Discrete Rebalancing in the Atlas Model

Ivan Wong
Columbia University

(Dated: October 7, 2024)

We examine trading strategies in the Atlas model, which sets the growth rate and volatility of stocks in an equity market to constants based off their rank. To do so, we simulate the Atlas model discretely to compute the discrete excess growth rate and compare its value with the continuous excess growth rate. Furthermore, we analyze how sensitive the discrete excess growth rate is to simulation parameters and investigate properties and possible error inherent in the discrete version of the Atlas model.

I. INTRODUCTION

Stochastic portfolio theory is a descriptive theory used to analyze the performance of portfolios in equity markets. The theory models stock prices in the market as Itô processes $X(t) = (X_1(t), \dots, X_n(t))$ and analyzes the performance of all-long portfolios $\pi(t) = (\pi_1(t), \dots, \pi_n(t))$, where $\pi_i(t)$ is the proportion of value of the portfolio invested in stock $X_i(t)$. SPT is used to both explain several different macroscopic pattern in the market as well as identify properties that harnessed in the construction of a portfolio.

In this way, Stochastic Portfolio Theory allows for the construction of portfolios that are relative arbitrages compared to the market. Indeed, the equally weighted or diversity weighted portfolios are known to outperform the market when continuously rebalanced for a reasonable definition of diversity. We prominently consider the excess growth rate γ^*, which is a predictor of the long-term performance of a portfolio. This value can be interpreted to be the value that portfolio rebalancing harnesses from diversity and the inherent chaos of the market, or as the Jensen gap between the average of logs or the log of averages in the discrete case.

As a descriptive theory, SPT additionally allows for the analysis of models that mimic select patterns that are observed in the real market. The capital distribution curve, which plots the log weight of stocks against their log rank, is known to maintain approximately the same curve in historical stock markets. As such, in attempting to model stock or equity markets, we choose models that can generate a stable capital distribution curve as seen in historical data.

The simplest of these is the Atlas model, a rank-based model of stocks in an equity market in which the growth rate and volatility of a stock at a given rank is constant. In the generalized Atlas model with n stocks with market capitalization $X_1(t), \dots, X_n(t)$, the stocks satisfy the stochastic differential equations,

$$d \log X_i(t) = \left(\gamma + \sum_{k=1}^{n} g_k \mathbb{1}_{\{rank_i(t)=k\}}\right) dt + \sum_{k=1}^{n} \sigma_k \mathbb{1}_{\{rank_i(t)=k\}} dW_i(t) \tag{I.1}$$

for $i = 1, \dots, n$, where $W_1, \dots, W_n$ are independent Brownian motions and $rank_i(t)$ denotes the rank of the ith stock at time t. This model is of particular interest when it satisfies,

$$g_1 < 0, g_1 + g_2 < 0, \dots, \sum_{k=1}^{n-1} g_k < 0, \sum_{k=1}^{n} g_k < 0 \tag{I.2}$$

$$\sigma_1^2 - \sigma_2^2 = \sigma_2^2 - \sigma_3^2 = \cdots = \sigma_{n-1}^2 - \sigma_n^2 \tag{I.3}$$

as these conditions guarantee that the model will have a stable capital distribution [2].

We consider the standard Atlas model, where $\gamma = g > 0$, $g_k = -g$ for $k = 1, \ldots, n-1$, and $g_n = (n-1)g$ for some constant g. We also require that $\sigma_k = \sigma > 0, \forall k$. Given the behavior of stocks in this model and a portfolio $\pi(t)$, we are able to calculate the excess growth rate γ^*, which can be used to analyze the long-term performance of the portfolio in the market. Furthermore, for an equally weighted portfolio in an equity market with uncorrelated stocks, the excess growth rate is known,

$$\gamma^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right) \tag{I.4}$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{n}\sigma_k^2 - \sum_{i=1}^{n} \frac{1}{n^2}\sigma_k^2\right) \tag{I.5}$$

Thus, when $\sigma_k = \sigma, \forall k$, $\gamma^*(t) = \frac{n-1}{2n}\sigma^2$; hence, the excess growth rate is constant and easily computable. However, given that markets and trades cannot operate on a continuous basis, it is of interest to investigate how the performance of the equally weighted portfolio balanced finitely many times compares to the excess growth rate. Thus, we simulate the Atlas model discretely and analyze the value of the discrete excess growth rate.

In this paper, we first describe how one may discretely simulate the Atlas model and estimate the expected value of the discrete excess growth rate of the equally weighted portfolio under this model [II]. We then investigate how the discrete excess growth rate changes with respect to the number of simulated points per trade, number of trades, number of stocks, volatility, and drift rate under this model III. We conclude that the discrete excess growth rate is monotonically increasing with the number of trades made in a given time interval and analyze whether or not the change in the discrete excess growth rate can be well-modelled by a regression III B. Then, we calculate the expected error between the discrete and continuous excess growth rate for specific values of σ and γ IV. At last, we discuss further directions of study, how these directions may be approached, and what results we may expect from these changes V.

II. METHODOLOGY

A. List of Variables

We discretize the Atlas model with a function $f(T, N, n, \sigma, \gamma, d)$ with variables listed in the table below,

T	The amount of time that the model simulates
N	The total number of points that the model simulates
n	The number of stocks in equity market
σ	The volatility of the stocks in the equity market
γ	The drift rate of the Atlas stock
d	The number of points simulated in each trade interval

We may conduct analysis on how sensitive the discrete excess growth rate of the Atlas model is to differences in an individual variable. It is also pertinent to observe relevant ratios and values derived from these variables, such as the ratio $\sigma : \gamma$, the number of trades $W = \frac{N}{d}$, and the size of the trading window in absolute time $\delta = \frac{Td}{N}$.

