Pit-Stop Revision Handwritten Notes

Feel free to pass this around!

PURE 2

① Algebraic Methods

Proof by Contradiction: Assume statement is untrue. Use logical steps to prove otherwise, that the original statement is actually true.

Partial fractions: $\dfrac{\ldots}{(\ldots)(\ldots)} = \dfrac{A}{(\ldots)} + \dfrac{B}{(\ldots)}$

$$\frac{\quad}{(\ldots)^2(\ldots)} = \frac{A}{(\ldots)} + \frac{B}{(\ldots)^2} + \frac{C}{(\ldots)}$$

② Functions

Modulus = $|\ldots|$ Anything in modulus results in a +ve value.

E.g. $|-2| = 2$

Solving modulus: $|3x-2| = 5$

$-$ve: $-(3x-2) = 5$

+ve: $3x - 2 = 5$

Domain = All possible x - values

Range = All possible y- values

Composite function: $fg(x)$

Apply g first, then f

(I like to think of it as the "g" function inside the "f" function)

Drawing Modulus = $y = |f(x)|$ (+ve y's)

$y = f(|x|)$
+ve x.
↳ Reflect the +ve x part into the -ve

(3) Sequences and Series

Arithmetic seq: $u_n = a + (n-1)d$

first term (a) — no. of terms (n) — common difference (d)

Arithmetic series: $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$S_n = \frac{n}{2}(a + l)$

last term (l)

Geometric seq: $u_n = ar^{n-1}$

common ratio (r)

Geometric series: $S_n = \dfrac{a(1-r^n)}{1-r}$ $\quad r \neq 1$

Use if $r<1$

Use if $r>1$ $\longrightarrow$ $S_n = \dfrac{a(r^n - 1)}{r-1}$ $\quad r \neq 1$

Sum to infinity: $S_\infty = \dfrac{a}{1-r}$ only if convergent

$\sum$ means 'Sum of'

"sigma"

Recurrence Relation = Plug-in previous value to get next value.

$-2, 1, -2, 1, \ldots$ = Periodic sequence (period 2).

④ Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

valid when $|x| < 1$, $n \in \mathbb{R}$

If $(1+bx)^n$: valid when $|bx| < 1$ or $|x| < \frac{1}{|b|}$

If $(a+bx)^n = a^n\left(1 + \frac{b}{a}x\right)^n$ valid when $\left|\frac{b}{a}x\right| < 1$ or $|x| < \left|\frac{a}{b}\right|$

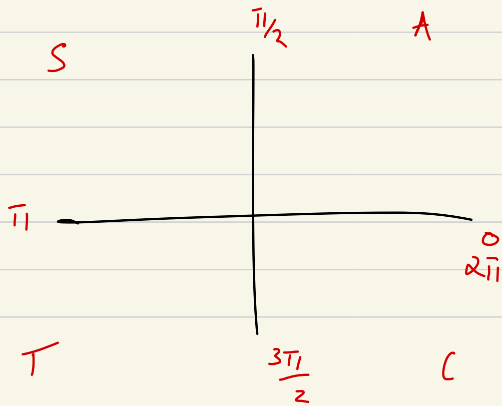
⑤ Radians (π)

Degrees $\longrightarrow$ Radians

$\div 180^\circ$ then $\times \pi$

Radians $\longrightarrow$ Degrees

$\div \pi$ then $\times 180^\circ$

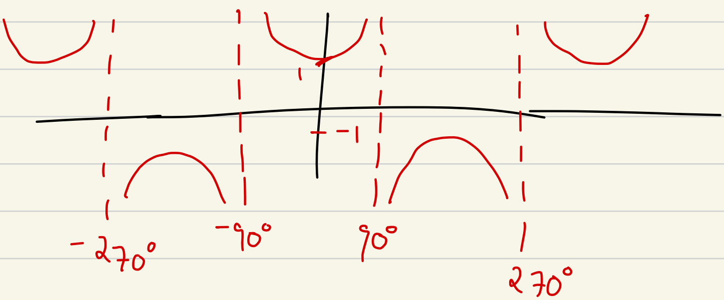


- Arc length : $l = r\theta$
- Area of Sector : $A = \frac{1}{2}r^2\theta$
- Area of Segment : $A = \frac{1}{2}r^2(\theta - \sin\theta)$
- Small Angle Approximations:
  - $\sin\theta \approx \theta$
  - $\tan\theta \approx \theta$
  - $\cos\theta \approx 1 - \frac{\theta^2}{2}$

