Chapter four

4.1 Method of images

Two dimensional image systems

The method of images is about getting flows that satisfy boundary conditions. It involves introduction of imaginary images (mirror image) into the flow field. This method is also applied in electrostatics.

Consider x=0 to be the plane of flow and suppose that there is a line source of strength m at z = z_0 where $\mathrm{Re}(z_1) \geq 0$. If the boundary is removed and an equal line source introduced at $z = -\overline{z_1}$ (the mirror image of z_1 through x=0) then the complex potential at p where $\overline{OP} = z$ is

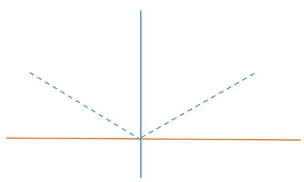

figure 1

$$w = \frac{m}{2\pi}\ln(z - z_1) + \frac{m}{2\pi}\ln\left(z + \overline{z_1}\right) = \frac{m}{2\pi}\ln\left(z^2 - z_1^2\right)$$

$$\frac{dw}{dz} = \frac{m}{2\pi}\frac{2z}{\left(z^2 - z_1^2\right)} = \frac{mz}{\pi\left(z^2 - z_1^2\right)}$$

On the plane x=0 we have $z = iy$ then $u - iv = -\frac{iym}{\pi\left(y^2 + z_1^2\right)}$ hence $u = 0$ and $v = \frac{my}{\pi\left(y^2 + z_1^2\right)}$

On the plane x=0 the velocity component u=0 and consequently no flow crosses this plane thus it can be replaced by a rigid barrier.

Definition

The image of a line source in a rigid infinite plane is a line source of equal strength at the optical image in the plane at which the line source is situated. e.g. image of a line source of strength m at (a,0) is a source of strength m at (-a,0) with the plane x=0.

4.1.1 Image of a doublet with respect to a plane

To find the image of a doublet AA' with respect to y axis, let the doublet AA' be a combination of a source of strength m at A' and a sink of strength (-m) at A with its axis AA' inclined at an angle α with x- axis as shown below:

figure 2

Image of A strength –m and image of A' strength m with respect to y axis are B strength –m and B' strength m respectively such that BC=AC and B'm=A' m hence the image of a doublet BB' of the same strength with its axis antiparallel to AA'.

Example

A two dimensional doublet of strength $-\mu$ is at the point $z = ia$ in a stream of velocity –vi in a semi-infinite liquid of constant density occupying the half plane y>0 and having y=0 as a rigid boundary. show that the complex potential of the motion is $F = vz + \dfrac{2\mu z}{z^2 + a^2}$

Shows also that for $0 < \mu < 4a^2 v$ there are no stagnation points on this boundary and that the pressure on it is a minimum at the origin and maximum at the point $z = \pm a\sqrt{3}$.

Solution

figure 3

Fig (3a) shows the physical model and fig (3b) the image system consisting of equal line doublets of strength μ at A(0,a), A' (0,-a) . The velocity potential due to uniform stream is vx and so the complex potential is vz.

From fig (b) $OP = z,\ \overline{AP} = z - ia,\ \ \overline{A'P} = z + ia$

The complex potential due to the line doublets at A, A' are $\dfrac{\mu}{z - ia}$ $\quad and \quad$ $\dfrac{\mu}{z + ia}$ respectively.

