

Three models of market impact

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Overview of this talk

- The optimal execution problem
- The square-root law of market impact
- Three models compatible with the square-root law
 - The continuous time propagator model
 - The Alfonsi and Schied order book model
 - The locally linear order book (LLOB) model
- Model-dependence of the impact profile

Overview of execution algorithm design

Typically, an execution algorithm has three layers:

- The macrotrader
 - This highest level layer decides how to slice the order: when the algorithm should trade, in what size and for roughly how long.
- The microtrader
 - Given a slice of the order to trade (a child order), this level decides whether to place market or limit orders and at what price level(s).
- The smart order router
 - Given a limit or market order, which venue should this order be sent to?

In this talk, we are concerned with the highest level of the algorithm: How to slice the order.

Statement of the problem

- Given a model for the evolution of the stock price, we would like to find an optimal strategy for trading stock, the strategy that minimizes some cost function over all permissible strategies.
- In all the models we will consider, the optimal strategy does not depend on the stock price and so may be determined in advance of trading.
 - The reason was given by Predoiu, Shaikhet and Shreve.

Predoiu, Shaikhet and Shreve

Suppose the cost associated with a strategy depends on the stock price only through the term

$$\int_0^T S_t dx_t.$$

with S_t a martingale. Integration by parts gives

$$\mathbb{E} \left[\int_0^T S_t dx_t \right] = \mathbb{E} \left[S_T x_T - S_0 x_0 - \int_0^T x_t dS_t \right] = -S_0 x_0$$

which is independent of the trading strategy and we may proceed as if $S_t = 0$.

Quote from [Predoiu, Shaikhet and Shreve]

“...there is no longer a source of randomness in the problem. Consequently, without loss of generality we may restrict the search for an optimal strategy to nonrandom functions of time”.

Practical implication

- Given a model which does not satisfy the conditions of Prediou, Shaikhet and Shreve, we can always find a similar model that does.
- Because the stock price does not mover very much over the course of a typical algorithmic execution, the optimal strategies will typically barely differ.
 - See [[Gatheral and Schied](#)] for a specific example of this.

The square-root formula for market impact

- For many years, traders have used the simple sigma-root-liquidity model described for example by Grinold and Kahn in 1994.
- Software incorporating this model includes:
 - Salomon Brothers, StockFacts Pro since around 1991
 - Barra, Market Impact Model since around 1998
 - Bloomberg, TCA function since 2005
- The model is always of the rough form

$$\Delta P = \text{Spread cost} + \alpha \sigma \sqrt{\frac{Q}{V}}$$

where σ is daily volatility, V is daily volume, Q is the number of shares to be traded and α is a constant pre-factor of order one.

Empirical question

So traders and trading software have been using the square-root formula to provide a pre-trade estimate of market impact for a long time.

Empirical question

Is the square-root formula empirically verified?

Impact of proprietary metaorders (from Tóth et al.)

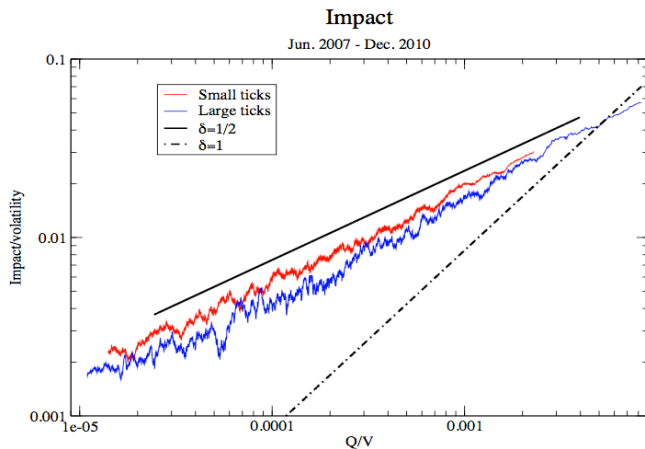


Figure 1: Log-log plot of the volatility-adjusted price impact vs the ratio Q/V

Notes on Figure 1

- In Figure 1 which is taken from [Tóth et al.], we see the impact of metaorders for CFM¹ proprietary trades on futures markets, in the period June 2007 to December 2010.
 - Impact is measured as the average execution shortfall of a meta-order of size Q .
 - The sample studied contained nearly 500,000 trades.
- We see that the square-root market impact formula is verified empirically for meta-orders with a range of sizes spanning two to three orders of magnitude!
- The square-root formula is so widely accepted as offering a good description of the data that we will often refer to it as the *square-root law*.

¹Capital Fund Management (CFM) is a large Paris-based hedge fund.

Some implications of the square-root formula

- The square-root formula refers only to the size of the trade relative to daily volume.
- It does not refer to for example:
 - The rate of trading
 - How the trade is executed
 - The capitalization of the stock
- Surely impact must be higher if trading is very aggressive?
 - The database of trades only contains sensible trades with reasonable volume fractions.
 - Were we to look at very aggressive trades, we would indeed find that the square-root formula breaks down.

