Esercizi su ricorrenze

Exercise 1.1 Risolvere, verificando la correttezza della soluzione trovata, la seguente equazione di ricorrenza valida *per ogni* $n \geq 0$:

$$T(n) = T(n-2) + 4n, \ n \geq 2, \quad T(1) = 4 \ \ , \ T(0) = 1$$

Answer: We solve by iteration, noticing that

$$T(n) = T(n-2) + 4n = T(n-2-2) + 4(n-2) + 4n = \ldots = T(n-2i) + 4\sum_{j=0}^{i-1}(n-2j).$$

Observe that when $i^\star = \lfloor n/2 \rfloor$, we have that $T(n) = T(n - 2\lfloor n/2 \rfloor) + 4\sum_{j=0}^{\lfloor n/2 \rfloor - 1}(n-2j)$. For this value we obtain

$$\begin{aligned} T(n) &= T(n-2\lfloor n/2 \rfloor) + 4(\lfloor n/2 \rfloor n - \lfloor n/2 \rfloor(\lfloor n/2 \rfloor - 1)) \\ &= T(n-2\lfloor n/2 \rfloor) + 4\lfloor n/2 \rfloor(n - \lfloor n/2 \rfloor + 1) \\ &= T(n-2\lfloor n/2 \rfloor) + 4\lfloor n/2 \rfloor(\lceil n/2 \rceil + 1) \end{aligned}$$

Observing that $\lfloor n/2 \rfloor = \lceil n/2 \rceil = n/2$ for even n, and that $\lfloor n/2 \rfloor = (n-1)/2$, and $\lceil n/2 \rceil = (n+1)/2$ for odd n, substituting the right value in the last equation and doing simple arithmetic manipulations, we obtain that $T(n) = (n+1)^2$ for all values of n, which can be easily proved correct by induction (omitted).

We remark that we could have easily obtained the closed formula by computing the initial values $(T(0) = 1, T(1) = 4, T(2) = T(0) + 8 = 9, T(3) = T(1) + 12 = 16, \ldots)$ and then formulating a plausible guess, and finally proving the guess correct via induction. □

Exercise 1.2 Si consideri la seguente equazione di ricorrenza quando il parametro n è della forma 2^{2^i}, con $i \geq 0$:

$$T(n) = \frac{\sqrt{n}}{2} T\left(\sqrt{n}\right), \; n > 2, \quad T(2) = 2.$$

(a) Risolvere la ricorrenza, determinando la formula esatta per $T(n)$.

(b) Verificare la soluzione ottenuta utilizzando l'induzione.

Answer: Recall that for $n = 2^{2^i}$ and $f(n) = \sqrt{n}$, we have that $f^{(j)}(n) = n^{1/2^j}$ and $f^\star(n,2) = \log_2 \log_2 n - 1$. We can then apply the general formula, keeping in mind that $s(n) = \sqrt{n}/2$, hence $s(f^j(n)) = \sqrt{n^{1/2^j}}/2 = n^{1/2^{j+1}}/2$, $T_0 = 2$, and $w(n) = 0$, which implies that the contribution of the internal nodes is null. We have:

$$\begin{aligned}
T(n) &= 2\left(\prod_{j=0}^{\log_2 \log_2 n - 1} n^{1/2^{j+1}}/2\right) \\
&= 2\left(n^{\sum_{j=0}^{\log_2 \log_2 n - 1} 1/2^{j+1}}\right)/2^{\log_2 \log_2 n} \\
&= 2\left(n^{\sum_{j'=1}^{\log_2 \log_2 n} 1/2^{j'}}\right)/2^{\log_2 \log_2 n} \\
&= 2\left(n^{(1 - 1/2^{\log_2 \log_2 n})}\right)/\log_2 n \\
&= 2\left(n/n^{1/\log_2 n}\right)\log_2 n \\
&= n/\log_2 n,
\end{aligned}$$

since $n^{1/\log_2 n} = n^{\log_n 2} = 2$.

To verify the above closed formula using induction, observe that $T(2) = 2 = (n/\log_2 n)|_{n=2}$. Under the hypothesis that the formula holds for values less than n, we have:

$$\begin{aligned}
T(n) &= (\sqrt{n}/2)T(\sqrt{n}) \\
&= (\sqrt{n}/2)(\sqrt{n}/\log_2(\sqrt{n})) \\
&= n/\log_2 n,
\end{aligned}$$

since $\log_2(\sqrt{n}) = \log_2(n)/2$. □

Exercise 1.3 Si consideri la seguente equazione di ricorrenza quando il parametro n è della forma 2^{2^i}, con $i \geq 0$:

$$T(n) = T\left(\sqrt{n}\right) + (\log_2 n)^2, \; n > 2, \quad T(2) = 0.$$

(a) Risolvere la ricorrenza, determinando la formula esatta per $T(n)$.

