1 Motivation

We build a simple model that estimates the impact of a forced unwind of large concentrated positions by a number of prime brokers that face the same defaulting counterparty simultaneously, taking into account, critically, the market impact of such unwind. We consider this as an add-on on top of the traditional Initial Margin (IM) model.

2 Background

A hedge fund (HF) enters synthetic positions referencing a risk factor $S = S(t)$ that, for simplicity, we call a stock, against a number of prime brokers (PBs). For simplicity we assume the HF wants to be long.

A typical derivatives position is a total return swap (TRS) where a HF would receive total returns on S in exchange for some funding rate. Derivatives are often preferred to cash equity as they are not subject to the same disclosure requirements as cash equity positions.

A PB would hedge its short position in S on a TRS by buying the equivalent amount of stock. While the HF is solvent, the position is "riskless" in the sense that price changes on the PB's holdings of the stock are just passed by the PB on to the client who bears all market risk. If, however, the HF defaults for whatever reason, the TRS will terminate, and the PB will be left with an unhedged position in the stock.

Naturally, it will try to unwind its hedge by selling the stock. If the movement in the stock make it so that the value is less than what the PB paid for the initial stock purchase, it would realize a loss.

Hence, a PB typically requires a margin, a form of collateral, from the HF. There are generally two kinds of margin. One is the *variation margin* which ensures that the PB's stock value+variation margin is (approximately) equal to the original price the PB paid to buy the stock. This can also be seen/interpreted as the MTM on the TRS that the PB would lose should the HF default (here and throughout we ignore MTM changes on the funding leg of the TRS).

The other kind is the *initial margin* that is designed to cover, with high probability, losses on the unwind of the stock position in orderly fashion over a number of days (MPOR), should the HF default and the stock move adversely over MPOR.

3 Setup

Let $S = S(t)$ be the stock. Let us assume the underlying annualized lognormal volatility of the stock is σ.

We assume that there are N PBs and the position of the HF vs the n-th PB is π_n units of S, with $\pi_n > 0$, $n = 1, \ldots, N$. That means the PB's exposure to

the HF is $-\pi_n S$, and hence it is holding stock as a hedge with the value of

$$\pi_n S(t)$$

at time t. Moreover, to acquire the hedge, the PB borrowed money (most likely internally, but we record it for proper accounting) and holds a cash position in the amount of the (negative of) initial stock purchase price:

$$-\pi_n S(0).$$

Hence the PB's MTM on the hedge portfolio at time t, is

$$\pi_n \left(S(t) - S(0)\right).$$

Again, under the assumption of negligible variation of the funding leg MTM, this is the same as the negative of the MTM change on the TRS from the PB's point of view.

We assume that each PB at time t holds $V_n(t)$ in variational margin and $I_n(t)$ in the initial margin, $n = 1, \ldots, N$. Under a strong margin agreement the VM will be set to keep the position at zero MTM at each point in time, i.e. by requiring that

$$V_n(t) + \pi_n \left(S(t) - S(0)\right) = 0. \tag{1}$$

The IM will be calculated as a high left-tail percentile of the future distribution of S over a specified MPOR. In reality (1) may only hold approximately due to the presence of thresholds, lags in posting the margin, etc. Similarly, IMs are dependent on a particular model that a given PB uses and could also be negotiated up-front as simply a percentage of notional, say.

The goal of our modelling is to estimate market feedback effects and its impact on a forced unwind from the point of view of one of the PBs. Each PB obviously knows (or at least should!) how its own VM and IM are calculated, but would have little idea about others'. Nor would it know the critical parameters such as actual exposures of other PBs to the risk factor of interest. With that in mind, we will eventually impose some reasonable distributions on $V_n(t)$'s and $I_n(t)$'s.

We will impose distribution on these parameters for all PBs, without singling out any particular one as "the" one we care about. But when specifying the distributions for the parameters we can obviously tighten the distribution of any one of them to adopt its point of view on the distributions.

With the intention of randomizing these later, we assume that $V_n(t)$'s and $I_n(t)$'s are given for $n = 1, \ldots, N$.