The total velocity potential at p is $F = vz + \dfrac{\mu}{z - ia} + \dfrac{\mu}{z + ia} = vz + \dfrac{2\mu z}{z^2 + a^2}$

Thus using quotient rule $\dfrac{dF}{dz} = v + \dfrac{2\mu\left(a^2 - z^2\right)}{\left(z^2 + a^2\right)^2}$

When x=z then $\dfrac{dF}{dz} = v + \dfrac{2\mu\left(a^2 - x^2\right)}{\left(x^2 + a^2\right)^2}$

Which is the speed of the fluid at any point (x,0) on the boundary y=0. Equating to zero gives

$$v\left(x^2+a^2\right)^2+2\mu\left(a^2-x^2\right)=0$$
$$v\left(x^4+2x^2a^2+a^4\right)+2\mu a^2-2\mu x^2=0$$
$$vx^4+2\left(va^2-\mu\right)x^2+\left(2\mu a^2+va^4\right)=0$$

Which is quadratic in x^2 and whose discriminant is

$$D=4\left\{\left(va^2-\mu\right)^2-v\left(va^4+2\mu a^2\right)\right\}$$
$$=4\left\{v^2a^4-2\mu va^2+\mu^2-v^2a^4-2v\mu a^2\right\}$$
$$=4\left\{\mu^2-4v\mu a^2\right\}$$
$$D=4\mu\left\{\mu-4va^2\right\}$$

Now D < 0 if $0<\mu<4va^2$ which means that the quadratic equation has no real roots which implies that there exists no stagnation points on the boundary y=0.

Applying the Bernoulli's equation along the stream line y = 0 at (x,0) on the boundary we have

$$\frac{P}{\rho}+\frac{1}{2}\left\{v+\frac{2\mu\left(a^2-x^2\right)}{\left(x^2+a^2\right)^2}\right\}^2=Const$$

P is a maximum value when $v+\dfrac{2\mu\left(a^2-x^2\right)}{\left(x^2+a^2\right)^2}$ is a minimum and conversely put, let

$$y=\left\{v+2\mu\left(a^2-x^2\right)\left(x^2+a^2\right)^{-2}\right\}^2$$
$$y^{\frac{1}{2}}=v+2\mu\left(a^2-x^2\right)\left(x^2+a^2\right)^{-2}$$

Differentiating with respect to x gives $\dfrac{1}{2}y^{-\frac{1}{2}}y'=-4\mu x\left(3a^2-x^2\right)\left(x^2+a^2\right)^{-3}$

$y'=0$ when x=0 or $x=\pm a\sqrt{3}$

Using first derivative test we note that $y'<0$ when $x=0^+$ and $y'>0$ when $x=0^-$ which implies that y is a maximum hence p is a minimum at the origin.

On the other hand $y'>0$, as $x\to\left(a\sqrt{3}\right)^+$ and $y'<0$ when $x\to\left(a\sqrt{3}\right)^-$ so that $x=a\sqrt{3}$, y is a minimum so that p is a maximum.

As $x\to\left(-a\sqrt{3}\right)^+$, $y'>0$ and $y'<0$, $x\to\left(-a\sqrt{3}\right)^-$ p is a maximum at $x=-\left(a\sqrt{3}\right)$.

Example 2

Suppose that a fluid fills the region of the space on the positive side of x axis, with rigid boundary and there is a source of strength +m at the point (0,a) and an equal sink at (0,b). If the pressure on the negative side of the boundary is the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is $P=\frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{4m^2x^2\left(b^2-a^2\right)^2}{\left(x^2+a^2\right)^2\left(x^2+b^2\right)^2}dx$

Solution

The given system consists of a source +m at A' and a sink at B' with respect to the positive ox line which is the rigid boundary as shown:

figure 4

The complex potential due to the system is

$$F(z)=m\ln(z-ia)-m\ln(z-ib)+m\ln(z+ia)-m\ln(z+ib)$$

$$=m\ln\left(\frac{z^2+a^2}{z^2+b^2}\right)$$

The speed of the fluid is $q=\left|\frac{dF}{dz}\right|=\frac{2mz\left|b^2-a^2\right|}{\left|z^2+a^2\right|\left|z^2+b^2\right|}$

For a point on the x axis we have $q=\frac{2mx\left|b^2-a^2\right|}{\left|x^2+a^2\right|\left|x^2+b^2\right|}$

From the Bernoulli's equation $\frac{P}{\rho}+\frac{1}{2}q^2=Const$ ....(i)