B. Initial Values

The general Atlas model is known to be ergodic [4] and has the following limiting distribution,

$$\log\left(\frac{X_{(k)}(t)}{X_{(k+1)}(t)}\right) \Rightarrow \text{Exp}\left(-4\frac{\sum_{j=1}^{k} g_k}{\sigma_k^2+\sigma_{k+1}^2}\right) \tag{II.1}$$

for $k = 1, \ldots, n-1$ where $X_{(k)}(t)$ is the market capitalization of the kth largest stock at time t [1]. As such, we may sample the initial values in the Atlas model from these characterizing distributions. Then, we may sample the difference between adjacent points in the model based off the characteristic stochastic differential equations.

C. Discrete Excess Growth Rate

Let $\mu_i(t)$ be the market weight of stock i at time t. For some observation window $\delta > 0$, we define the value $R_i(t;\delta)$ by,

$$R_i(t;\delta) = \frac{\mu_i(t+\delta)}{\mu_i(t)} \tag{II.2}$$

to be the return of weight i from time t to $t+\delta$ [5]. Then, we compute the discrete excess growth rate γ_d^* form the following formula:

$$\gamma_d^*(\pi, R;\delta) = \log\left(\sum_{i=1}^{n} \pi_i R_i\right) - \sum_{i=1}^{n} \pi_i \log(R_i) \tag{II.3}$$

$$= \log\left(\sum_{i=1}^{n} \frac{\mu_i(t+\delta)}{\mu_i(t)}\right) - \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{\mu_i(t+\delta)}{\mu_i(t)}\right) - \log(n) \tag{II.4}$$

$$= \log\left(\sum_{i=1}^{n} \frac{X_i(t+\delta)}{X_i(t)}\right) - \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{X_i(t+\delta)}{X_i(t)}\right) - \log(n) \tag{II.5}$$

We may note that the third term, $\log(n)$, is obviously determined by n; additionally, the expected value of the second term is precisely $\frac{\gamma\delta}{n}$, which can be derived from observing that Atlas model is ergodic with average speed $\frac{\delta}{n}$ [4] or finding the log differences of the stocks from their SDEs.

We then investigate the value

$$\delta \mapsto \delta^{-1}\,\mathbb{E}[\gamma_d^*(\pi, R;\delta)] \tag{II.6}$$

which can be estimated for fixed δ by averaging the realized excess growth rate over the path of the simulation. This primarily involves the evaluation of the value expected value $\mathbb{E}\left[\log\left(\sum_{i=1}^{n} \frac{X_i(t+\delta)}{X_i(t)}\right)\right]$.

III. RESULTS AND ANALYSIS

We analyze the sensitivity of discrete excess growth rate with respect to the number of simulated points per trade, number of trades, number of stocks, volatility, and drift rate. To do so, we vary a single variable while holding all other variables constant and estimate $\mathbb{E}[\gamma_d^*]$ by discretely simulating 100 equity markets using the Atlas model for each unique value of the desired variable. We additionally observe if the change in the discrete excess growth rate from change in our desired variable can be predicted and modelled via regression.

A. Simulated Points Per Trade

We preface the analysis by varying the number of points simulated per trade. In order to do this, we hold the values $T = 10, \sigma = 5, \gamma = 5, n = 10$ as constant, while varying d. Additionally, we choose to trade 100 times per simulation, thereby giving a total of $N = 100d$ simulated points.

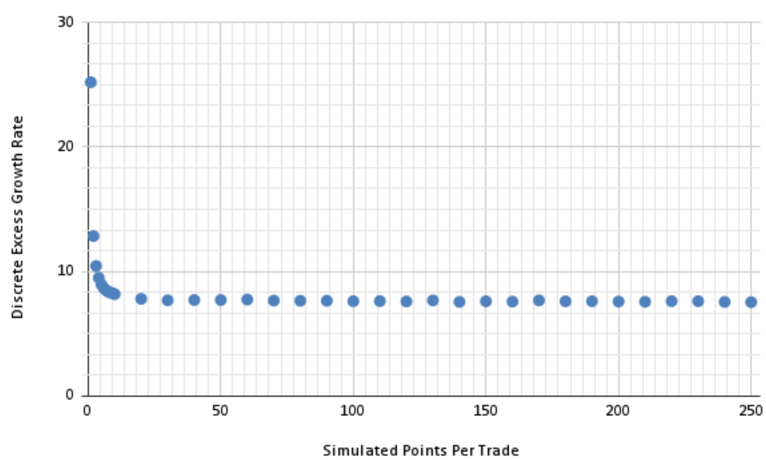

FIG. 1. Discrete Excess Growth Rate by Simulated Points Per Trade

In Figure 1, we may observe that, for small values of d (i.e. $d < 10$), the discrete excess growth rate is far larger than for all other values of d and far exceeds the continuous excess growth rate. Specifically, for $d = 1$, we simulate an expected value of $\mathbb{E}[\gamma_d^*] = 25.19$, whereas the continuous value $\gamma^* = 11.25$ is less than half of that.

As such, we remark that the simulated discrete Atlas model does not appropriately capture the behavior of the traditional Atlas model; rather, there is error that builds up (and is later quantified IV A) when there are not enough points simulated between each trade.

We instead observe that for sufficiently high values of d, the discrete excess growth rate remains mostly constant and indicates a more stable and reliable approximation of the Atlas model. As such, in the following analyses, we choose to hold d to a constant value of 50 for more accurate values for the discrete excess growth rate.

B. Number of Trades

For this simulation, we again hold the variables $T = 10, \sigma = 5, \gamma = 5, n = 10$ as constant, as well as the value $d = 50$. Instead, we vary the number of trades W, and thus simulate a total number of $N = 50W$ points in the model.

