MATRICS AND CALCULUS ASSIGNMENT

Sec2sit097

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It - E

Assignment - Questions :

1. Find the directional derivative of $\phi = 2xy + z^2$ at the point $(1, -1, 3)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$.

Let:

$$D = \vec{\nabla}.\phi. \frac{\vec{a}}{|\vec{a}|} \quad \text{——} ①$$

given:

$$\phi = 2xy + z^2 \quad , \quad \vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{\nabla}.\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 2y$$

$$\frac{\partial \phi}{\partial y} = 2x$$

$$\frac{\partial \phi}{\partial z} = 2z$$

$$\vec{\nabla}.\phi = \vec{i}\,2y + \vec{j}\,2x + \vec{k}\,2z$$

Put $(1, -1, 3)$ points

$$\vec{\nabla}.\phi = -2\vec{i} + 2\vec{j} + 6\vec{k}$$

$$① \Rightarrow D = \vec{\nabla}\phi . \frac{\vec{a}}{|\vec{a}|}$$

$$D = -2\vec{i} + 2\vec{j} + 6\vec{k} . \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{(1)^2 + (2)^2 + (2)^2}}$$

$$D = \frac{-2 + 4 + 12}{\sqrt{9}}$$

$$D = \frac{14}{3}$$

2. Find the angle between the surfaces $x^2-y^2-z^2=11$ and $xy+yz-zx-18=0$ at point $(6,4,3)$

Given:

$\phi_1 \Rightarrow x^2-y^2-z^2-11=0$

$\phi_2 \Rightarrow xy+yz-zx-18=0$

WKT:

$$\cos\theta = \frac{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1||\vec{\nabla}\phi_2|} \quad \text{——①}$$

$$\vec{\nabla}\phi_1 = \vec{i}\,\frac{\partial\phi_1}{\partial x} + \vec{j}\,\frac{\partial\phi_1}{\partial y} + \vec{k}\,\frac{\partial\phi_1}{\partial z}$$

$$\vec{\nabla}\phi_1 = \vec{i}\,(2x) + \vec{j}\,(-2y) + \vec{k}\,(-2z)$$

$$\vec{\nabla}\phi_1 = 12\vec{i} - 8\vec{j} - 6\vec{k} \quad \text{——②} (\because \text{Put }(6,4,3))$$

$$\vec{\nabla}\phi_2 = \vec{i}\,\frac{\partial\phi_2}{\partial x} + \vec{j}\,\frac{\partial\phi_2}{\partial y} + \vec{k}\,\frac{\partial\phi_2}{\partial z}$$

$$\vec{\nabla}\phi_2 = \vec{i}\,(y-z) + \vec{j}\,(x+z) + \vec{k}\,(y-x)$$

Put $(6,4,3)$ points

$$\vec{\nabla}\phi_2 = \vec{i} + 9\vec{j} - 2\vec{k} \quad \text{——③}$$

Substituting ③ & ② in ① we get

$$\cos\theta = \frac{\left(12\vec{i}-8\vec{j}-6\vec{k}\right)\cdot\left(\vec{i}+9\vec{j}-2\vec{k}\right)}{\left(\sqrt{(12)^2+(-8)^2+(-6)^2}\right)\left(\sqrt{(1)^2+(9)^2+(-2)^2}\right)}$$

$$\cos\theta = \frac{12-72+12}{\sqrt{244}\,\sqrt{86}}$$

$$\theta = \cos^{-1}\left(\frac{-48}{\sqrt{244}\,\sqrt{86}}\right) \text{ (or) } \cos^{-1}\left(\frac{-48}{\sqrt{21000}}\right)$$

3. Find the scalar point function whose gradient is
$$(2xy-z^2)\vec{i} + (2yz+x^2)\vec{j} + (y^2-2zx)\vec{k}$$

Let:
$$F = (2xy-z^2)\vec{i} + (2yz+x^2)\vec{j} + (y^2-2zx)\vec{k}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Equating the both coefficient $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial\phi}{\partial x} = 2xy-z^2 \qquad ①$$

$$\frac{\partial\phi}{\partial y} = 2yz+x^2 \qquad ②$$

$$\frac{\partial\phi}{\partial z} = y^2-2zx \qquad ③$$

Integrating with respect to x in ①
$$\int d\phi = \int(2xy-z^2)dx$$
$$\phi = \frac{2x^2y}{2} - z^2x$$
$$\phi = x^2y - xz^2 + c \qquad ④$$

Integrating w.r to y in ②
$$\int d\phi = \int(2yz+x^2)dy$$
$$\phi = \frac{2y^2}{2}z + x^2y$$
$$\phi = y^2z + x^2y + c \qquad ⑤$$