Compatible dynamics

- We will now present three different models whose dynamics are compatible with the square-root formula
 - The continuous time propagator model
 - The Alfonsi and Schied order book model
 - The locally linear order book (LLOB) model
- In particular, for each of these models, we will focus on qualitative features of the optimal liquidation strategy.

Price manipulation

A trading strategy $\Pi = \{x_t\}$ is a *round-trip trade* if

$$\int_0^T \dot{x}_t dt = 0$$

Definition

A *price manipulation* is a round-trip trade Π whose expected cost $C[\Pi]$ is negative.

- You would want to repeat such a trade over and over.
- If there is price manipulation, there is no optimal strategy.

Transaction-triggered price manipulation

Definition (Alfonsi, Schied, Slynko (2009))

A market impact model admits *transaction-triggered price manipulation* if the expected costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

As discussed in [[Alfonsi, Schied and Slynko](#)], transaction-triggered price manipulation can be regarded as an additional model irregularity that should be excluded. Transaction-triggered price manipulation can exist in models that do not admit standard price manipulation in the sense of the Huberman and Stanzl definition.

The continuous time propagator model

- In this model from [Gatheral], the stock price S_t at time t is given by

$$S_t = S_0 + \int_0^t f(\dot{x}_s) G(t-s) ds + \int_0^t \sigma dZ_s \quad (1)$$

where \dot{x}_s is our rate of trading in dollars at time $s < t$, $f(\dot{x}_s)$ represents the impact of trading at time s and $G(t-s)$ is a decay factor.

- S_t follows an arithmetic random walk with a drift that depends on the accumulated impacts of previous trades.
- The cumulative impact of (others') trading is implicitly in S_0 and the noise term.
- Drift is ignored.

- We refer to $f(\cdot)$ as the *instantaneous market impact function* and to $G(\cdot)$ as the decay kernel.
- (1) is a generalization of processes previously considered by Almgren, Bouchaud and Obizhaeva and Wang.

Remark

The price process (1) is not the only possible generalization of price processes considered previously. On the one hand, it seems like a natural generalization. On the other hand, it is not motivated by any underlying model of the order book.

Cost of trading

- Denote the number of shares outstanding at time t by x_t . Then from (1), the expected cost \mathcal{C} associated with a given trading strategy is given by

$$\mathcal{C} = \int_0^T \dot{x}_t dt \int_0^t f(\dot{x}_s) G(t-s) ds \quad (2)$$

- The $dx_t = \dot{x}_t dt$ shares liquidated at time t are traded on average at a price

$$S_t = S_0 + \int_0^t f(\dot{x}_s) G(t-s) ds$$

which reflects the residual cumulative impact of all prior trading.

The square-root model

Consider the following special case of (1) with $f(v) = \frac{3}{4}\sigma \sqrt{v/V}$ and $G(\tau) = 1/\sqrt{\tau}$:

$$S_t = S_0 + \frac{3}{4} \sigma \int_0^t \sqrt{\frac{v_s}{V}} \frac{ds}{\sqrt{t-s}} + \text{noise} \quad (3)$$

which we will call the *square-root process*.

It is easy to verify that under the square-root process, the expected cost of a VWAP execution is given by the square-root law for market impact:

$$\frac{C}{Q} = \sigma \sqrt{\frac{Q}{V}} \quad (4)$$

- Of course, that doesn't mean that the square-root process is the true underlying process!

The optimal strategy under the square-root process

- In [Curato et al.], we show that this model admits both transaction-triggered price manipulation and price manipulation.
 - There is no optimal strategy.
- We show numerical evidence that this problem may be mitigated by introducing a bid-ask spread cost or by imposing convexity of the instantaneous market impact function for large trading rates
 - The objective in each case is to robustify the solution in a parsimonious and natural way.

The lowest cost strategies

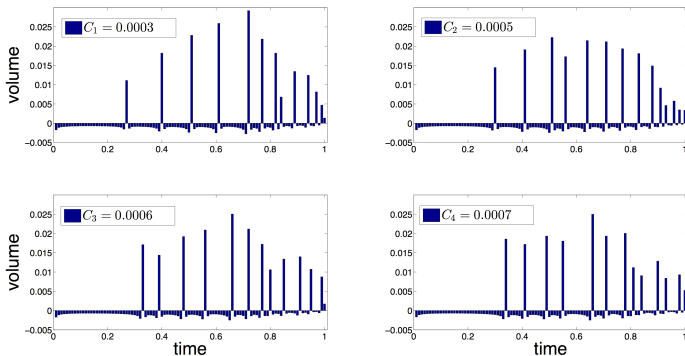


Figure 2: The four lowest cost solutions from brute-force minimization of the square-root model cost functional (2) with 10% participation rate. The costs are reported in the insets.

