Brownian motion and stochastic integrals

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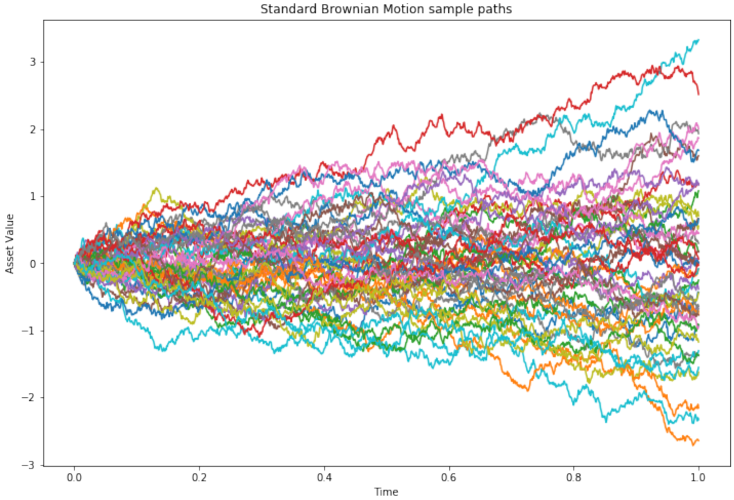
Brownian motion



Brownian motion, mathematically

Real valued Brownian motion indexed by time $t \in \mathbb{R}_+$:

- A stochastic process $(B_t)_{t \in \mathbb{R}_+} : \mathbb{R}_+ \to \Omega \to \mathbb{R}$ such with a specific distribution
- Almost surely continuous paths: $t \mapsto B_t(\omega)$ (in Lean: $(B \cdot \omega)$) is continuous for almost all ω

Alternatively: the increments $B_t - B_s$ are Gaussian with mean 0 and variance $t - s$, independent of B_u for $u \leq s$, and $B_0 = 0$. $+$ continuity.

Genesis of the project

With Peter Pfaffelhuber, we formalized Kolmogorov's extension theorem in 2023.

- A family (P_J) of probability measures on E^J for finite $J \subseteq T$, is projective if for all $J \subseteq I$, the pushforward of P_I by restriction to E^J is P_J.
- Extension theorem: a projective family (P_J) defines a unique measure on E^T whose finite dimensional marginals are the P_J.

Finite dimensional distributions of a process characterize its distribution.

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We decided to turn it into a collaborative project.

Construction overview

First construct a “pre-Brownian”, with the right distributions but not necessarily continuous paths.

- Kolmogorov extension: sufficient to build a projective family of measures.
- For the Brownian motion, finite dimensional distributions are multivariate Gaussians with mean 0 and covariance $\mathrm{cov}(B_{t_i}, B_{t_j}) = \min(t_i, t_j)$.

Then get a continuous modification.

$\to$ Build multivariate Gaussians (only Gaussians on $\mathbb{R}$ in Mathlib at the time).
$\to$ prove that family is projective
$\to$ prove the extension theorem to get a process with the right finite distributions.
$\to$ use the Kolmogorov-Chentsov continuity theorem to get a modification with continuous paths.

Project organization

Preparation phase

- Choose a general proof strategy
- Identify references (textbooks, articles)
- Write a detailed blueprint

Collaboration/formalization phase

- Zulip channel for discussions
- Public project on GitHub, with continuous integration
- Contributors make pull requests

PR phase (ongoing): move everything to Mathlib. 60+ pull requests so far.

Project organization

Preparation phase (3+ weeks, 2 contributors)

- Choose a general proof strategy
- Identify references (textbooks, articles)
- Write a detailed blueprint

Collaboration/formalization phase (3 weeks, 9 contributors)

- Zulip channel for discussions
- Public project on GitHub, with continuous integration
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Blueprint

Developed by Patrick Massot for the Sphere Eversion project.

$\rightarrow$ Let's look at it!

Writing a blueprint:

- detailed definitions, many small lemmas
- for people who don't know the full proof
- from Mathlib to the goal
- the dependency graph is very important

Bottleneck of the project!

What did we formalize?

- Gaussian measures (in Banach spaces, Hilbert spaces, Euclidean spaces)
- Finite distributions of stochastic processes, modifications, indistinguishability
- Kolmogorov extension theorem
- Kolmogorov–Chentsov continuity theorem [Krätschmer & Urusov, 2023]
  - Covering numbers, chaining
- Putting it all together: real valued Brownian motion on $\mathbb{R}_+$, Wiener measure

Brownian motion on $\mathbb{R}_+$: a stochastic process $(B_t)_{t \in \mathbb{R}_+}$ such that

- for $(t_1, \ldots, t_n) \in \mathbb{R}_+^n$, $B_{t_1}, \ldots, B_{t_n}$ are jointly Gaussian with mean 0 and covariance $\mathrm{cov}(B_{t_i}, B_{t_j}) = \min(t_i, t_j)$,
- B has continuous paths.

