Gen 2: Quaternions and Rotation Cheat Sheet

Quaternions are points on the 4D unit hypersphere. Four-dimensional complex numbers are always of the form:

$$a + bi + cj + dk$$

…with one real part 'a', and 3 *imaginary* or *vector* parts 'b', 'c', and 'd'. Since all quaternions fall on the unit hypersphere, it will always have a distance 1 from the origin. They therefore maintain the following relationship:

$$a^2 + b^2 + c^2 + d^2 = 1$$

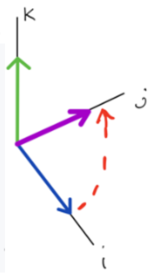
If X and Y are two quaternions that satisfy the above rule, XY will also satisfy it.

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

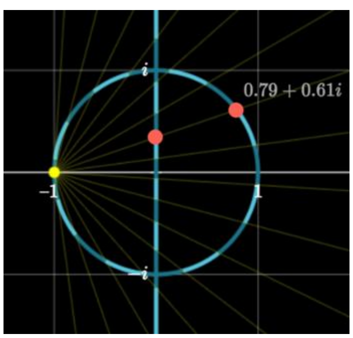


Since quaternions are extensions of complex numbers, we can multiply them by distribution, but this requires stronger definitions of i, j, and k and their multiplication. These relationships can be easily remembered with the right-hand-rule.

Now we can multiply the quaternions by distribution. This can be simplified to the following equation, known as the *Hamilton product*:

$$(a_1 + b_1 i + c_1 j + d_1 k) * (a_2 + b_2 i + c_2 j + d_2 k) = \begin{aligned} &a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ &+ (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\ &+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j \\ &+ (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k. \end{aligned}$$

In order to form a 3D representation of our 4D quaternion, we use a *stereographic projection*, which draws lines through the point (-1, 0, 0, 0) and every other one on the hypersphere. Wherever these lines intersect the 3D space is their projection onto it (2D projection into a 1D space shown below):



Yellow lines are drawn originating at -1 + 0*i* and intersect with every point on the unit circle. The point at which the line intersects the *i*-line is where the point is projected onto the 1D line. Here, you can see the 2D point 0.79 + 0.61*i* onto the point ~0.4*i*.

All these elements combined allow us to use quaternions to define a robot's (or any other 3D object's) orientation in 3D space. Instead of adding rotations, we use the *Hamilton product* to combine quaternions (since we are working one dimension up). The dimensionality of the rotation is best visualized as a rotation Θ around a *Euler axis* (a 3D unit vector). A quaternion *q*, which describes a rotation Θ around the unit vector *u*, is given as:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Helpful links:
https://eater.net/quaternions
https://en.wikipedia.org/wiki/Quaternion
https://en.wikipedia.org/wiki/Quaternions_and_spatial_rotation
https://en.wikipedia.org/wiki/Stereographic_projection