(b) Verificare la soluzione ottenuta utilizzando l'induzione.

Answer: Recall that for $n = 2^{2^i}$ and $f(n) = \sqrt{n}$, we have that $f^{(\ell)}(n) = n^{1/2^\ell}$ and $f^\star(n, 2) = \log_2 \log_2 n - 1$. We can then apply the general formula, keeping in mind that $s(n) = 1$, $T_0 = 0$, and $w(n) = \log_2^2 n$, whence $w(f^{(\ell}(n)) = (\log_2(n^{1/2^\ell}))^2 = (1/2^{2\ell}) \log_2^2 n$. We have:

$$
\begin{aligned}
T(n) &= \sum_{\ell=0}^{\log_2 \log_2 n - 1} (1/2^{2\ell}) \log_2^2 n \\
&= \log_2^2 n \sum_{\ell=0}^{\log_2 \log_2 n - 1} (1/4^\ell) \\
&= (4/3)(\log_2^2 n)(1 - (1/2^{2\log_2 \log_2 n})) \\
&= (4/3)(\log_2^2 n)(1 - (1/2^{\log_2 \log_2^2 n})) \\
&= (4/3)(\log_2^2 n)(1 - 1/\log_2^2 n) \\
&= (4/3)(\log_2^2 n - 1).
\end{aligned}
$$

The correctness of the above closed formula can be easily proved by induction (omitted). □

Exercise 1.4 Si risolva la seguente equazione di ricorrenza per valori del parametro $n = 3^{3^i}$:

$$
T(n) = \begin{cases} T\left(\sqrt[3]{n}\right) + 2\log_3 \log_3 n, & n > 3, \\ 0 & n = 3. \end{cases}
$$

Answer: Recall that for $n = 3^{3^i}$ and $f(n) = \sqrt[3]{n}$, we have that $f^{(\ell)}(n) = n^{1/3^\ell}$ and $f^\star(n, 3) = \log_3 \log_3 n - 1$. We can then apply the general formula, keeping in mind that $s(n) = 1$, $T_0 = 0$, and $w(n) = 2\log_3 \log_3 n$, whence $w(f^{(\ell)}(n)) = 2\log_3 \log_3(n^{1/3^\ell}) = 2\log_3((log_3 n)/3^\ell) = 2(\log_3 \log_3 n - \ell)$. We have:

$$
\begin{aligned}
T(n) &= \sum_{\ell=0}^{\log_3 \log_3 n - 1} 2(\log_3 \log_3 n - \ell) \\
&= 2 \sum_{\ell'=1}^{\log_3 \log_3 n} \ell' \\
&= (\log_3 \log_3 n)((\log_3 \log_3 n + 1).
\end{aligned}
$$

The correctness of the above closed formula can be trivially proved by induction (omitted). □

Exercise 1.5 Si consideri la seguente equazione di ricorrenza per valori di n che siano potenze di due:

$$T(n) = \begin{cases} T(n/2) + T(n/4) + 2T(n/8) + n, & n \geq 8, \\ 4 & n < 8. \end{cases}$$

Si determini una costante $c > 0$ tale che $\forall n, T(n) \geq cn \log_2 n$.

Answer: Let us set up a parametric induction framework. For the base cases $n = 1, 2, 4$, we obtain the three constraints

$$4 \overset{?}{\geq} c \cdot 1 \log_2 1 = 0 \text{ (empty constraint)}$$
$$4 \overset{?}{\geq} c \cdot 2 \log_2 2 \Leftrightarrow c \leq 2$$
$$4 \overset{?}{\geq} c \cdot 4 \log_2 4 \Leftrightarrow c \leq 4/8 = 1/2.$$