4 The model, assumptions and motivation

4.1 Information on default

An important element of our model is the assumption on how the default of HF is revealed to the market, here meaning all relevant PBs. We assume that

if the HF defaults at time t, all PBs know about it at the same time. One can contemplate more elaborate mechanisms such as the n-th PB only finds out about HF's inability to post margin when it actually makes a margin call, which could be some time after t due to how the margin agreement is structured, e.g. with thresholds, grace periods, etc. We will leave these complications for later research as our aim is to have a simple model that captures main features of the situation we are trying to model.

4.2 Default trigger

IM is typically calculated on each day (or at least regularly) assuming immediate default of the client (HF). We view our model as a sort of an add-on to that so make the same assumption – our calculations as detailed below are on each day assuming immediate default of the HF for whatever reason (that forces liquidation of stock hedges by all PBs that push the stock lower thus leading to additional loss beyond the IM percentile loss in "normal" unwind that we are trying to estimate).

It is of course likely that, with a large position, an initial downward move in the stock price triggered by some exogenous factor led to margin calls on the stock that led to the default of the HF.

The "immediate exogenous default" concept is appealing as it corresponds to the IM setup and hence directly comparable, and of course is rather simple in concept. We probably need to think a bit more about this as possible future enhancements. There are a couple of angles I can think of right now. One, if the default was triggered by a fall in price of the stock, that drop will also contribute to the overall loss of PBs. This is easy to accommodate as we just assume that the stock has a downward jump at time t (the time of calculation, assumed to be the default time of the HF)

$$\frac{S(t) - S(t-)}{S(t-)} = \delta \leq 0 \tag{2}$$

where δ is yet another model parameter.

Another angle here could be to try to estimate the size of the jump δ that would *actually trigger* the default of the HF. So the setup could be somewhat different, and the basic question that the model is answering would be "what if there is a jump δ at time t" rather than "what if there is a default of the HF at time t". More assumptions/unknown parameters would be required to estimate the jump that would trigger a default of the HF, however. So I think we should leave this for further research (?)

4.3 Decision lag

Another important characteristic of our model is the fact that there is a lag between the realization that the HF is possibly insolvent (e.g. hearing in the news that a margin call from some other PB was missed) and a decision to unwind the position. The behavioral motivation here is the empirically observed

fact that for a large position with, presumably, an important client, a PB would need to reach certain level of consensus internally, and perhaps have various discussions with the HF, before getting required senior sign-offs for the unwind. Some PBs are more decisive here while others may have more internal layers to navigate. We model this "decision lag" as a random time $\omega_n(t)$ for the n-th PB, interpreted as the time when the unwind starts should the HF default at time t. We consider this time as stochastic because of inherent uncertainty of how long the decision would take. Moreover, the parameters of this uncertainty are also uncertain due to imperfect information about the speed of decision making in PBs [XXX is this too complicated? random variable with random parameters? maybe we simplify this later].

4.4 Model for market impact

The model for the stock in the "normal" conditions is of the form

$$dS(t)/S(t) = \sigma \; dW(t). \tag{3}$$

Note that we ignore the drift because of short timeliness involved. However, we introduce another key part of our model and that is of market impact. The standard model of market impact is the so-called square root law [many refs here, see "articles" folder] that basically says that the impact (downward from selling in our case) is proportional to the square root of the number of shares sold vs. daily volume:

$$\frac{S(u+dt) - S(u)}{S(u)} \sim \text{sign}(Q(u)) \times c \times \hat{\sigma}\sqrt{\frac{|Q(u)|}{\nu}}, \tag{4}$$

where $Q(u)$ is the number of shares transacted at u, ν is the daily volume in shares, $\hat{\sigma}$ is the **daily** volatility and c some constant empirically observed to be close to 1, and dt is one day, $dt = 1/250$.

Eventually we will combine (3) and (4) where, for each day of selling, we combine the contributions from those PBs that are actually selling in (4).

4.5 MPOR and execution strategy

For the initial version of the model we assume that, given HF default at time t, each PB would be selling its shares in equal daily amounts over the time period

$$[\omega_n(t), \omega_n(t) + \mu_n]$$

where μ_n, $n = 1, \ldots, N$, is the MPOR for he n-th PB. The collection $\{\mu_n\}$ is yet another input to the model that, from the point of view of one of the PBs, is random as it is not known with certainty.