In the problem the boundary conditions are $P = P_0$ a constant $q = 0 \;\; as \;\; x \to \infty$ so that

$$\frac{P_0}{\rho} = Const \quad ....(ii)$$

Subtracting (i) from (ii) we have $P_0 - P = \frac{1}{2}\rho q^2$ .... (iii)

Let p' be the total resultant pressure on the boundary then

$$P' = \int_{-\infty}^{\infty} (P_0 - P)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}\rho q^2 \, dx$$

$$P = \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{4m^2 x^2 (b^2 - a^2)}{(x^2 + a^2)^2 (x^2 + b^2)^2}\, dx$$

This can be evaluated using partial fractions.

4.2 Blasius' Theorem

If the complex potential for a flow around a body is known, the Blasius laws provides a convenient way of evaluating the forces and turning moments on the body by means of simple contour integration. The contour integrals can in turn be evaluated using residue theorem.

Statement

Let $F = \phi + i\psi$ be the complex velocity potential for a two dimensional steady flow of an incompressible fluid, then the force $X - iY$ due to hydrodynamic pressure is given by

$$X - iY = \frac{1}{2} i\rho \oint_c \left| \frac{dF}{dz} \right|^2 dz$$

And the moment M of the force about the origin on the surface is $M = -\frac{1}{2}\rho \operatorname{Re} \oint_c z \left| \frac{dF}{dz} \right|^2 dz$

Where ρ is the density and c is the contour on the surface of the cylinder.

Proof

Consider some flow pattern in the complex z-plane that is specified by the complex velocity potential $F(z)$. Let C be some closed curve in the complex z-plane and dz be an element on C. Then idz is perpendicular to dz.

The fluid pressure on this curve is determined from the Bernoulli's equation $\frac{P}{\rho}+\frac{q^2}{2}=\frac{P_0}{\rho}$

where the complex velocity $\frac{dF}{dz}=u-iv$ so that

$$P=P_0-\frac{1}{2}\rho\left|\frac{dF}{dz}\right|^2 \qquad (1)$$

Let us evaluate the resultant force (per unit length), and the resultant moment (per unit length), acting on the fluid within the curve as a consequence of this pressure distribution. Consider a magnification of dz as shown below

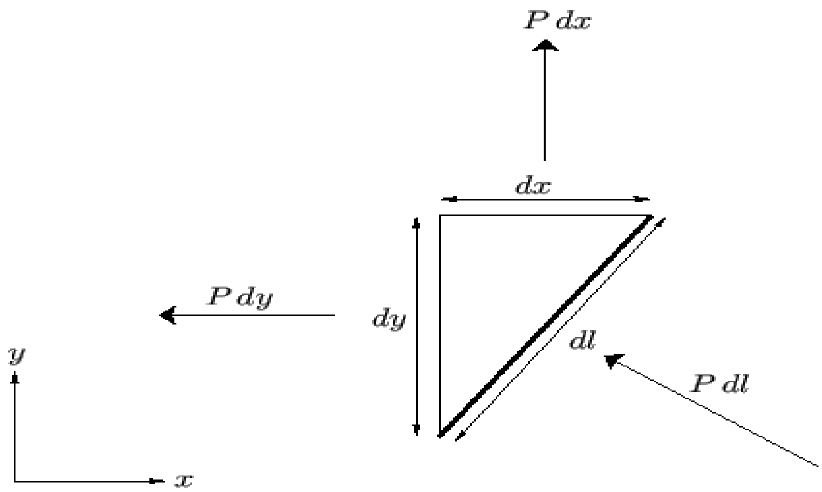

Figure 5: Force acting across a short section of a curve.