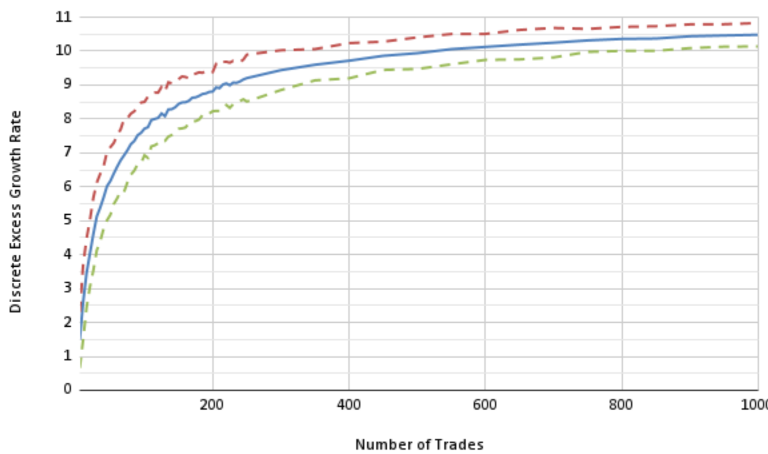

FIG. 2. Excess Growth Rate by Number of Trades

The graph displayed in Figure 2 shows the growth of the discrete excess growth rate with the number of trades as well as error lines that are 2 standard deviations away from the calculated expected value. It is sufficiently clear that the discrete excess growth rate increases with the number of trades, reaching an expected value of 10.48 when traded a total of 1000 times in the trading window, with upper error of 10.83. This falls short of the continuous excess growth rate of $\gamma^* = 11.25$. However, the graph appears to even out and converge towards a value as the number of trades increases; hence, it follows the expected behavior of approaching the continuous excess growth rate in the limit.

We additionally observed from Figure 3 that the growth of the discrete excess growth rate appears to be well approximated by the logarithmic function for small enough W. Thus, we plotted a logarithmic regression with $y = -0.44 + 1.70 \log(x)$. The correlation coefficient is $r = 0.9827$, with an R-squared value of $r^2 = 0.9657$, confirming the strength of this approximation for the simulated values.

At the same time, according to Figure 4, there are clear trends present in the residuals. Specifically, the residual plot appears to be piecewise linear, with a large positive slope up to $W = 100$ and a smaller negative slope for the rest of the values of W. This downward trend is to be expected: we do not expect the growth of the discrete excess growth rate to be logarithmic for all values of W as $\log(x)$ grows unbounded with x, while the discrete excess growth rate should converge.

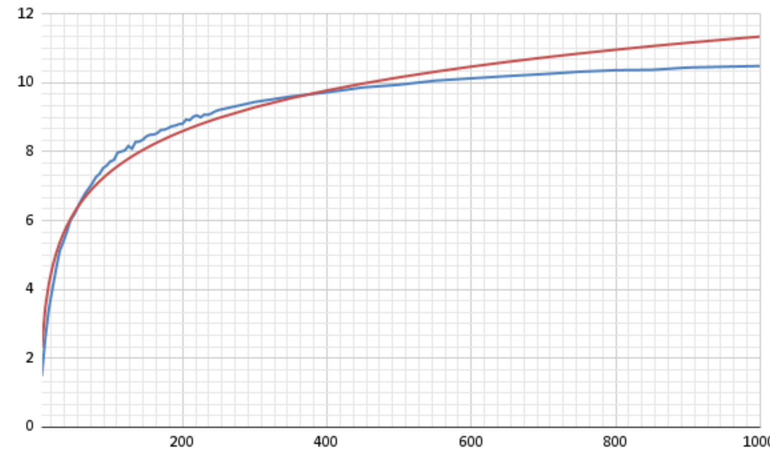

FIG. 3. Logarithmic Regression of Discrete Excess Growth Rate as Function of Number of Trades

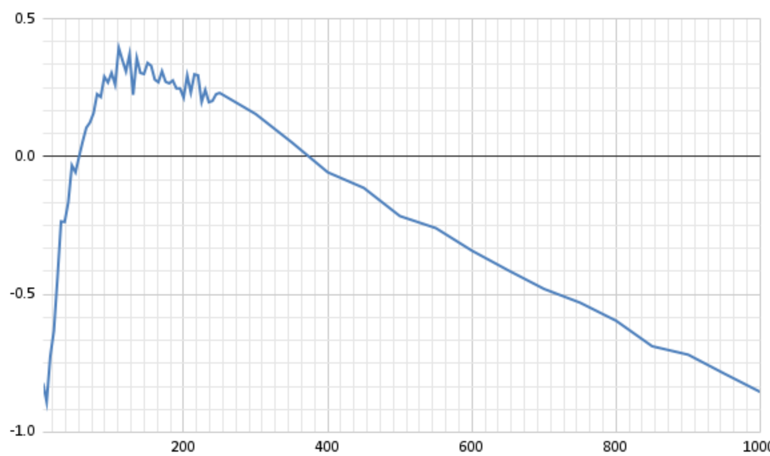

FIG. 4. Residual Values of Logarithmic Regression

C. Number of Stocks

We hold the values $T = 10, \sigma = 5, \gamma = 5, d = 50, W = 100$ constant, and vary the number of stocks n.

We observe in Figure 5 that both the discrete and continuous excess growth rate increase at approximately the same rate, with the difference $\gamma^* - \gamma_d^*$ slightly decreasing and reaching an approximate value of 3.3 when $n = 50$. We note that the continuous excess growth rate given by $\gamma^* = \frac{n-1}{2n}\sigma^2$ converges to the value $\frac{\sigma^2}{2} = 12.5$ as n approaches ∞; hence, we may similarly expect γ_d^* to converge as it is expected to be increasing and bounded from above by γ^*. However, the value that γ_d^* converges to is not evident.

