

# Swinging Atwood Machine

Michael Kister, Alli Behr

## Lagrangian Function

$$\mathcal{L} = \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = T - U$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

## Hamiltonian

$$\mathcal{H} = \sum_{i=1}^n p_i \dot{q}_i - \mathcal{L}$$

$$\mathcal{H} = \mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n)$$

$$\mathcal{H} = T + U$$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

## Swinging Atwood Machine

$$\begin{aligned} T &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_1 v_2^2 \\ &= \frac{1}{2} m_1 \left( \dot{r}_1^2 + (r_1 \dot{\theta}_1)^2 \right) + \frac{1}{2} m_2 \left( \dot{r}_2^2 + (r_2 \dot{\theta}_2)^2 \right) \\ U &= m_1 g [r_1 (1 - \cos \theta_1) + (R - r_1)] + m_2 g [r_2 (1 - \cos \theta_2) + (R - r_2)] \end{aligned}$$

$$\begin{aligned} r_2 = R - r_1 & \quad \text{and} \quad \dot{r}_2 = -\dot{r}_1 \\ \dot{\theta}_2 = 0 & \quad \text{and} \quad \theta_2 = 0 \end{aligned}$$

$$\mathcal{H} = \frac{1}{2}m_1 \left( \dot{r}_1^2 + (r_1 \dot{\theta}_1)^2 \right) + \frac{1}{2}m_2 \dot{r}_1^2 - m_1 g r_1 \cos \theta_1 + m_2 g r_1$$

or

$$\mathcal{H} = \frac{1}{2}m_1 \left( \dot{r}^2 + (r \dot{\theta})^2 \right) + \frac{1}{2}m_2 \dot{r}^2 - m_1 g r \cos \theta + m_2 g r$$

$$\begin{aligned} p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial T}{\partial \dot{r}} \\ &= \frac{\partial}{\partial \dot{r}} \left( \frac{1}{2}m_1 \left( \dot{r}^2 + (r \dot{\theta})^2 \right) + \frac{1}{2}m_2 \dot{r}^2 \right) \\ &= \dot{r}(m_1 + m_2) \implies \\ \dot{r} &= \frac{p_r}{(m_1 + m_2)} \end{aligned}$$

and:

$$\begin{aligned} p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} \\ &= \frac{\partial}{\partial \dot{\theta}} \left( \frac{1}{2}m_1 \left( \dot{r}^2 + (r \dot{\theta})^2 \right) + \frac{1}{2}m_2 \dot{r}^2 \right) \\ &= m_1 r^2 \dot{\theta} \implies \\ \dot{\theta} &= \frac{p_\theta}{m_1 r^2} \end{aligned}$$

$$\mathcal{H} = \frac{1}{2}m_1 \left[ \left( \frac{p_r}{(m_1 + m_2)} \right)^2 + \left( r \left( \frac{p_\theta}{m_1 r^2} \right) \right)^2 \right] + \frac{1}{2}m_2 \left( \frac{p_r}{(m_1 + m_2)} \right)^2 - m_1 g r \cos \theta + m_2 g r$$

which simplifies to:

$$\mathcal{H} = \frac{p_r^2}{2(m_1 + m_2)} + \frac{p_\theta^2}{2m_1 r^2} + m_2 g r - m_1 g r \cos \theta$$

$$\begin{aligned}
\dot{\theta} &= \frac{\partial \mathcal{H}}{\partial p_{\theta}} = \frac{p_{\theta}}{m_1 r^2} & \dot{p}_{\theta} &= -\frac{\partial \mathcal{H}}{\partial \theta} = -m_1 g r \sin \theta \\
\dot{r} &= \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{(m_1 + m_2)} & \dot{p}_r &= -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_{\theta}^2}{m_1 r^3} - m_2 g + m_1 g \cos \theta
\end{aligned}$$