

Question

Define $p = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and $q = \sum_{k=1}^{\infty} \frac{1}{k^3}$. Find a way to write $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$

Ours Longest: 1724 Tokens

Ground Truth Answer: $p - q$

NRP: 1191 Tokens

Okay, so I have this problem where I need to find an expression for the double sum $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$ in terms of (p) and (q) , where $(p = \sum_{k=1}^{\infty} \frac{1}{k^2})$ and $(q = \sum_{k=1}^{\infty} \frac{1}{k^3})$. Hmm, okay, let's break this down step by step.
First, I know that (p) is the well-known Basel problem, which sums to $\frac{\pi^2}{6}$, but I don't know if that's relevant here. Maybe not directly, but it's good to remember. The other sum, (q) , is the Riemann zeta function at 3, so it's $\zeta(3)$, but again, not sure if that's useful yet.
The main goal is to express the double sum $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$ in terms of (p) and (q) . I need to find a clever way to relate this double sum to the single sums (p) and (q) .
I remember that sometimes, for sums like this, changing variables or using symmetry can help. Let me try to visualize the double sum. It's over all positive integers (j) and (k) , and the term is $\frac{1}{(j+k)^3}$. So, for each pair $((j, k))$, we're adding up the reciprocal of the cube of their sum.
I wonder if I can express this double sum as a single sum by changing the order of summation or by some substitution. Maybe if I let $(n = j + k)$, then for each (n) , I can sum over all pairs $((j, k))$ such that $(j + k = n)$. That seems promising.
So, let me try that. Let me fix (n) and consider all pairs $((j, k))$ where $(j + k = n)$. For each (n) , (j) can range from 1 to $(n - 1)$, with $(k = n - j)$. Therefore, the double sum can be rewritten as:
$$\sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{1}{n^3}$$
Because $(j + k = n)$ implies $(k = n - j)$, and since both (j) and (k) are at least 1, (n) starts from 2 (since the smallest (j) and (k) can be is 1, so $(1 + 1 = 2)$).
simplifying the inner sum, it's just $\sum_{j=1}^{n-1} \frac{1}{n^3}$. Since $\frac{1}{n^3}$ is constant with respect to (j) , the inner sum is just $\frac{n-1}{n^3}$.Therefore, the entire double sum becomes:
$$\sum_{n=2}^{\infty} \frac{n-1}{n^3}$$
I can split this fraction:
$$\sum_{n=2}^{\infty} \left(\frac{n-1}{n^3} - \frac{1}{n^3} \right)$$
Now, I can split this into two separate sums:
$$\sum_{n=2}^{\infty} \left(\frac{n-1}{n^3} - \frac{1}{n^3} \right) = \sum_{n=2}^{\infty} \frac{1}{n^3} - \sum_{n=2}^{\infty} \frac{1}{n^3}$$
I know that the original definitions of (p) and (q) start from $(n = 1)$, so I can write these sums in terms of (p) and (q) by subtracting the $(n=1)$ terms.
So, $\sum_{n=2}^{\infty} \frac{1}{n^2} = p - \frac{1}{1^2} = p - 1$, and similarly, $\sum_{n=2}^{\infty} \frac{1}{n^3} = q - \frac{1}{1^3} = q - 1$.Therefore, substituting back into our expression:
$$(p - 1) - (q - 1) = p - 1 - q + 1 = p - q$$
So, the double sum simplifies to $(p - q)$.I think that makes sense. So, the key steps were changing variables to $(n = j + k)$, then rewriting the double sum as a single sum over (n) , evaluating the inner sum, and then expressing the result in terms of (p) and (q) . The important part was recognizing that the inner sum simplifies nicely and then using the definitions of (p) and (q) to relate it back.
I don't see any mistakes in the steps, so I'm pretty confident the answer is $(p - q)$.
Final Answer**
The double sum can be written as $\boxed{(p - q)}$.

Remaining: None

LC-R1 Longest: 4287 Tokens

NRP Ratio: 100%

NRP: 3283 Tokens

Okay, so I need to find a way to express the double sum $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$ in terms of (p) and (q) , where $(p = \sum_{k=1}^{\infty} \frac{1}{k^2})$ and $(q = \sum_{k=1}^{\infty} \frac{1}{k^3})$. Hmm, let me think about how to approach this.
First, I know that (p) is the well-known Basel problem, which sums to $\frac{\pi^2}{6}$, but I don't think I need the exact value here. Similarly, (q) is the sum of reciprocals of cubes, which converges to Apéry's constant, approximately 1.202..., but again, exact value isn't necessary here.
The double sum given is over all positive integers (j) and (k) , and the term inside is $\frac{1}{(j+k)^3}$. So, it's a double series where each term depends on the sum of two variables. This seems a bit tricky because it's a two-dimensional sum, unlike single sums which are straightforward.
Maybe I can change variables or find a way to express this double sum in terms of (p) and (q) . Let me consider switching the order of summation or using some symmetry.
Another idea: perhaps express $\frac{1}{(j+k)^3}$ as an integral. Since $\frac{1}{n^3}$ can be written as an integral, maybe I can use that to interchange the sum and the integral.
Recall that for positive integers (n) , we have:
$$\sum_{m=1}^{\infty} \frac{1}{m^3} = \int_0^{\infty} x^{-2} dx$$
Wait, is that correct? Let me check. Actually, [omitted...]. So, that gives us $\boxed{(p - q)}$.

Remaining: 492 Tokens

Hmm, that seems too straightforward.
Wait, but is that correct? [omitted...]
Which is equal to:
$$\left(\sum_{m=1}^{\infty} \frac{1}{m^2} - 1 \right) - \left(\sum_{m=1}^{\infty} \frac{1}{m^3} - 1 \right) = (p - 1) - (q - 1) = p - q$$
So, is that correct? Hmm. Let me verify.[omitted...] which is indeed $\boxed{(p - q)}$.
Therefore, the double sum is equal to $(p - q)$.
So, the answer is $\boxed{(p - q)}$.
Final Answer**
$$\boxed{(p - q)}$$

NRP Ratio: 87%