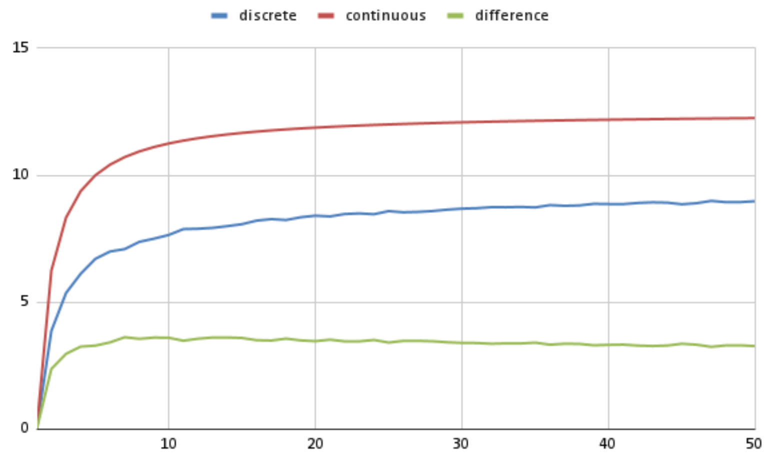

FIG. 5. Excess Growth Rate as a Function of Number of Stocks

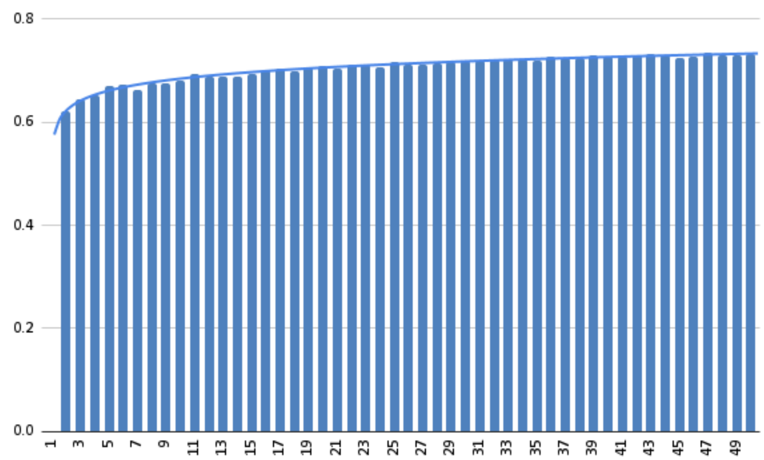

FIG. 6. Logarithmic Regression of Ratio of Discrete to Continuous Excess Growth Rate With Number of Stocks

From Figure 6 and Figure 7, we find that the ratio γ_d^*/γ^* appears to be monotonically increasing with the value of n. We perform a logarithmic regression with equation $y = 0.621 + 0.0287\log(x)$ and r^2 value of 0.98. Furthermore, aside from $n = 0$, the maximal magnitude of the residuals is 0.02090 at $n = 1$. As such, the ratio of the discrete excess growth rate to the continuous excess growth rate can be very well approximated using logarithms for sufficiently small values of n. However, given that we expect the discrete excess growth rate to be strictly smaller than the continuous excess growth rate, we have that $\gamma_d^*/\gamma^* < 1$. As such, γ_d^*/γ^* cannot grow logarithmically for all values of n as $\log(x)$ grows unbounded as the value of x increases.

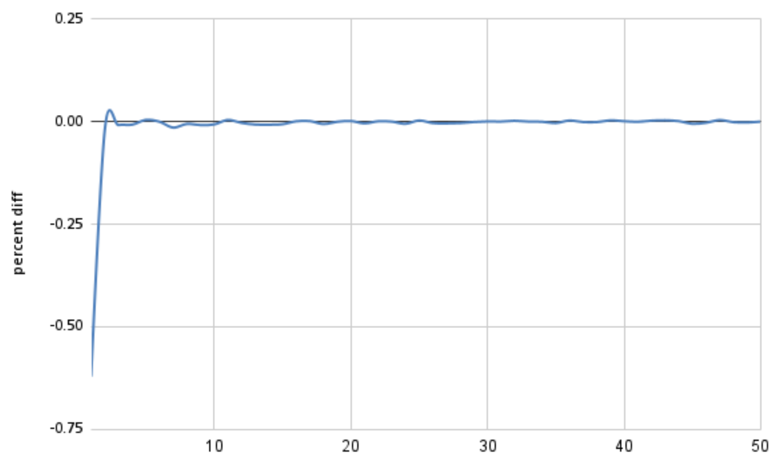

FIG. 7. Residuals of Logarithmic Regression of Ratio

D. Volatility

We hold the values $T = 10, \gamma = 5, d = 50, W = 100, n = 10$ constant, and vary the volatility of the stocks σ.