Assuming that the hypothesis holds for values of the parameter which are powers of two strictly less than n, we obtain:

$$\begin{aligned} T(n) &= T(n/2) + T(n/4) + 2T(n/8) + n \\ &\geq c(n/2)(\log_2(n/2)) + c(n/4)(\log_2(n/4)) + 2c(n/8)(\log_2(n/8)) + n \\ &= c(n/2)(\log_2 n - 1) + c(n/4)(\log_2 n - 2) + 2c(n/8)(\log_2 n - 3) + n \\ &= c(n/2 + n/4 + 2n/8) \log_2 n - cn(1/2 + 1/2 + 3/4) + n \\ &= cn \log_2 n - (7c/4)n + n \overset{?}{\geq} cn \log_2 n \\ &\Leftrightarrow -(7c/4)n + n \geq 0 \\ &\Leftrightarrow c \leq 4/7 \end{aligned}$$

To let the parametric induction go through we intersect all the constraints. Hence, it suffices to choose $c = \min\{2, 1/2, 4/7\} = 1/2$, hence $T(n) \geq (1/2)n \log_2 n$. □

Exercise 1.6 Si consideri la seguente relazione di ricorrenza definita per valori arbitrari positivi del parametro n:

$$T(n) = \begin{cases} 3, & n \leq 6, \\ T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + n, & n > 6. \end{cases}$$

Si dimostri che $T(n) = O(n)$.

Answer: Let us set up a parametric induction framework to determine a constant c such that $T(n) \leq cn$. For the base case $n \leq 6$, we obtain the generic constraint $3 \stackrel{?}{\leq} cn \Leftrightarrow c \geq 3/n$. Thus the more stringent constraint is for $n = 1$, which yields $c \geq 3$. Assuming that the hypothesis holds for values of the parameter which are less than n, we obtain:

$$\begin{array}{rcl} T(n) & = & T(\lfloor n/3 \rfloor) + T(\lceil n/2 \rceil - 1) + n \\ & \leq & c\lfloor n/3 \rfloor + c((\lceil n/2 \rceil - 1) + n \\ & \leq & cn/3 + c(n/2 + 1 - 1) + n \\ & = & c(5/6)n + n \stackrel{?}{\leq} cn \\ & \Leftrightarrow & cn/6 \geq n \\ & \Leftrightarrow & c \geq 6. \end{array}$$

To let the parametric induction go through, we intersect the constraints. Hence, it suffices to choose $c = \max\{3, 6\} = 6$, hence $T(n) \leq 6n$. □

Exercise 1.7 Si risolva la seguente equazione di ricorrenza per valori del parametro $n = 2^i$:

$$T(n) = \begin{cases} 3T\left(\frac{n}{2}\right) + 7n^{\log_2 3}, & n > 1, \\ 1 & n = 1. \end{cases}$$

Answer: Recall that for $n = 2^i$ and $f(n) = n/2$, we have that $f^{(j)}(n) = n/2^j$ and $f^\star(n, 1) = \log_2 n - 1$. We can then apply the general formula, keeping in mind that $s(n) = 3$, hence the number of nodes at level ℓ is 3^ℓ, $T_0 = 1$, and $w(n) = 7n^{\log_2 3}$, whence $w(f^{(\ell}(n)) = 7(n/2^\ell)^{\log_2 3} = 7n^{\log_2 3}/3^\ell$. We have:

$$\begin{array}{rcl} T(n) & = & \displaystyle\sum_{\ell=0}^{\log_2 n - 1} 3^\ell \cdot 7(n/2^\ell)^{\log_2 3} + 3^{\log_2 n} \\ & = & 7n^{\log_2 3} \left(\displaystyle\sum_{\ell=0}^{\log_2 n - 1} 3^\ell/3^\ell \right) + n^{\log_2 3} \\ & = & n^{\log_2 3}(7\log_2 n + 1). \end{array}$$

The correctness of the above closed formula can be easily proved by induction (omitted). □

Exercise 1.8 Si risolva e si verifichi la seguente equazione di ricorrenza per valori del parametro $n = 2^{2^i}$:

$$T(n) = \begin{cases} \sqrt{2}T(\sqrt{n}) + \sqrt{\log n}, & n > 2, \\ 0 & n = 2. \end{cases}$$

Answer: Recall that for $n = 2^{2^i}$ and $f(n) = \sqrt{n}$, we have that $f^{(\ell)}(n) = n^{1/2^\ell}$ and $f^\star(n, 2) = \log_2 \log_2 n - 1$. We can then apply the general formula, keeping in mind that $s(n) = \sqrt{2}$, whence $\prod_{j=0}^{\ell-1} s(f^{(j)}(n)) = (\sqrt{2})^\ell = 2^{\ell/2}$, $T_0 = 0$, and $w(n) = \sqrt{\log_2 n}$, whence $w(f^{(\ell}(n)) = \sqrt{\log_2 n^{1/2^\ell}} = \sqrt{\log_2 n}/2^{\ell/2}$. We have:

$$\begin{aligned} T(n) &= \sum_{\ell=0}^{\log_2 \log_2 n - 1} 2^{\ell/2} \sqrt{\log_2 n}/2^{\ell/2} \\ &= \sqrt{\log_2 n} \sum_{\ell=0}^{\log_2 \log_2 n - 1} 1 \\ &= \sqrt{\log_2 n} \log_2 \log_2 n. \end{aligned}$$

