Polychromatic Colourings of Integers

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December 11, 2025

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1 Polychromatic colourings

2 Formalisation

3 Computation

Definitions

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Must have $1 \leq p(S) \leq \#S$

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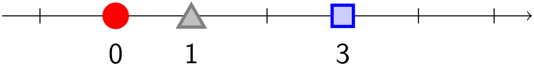

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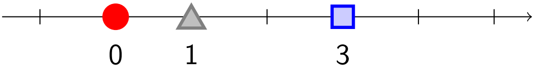

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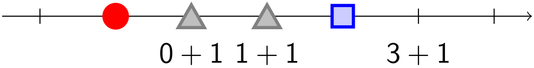

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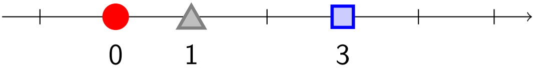

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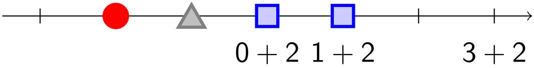

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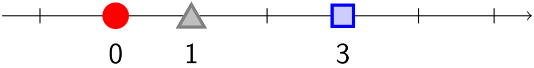

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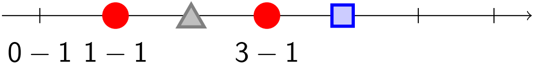

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- With two colours, sets of size 2 can always have a polychromatic colouring, but not size 3.
- Strauss asked: If we fix the number of colours k, is there a large enough size m such that every S with $\#S \geq m$ admits a polychromatic k-colouring? In other words, such that every S with $\#S \geq m$ has $p(S) \geq k$?

First bound

Theorem (Erdős, Lovász 1975)

Yes, $m \leq (3 + o(1))k \log k$ works. In particular, some finite m works.

Equivalently, $p(S) \geq \frac{\#S}{(3+o(1)) \log \#S}$.

Small sets

In the case $\#S = 4$, Axenovich, Goldwasser, Lidický, Martin, Offner, Talbot and Young (2021) proved $p(S) \geq 3$, and hence that $m(3) = 4$.

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We will discuss the work-in-progress formalisation of these two results.

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Definitions

```
def IsPolychrom (S : Finset G) (χ : G → K) :
    Prop :=
  ∀ n : G, ∀ k : K, ∃ a ∈ S, χ (n + a) = k

lemma IsPolychrom.nonempty (h : IsPolychrom S χ) :
    S.Nonempty := by
  obtain ⟨i, hi, hi'⟩ := h 0 (χ 0)
  use i

lemma IsPolychrom.finite (hχ : IsPolychrom S χ) :
    Finite K := ...
```

More definitions


```
def HasPolychromColouring (K : Type*)
    (S : Finset G) : Prop :=
  ∃ χ : G → K, IsPolychrom S χ

lemma HasPolychromColouring.nonempty_set
    (h : HasPolychromColouring K S) :
    S.Nonempty := by
  obtain ⟨χ, hχ⟩ := h
  exact hχ.nonempty
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Local lemma

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- Given an independent collection A_i of 'bad' events, if each $\mathbb{P}(A_i) < 1$, then the probability of all of them failing is nonzero
- But what if they're not independent?
- We can still show that the probability of all A_i failing is nonzero, if each individual one is unlikely enough and they are 'mostly' independent

Formal Local Lemma


```
theorem symmetricLocalLemma
    {h0 : MeasurableSpace Ω} {P : Measure Ω}
    [IsProbabilityMeasure P] [Fintype ι]
    (hA : ∀ i, MeasurableSet (A i))
    {p : ℝ} (hAp : ∀ i, P (A i) ≤ p)
    {N : ι → Finset ι}
    (h : lopsidedCondition P A N)
    {d : ℕ} (hd : d ≠ 0)
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    0 < P (⋂ i, (A i)ᶜ) := by ...
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- Events aren't independent, but they don't overlap too much
- Works reasonably well, we can estimate A_i easily, and bound d enough
- But this only works for finitely many events!
- So the theorem only works if we only care about finitely many translations...
- It works for finite groups, which is nice, but not good enough

Selection Lemma

Instead, uncover Rado's Selection Lemma

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```
theorem Finset.rado_selection {α β : Type}
    [Finite β] (g : Finset α → α → β) :
    ∃ χ : α → β, ∀ s : Finset α,
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Proof uses Tychonoff, and that the intersection of closed sets in a compact space having the finite intersection property is nonempty.
This 'compactness' principle appears often throughout combinatorics, so isolating it into Rado's Selection Lemma helps formal re-use.

Analysis

Putting those pieces together, we get

```
lemma exists_of_le {k m : ℕ}
    {S : Finset G} (hm : #S = m)
    (hm₂ : 2 ≤ m) (hk : k ≠ 0)
    (hkm : Real.exp 1 * (m * (m - 1) + 1) *
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The bound doesn't depend on the group!
It turns out to be satisfied if $m \geq 3k^2$, or with a bit of calculus, $m \geq k(3 \log k + 2 \log \log k + 5.2) = (3 + o(1)) k \log k$.

Asymptotic answer

```
theorem exists_colouring_asymptotic
    {ε : ℝ} (hε : 0 < ε) :
    ∀ᶠ k : ℕ in atTop, ∀ S : Finset G,
      (3 + ε) * k * log k ≤ #S →
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Filters are great!

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- Looking for general results encouraged me to find Rado's Selection Lemma
- Automation helped prove the results in calculus (but this one could be better!)

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How many such S are there? $\binom{288}{3} \approx 4$ million.

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- Can we salvage the proof?

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- I got Z3 (with Python) to help the search get over the line, and print out the colourings generated

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- To check an individual one, write kernel-optimised bitwise operations

```
example :
    ∀ (a b c : ℤ),
      0 < a → a < b → b < c → c < 289 →
    HasPolychromColouring (Fin 3) {0, a, b, c}
```

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- Mathlib enables doing this increasingly quickly, as it grows
- Nothing is impossible!

Acknowledgments

Joachim Breitner, Mario Carneiro, David Thrane Christiansen, Rémy Degenne, Marc Huisinga, Kenny Lau, Jason Reed, John Talbot, Sebastian Ullrich, Eric Wieser, Andrew Yang

Thank you for your attention!