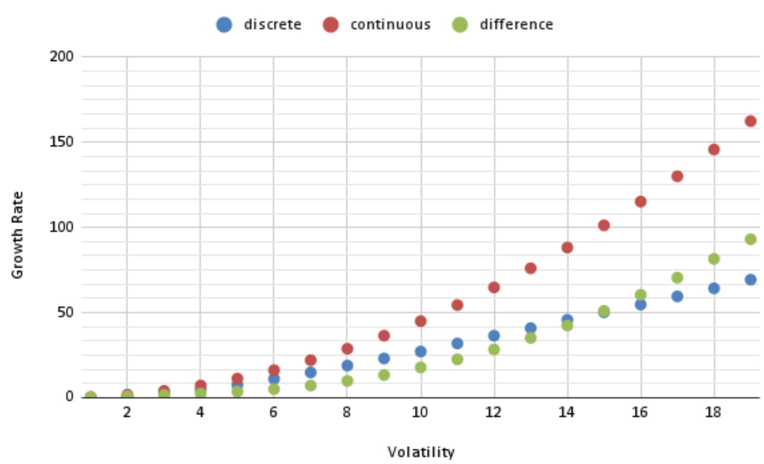

FIG. 8. Discrete Excess Growth Rate as a Function of Stock Volatility

As seen in Figure 9, the discrete and continuous excess growth rate appear to grow quadratically as we expect; however, we find that the difference $\gamma^* - \gamma_d^*$ overtakes γ_d^* when $\sigma \geq 15$. We verify precisely how the discrete excess growth rate grows with the following quadratic regression,

The quadratic regression in Figure 9 yields an equation of $y = -4.64 + 2.3x + 0.0873x^2$, with an r^2 value of 0.998. In comparison, the continuous excess growth rate can be calculated with

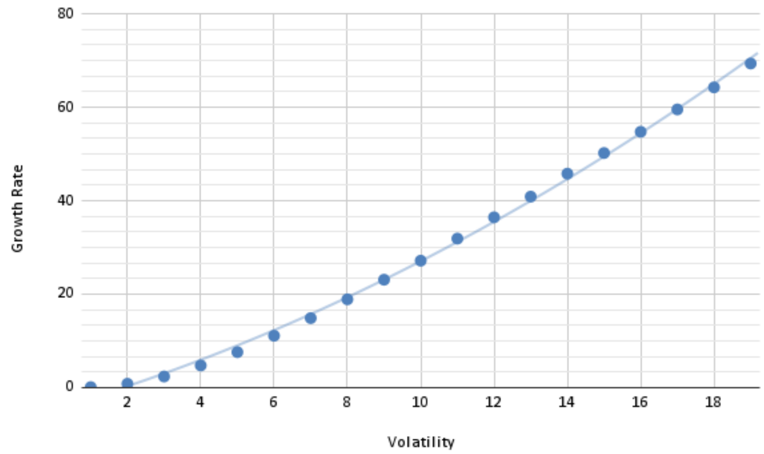

FIG. 9. Quadratic Regression of Discrete Excess Growth Rate as Function of Volatility

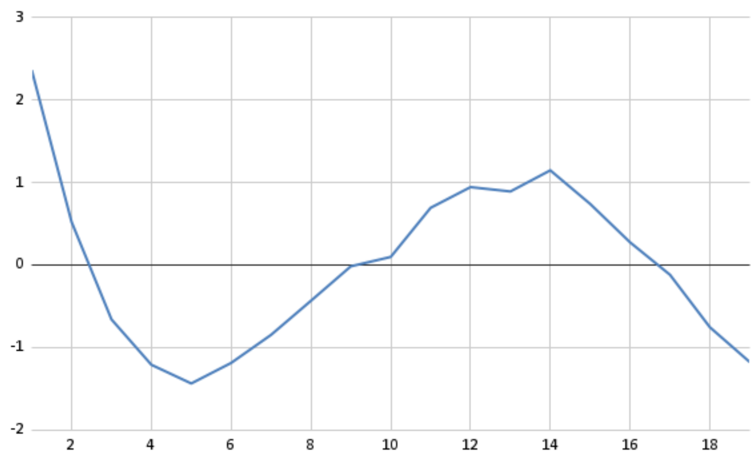

FIG. 10. Residuals of Quadratic Regression of Discrete Excess Growth Rate

the equation $\gamma^* = \frac{n-1}{2n}\sigma^2 = 0.45\sigma^2$. Thus, we observe that the coefficient of the quadratic term of the discrete excess growth rate is much smaller than that of the continuous excess growth rate; the ratio is valued at $0.0873/0.45 = 0.194$.

We note that the residual values shown in Figure 10 of the regression appear to cycle between being positive and negative. As the residuals are not random and appear to be wavelike in nature, this suggests that the discrete excess growth rate does not only grow quadratically with σ. However given that the maximal magnitude of the residuals occurs at $\sigma = 0$ with a value of 2.36, the inaccuracy of the error of this regression appears to be bounded. Therefore, we intuit that the quadratic regression can approximate the discrete excess growth rate well for all values of σ.

As mentioned above and displayed in Figure 11, we find that $\gamma^* - \gamma_d^* > \gamma_d^*$ for sufficiently

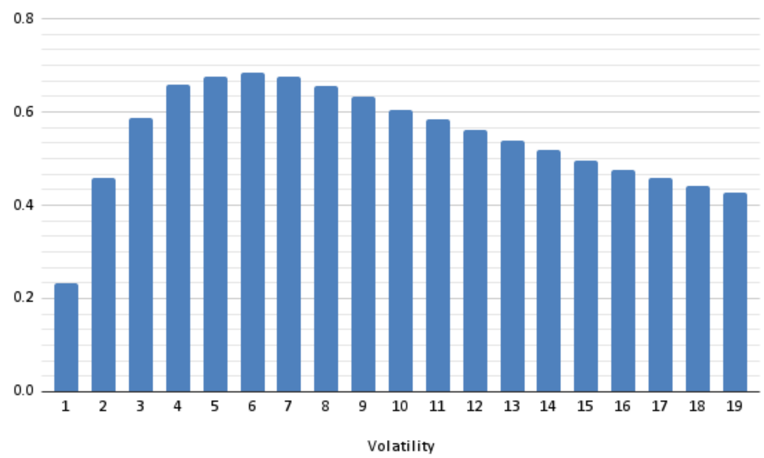

FIG. 11. Ratio of Discrete to Continuous Excess Growth Rate With Volatility

large values of σ. We investigate the exact value of γ_d^*/γ^*, noting that the ratio falls below the value 0.5 for $\sigma \geq 15$. It is notable that there appears to be a local maxima at $\sigma = 6$ which gives a peak value of 0.685 for the ratio between the discrete and continuous excess growth rate. As the ratio appears to be decreasing for $\sigma > 6$, this suggests that there is an optimal value for σ to maximize the efficiency of discrete rebalancing compared to continuous rebalancing.