Consider a small element of the curve C, lying between (x,y) and (x+dx, y+dy), which is sufficiently short that it can be approximated as a straight-line. Let P be the local fluid pressure on the outer (i.e., exterior to the curve) side of the element. As illustrated in the figure 1 above, the pressure force (per unit length) acting inward (i.e., toward the inside of the curve) across the element has a component **Pdy** in the negative x-direction, and a component Pdx in the positive **y**-direction. Thus, if X and Y are the components of the resultant force (per unit length) in the **x**- and **y**-directions, respectively, then

$$dX=-Pdy$$
$$dY=Pdx$$

The pressure force (per unit length) acting across the element also contributes to a moment (per unit length), M, acting about the z-axis, where $M=xdY-ydX=P(xdx+ydy)$
Thus, the x and y -components of the resultant force (per unit length) acting on the of the fluid within the curve, as well as the resultant moment (per unit length) about the z-axis, are given by

$$X = -\oint_C P dy \qquad (2)$$

$$Y = \oint_C P dx \qquad (3)$$

$$M = \oint_C P(xdx + ydy) \qquad (4)$$

respectively, where the integrals are taken (counter-clockwise) around the curve C. Finally, given that the pressure distribution on the curve takes the form (1), and that a constant pressure (P_0) obviously yields zero force and zero moment, we find that equations (2), (3) and (4) become

$$X = \frac{1}{2}\rho\oint_C \left|\frac{dF}{dz}\right|^2 dy \qquad (5)$$

$$Y = -\frac{1}{2}\rho\oint_C \left|\frac{dF}{dz}\right|^2 dx \qquad (6)$$

$$M = -\frac{1}{2}\rho\oint_C \left|\frac{dF}{dz}\right|^2 (xdx + ydy) \qquad (7)$$

Now, $z = x + iy$, and the conjugate $\bar{z} = x - iy$. Hence, $d\bar{z} = dx - idy$ and $id\bar{z} = dy + idx$ and $|z|^2 = z \cdot \bar{z}$. It follows that

$$X - iY = \frac{1}{2}i\rho\oint_C \left|\frac{dF}{dz}\right|^2 d\bar{z} \qquad (8)$$

However, $\left|\frac{dF}{dz}\right|^2 d\bar{z} = \frac{dF}{dz}\frac{d\bar{F}}{d\bar{z}}d\bar{z} = \frac{dF}{dz}d\bar{F} \qquad (9)$

where $dF = d\phi + id\psi$ and $d\bar{F} = d\phi - id\psi$. Suppose that the curve C corresponds to a streamline of the flow, in which case $\psi = Constant$ on C. Thus, $d\psi = 0$ on C, and so $d\bar{F} = dF$. Hence, on C, from (9) we have $\frac{dF}{dz}d\bar{F} = \frac{dF}{dz}dF = \left(\frac{dF}{dz}\right)^2 dz$ so that

$$\left|\frac{dF}{dz}\right|^2 d\bar{z} = \left(\frac{dF}{dz}\right)^2 dz \qquad (10)$$

which implies that equation (8) becomes

$$X - iY = \frac{1}{2}i\rho\oint_C \left(\frac{dF}{dz}\right)^2 dz \qquad (11)$$

Which is the *Blasius theorem* of forces.

Now, $zd\bar{z} = (x + iy)(dx - idy) = xdx + ydy + i(ydx - xdy)$ so that $xdx + ydy = \mathrm{Re}(zd\bar{z})$. Hence, equation (7) becomes

$$M = \mathrm{Re}\left(-\frac{1}{2}\rho \oint_C \left| \frac{dF}{dz} \right|^2 z d\bar{z} \right)$$

or, making use of equation (10) then the Blasius theorem of moments is given by,

$$M = \mathrm{Re}\left(-\frac{1}{2}\rho \oint_C \left(\frac{dF}{dz} \right)^2 z dz \right) \qquad (12)$$

In fact, Equations (11) and (12) hold even when ψ is not constant on the curve C, as long as C can be continuously deformed into a constant- ψ curve without leaving the fluid or crossing over a singularity of $\left(\frac{dF}{dz} \right)^2$.