6 Trig Functions

- $\sec x = \frac{1}{\cos x}$
- $\text{cosec}\, x = \frac{1}{\sin x}$
- $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

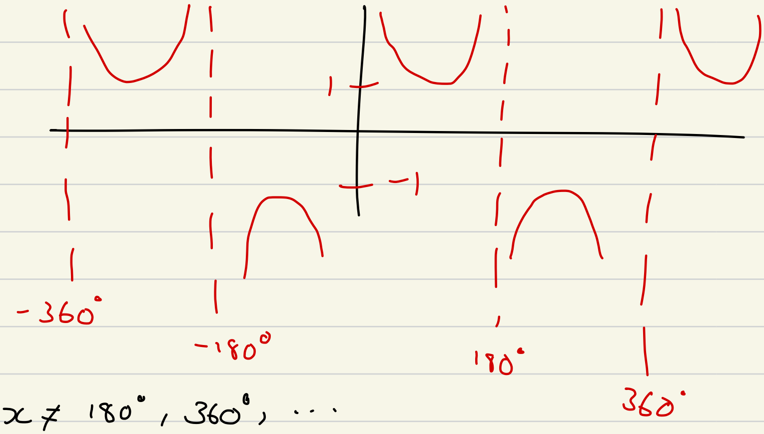
$y = \sec x$



Domain : $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$

Range: $y \geq 1$ or $y \leq -1$

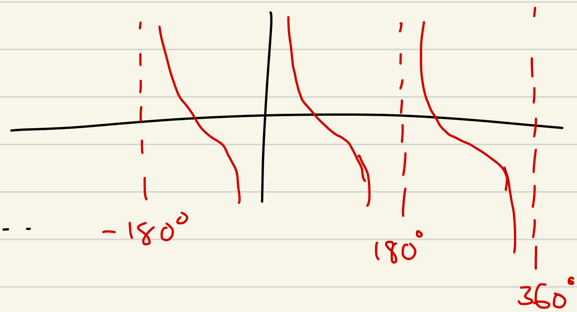
$y = \text{cosec}\, x$



Domain: $x \neq 180°, 360°, \ldots$

Range: $y \geqslant 1$ or $y \leqslant -1$

$y = \cot x$



Domain: $x \in \mathbb{R}$,
$x \neq 0, 180°, 360°, \ldots$

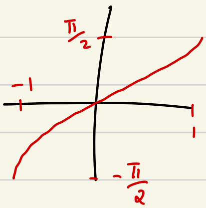
Range: $y \in \mathbb{R}$.

$\sec x = k$ and $\text{cosec}\, x = k$ have no solutions for $-1 < k < 1$

- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \text{cosec}^2 x$

Inverse functions: arc...

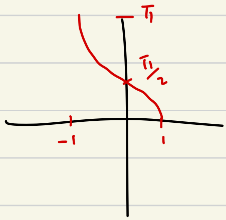
arsinx



Domain. $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq \text{arsinx} \leq \frac{\pi}{2}$

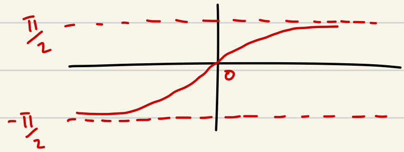
arccosx



Domain: $-1 \leq x \leq 1$

Range: $0 \leq \arccos x \leq \pi$

arctan x



Domain: $x \in \mathbb{R}$

Range: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

7 Trig and Modelling

Addition Formulae:

- $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double Angle Formulae:

- $\sin 2A \equiv 2\sin A\cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$

- $a\sin x \pm b\cos x \longrightarrow R\sin(x \pm \alpha)$
- $a\cos x \pm b\sin x \longrightarrow R\cos(x \mp \alpha)$
- $R\cos\alpha = a$ - $R\sin\alpha = b$ - $R = \sqrt{a^2 + b^2}$

8 Parametric Equations

For parametrics $x = p(t)$ and $y = q(t)$ with Cartesian $y = f(x)$:

Domain of $f(x)$ = Range of $p(t)$

Range of $f(x)$ = Range of $q(t)$

Curve sketching: Form a table.