E. Drift Rate

We hold the values $T = 10, \sigma = 5, d = 50, W = 100, n = 10$ constant, and vary the drift rate of the Atlas stock γ.

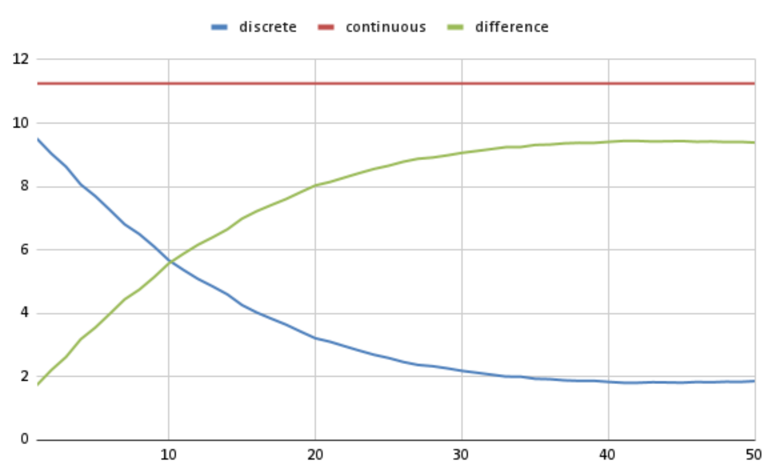

FIG. 12. Excess Growth Rate as a Function of Drift Rate

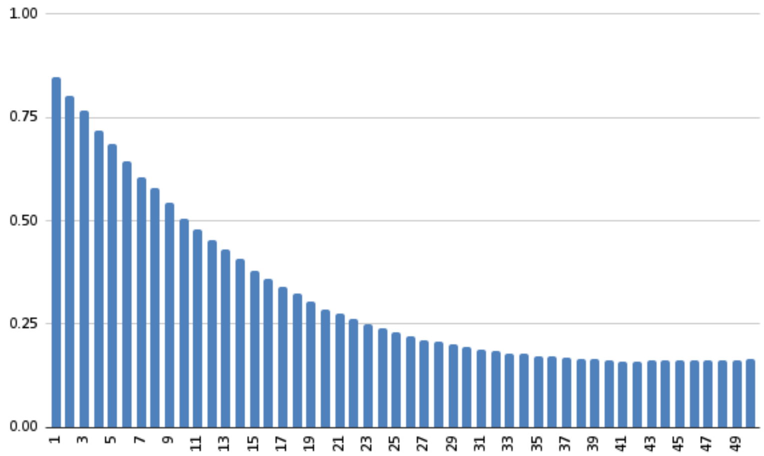

FIG. 13. Ratio of Discrete to Continuous Excess Growth Rate as a Function of Drift Rate

Recall that the value of γ does not affect the value of γ^*, so the continuous excess growth rate remains constant regardless of the value of γ. As such, the value of γ_d^* and the ratio γ_d^*/γ^* behave in the same way. Hence, they can be analyzed the same way. Thus, it is sufficient to analyze Figure 13.

The ratio of the discrete and continuous excess growth rate appears to be monotonically decreasing with the value of γ. However, for γ sufficiently large, the ratio appears to stabilize at a value of 0.161. This implies that to maximize efficiency, one should choose equity markets modeled by Atlas models with drift rates as small as possible; on the flipside, for large enough γ, further increases in the drift rate does not decrease the efficiency of discrete rebalancing much.

IV. QUANTIFYING ERROR

Although the discretely simulated Atlas model and discrete excess growth rate converge to their continuous analogue as the number of simulated points, N, approaches infinity, we note that there is inherent error between γ^* and γ_d^* for finite N. We find explicit values or bounds on the expected error $\mathbb{E}[\gamma_d^* - \gamma^*]$ for special values of σ and γ.

A. Error in Low Volatility

We note that the discrete simulation of the Atlas model inherently assumes that the rank of stocks will not drastically change between adjacent points. Specifically, numerous changes in the Atlas stocks will build inaccuracy in the model; thus, the model grows more unreliable when the value of stocks are close to each other.

This situation can be easily observed when the ratio between σ and γ is extremely small. Take the most extreme instance: $\sigma = 0$. It is sufficiently clear that the continuous excess growth rate $\gamma^* = 0$. However, when we calculate the discrete excess growth rate, we can observe

$$\mathbb{E}[\gamma_d^*] = \mathbb{E}\left[\log\left(\sum_{i=1}^n \frac{X_i(t+\delta)}{X_i(t)}\right) - \frac{1}{n}\sum_{i=1}^n \log\left(\frac{X_i(t+\delta)}{X_i(t)}\right) - \log(n)\right] \tag{IV.1}$$

$$= \mathbb{E}\left[\log\left(\sum_{i=1}^n e^{\log(X_i(t+\delta))-\log(X_i(t))}\right)\right] - \frac{\gamma\delta}{n} - \log(n) \tag{IV.2}$$

$$= \log\left(e^{\gamma\delta} + n - 1\right) - \frac{\gamma\delta}{n} - \log(n) \tag{IV.3}$$

$$= \log\left(\frac{e^{\gamma\delta}}{n} + \frac{n-1}{n}\right) - \frac{\gamma\delta}{n} \tag{IV.4}$$

which is clearly positive and grows unbounded as the value of $\gamma\delta$ increases.