Integrating w. r to z in ③
$$\int d\phi = \int(y^2-2zx)dz$$
$$\phi = zy^2 - \frac{2z^2x}{2}$$

$$= zy^2 - xz^2 + C \quad -\textcircled{6}$$

From ④,⑤,⑥

$$\phi = x^2y + y^2z - z^2x + C$$

4. Show that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential.

WKT:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\dfrac{\partial}{\partial y}(3xz^2) - \dfrac{\partial}{\partial z}(2y\sin x - 4)\right] - \vec{j}\left[\dfrac{\partial}{\partial x}(3xz^2) - \dfrac{\partial}{\partial z}(y^2\cos x + z^3)\right]$$

$$+ \vec{k}\left[\dfrac{\partial}{\partial x}(2y\sin x - 4) - \dfrac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$$= \vec{i}(0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2y\cos x)$$

$$\nabla \times \vec{F} = 0$$

∴ $\vec{F}$ is irrotational.

Then :

$$\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$$

$$F = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$$

Equating the both coefficient $\vec{i}, \vec{j}, \vec{k}$

$$\dfrac{\partial\phi}{\partial x} = y^2\cos x + z^3 \quad -\textcircled{1}$$

$$\dfrac{\partial\phi}{\partial y} = 2y\sin x - 4 \quad -\textcircled{2}$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 \qquad —③$$

Integrating w. r to x in ①

$$\int d\phi = \int (y^2 \cos x + z^3)\, dx$$

$$\phi = y^2 \sin x + xz^3 + C \qquad —④$$

Integrating . w.r to y in ②

$$\int d\phi = \int (2y \sin x - 4)\, dy$$

$$\phi = \frac{2y^2 \sin x}{2} - 4y$$

$$\phi = y^2 \sin x + 4y + C \qquad —⑤$$

Integrating . w.r to z in ③

$$\int d\phi = \int (3xz^2)\, dz$$

$$\phi = \frac{3xz^3}{3}$$

$$\phi = xz^3 + C \qquad —⑥$$

Let: From ④, ②, ⑤

$$\phi = y^2 \sin x + xz^3 - 4y + C$$

Hence showed

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5. Show that $r^n \vec{r}$ is an irrotational vector for any value of n but it is solenoidal only if $n = 3$

Let:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \qquad \Rightarrow r^2 = x^2 + y^2 + z^2$$

$$r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$(r^n \vec{r}) = r^n (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

Let:

$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{\partial r}{\partial x} = 2x \qquad 2r \frac{\partial r}{\partial y} = 2y \qquad 2r \frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y) \right] - \vec{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x) \right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x) \right]$$

$$= \vec{i}\left[z n r^{n-1} \frac{\partial r}{\partial y} - y n r^{n-1} \frac{\partial r}{\partial z} \right] - \vec{j}\left[z n r^{n-1} \frac{\partial r}{\partial x} \right.$$

$$\left. - x n r^{n-1} \frac{\partial r}{\partial z} \right] - \vec{k}\left[y n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial y} \right]$$

$$= \vec{i}\left[z n r^{n-1}\left(\frac{y}{r}\right) - y n r^{n-1}\left(\frac{z}{r}\right) \right] - \vec{j}\left[z n r^{n-1}\frac{x}{r} - x n r^{n-1}\frac{z}{r} \right]$$

$$+ \vec{k}\left[y n r^{n-1}\frac{x}{r} - x n r^{n-1}\left(\frac{y}{r}\right) \right]$$

$$= 0$$

$r^n \vec{r}$ is irrotational for all values of n

$$\nabla \cdot (r^n \vec{r}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right) \cdot \left(r^n (x\vec{i} + y\vec{j} + z\vec{k}) \right)$$

$$= \frac{\partial}{\partial x} r^n x + \frac{\partial}{\partial y} r^n y + \frac{\partial}{\partial z} r^n z$$

$$= r^n + xnr^{n-1}\frac{\partial r}{\partial x} + r^n + ynr^{n-1}\frac{\partial r}{\partial y} + r^n + znr^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + xnr^{n-1}\frac{x}{r} + ynr^{n-1}\frac{y}{r} + znr^{n-1}\frac{z}{r}$$

$$= 3r^n + nr^{n-1}r^{-1}x^2 + nr^{n-1}r^{-1}y^2 + nr^{n-1}r^{-1}z^2$$

$$= 3r^n + nr^{n-2}x^2 + nr^{n-2}y^2 + nr^{n-2}z^2$$

$$= 3r^n + nr^{n-2}\left[x^2 + y^2 + z^2\right]$$

$$= 3r^n + nr^{n-2}(r^2)$$

$$= 3r^n + nr^n$$

$$\nabla\cdot(r^n\vec{r}) = (3+n)r^n$$

$\nabla\cdot r^n\vec{r} = 0$ it is solenoidal

$$(3+n)r^n = 0$$

$$3+n = 0$$

$$\boxed{n = -3}$$

Hence   showed