Comments on Figure 2

- All of the lowest cost solutions are characterized by a few intense positive spikes, separated by periods of slow selling.
- If we impose that a strategy should be monotone (no wrong-way trading), qualitatively similar strategies involving short bursts of trading separated by periods of inactivity have significantly lower expected cost than VWAP in the propagator model.

The model of Alfonsi, Fruth and Schied

[Alfonsi, Fruth and Schied] consider the following (AS) model of the order book:

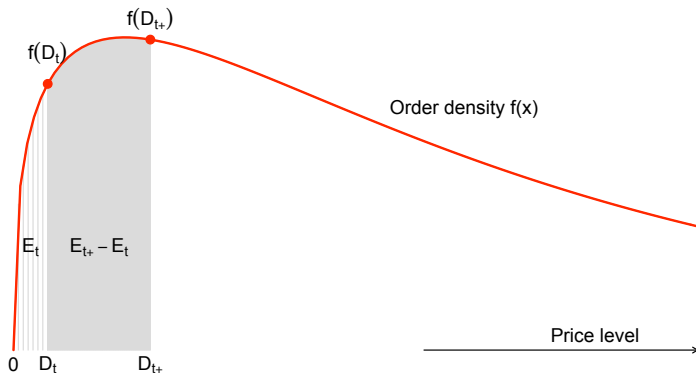
- There is a continuous (in general nonlinear) density of orders $f(x)$ above some martingale ask price A_t . The cumulative density of orders up to price level x is given by

$$F(x) := \int_0^x f(y) dy$$

- Executions eat into the order book (*i.e.* executions are with market orders).
- A purchase of ξ shares at time t causes the ask price to increase from $A_t + D_t$ to $A_t + D_{t+}$ with

$$\xi = \int_{D_t}^{D_{t+}} f(x) dx = F(D_{t+}) - F(D_t)$$

Schematic of the model



When a trade of size ξ is placed at time t ,

$$E_t \mapsto E_{t+} = E_t + \xi$$

$$D_t = F^{-1}(E_t) \mapsto D_{t+} = F^{-1}(E_{t+}) = F^{-1}(E_t + \xi)$$

Optimal liquidation strategy in the AS model

- [Alfonsi, Fruth and Schied] show that the optimal liquidation strategy is to trade a block at the beginning, a block at the end and at a constant rate in-between.
- Specifically, the optimal trading rate is given for $t \in (0, T)$ by

$$u_t = \xi_0 \delta(t) + \xi_0 \rho + \xi_T \delta(T - t).$$

- The optimal strategy involves only purchases of stock, no sales.
- Thus there cannot be price manipulation in the AS model.

When is the bucket-shaped strategy optimal?

- [Predoiu, Shaikhet and Shreve] showed that the bucket-shaped strategy is optimal under more general conditions than exponential resiliency.
 - Specifically, if resiliency is a function of E_t (or equivalently D_t) only, the optimal strategy has a block trades at inception and completion and continuous trading at a constant rate in-between.

The LLOB model

- Let $\rho_{\pm}(x, t)$ denote the average *latent* order density on the bid and ask side of the latent order book and define

$$\varphi(x, t) = \rho_+(x, t) - \rho_-(x, t)$$

where x is the price. Further define the relative price

$$y = x - \hat{p}_t$$

where \hat{p}_t is the efficient price where supply meets demand and $\rho_{\pm} = 0$.

Evolution of φ in the presence of a metaorder

[Donier et al.] argue that the resulting latent order density is *linear* in the neighborhood of the efficient price (*i.e.* locally). They posit the following equation for the evolution of φ (for y close to y_t) in the presence of a metaorder with signed trading rate m_t :

$$\frac{\partial}{\partial t} \varphi(y, t) = \mathcal{D} \frac{\partial^2}{\partial y^2} \varphi(y, t) + m_t \delta(y - y_t) \quad (5)$$

with the boundary condition

$$\lim_{y \rightarrow \pm\infty} \frac{\partial}{\partial y} \varphi(y, t) = -\mathcal{L}$$

and where $y_t = p_t - \hat{p}_t$ represents the difference between the impacted and unimpacted market prices.