B. Error in Low Drift Rate

Instead taking $\gamma = 0$, which does not yield a defined value for the ration $\sigma : \gamma$ and simulates a degenerate version of the Atlas model, gives us the expected discrete excess growth rate,

$$\mathbb{E}[\gamma_d^*] = \mathbb{E}\left[\log\left(\sum_{i=1}^n \frac{X_i(t+\delta)}{X_i(t)}\right) - \frac{1}{n}\sum_{i=1}^n \log\left(\frac{X_i(t+\delta)}{X_i(t)}\right) - \log(n)\right] \tag{IV.5}$$

$$= \mathbb{E}\left[\log\left(\sum_{i=1}^n e^{\log(X_i(t+\delta))-\log(X_i(t))}\right)\right] - \frac{\gamma\delta}{n} - \log(n) \tag{IV.6}$$

$$= \mathbb{E}\left[\log\left(\sum_{i=1}^n Y_i\right)\right] - \log(n) \tag{IV.7}$$

where $Y_i \overset{iid}{\sim} \text{Lognormal}(0, \sigma^2)$. The sum of independent lognormals is often approximated to itself have a lognormal distribution; furthermore, the log of this sum is generally known to approach the normal distribution asymptotically [3]. This gives that for sufficiently large n,

$$\sum_{i=1}^n Y_i \sim \mathcal{N}(\log(n) + \frac{\sigma^2}{2} - \frac{\sigma_Z^2}{2}, \sigma_Z^2) \tag{IV.8}$$

$$\sigma_Z^2 = \log\left(\frac{e^{\sigma^2} + n - 1}{n}\right) \tag{IV.9}$$

Hence, we may approximate the discrete excess growth rate as,

$$\mathbb{E}[\gamma_d^*] = \log(n) + \frac{\sigma^2}{2} - \frac{(\log\left(e^{\sigma^2} + n - 1\right) - \log(n))^2}{2} - \log(n) \tag{IV.10}$$

$$= \frac{\sigma^2}{2} - \frac{(\log\left(e^{\sigma^2} + n - 1\right) - \log(n))^2}{2} \tag{IV.11}$$

If we compare this with the continuous excess growth rate $\gamma^* = \frac{\sigma^2}{2} - \frac{\sigma^2}{2n}$, we have the difference,

$$\mathbb{E}[\gamma_d^* - \gamma^*] = \frac{\sigma^2}{2n} - \frac{(\log\left(e^{\sigma^2} + n - 1\right) - \log(n))^2}{2} \tag{IV.12}$$

which is approximately 0 for small enough σ and large enough n, but decreases quickly outside of these conditions.

V. CONCLUSION AND FURTHER DIRECTIONS

As can be seen from the results of the simulation, the discrete excess growth rate appears to be strictly less than the continuous excess growth rate for nondegenerate simulations (i.e. d sufficiently large) and approaches the value of the continuous excess growth rate from below. As such, forgoing technical difficulties, trading costs, or other complications, it would be in one's best interest to trade as frequently as possible in equity markets well-approximated by the Atlas model. For portfolios that are rebalanced highly infrequently, increasing the number of trades can improve the excess growth rate of the portfolio approximately logarithmicly.

Furthermore, based off the sensitivity of the discrete excess growth rate with regards to the number of stocks, volatility, and drift rate, we find that it is best to trade with as many stocks and as large a volatility as possible in the equity market, while having a lower drift rate. Based off of efficiency compared to the continuous excess growth rate, we find that trading in the discrete model is most efficient for a higher amount of stocks and a lower drift rate; the volatility, on the other hand, appears to have a maxima that does not occur at absolute highs or lows.

Though we give simple commentary on precisely how the discrete excess growth rate grows with respect to each of the variables mentioned prior, it is not explicitly clear (and oftentimes doubtful) whether or not the documented behavior holds true in the limit. Thus, it would be of interest to clarify precisely how quickly the discrete excess growth rate grows with the number of trades, number of stocks, volatility, and drift rate of the stocks in the equity market; additionally, it would be valuable to establish how well we can approximate this growth rate for sufficiently small or large values of the aforementioned variables. Ideally, it would be possible to find values or ratios for said variables such that the discrete excess growth rate approaches the continuous excess growth rate as quickly as possible.

The discrete simulation of the Atlas model shown in this paper is only for equally weighted portfolios; however, the results can be further generalized for functionally generated portfolios by passing a function π into the simulation. Note that the formulas used for calculating the discrete and continuous excess growth rate should be adjusted for this change, and will no longer be as simple to calculate. Moreover, given that the SDE of each stock in the market is the same, it would only be appropriate that the portfolio generating function takes rank as a variable. We observe that there is interest and several known properties for portfolios that do not have the Atlas stock [1].

It is also reasonable to expand the simulation to the generalized Atlas model, with less restriction on the values of g_k and σ_k. Specifically, one may only restrict g_k and σ_k to guarantee a stable capital distribution [2]. As this change no longer requires that the SDEs of the stocks in the market are the same, it would be valuable to look to functionally generated portfolios that that takes in the drift rates and volatilities as variables, either in addition to or in lieu of rank. The problem may further be expanded to Atlas models with countably many stocks; however, we acknowledge that the analysis of such a model does not necessarily lend itself towards simulation and must be approached differently.

It would be similarly interesting to investigate and discretely simulate other models of equity markets. As there is inherent error associated with the discrete simulation of rank-based models where the stochastic behavior of individual stocks change, we recommend further analysis into bounding the difference between the discrete and continuous excess growth rate. This analysis would ideally be applied to a general rank-based model with parameters that can be used to approximate historical data and the real-life capital distribution.