4.6 Optimal execution

In future developments we could explore the question of what is the optimal execution strategy for a given PB given our model (and the uncertainty over parameters). In the "standard" optimal liquidation of a large order type problem, the optimal is VWAP [XXX? is this true, need to check], with the optimal strategy balancing market impact vs speed of execution. The problem would be more complicated here as there will be multiple competing PBs.

4.7 Losses

The total value realized by the n-th PB by selling its stock, assuming uniform execution strategy, is given by

$$\frac{\pi_n}{\mu_n} \int_{\omega_n(t)}^{\omega_n(t)+\mu_n} S(t)\ dt.$$

The loss, assuming the HF defaults at t, is then defined as

$$L_n = \frac{\pi_n}{\mu_n} \int_{\omega_n(t)}^{\omega_n(t)+\mu_n} S(t)\ dt + V_n(t) + I_n(t) - \pi_n S(0). \tag{5}$$

5 The model, inner simulation

Given the above assumptions and definitions, it is pretty obvious how we would calculate the distribution of losses.

1. There will be an outer loop to simulate the values of various parameters (positions, VMs, IMs, MPORs, decision lags, etc.)
2. For each realization of the model parameters:
3. Assume immediate default of the HF. If using (2), calculate $S(t)$ from the observed $S(t-)$, and assume all IMs, VMs etc. are associated with $S(t-)$, thus baking in a degree of immediate loss into the model
4. For each day u after the default until the last $\max_{n=1,\ldots,N} \{\omega_n(t) + \mu_n\}$:
   (a) Figure out the total amount of stock being sold:
   $$Q(u) = \sum_{n=1}^{N} 1_{\{u \in [\omega_n(t), \omega_n(t)+\mu_n]\}} \frac{\pi_n}{\mu_n}$$
   (b) Simulate the stock price to next day using (here $dt = 1$ day $= 1/250$)
   $$S(u+dt) = S(u) \left(1 - c\sigma \sqrt{\frac{Q(u)}{\nu} dt} + \sigma\ \zeta(u) \sqrt{dt} \right). \tag{6}$$

Here as we recall σ is the annual volatility, c is an empirical constant close to one (model parameter), ν is estimated daily volume in shares, and $\zeta(u)$'s are i.i.d. Gaussian.

5. After each simulated path of S is calculated, each PB's losses are calculated using (5).

6. By running multiple paths we build a loss distribution for each PB

7. We can also calculate the loss under the standard IM model for comparison by getting rid of the drift term in (6)

Remark 1 *As written, the model requires both $S(0)$ (in (5)) and $S(t-)$ (in (2), (6)). On the other hand, assuming perfect collateralization with the variation margin, we would have*

$$V_n(t) + \pi_n \left(S(t-) - S(0)\right) = 0 \tag{7}$$

so we can eliminate $S(0)$ from (5) and write

$$L_n = \frac{\pi_n}{\mu_n} \int_{\omega_n(t)}^{\omega_n(t)+\mu_n} S(t)\ dt + I_n(t) - \pi_n S(t-).$$

Remark 2 *The assumption (7) could be too strong, and we can replace it with*

$$V_n(t) + \pi_n \left(S(t-) - S(0)\right) = G_n(t), \tag{8}$$

where the "gap" $G_n(t)$ could be yet another model parameter, generally bounded by the threshold R_n of the n-th PB margin account,

$$|G_n(t)| \leq R_n$$

for $n = 1, \ldots, N$. The last equation could be used in simulating./randomizing these gaps. With (8) in mind, $S(0)$ can be eliminated from (5) so that we can be write

$$L_n = \frac{\pi_n}{\mu_n} \int_{\omega_n(t)}^{\omega_n(t)+\mu_n} S(t)\ dt + I_n(t) + G_n(t) - \pi_n S(t-).$$

6 The model, outer simulation

Here we need to think about how to parameterize distributions of various model parameters in a reasonable and flexible way. TBC