Solving for φ and the impacted price y_t

It is straightforward to verify that the solution of (5) is given by

$$\varphi(y, t) = -\mathcal{L} y + \int_0^t \frac{ds m_s}{\sqrt{4\pi D(t-s)}} \exp\left\{-\frac{(y-y_s)^2}{4D(t-s)}\right\}.$$

The price solves $\varphi(y_t) = 0$. Thus the impacted relative price y_t satisfies

$$y_t = \frac{1}{\mathcal{L}} \int_0^t \frac{m_s ds}{\sqrt{4\pi D(t-s)}} \exp\left\{-\frac{(y_t-y_s)^2}{4D(t-s)}\right\}. \quad (6)$$

Cost of liquidation in the LLOB model

The expected cost of liquidation is then given by

$$\mathcal{C} = \frac{1}{\mathcal{L}} \int_0^T dt m_t \int_0^t \frac{m_s ds}{\sqrt{4\pi D(t-s)}} \exp \left\{ -\frac{(y_t - y_s)^2}{4D(t-s)} \right\}. \quad (7)$$

- \mathcal{C} is positive definite so price manipulation is not possible in the LLOB model.

Intuition for no price manipulation in the LLOB model

Some intuition for no price manipulation in the LLOB model is as follows.

- Consider a buy metaorder.
- As execution proceeds, the slope of the order book on the ask side steepens relative to the slope on the bid side.
 - Consequently, if the trade is reversed, the resulting sell metaorder causes higher price impact.
- This feature of the LLOB model is reminiscent of the behavior of the AS model where the spread widens as the metaorder eats into the order book.
- In the context of the propagator model, this is as if the instantaneous market impact function f were to depend on the history of order flow.

Simplify notation

- Define

$$D = \sigma^2/2, \mathcal{L} = \frac{V}{\sigma^2}$$

where V is market volume per unit time and σ is price volatility.

- If some traders' intentions are relative to the market price rather than at fixed prices, we would have $D = \alpha \sigma^2/2$ for some $\alpha < 1$.
- Further define the normalized impact

$$z_t = \frac{y_t}{\sigma \sqrt{T}}$$

and the participation rate $\eta_s = \frac{m_s}{V}$.

Then, with t now as a proportion of the terminal time T , (6) becomes

$$z_t = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{\eta_s ds}{\sqrt{t-s}} \exp \left\{ -\frac{(z_t - z_s)^2}{2(t-s)} \right\} \quad (8)$$

and (7) becomes

$$\mathcal{C} = \sigma \sqrt{T} V T \int_0^1 \eta_t dt \int_0^t \eta_s ds \frac{1}{\sqrt{2\pi|t-s|}} \exp \left\{ -\frac{(z_t - z_s)^2}{2|t-s|} \right\}. \quad (9)$$

- Since the impacted price is the solution of the integral equation (8), it's not easy to find the optimal strategy that minimizes the cost (9).
- Nevertheless [Donier et al.] show how to do asymptotic analysis.

Small η

In the limit $\eta \rightarrow 0$, where the participation rate of the metaorder is small, the exponent is small and we obtain

$$z_t \approx \frac{1}{\sqrt{2\pi}} \int_0^t \frac{\eta_s ds}{\sqrt{t-s}}. \quad (10)$$

This is nothing other than the propagator model with linear instantaneous market impact and a power-law decay kernel.

Small η optimal strategy

The optimal strategy was computed in
[Gatheral, Schied and Slynko] as

$$\eta_s = \frac{A}{[s(1-s)]^{1/4}}$$

The normalizing factor A is given by

$$\int_0^1 \eta_s ds = \frac{Q}{V T} = A \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{4}\right)^2$$

The optimal strategy is absolutely continuous with no block trades.
However, it is singular at $s = 0$ and $s = 1$.

- The optimal strategy looks very close to the bucket strategy of Alfonsi and Schied.

Schematic of the small η optimal strategy

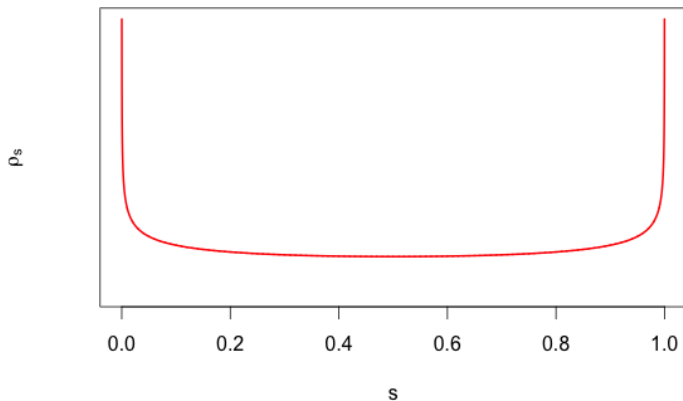


Figure 3: The LLOB optimal strategy for low trading rates.