Additionally, it may be worthwhile to observe if the behavior of the discretized Atlas model holds when the simulated Brownian motions $W_1, \ldots, W_n$ are correlated, which can be done via Cholesky decomposition. We note that many of the approximations for the log of sum of lognormal random variables require a condition of independence; we can thus expect that the expected value of the discrete excess growth rate may differ when the Brownian motions in the model are correlated with each other.

VI. ACKNOWLEDGEMENTS

We would like to thank Steven Campbell for proposing this problem as well as providing his continued guidance and advice throughout this research experience.

VII. APPENDIX

```python
import numpy as np
import matplotlib.pyplot as plt

def simulate_atlas(T, N, i, init_value, sigma, gamma, g):
    """
    simulate_atlas is a function that simulates the capital
    distributions of stocks in a general Atlas model

    :param T: time interval
    :param N: number of points per path
    :param i: number of stocks simulated
    :param init_value: vector of initial values of stocks
    :param sigma: vector of volatilities of stocks
    :param gamma: base drift of stocks
    :param g: vector of drifts of stocks

    :return: vector of capital distributions of stocks sorted by rank
    """

    X = np.empty((i, N))
    X[:,0] = init_value
    drifts = np.add(gamma, g)
    dt = T/N

    for t in range(1, N):
        arr = X[:, t-1]
        order = arr.argsort()
        ranks = order[::-1].argsort()
        Zn = np.multiply(np.sqrt(dt), np.random.randn(i))

        X[:,t] = np.add( np.add(X[:,t-1], np.multiply(dt, drifts[ranks])), np.
    multiply(Zn, sigma[ranks]))
        # X[:,t-1] + dt * drifts[ranks] + sqrt(dt) * Z * sigma(ranks)

    return np.exp(X)

```

```python
def generate_init_values(i, g, sigma, rand_value):
    """
    generate_init_values creates a vector of initial values based
    off of the limiting distribution of the capital distribution
    of stocks in the general Atlas model

    :param i: number of stocks simulated
    :param g: vector of drifts of stocks
    :param sigma: vector of volatilities of stocks
    :param rand_value: value of capital distribution of the weakest stock

    :return: vector of initial value of Atlas model stocks
    """

    init_value = np.empty(i)
    init_value[i-1] = rand_value
    for j in range(i-2, -1, -1):
        # Z ~ Exp((sum_{j=1}^k g_k) / (sigma_k^2 + sigma_{k+1}^2))
        Z = np.random.exponential( (sigma[j]^2 + sigma[j+1]^2) / (-4 * np.sum(g
    [:j+1])) )
        init_value[j] = np.exp(Z) * init_value[j+1]

    return np.log(init_value)

def discrete_excess_growth_rate(pi, delta, X, T, N):
    """
    discrete_excess_growth_rate calculates the discrete excess growth rate of
    a given portfolio and a given path traded at delta intervals of time

    :param pi: vector of portfolio weights
    :param delta: time interval over which the portfolio is rebalanced
    :param X: capital distribution of stocks
    :param T: time interval
    :param N: number of points per path
    :param i: number of stocks simulated

    :return: the average of the realized excess growth rate
    """
    excess = 0
    next_weights = np.divide(X[:,0], np.sum(X[:,0]))
    # print(next_weights)
    for t in range(delta, N, delta):
        # calculate the returns of each stock
        curr_weights = next_weights
        next_weights = np.divide(X[:,t], np.sum(X[:,t]))
        R = np.divide(next_weights, curr_weights)

        # print("curr_weights: ")
        # print(curr_weights)
        # print("next_weights: ")
        # print(next_weights)

        excess += np.log(np.sum(np.multiply(pi, R)))
        excess -= np.sum(np.multiply(pi, np.log(R)))

    return excess / T

def continuous_excess_growth_rate(pi, sigma, i):
    """
    continuous_excess_growth_rate calculates the continous excess growth rate
    of a given portfolio and a given path over a finite amount of time

```

```
    :param pi: vector of portfolio weights
    :param sigma: vector of volatilities of stocks
    :param i: number of stocks in world
    :param T: time interval

    :return: the average of the realized excess growth rate
    """

    # Formulas for continous time variables
    # gamma*(t) = 1/2 (sum_{i=1}^{n} \pi_i(t) \sigma_{ii}(t) - sigma_{\pi \pi}(t))
    # sigma_{\pi \pi}(t) = \sum_{i,j = 1}^{n} \pi_i(t) \pi_j(t) \sigma_{ij}(t)
    # sigma_{ij}(t) dt = d<logX_i, logX_j>_t

    sigma_pipi = 0
    for j in range(i):
        sigma_pipi += np.power(pi[j] * sigma[j], 2)

    excess = 0
    for k in range(i):
        excess += pi[k] * sigma[k] * sigma[k]
    excess -= sigma_pipi
    excess *= 1/2

    return excess
```


[1] A. D. Banner, R. Fernholz, and I. Karatzas. Atlas models of equity markets. *The Annals of Applied Probability*, 15(4), Nov. 2005.

[2] T. Ichiba, V. Papathanakos, A. Banner, I. Karatzas, and R. Fernholz. Hybrid atlas models. *The Annals of Applied Probability*, 21(2), Apr. 2011.

[3] N. A. Marlow. A normal limit theorem for power sums of independent random variables. *The Bell System Technical Journal*, 46(9):2081–2089, 1967.

[4] S. Pal and J. Pitman. One-dimensional brownian particle systems with rank dependent drifts, 2008.

[5] S. Pal and T.-K. L. Wong. Energy, entropy, and arbitrage, 2016.