(M+1)$, we only need to solve for the middle $M-1$ entries (i.e., $\widetilde{h}_0^{H,i}$ and $\widetilde{h}_M^{H,i}$ are given by the boundary conditions) at every time iteration.

5.6.3 Numerical Experiments

To analyze the effectiveness of this numerical method, we will provide a comparison of the numerical solution to the closed-form solution of Model 1.

Length of Time Interval	T	1
Time Steps	N	1000

Table 1: Parameters used in the implementation of the numerical schemes.

In the left plot[15] given in Figure 4, we can observe that it is difficult to observe the discrepancies between the true solution of the HJB equation which is a decreasing function of time t, and the numerical solution provided by our scheme. The plots of the absolute error and the corresponding relative error between the two solutions shows that, although there is some difference between the solutions, this difference has a magnitude of 10^{-4} which is negligible given that the actual solution is of magnitude 100. This motivates us to apply the numerical scheme to the optimal liquidation problem with default risk.

5.6.4 Discretization of the Continuous Process

Consider the stock issuer's market value Y_t. Denote

$$\wp_k = \left\{ (k - \frac{1}{2})\Delta x < \log\left(\frac{Y_t}{\alpha^*}\right) < (k + \frac{1}{2})\Delta x \right\}.$$

Set $\log\left(\frac{Y_t}{\alpha^*}\right) = k\Delta x$, while $\log\left(\frac{Y_t}{\alpha^*}\right) \in \wp_k$. Suppose the initial market value is $y = 1000\ m$, where 'm' represents the unit 'million'. We assume that the barrier $\alpha^* = 10\ m$. Table 2 displays the parameters used in the implementation of the numerical schemes. Other model parameters not listed in Table 2 are the same with those used in Figure 1.

[15] Corresponding to the strategy of inactive traders, such as buy and hold strategy.

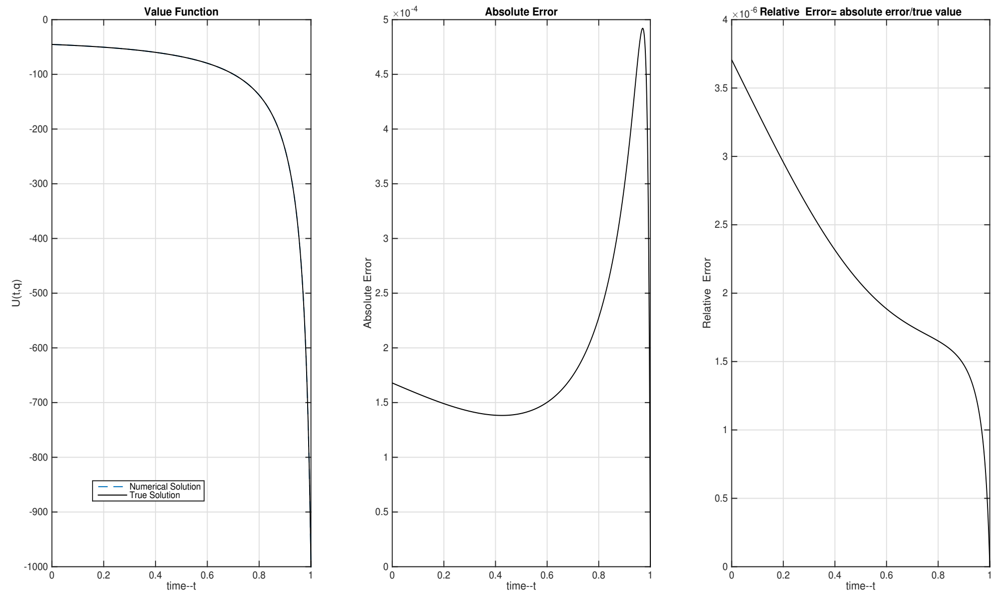

Figure 4: Comparison of the numerical and true solutions for Model 1 with $X_t \equiv Q = 100$, for any $t \in [0, T]$. All parameters are the same with those used in Figure 1.

Length of Time Interval	T	1
Time Steps	N	1000
Space Steps	M	10000
Drift of Stock issuer's Market Value	β	-0.5
Volatility of Stock issuer's Market Value	ξ	2

Table 2: Parameters used in the implementation of the numerical schemes.

Figure 5 shows that the optimal liquidation strategy and the corresponding optimal number of shares in the risky asset for one simulation of the market value path, $\{\log\left(\frac{Y_t}{\alpha^*}\right), t \in [0, T]\}$. We observe that strategies under default risk are inter-temporarily updated. This update process, different from the first model, depends not only on the remaining time to liquidate, but also on the stock issuer's market value. Notice that, at time $t = 0.6$, there is a sharp drop in the stock issuer's market value:

$$\Delta \log\left(\frac{Y_{0.6}}{\alpha^*}\right) = \log\left(\frac{Y_{0.6+}}{\alpha^*}\right) - \log\left(\frac{Y_{0.6}}{\alpha^*}\right) = 2.5 - 4 = -1.5,$$

about 424 millions[16]. To reduce the exposure to the potential position risk incurred by the counter-party's default risk, the agent would therefore speed up his/her liquidation speed. This explains the local peak near $t = 0.6$ in the middle plot of Figure 5.