Large η

- In the limit $\eta \rightarrow \infty$, where the participation rate of the metaorder is large, the exponent is dominated by times s close to t .
- Assuming the trading strategy is differentiable, using a saddle-point approximation, we obtain

$$\begin{aligned}
 z_t &= \frac{1}{\sqrt{2\pi}} \int_0^t \frac{\eta_s ds}{\sqrt{t-s}} \exp \left\{ -\frac{(z_t - z_s)^2}{2(t-s)} \right\} \\
 &\approx \frac{\eta_t}{\sqrt{2\pi}} \int_0^\infty \frac{du}{\sqrt{u}} \exp \left\{ -\frac{\dot{z}_t^2}{2} u \right\} \\
 &= \frac{\eta_t}{|\dot{z}_t|}.
 \end{aligned} \tag{11}$$

Large η execution cost

- Integrating (11) assuming $\eta_s \geq 0$ then gives

$$z_t^2 \approx 2 \int_0^t \eta_s ds = 2 \frac{Q_t}{V T}$$

where Q_t is quantity executed up to time t .

- Then

$$\begin{aligned} \mathcal{C} &\approx \sigma \sqrt{T} V T \int_0^1 \eta_t z_t dt \\ &= \sqrt{2} \frac{\sigma}{\sqrt{V}} \int_0^Q \sqrt{Q_t} dQ_t \\ &= Q \frac{2}{3} \sqrt{2} \sigma \sqrt{\frac{Q}{V}}. \end{aligned} \tag{12}$$

- Expected cost per share seems to be *independent* of strategy and square-root in the executed quantity.
 - See later for a computation of the cost of a VWAP execution that gives the same result.

VWAP and the market impact profile

- There have been many empirical studies of the *impact profile*, the evolution of the market price over time during and after the execution of a metaorder.
 - Two recent such studies are by [Bacry, Iuga et al.] and [Zarinelli et al.].
- It is a stylized fact that most metaorders look like VWAP. Quoting from [Bacry, Iuga et al.]:

A VWAP (i.e. Volume Weighted Average Price) is a trading algorithm parameterized by a start time and an end time, which tries to make the integrated transaction volume to be as close as possible to the average intraday volume curve of the traded security.

- In other words, VWAP orders trade at a constant rate in volume time.

Empirical market impact profiles from [Bacry, Iuga et al.]

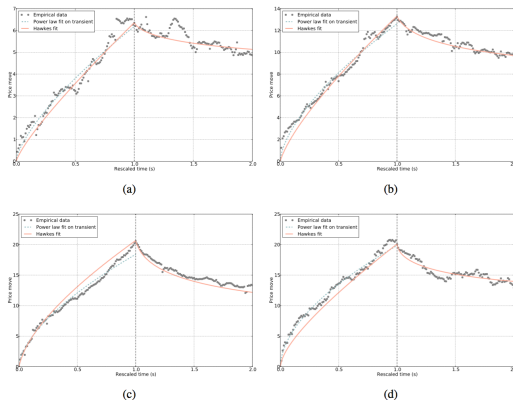


Figure 4: Figure 8 of [Bacry, Iuga et al.]. Fixing participation rate ρ at around 2%, the market impact profile during execution gets less concave (more linear) as the order duration T decreases.

Remarks on Figure 4

- In each subplot, metaorder sizes are 1 – 3% of daily volume but with different durations T and thus different participation rates

$$\eta = \frac{Q}{V T}.$$

- The power-law

$$\Delta P \propto s^\gamma, s \in [0, 1]$$

is fitted, obtaining a different γ for each bucket.

- Results are

	T (min.)	γ	η
(a)	[3,15)	0.80	1.16
(b)	[15,30)	0.66	0.38
(c)	[30,60)	0.62	0.18
(d)	[60,90)	0.55	0.11

Overall results of [Bacry, Iuga et al.]

- The square-root law is confirmed again, at least approximately.
- There is a duration effect on cost of the rough form $1/T^{0.25}$.
- Prior to completion, the market impact profile is power-law with an exponent of around 0.6.
- Decay of market impact after metaorder completion has a slow initial phase followed by a power-law decay with exponent of around 0.6.
- However, the higher the participation rate, the more linear the market impact profile.
 - Intuition: At high trading rates, the order book has insufficient time to refresh.

Empirical market impact profiles from [Zarinelli et al.]

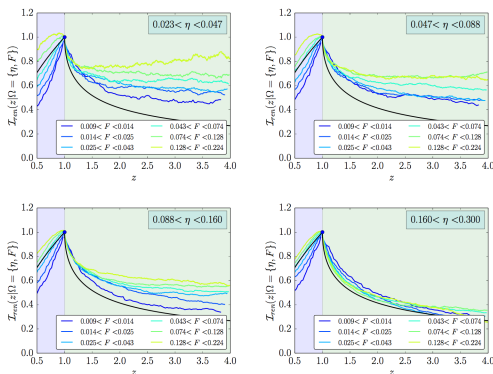


Figure 13: Decay of temporary market impact after the execution of a metaorder. We follow the normalised market impact path $\mathcal{I}_{ren}(z|\Omega = \{\eta, F\})$ as a function of the rescaled variable $z = v/F$. Within each panel the solid lines correspond to the average market impact trajectory for metaorders with different durations F ; the four panels correspond to different participation rates η . We consider the price dynamics up to the end of day when the metaorder was placed. The black line corresponds to the prediction of the propagator model with $\delta = 0.5$. Overnight returns and the price path of subsequent days are not considered in our analysis.

