Swinging Atwood Machine

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Lagrangian Function

$$\mathcal{L} = \mathcal{L}(q_1, \ldots, q_n, \dot{q_1}, \ldots, \dot{q_n}, t) = T - U$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q_{ix}}}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q_i}}$$

Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

$$\mathcal{H} = \mathcal{H}(q_1, \ldots, q_n, p_1, \ldots, p_n)$$

$$\mathcal{H} = T + U$$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \text{and} \qquad \dot{p_i} = -\frac{\partial \mathcal{H}}{\partial q_i}$$

Swinging Atwood Machine

$$\begin{aligned}
T &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_1 v_2^2 \\
&= \frac{1}{2} m_1 \left(\dot{r_1}^2 + (r_1 \dot{\theta_1})^2 \right) \; + \; \frac{1}{2} m_2 \left(\dot{r_2}^2 + (r_2 \dot{\theta_2})^2 \right) \\
U &= m_1\, g\, \left[r_1 \left(1 - \cos\theta_1\right) + (R - r_1) \right] \; + \; m_2\, g\, \left[r_2 \left(1 - \cos\theta_2\right) + (R - r_2) \right]
\end{aligned}$$

$$r_2 = R - r_1 \qquad \text{and} \qquad \dot{r_2} = -\dot{r_1}$$
$$\dot{\theta_2} = 0 \qquad \text{and} \qquad \theta_2 = 0$$

$$\mathcal{H} = \frac{1}{2}m_1\left(\dot{r_1}^2 + (r_1\dot{\theta_1})^2\right) \,+\, \frac{1}{2}m_2\dot{r}_1^2 \,-\, m_1\,g\,r_1\,\cos\theta_1 \,+\, m_2\,g\,r_1$$

or

$$\mathcal{H} = \frac{1}{2}m_1\left(\dot{r}^2 + (r\,\dot{\theta})^2\right) \,+\, \frac{1}{2}m_2\,\dot{r}^2 \,-\, m_1\,g\,r\,\cos\theta \,+\, m_2\,g\,r$$

$$\begin{aligned}
p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} &= \frac{\partial T}{\partial\dot{r}} \\
&= \frac{\partial}{\partial\dot{r}}\left(\frac{1}{2}m_1\left(\dot{r}^2 + (r\,\dot{\theta})^2\right) \,+\, \frac{1}{2}m_2\,\dot{r}^2\right) \\
&= \dot{r}(m_1 + m_2) \implies \\
\dot{r} &= \frac{p_r}{(m_1 + m_2)}
\end{aligned}$$

and:

$$\begin{aligned}
p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{\partial T}{\partial\dot{\theta}} \\
&= \frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}m_1\left(\dot{r}^2 + (r\,\dot{\theta})^2\right) \,+\, \frac{1}{2}m_2\,\dot{r}^2\right) \\
&= m_1\,r^2\,\dot{\theta} \implies \\
\dot{\theta} &= \frac{p_\theta}{m_1\,r^2}
\end{aligned}$$

$$\mathcal{H} = \frac{1}{2}m_1\left[\left(\frac{p_r}{(m_1 + m_2)}\right)^2 + \left(r\left(\frac{p_\theta}{m_1\,r^2}\right)\right)^2\right] \,+\, \frac{1}{2}m_2\left(\frac{p_r}{(m_1 + m_2)}\right)^2 \,-\, m_1\,g\,r\,\cos\theta \,+\, m_2\,g\,r$$

which simplifies to:

$$\mathcal{H} = \frac{p_r^2}{2(m_1 + m_2)} \,+\, \frac{p_\theta^2}{2\,m_1\,r^2} \,+\, m_2\,g\,r \,-\, m_1\,g\,r\,\cos\theta$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{m_1\, r^2} \qquad\qquad \dot{p_\theta} = -\frac{\partial \mathcal{H}}{\partial \theta} = -m_1\, g\, r\, \sin\theta$$

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{(m_1 + m_2)} \qquad\qquad \dot{p_r} = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{m_1\, r^3} \,-\, m_2\, g \,+\, m_1\, g\, \cos\theta$$