The correctness of the above closed formula can be easily proved by induction (omitted). □

Exercise 1.9 Si consideri la seguente equazione di ricorrenza definita per ogni valore positivo del parametro n:

$$T(n) = \begin{cases} 7T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n^3, & n > 1, \\ 8 & n = 1. \end{cases}$$

Utilizzando l'induzione parametrica, si dimostri che $T(n) = O(n^3)$.

Answer: Let us set up a parametric induction framework to determine a constant c such that $T(n) \le cn^3$. For the base case $n = 1$, we obtain the simple constraint $c \ge 8$ from $8 \stackrel{?}{\le} cn^3|_{n=1}$. Assuming that the hypothesis holds for values of the parameter which are less than n, we obtain:

$$\begin{aligned} T(n) &= 7T(\lfloor n/2 \rfloor) + n^3 \\ &\le 7c(\lfloor n/2 \rfloor)^3 + n^3 \\ &\le 7cn^3/8 + n^3 \stackrel{?}{\le} cn^3 \\ &\Leftrightarrow 7c/8 + 1 \le c \\ &\Leftrightarrow c \ge 8. \end{aligned}$$

To let the parametric induction go through, we intersect the constraints. Hence, it suffices to choose $c = \max\{8, 8\} = 8$, hence $T(n) \leq 8n^3$. $\square$

Exercise 1.10 Al variare dei due parametri interi positivi a e b, si consideri la seguente famiglia di ricorrenze, definite per $n = 2^{2^i}$:

$$T_{a,b}(n) = \begin{cases} 2^a T_{a,b}(\sqrt{n}) + \log_2^b n, & n > 2, \\ 0 & n = 2. \end{cases}$$

Si determini l'ordine di grandezza di $T_{a,b}(n)$ in funzione di n e dei due parametri a e b. (*Suggerimento:* Si applichi la formula generale e si discutano vari casi a seconda dei valori di a e b ...)

Answer: Let us apply the general formula with $s(n) = 2^a$, $f(n) = \sqrt{n}$, $w(n) = \log_2^b n$, $T_0 = 0$ and $n_0 = 2$. We have $f^{(i)}(n) = n^{1/2^i}$ and $f^\star(n, 2) = \log_2 \log_2 n - 1$. Moreover, $\prod_{j=0}^{\ell-1} 2^a = 2^{a\ell}$ and $w(f^{(\ell)}(n)) = \log_2^b n^{1/2^\ell} = (1/2^{b\ell}) \log_2^b n$. Therefore:

$$\begin{aligned} T(n) &= \sum_{\ell=0}^{\log_2 \log_2 n - 1} 2^{(a-b)\ell} \log_2^b n \\ &= \log_2^b n \sum_{\ell=0}^{\log_2 \log_2 n - 1} (2^{a-b})^\ell. \end{aligned}$$

Let $S(n) = \sum_{\ell=0}^{\log_2 \log_2 n - 1} (2^{a-b})^\ell$. We have three cases, depending on the values of a and b. If $a < b$, then $S(n) = \Theta(1)$, hence $T_{a,b}(n) = \Theta(\log^b n)$. If $a = b$, then $S(n) = \log_2 \log_2 n$, hence $T_{a,b}(n) = \Theta(\log^b n \log \log n)$. Finally, if $a > b$, then $S(n) = \Theta((2^{a-b})^{\log_2 \log_2 n}) = \Theta(\log^{a-b} n)$, hence $T_{a,b}(n) = \Theta(\log^a n)$.

Observe that when the logarithm appears within an asymptotic notation (and is not at the exponent), we omit the constant base since in this case the base cannot affect the asymptotics. $\square$

Exercise 1.11 Si consideri la seguente equazione di ricorrenza, definita per valori arbitrari parametro $n > 0$:

$$T(n) = \begin{cases} \lfloor \sqrt{n} \rfloor T(\lfloor \sqrt{n} \rfloor) + n^2, & n > 2, \\ 1 & n = 1, 2. \end{cases}$$

Si dimostri, utilizzando l'induzione parametrica, che $T(n) \in O(n^2)$.