Remarks on the [Zarinelli et al.] results

- Again, overall results are more or less consistent with the square-root law.
 - Though the authors say logarithmic is better.
- The top-left subplot in their Figure 13 shows qualitative agreement with [Bacry, Iuga et al.].
 - Fixing $\eta \approx 2\%$, we see that the impact profile becomes more linear as duration decreases.

The effect of other metaorders

- A number of authors point out that estimates of the market impact profile are biased by the presence of other metaorders trading at the same time as the metaorder of interest.
- In particular, to quote [Zarinelli et al.],

The positive autocorrelation of metaorder signs qualitatively explains the findings on price decay. Market impact trajectories of metaorders with very large participation rate are negligibly perturbed by the other metaorders and their trajectories are roughly independent of duration (bottom right panel of Figure 13). Moreover, the market impact trajectory is quite well described by the propagator model...

Figure 1 of [Zarinelli et al.]

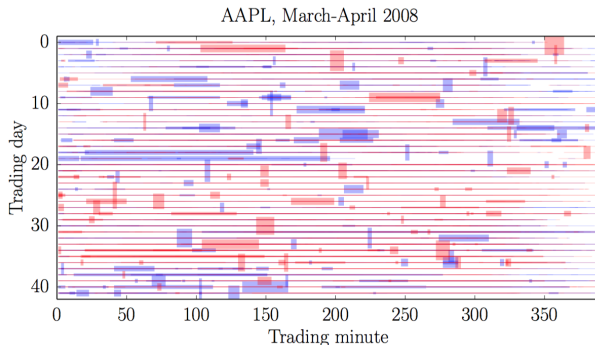


Figure 1: Time series of metaorders active on the market for AAPL in the period March-April 2008. Buy (Sell) metaorders are depicted in blue (red). The thickness of the line is proportional to the metaorder participation rate. More metaorders in the same instant of time give rise to darker colours. Each horizontal line is a trading day. We observe very few blanks, meaning that there is almost always an active metaorder from our database, which is of course only a subset of the number of orders that are active in the market.

Consistency between models and data

We are now in a position to study consistency between these three models and empirical observation. Specifically, for a VWAP order:

- Is the expected cost of execution consistent with the square-root law?
- Is the impact profile consistent with observation?

VWAP in the propagator model

From (2), fixing $\dot{x}_t = \eta$, we have that

$$\mathcal{C} = \eta f(\eta) \int_0^T dt \int_0^t G(t-s) ds =: \eta f(\eta) H(T).$$

- We already know this model can be made consistent with the square-root law.
- Fixing the participation rate, we see that the impact profile can depend on duration, consistent with the results of both [Bacry, Luga et al.] and [Zarinelli et al.].
 - For fixed T , bucketing the data by η gives an estimate of $f(\eta)$.
 - For fixed η , bucketing the data by T gives an estimate of $H(T)$ and so of $G(\tau)$.

VWAP impact profile in the square-root model

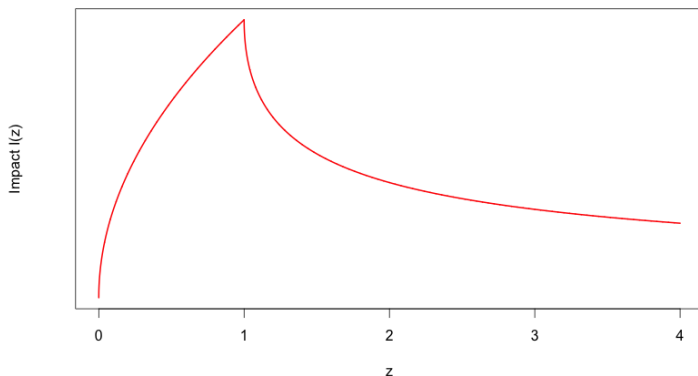


Figure 5: The VWAP impact profile in the propagator model with $f(\eta) = \sqrt{\eta}$ and $G(\tau) = 1/\sqrt{\tau}$.

VWAP in the AS model

In the AS model, the current spread D_t and the volume impact process E_t are related as $D_t = F^{-1}(E_t)$ so effectively, for continuous trading strategies,

$$\mathbb{E}[S_t] = F^{-1} \left(\int_0^t \dot{x}_s e^{-\rho(t-s)} ds \right)$$

Thus, in the case $\dot{x}_t = \eta$,

$$\mathcal{C} = \eta \int_0^T dt F^{-1} \left(\int_0^t \rho e^{-\rho(t-s)} ds \right).$$

- We can make this model consistent with the square-root law by appropriately specifying F (see [\[Gatheral, Schied and Slynko \(2011\)\]](#)).
- There seems to be enough flexibility to enforce consistency with the empirical impact profiles.