E.g.

$t =$				
x parametric $=$				
y parametric $=$				

9 Differentiation

- $y = \sin kx$ $\qquad \dfrac{dy}{dx} = k\cos kx$
- $y = \cos kx$ $\qquad \dfrac{dy}{dx} = -k\sin kx$
- $y = \tan kx$ $\qquad \dfrac{dy}{dx} = k\sec^2 kx$
- $y = \sec kx$ $\qquad \dfrac{dy}{dx} = k\sec kx\tan kx$
- $y = \operatorname{cosec} kx$ $\qquad \dfrac{dy}{dx} = -k\operatorname{cosec} kx\cot kx$
- $y = \cot kx$ $\qquad \dfrac{dy}{dx} = -k\operatorname{cosec}^2 kx$

- $y = e^{kx}$

  $\dfrac{dy}{dx} = ke^{kx}$

- $y = \ln x$

  $\dfrac{dy}{dx} = \dfrac{1}{x}$

- $y = a^{kx}$

  $\dfrac{dy}{dx} = a^{kx}k\ln a$

Chain Rule = $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Product Rule : $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$y = uv$

Quotient Rule : $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$y = \frac{u}{v}$

$$\boxed{\frac{dy}{dx} = \frac{1}{dx/dy}}$$

Parametrics: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Implicit : Add " $\frac{dy}{dx}$ " after differentiating a y-term.

E.g.

$x^3 + y^2 = 2$

$\hookrightarrow 3x^2 + 2y\frac{dy}{dx} = 0$

If : $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

Differentiating this function

Second Derivative

Concave : $f''(x) \leq 0$ Convex : $f''(x) \geq 0$

Point of Inflection = $f''(x) = 0$.
(aka turning point) Changes sign on either side.

(10) Numerical Methods

Iteration: Rearrange $f(x) = 0$ into $x = g(x)$

$$\downarrow$$

$$x_{n+1} = g(x_n)$$

Newton Raphson method : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(11) Integration

- $\int e^x \, dx = e^x + c$
- $\int \sin x \, dx = -\cos x + c$
- $\int \frac{1}{x} \, dx = \ln|x| + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int \cos x \, dx = \sin x + c$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int \tan x \, dx = \ln|\sec x| + c$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + c$
- $\int \cot x \, dx = \ln|\sin x| + c$
- $\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + c$
- $\int f'(ax+b) = \frac{1}{a} f(ax+b) + c$

Reverse Chain Rule:

- $\int k \frac{f'(x)}{f(x)} \, dx$ = try $\ln|f(x)|$, differentiate and adjust where needed.
- $\int k f'(x)(f(x))^n \, dx$ = try $(f(x))^{n+1}$, differentiate and adjust.

Integration by substitution: use $u = \ldots$ and change the whole integral to u terms.

Integration by parts: $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$

Area between two curves: $\int_a^b \left(f(x) - g(x)\right) dx$

Trapezium Rule: $\int_a^b y\, dx \approx \frac{1}{2} h \left(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right)$

(will always have an x and y table)

$h = \frac{b-a}{n}$

no. of trapezia

Differential Equations: When $\frac{dy}{dx} = f(x)g(y) \rightarrow \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$

12 3D Vectors

Now working with x, y, z axis.

distance from origin: $\sqrt{x^2 + y^2 + z^2}$

distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $p\underline{i} + q\underline{j} + r\underline{k}$

vector $\underline{a} = x\underline{i} + y\underline{j} + z\underline{k}$ makes an angle.

- with x-axis = $\cos\theta_x = \frac{x}{|\underline{a}|}$ - y-axis $\cos\theta_y = \frac{y}{|\underline{a}|}$

- z-axis = $\cos\theta_z = \frac{z}{|\underline{a}|}$

$p\underline{i} + q\underline{j} + r\underline{k} = u\underline{i} + v\underline{j} + w\underline{k}$

$\hookrightarrow$ $p = u$ $q = v$ $r = w$

Resultant force $(R) = F_1 + F_2 + F_3$

$F = ma$