[16] $Y_{0.6} \times (1 - e^{-1.5}) = \alpha^* e^4 \times (1 - e^{-1.5})$.

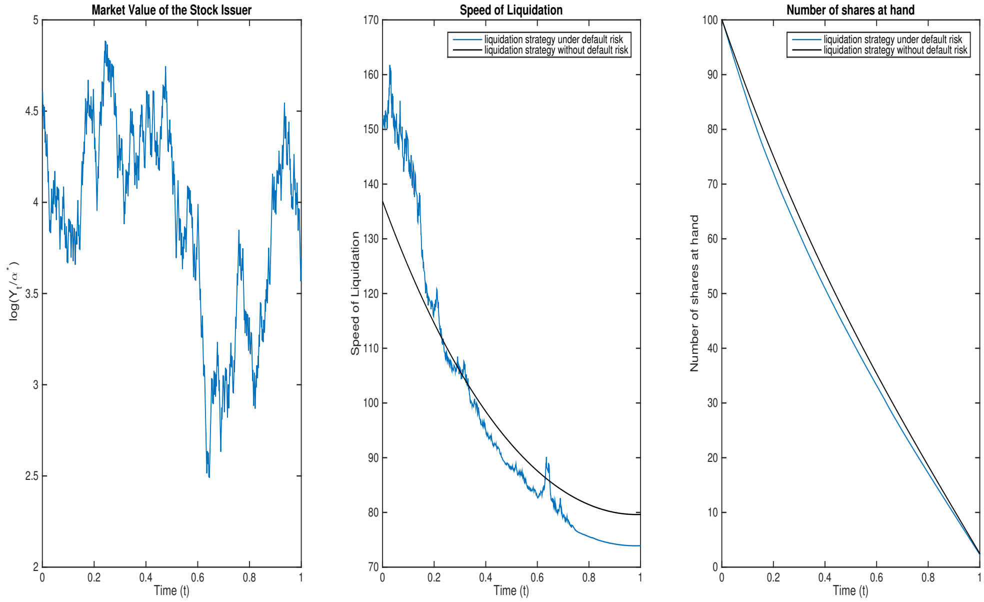

Figure 5: A comparison of liquidation strategies with/without counter-party default risk.

6 Conclusions

In this paper, we adopt Almgren-Chriss's market impact model and relax the assumption of a known pre-determined time horizon to study the optimal liquidation problem under a randomly-terminated setting. In some situation, it is more realistic to assume that the liquidation horizon depends on some of the stochastic factors of the model. For example, some financial markets adopt the circuit-breaking mechanism, which makes the horizon of the investor subject to stock price movement. Once the stock price touches the daily limits, all transactions of the stock will be suspended.

Optimal liquidation strategy of large trades in a known pre-determined time horizon is first discussed as a benchmark case. We then extend our basic model to a randomly-terminated time horizon. In particular, two different liquidation scenarios are analyzed to shed light on the relation between optimal liquidation strategies and potential liquidity risk subject to either

1. an exogenous trigger event controlled by the hazard rate $\{l(t), t \geq 0\}$; or
2. a counterparty risk.

For cases where no closed-form solutions can be obtained, we study the problem via the stochastic control approach. By combining our results with comparison principles for viscosity solutions, we characterize the value function as the unique viscosity solution of the associated HJB equation, and hence, the optimal liquidation strategies that we found numerically serve as good approximations of the unique solutions according to the theory of viscosity solutions.

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Appendix. A

In this appendix we will prove Inequality (56), which is a technical and crucial inequality that we used in the proof of Theorem 9.

Proof of Inequality (56).
There are three cases that we need to consider:

Case 1. Both $\widehat{\mathcal{H}}(t,q,p_2)$ and $\widehat{\mathcal{H}}(t',q',p_2)$ are equal to zero;

Case 2. Neither $\widehat{\mathcal{H}}(t,q,p_2)$ nor $\widehat{\mathcal{H}}(t',q',p_2)$ are equal to zero;

Case 3. Only one of $\widehat{\mathcal{H}}(t,q,p_2)$ and $\widehat{\mathcal{H}}(t',q',p_2)$ equals to zero.

It is obvious that Inequality (56) holds under **Case 1**, because the left hand side of (56) equals to 0 in this case. Therefore, we will only focus on the proofs for Cases 2 and 3.