VWAP impact profiles in the Alfonsi-Schied model

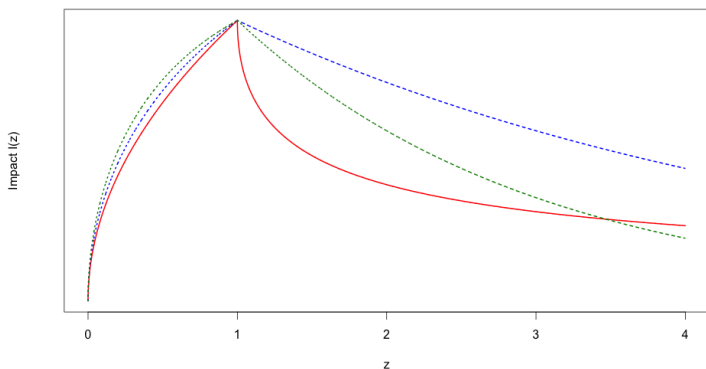


Figure 6: The VWAP impact profile in the AS model with $F^{-1}(x) = \sqrt{x}$. The red line is the square-root propagator model; the blue and green dashed lines, the AS model with $\rho = 1/2$ and $\rho = 1$ respectively.

VWAP in the LLOB model

When $\eta_s = \eta$ a constant, we get the equation

$$z_t = \frac{\eta}{\sqrt{2\pi}} \int_0^t \frac{ds}{\sqrt{t-s}} \exp \left\{ -\frac{(z_t - z_s)^2}{2(t-s)} \right\}. \quad (13)$$

The *exact* solution to (13) is $z_t = A(\eta) \sqrt{t}$ where $A(\eta)$ solves

$$A(\eta) = \frac{\eta}{\sqrt{2\pi}} \int_0^1 \frac{du}{\sqrt{1-u}} \exp \left\{ -\frac{A(\eta)^2 (1 - \sqrt{u})}{2(1 + \sqrt{u})} \right\}.$$

- Thus, for fixed η , market impact is square-root in time.

The expected cost is given by

$$\mathcal{C} = \sigma \sqrt{T} V T \int_0^1 \eta_t z_t dt = \frac{2}{3} Q \sigma \sqrt{T} A(\eta). \quad (14)$$

If $A(\eta) \ll 1$, we get

$$A(\eta) \approx \sqrt{\frac{2}{\pi}} \eta$$

and if $A(\eta) \gg 1$,

$$A(\eta) \approx \frac{2\sqrt{2\pi}}{A(\eta)} \frac{\eta}{\sqrt{2\pi}} = \frac{2\eta}{A(\eta)}.$$

Then in this case,

$$A(\eta) \approx \sqrt{2\eta}.$$

Consistency with the square-root law

In general, the LLOB model does not seem to be consistent with the square-root law. For example

- For small η , from (14),

$$\frac{C}{Q} \approx \sigma \sqrt{T} \sqrt{\frac{2}{\pi}} \eta = \frac{2}{3} \sqrt{\frac{2}{\pi}} \sigma \frac{Q}{V \sqrt{T}}$$

which is definitely not consistent with the square-root law.

- On the other hand, for large η ,

$$\frac{C}{Q} \approx \sigma \sqrt{T} \sqrt{2\eta} = \frac{2}{3} \sqrt{2} \sigma \sqrt{\frac{Q}{V}}$$

which is consistent with (12) and with the square-root law.

- To be useful in practice, we therefore need η to be large, and to reinterpret D as $\alpha \sigma^2/2$ and V as αV with $\alpha \ll 1$.
 - “Include only true investors” maybe?

VWAP impact profiles in the LLOB model

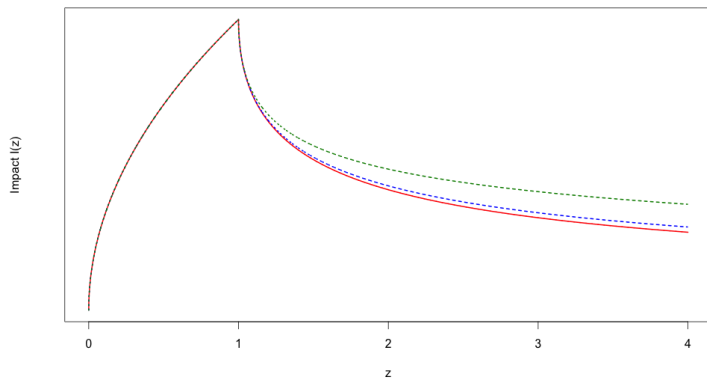


Figure 7: VWAP impact profiles in the LLOB model. The red line is the square-root propagator model; the blue and green dashed lines, the LLOB model with $\eta = 2$ and $\eta = 10$ respectively.