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2 The Kolmogorov-Chentsov theorem

Continuous modifications

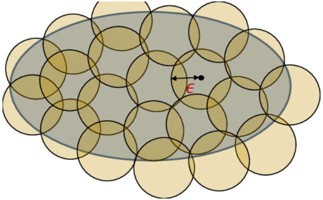
Modification:

- $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ are modifications of each other if for all $t \in T$, $X_t = Y_t$ almost surely.
- $\rightarrow$ Same distribution.
- Not necessarily indistinguishable: $\mathbb{P}(\forall t \in T, X_t = Y_t)$ may be < 1.

Kolmogorov-Chentsov theorem: sufficient conditions for a process to have a modification with continuous paths.

We followed the proof from
[Krätschmer and Urusov, A Kolmogorov–Chentsov type theorem on general metric spaces with applications to limit theorems for Banach-valued processes, Journal of Theoretical Probability, 2023]

Assumptions of the theorem



T, E extended pseudo-metric spaces with metrics d_T, d_E. $(X_t)_{t\in T}$ with values in E.

Assumptions:

- T has bounded covering number with exponent $d > 0$: for some $C > 0$ and all $\varepsilon > 0$, $N_\varepsilon(T) \leq C\varepsilon^{-d}$
- Kolmogorov condition for exponents (p, q), with $q > 0$ and $p > 0$: (X_s, X_t) measurable for the Borel σ-algebra on $E \times E$, and

$$\exists M, \forall s, t \in T, \qquad \mathbb{E}\left[d_E(X_s, X_t)^p\right] \leq M d_T(s,t)^q \,.$$

The Kolmogorov-Chentsov theorem

T, E extended pseudo-metric spaces with metrics d_T, d_E. $(X_t)_{t\in T}$ with values in E.

Theorem (Kolmogorov-Chentsov, Krätschmer & Urusov, 2023)

Suppose that T has bounded covering number with exponent $d > 0$ and that $(X_t)_{t\in T}$ satisfies the Kolmogorov condition for exponents (p, q), with $q > d$ and $p > 0$. Then for all $\beta \in (0, (q-d)/p)$ there exists a finite constant L such that for every countable subset $T' \subseteq T$,

$$\mathbb{E}\left[\sup_{s,t\in T'} \frac{d_E(X_s, X_t)^p}{d_T(s,t)^{\beta p}}\right] \leq L\,.$$

If furthermore E is complete, and T is second-countable, then the process X has a modification with Hölder continuous paths of exponent β for all $\beta \in (0, (q-d)/p)$.

Formalisation insights

It's useful to have definitions focused on random variables

- `HasLaw X P Q`: X has law P on (Ω, Q).
- Could be `Q.map X = P`, but `HasLaw` also contains `AEMeasurable X Q`, to alow composition.
- `HasGaussianLaw`, random variable version of `IsGaussian`.

Constants are hard on paper, easy in Lean.

1. Constant improved in Lemma 1
2. error in lemma 8 where it changes a value
3. quick to fix!

Formalisation insights

Sometimes you don't use the theorem, but actually the particular proof.

Important in the K.C. theorem: the modification is built from limits.

E not second-countable:

- Measurability of $d_E(X_s, X_t)$ needs measurability of pairs (X_s, X_t).
- If we build 2 modifications $Y^{(1)}$, $Y^{(2)}$, and want to combine them, we want measurability of $(Y_s^{(1)}, Y_t^{(2)})$.
- The theorem as stated does not give that. Our formal theorem does:
  `IsLimitOfIndicator Y X P U`

Generalization

Easy to track where assumptions are used.

T and E not true metric spaces but pseudo-metric spaces (distance can be 0 between different points).

- The paper we follow has metric spaces
- We used that in a first formalization
- Then: change into pseudo-metric, see what breaks
- Many minor breaks. Two places needs thought.
- Focus on those two: zulip discussions. Adaptation of the proof, thanks to Sébastien Gouëzel.

Did we formalize the right thing?

The computer checks the proofs, but not the definitions

If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

- B has stationary increments: for all $s \leq t \in \mathbb{R}_+$, $B_t - B_s$ has law $N(0, t - s)$,
- B has independent increments,
- a process with independent increments and such that X_t has law $N(0, t)$ for all $t \in \mathbb{R}_+$ has the same distribution as B,
- for $c > 0$, the process $B_{ct}/\sqrt{c}$ is a Brownian motion,
- for $t_0 \in \mathbb{R}_+$, the process $B_{t_0+t} - B_{t_0}$ is a Brownian motion,
- $tB_{1/t}$ is a Brownian motion,
- B_t tends to 0 almost surely as t tends to 0.

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3 Stochastic integrals

We are not done

Goal: Stochastic integrals

$$\int X dB$$

- Project ongoing, accepting contributions.
- Change in organization: blueprint and code in parallel. Both collaboratively.
- Goal: Itô's lemma for integrals against cadlag semimartingales.
- Some steps: Début theorem, Doob-Meyer decomposition, quadratic variation, square integrable martingales...
- Recent progress: right-continuous filtrations, local properties, local martingales.

Summary

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Thank you!