Case 2. Without loss of generality, we assume that $\widehat{\mathcal{H}}(t,q,p_2) \geq \widehat{\mathcal{H}}(t',q',p_2) > 0$. Hence,

$$\begin{aligned}\widehat{\mathcal{H}}(t,q,p_2) - \widehat{\mathcal{H}}(t',q',p_2) &= \frac{1}{4\nu}e^{\varrho t}[\eta q + e^{-\varrho t}p_2]^2 - \frac{1}{4\nu}e^{\varrho t'}[\eta q' + e^{-\varrho t'}p_2]^2 \\ &= \frac{1}{4\nu}\left[\eta^2[q^2e^{\varrho t} - (q')^2e^{\varrho t'}] + 2\eta p_2(q-q') + p_2^2\cdot[e^{-\varrho t} - e^{-\varrho t'}]\right].\end{aligned} \tag{62}$$

Notice that

$$q^2e^{\varrho t} - (q')^2e^{\varrho t'} = q^2[e^{\varrho t} - e^{\varrho t'}] + e^{\varrho t'}(q-q')(q+q').$$

It follows from the mean value theorem that there exist $\tilde{t}$ and $\hat{t}$ in between t and t' such that

$$\begin{cases} |e^{\varrho t} - e^{\varrho t'}| = \left[\dfrac{d}{dt}e^{\varrho t}\right]\Big|_{t=\tilde{t}}|t-t'| \leq \varrho e^{\varrho T}|t-t'| \\ |e^{-\varrho t} - e^{-\varrho t'}| = \left|\left[\dfrac{d}{dt}e^{-\varrho t}\right]\Big|_{t=\hat{t}}\right||t-t'| \leq \varrho|t-t'|. \end{cases} \tag{63}$$

Since $(t,t',(y,q),(y',q'))$ belongs to the bounded set $[0,T)^2 \times \bar{\mathcal{O}}^2$, there exists a constant $\hat{Q}$ so that $|q|,|q'| \leq \hat{Q}$. Therefore,

$$|q^2e^{\varrho t} - (q')^2e^{\varrho t'}| \leq \varrho e^{\varrho T}\hat{Q}^2|t-t'| + 2\hat{Q}e^{\varrho T}|q-q'|. \tag{64}$$

Combining the results in Eqs. (62), (63), and (64), we conclude that

$$\widehat{\mathcal{H}}(t,q,p_2) - \widehat{\mathcal{H}}(t',q',p_2) \leq C((1+|p_2|^2)|t-t'| + (1+|p_2|)|q-q'|),$$

where C is a constant depending on η, ν, ρ, T and $\hat{Q}$.

Case 3. Without loss of generality, we assume that

$$\widehat{\mathcal{H}}(t', q', p_2) = 0 \quad \text{and} \quad \widehat{\mathcal{H}}(t, q, p_2) > 0.$$

That is,

$$-\eta q' e^{\varrho t'} \leq p_2 < -\eta q e^{\varrho t} < 0. \tag{65}$$

Recall that

$$\begin{aligned}
\widehat{\mathcal{H}}(t, q, p_2) &= \frac{1}{4\nu} e^{\varrho t} [\eta q + e^{-\varrho t} p_2]^2 \\
&= \frac{1}{4\nu} \left[e^{\varrho t} \eta^2 q^2 + 2\eta q p_2 + e^{-\varrho t} p_2^2 \right] \\
&= \frac{1}{4\nu} \left[\eta q [e^{\varrho t} \eta q + p_2] + p_2 [\eta q + p_2 e^{-\varrho t'}] + p_2^2 [e^{-\varrho t} - e^{-\varrho t'}] \right].
\end{aligned}$$

From the second inequality in Eq. (65), we have $e^{\varrho t} \eta q + p_2 < 0$. From the first inequality in Eq. (65), we have $\eta q + p_2 e^{-\varrho t'} \geq \eta(q - q')$. Since $p_2 < 0$, we obtain

$$p_2(\eta q + p_2 e^{-\varrho t'}) \leq \eta p_2 (q - q').$$

Therefore,

$$\begin{aligned}
\widehat{\mathcal{H}}(t, q, p_2) &\leq \frac{1}{4\nu} \left[\eta p_2 (q - q') + p_2^2 [e^{-\varrho t} - e^{-\varrho t'}] \right] \\
&\leq \frac{1}{4\nu} \left[\eta |p_2| |q - q'| + \varrho |p_2|^2 |t - t'| \right].
\end{aligned} \tag{66}$$

The last inequality in Eq. (66) is due to the result in Eq. (63). Hence, $\widehat{\mathcal{H}}$ satisfies the inequality

$$\left| \widehat{\mathcal{H}}(t, q, p_2) - \widehat{\mathcal{H}}(t', q', p_2) \right| \leq \frac{\eta}{4\nu} |p_2| |q - q'| + \frac{\rho}{4\nu} |p_2|^2 |t - t'|,$$

which implies Inequality (56).

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