Algorithm Design and Analysis
Assignment 2

1. (20 points) Here is a proposal to find the length of the shortest cycle in an unweighted undirected graph:

   DFS the graph, when there is a back edge (v, u), it forms a cycle from u to v, and the length is $level[v] - level[u] + 1$, where the level of a vertex is its distance in the DFS tree from the root. This suggests the following algorithm:

   - Do a DFS and keep tracking the level.
   - Each time we find a back edge, compute the cycle length, and update the smallest length.

   Please justify the correctness of the algorithm, prove it or provide a counterexample.

2. (20 points) Given a directed graph $G = (V, E)$ on which each edge $(u, v) \in E$ has a weight $p(u, v)$ in range $[0, 1]$, that represents the reliability. We can view each edge as a channel, and $p(u, v)$ is the probability that the channel from u to v will not fail. We assume all these probabilities are independent. Give an efficient algorithm to find the most reliable path from two given vertices s and t. Hint: it makes a path failed if any channel on the path fails, and we want to find a path with minimized failure probability.

3. (20 points) We have a connected undirected graph $G = (V, E)$, and a specific vertex $u \in V$. Suppose we compute a depth-first search tree rooted at u, and obtain a T that includes all nodes of G. Suppose we then compute a breath-first search tree rooted at u, and obtain the same tree T. Prove that $G = T$. (In other words, if T is both a DFS tree and a BFS tree rooted at u, then G cannot contain any edges that do not belong to T.)

4. (20 points) Given a directed graph $G(V, E)$ where each vertex can be viewed as a port. Consider that you are a salesman, and you plan to travel the graph. Whenever you reach a port v, it earns you a profit of p_v dollars, and it cost you c_{uv} if you travel from u to v. For any directed cycle in the graph, we can define a profit-to-cost ratio to be

$$r(C) = \frac{\sum_{(u,v)\in C} p_v}{\sum_{(u,v)\in C} c_{uv}}.$$

As a salesman, you want to design an algorithm to find the best cycle to travel with the largest profit-to-cost ratio. Let r^* be the maximum profit-to-cost ratio in the graph.

(a) (10 points) If we guess a ratio r, can we determine whether $r^* > r$ or $r < r^*$ efficiently?

(b) (10 points) Based the guessing approach, given a desired accuracy $\epsilon > 0$, design an efficient algorithm to output a good-enough cycle, where $r(C) \geq r^* - \epsilon$. Justify the correctness and analyze the running time in terms of $|V|$, ϵ, and $R = \max_{(u,v)\in E}(p_u/c_{uv})$.

5. (20 points) Consider if we want to run Dijkstra on a bounded weight graph $G = (V, E)$ such that each edge weight is integer and in the range from 1 to C, where C is a relatively small constant.

(a) (10 points) Show how to make Dijkstra run in $O(C|V| + |E|)$.

(b) (10 points) Show how to make Dijkstra run in $O(\log C(|V| + |E|))$. Hint: Can we use a binary heap with size C but not $|V|$?

6. How long does it take you to finish the assignment (include thinking and discussing)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Write down their names here.