Answer:

We postulate that there exists a positive constant c for which $T(n) \leq cn^2$, for any value of $n > 0$, and then proceed to find an appropriate value for c using parametric induction. Consider the base cases first. It must be $T(1) = 1 \leq c \cdot 1^2$ and $T(2) = 1 \leq c \cdot 2^2$, whence $c \geq \max\{1, 1/4\} = 1$. For $n \geq 3$, assume that $T(k) \leq ck^2$ for $k < n$. When $k = n$ we obtain:

$$\begin{aligned} T(n) &\leq \lfloor\sqrt{n}\rfloor T(\lfloor\sqrt{n}\rfloor) + n^2 \\ &\leq cn\sqrt{n} + n^2 \end{aligned}$$

It then suffices to choose a constant c such that for all $n \geq 3$,

$$\begin{aligned} cn\sqrt{n} + n^2 &\leq cn^2 \Leftrightarrow \\ c + \sqrt{n} &\leq c\sqrt{n} \Leftrightarrow \\ c &\geq \sqrt{n}/(\sqrt{n} - 1) \end{aligned}$$

Note that $f(n) = \sqrt{n}/(\sqrt{n} - 1)$ is a strictly decreasing function over its domain, hence it suffices to choose $c \geq f(3) = \sqrt{3}/(\sqrt{3} - 1) > 1$. Putting it all together, we have proved that

$$T(n) \leq \frac{\sqrt{3}}{\sqrt{3} - 1} n^2.$$

□

Exercise 1.12 Sia n una potenza di due. Dato un problema Π, sia A^1_Π un algoritmo ricorsivo per Π la cui complessità in tempo $T_1(n)$ obbedisce alla seguente ricorrenza:

$$T_1(n) = \begin{cases} 1 & n = 1, \\ 2T_1\left(\frac{n}{2}\right) + 2n & n > 1. \end{cases}$$

Sia A^2_Π un ulteriore algoritmo per Π di complessità $T_2(n) = n^{3/2}$. Si determini la complessità in tempo del migliore algoritmo ibrido per Π ottenibile combinando A^1_Π e A^2_Π.

Answer: Let $i \in \mathcal{I}_\Pi$ be an instance of Π, with $\text{size}(i) = n$, and let n_0 be a constant parameter (power of two) to be determined. In order to obtain a hybrid algorithm by combining A^1_Π and A^2_Π, we change the base case in the code of A^1_Π, as follows.

HYBRID(i)
$n \leftarrow$ size(i)
if $n \leq n_0$
    then return $A^2_{\Pi}(i)$
...
{ same code as in A^1_{Π}}
...

The recurrence associated with the new algorithm is parametric in both n and n_0:

$$T_{n_0}(n) = \begin{cases} n^{3/2} & n \leq n_0, \\ 2T_{n_0}\left(\frac{n}{2}\right) + 2n & n > n_0. \end{cases}$$

Let $n \geq n_0$. By using the iterated method, we obtain:

$$T_{n_0}(n) = 2^i T_{n_0}\left(\frac{n}{2^i}\right) + i \cdot 2n, \quad 1 \leq i \leq \log(n/n_0).$$

By substituting $i = \log(n/n_0)$ in the above formula, we obtain:

$$T_{n_0}(n) = \left(\frac{n}{n_o}\right) n_0^{3/2} + \log(n/n_0) \cdot 2n = 2n \log n + n(\sqrt{n_0} - 2\log n_0)$$

(note that $T_{n_0}(n_0) = n_0^{3/2}$). In order to obtain the optimal choice for n_0, it suffices to compute the partial derivative of $T(n, n_0)$ with respect to n_0 and equate it to zero:

$$\frac{\partial T_{n_0}(n)}{\partial n_0} = n\left(\frac{1}{2\sqrt{n_0}} - \frac{2}{n_0 \ln 2}\right) = 0 \Leftrightarrow \sqrt{n_0} = \frac{4}{\ln 2} \Leftrightarrow n_0 = \frac{16}{\ln^2 2} \simeq 33.3.$$

Therefore, the best choice is either $\bar{n}_0 = 32$ or $\bar{n}_0 = 64$. Since $\sqrt{32} - 2\log 32 < \sqrt{64} - 2\log 64$, we choose $\bar{n}_0 = 32$, which yields the best hybrid algorithm. $\square$