Consistency of LLOB with [Bacry, Iuga et al.]

- It is straightforward to check that the large η regime is $\eta > 10$ or so – too high to be reasonable.
 - Although this is fixable by reinterpreting \mathcal{D} as $\alpha \sigma^2$ with $\alpha \ll 1$, keeping \mathcal{L} constant.
- Nor is LLOB consistent with the impact profiles estimated by [Bacry, Iuga et al.] and [Zarinelli et al.].
 - For fixed η , the LLOB impact profile prior to completion is always square root in normalized time z .
 - The decay rate after completion depends on η .

Pros and cons

We may summarize our discussion as follows:

	Microstructural foundation	Practicality /realism	Consistency (no manipulation)
Propagator	✗	✓	✗
Alfonsi-Schied	✓	✗	✓
LLOB	✓	✗	✓

- A realistic, practical, self-consistent model of market impact is still lacking.

Optimal strategies

- The practical problem of interest is how to trade optimally so as to minimize expected (impact) cost.
 - Recall that in Almgren-Chriss style models, VWAP is optimal if there is no risk penalty.
- Optimal strategies in the three models are as follows:
 - In the propagator model, the optimal monotone strategy (no wrong-way trading) consists of bursts of trading separated by periods of non-trading.
 - In the Alfonsi-Schied order book model, the optimal strategy is bucket-shaped.
 - In the LLOB model, asymptotic analysis suggests that there is little saving from trading optimally.

Summary

- Empirically, the simple square-root model of market impact (law) turns out to be a remarkably accurate description for reasonably sized meta-orders.
- We presented three models, some more realistic than others, that are potentially consistent with the square-root law.
- Each of these models has deficiencies:
 - The (unregularized) propagator model is *ad hoc* and admits price manipulation.
 - The Alfonsi-Schied model is not consistent with empirical estimates of impact profiles.
 - The LLOB model seems not to be consistent with empirical estimates of impact profiles, nor is it naturally consistent with the square-root law.
- A realistic and practical model of market impact is still lacking.

References



[Alfonsi, Fruth and Schied] Aurélien Alfonsi, Antje Fruth and Alexander Schied, Optimal execution strategies in limit order books with general shape functions, *Quantitative Finance* **10**(2) 143–157 (2010).



[Alfonsi, Schied and Slynko] Aurélien Alfonsi, Alexander Schied and Alla Slynko, Order book resilience, price manipulation, and the positive portfolio problem, *SIAM Journal on Financial Mathematics* **3**(1) 511–533 (2012).



[Bacry, Iuga et al.] E. Bacry, A. Iuga, M. Lasnier, and C.-A. Lehalle, Market impacts and the life cycle of investors orders, *Market Microstructure and Liquidity*, 011550009 (2014).



[Curato et al.] Gianbiagio Curato, Jim Gatheral and Fabrizio Lillo, Optimal execution with nonlinear transient market impact, *Quantitative Finance*, forthcoming (2016).



[Donier et al.] Jonathan Donier, Julius Bonart, Iacopo Mastromatteo, and J-P Bouchaud, A fully consistent, minimal model for non-linear market impact, *Quantitative Finance* **15**(7):1109–1121 (2015).



[Gatheral] Jim Gatheral, No-dynamic-arbitrage and market impact, *Quantitative Finance* **10**(7) 749–759 (2010).

References



[Gatheral and Schied] Jim Gatheral and Alexander Schied, Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework, *International Journal of Theoretical and Applied Finance* **14**(3) 353–368 (2011).



[Gatheral, Schied and Slynko (2011)] Jim Gatheral, Alexander Schied, and Alla Slynko. Exponential resilience and decay of market impact, in *Econophysics of order-driven markets*, 225–236, Springer Milan (2011).



[Gatheral, Schied and Slynko] Jim Gatheral, Alexander Schied and Alla Slynko, Transient linear price impact and Fredholm integral equations, *Mathematical Finance* **22**(3) 445–474 (2012).



[Predoiu, Shaikhet and Shreve] Silviu Predoiu, Gennady Shaikhet and Steven Shreve, Optimal execution in a general one-sided limit-order book, *SIAM Journal on Finance Mathematics* **2** 183–212 (2011).



[Tóth et al.] Bence Tóth, Yves Lempérière, Cyril Deremble, Joachim de Lataillade, Julien Kockelkoren, and Jean-Philippe Bouchaud, Anomalous price impact and the critical nature of liquidity in financial markets, *Physical Review X* 021006, 1-11(2011).



[Zarinelli et al.] Elia Zarinelli, Michele Treccani, J. Doyne Farmer, and Fabrizio Lillo, Beyond the square root: Evidence for logarithmic dependence of market impact on size and participation rate, *Market Microstructure and Liquidity* 1550004 (2015).