

## Question

Define  $p = \sum_{k=1}^{\infty} \frac{1}{k^2}$  and  $q = \sum_{k=1}^{\infty} \frac{1}{k^3}$ . Find a way to write  $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$

Ours Longest: 1724 Tokens

Ground Truth Answer:  $p-q$

### NRP: 1191 Tokens

Okay, so I have this problem where I need to find an expression for the double sum  $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$  in terms of  $\langle p \rangle$  and  $\langle q \rangle$ , where  $\langle p \rangle = \sum_{k=1}^{\infty} \frac{1}{k^2}$  and  $\langle q \rangle = \sum_{k=1}^{\infty} \frac{1}{k^3}$ . Hmm, okay, let's break this down step by step.

First, I know that  $\langle p \rangle$  is the well-known Basel problem, which sums to  $\frac{\pi^2}{6}$ , but I don't know if that's relevant here. Maybe not directly, but it's good to remember. The other sum,  $\langle q \rangle$ , is the Riemann zeta function at 3, so it's  $\zeta(3)$ , but again, not sure if that's useful yet.

The main goal is to express the double sum  $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$  in terms of  $\langle p \rangle$  and  $\langle q \rangle$ . I need to find a clever way to relate this double sum to the single sums  $\langle p \rangle$  and  $\langle q \rangle$ .

I remember that sometimes, for sums like this, changing variables or using symmetry can help. Let me try to visualize the double sum. It's over all positive integers  $\langle j \rangle$  and  $\langle k \rangle$ , and the term is  $\frac{1}{(j+k)^3}$ . So, for each pair  $\langle (j, k) \rangle$ , we're adding up the reciprocal of the cube of their sum.

I wonder if I can express this double sum as a single sum by changing the order of summation or by some substitution. Maybe if I let  $\langle n \rangle = j+k$ , then for each  $\langle n \rangle$ , I can sum over all pairs  $\langle (j, k) \rangle$  such that  $\langle j+k = n \rangle$ . That seems promising.

So, let me try that. Let me fix  $\langle n \rangle$  and consider all pairs  $\langle (j, k) \rangle$  where  $\langle j+k = n \rangle$ . For each  $\langle n \rangle$ ,  $\langle j \rangle$  can range from 1 to  $\langle n-1 \rangle$ , with  $\langle k = n-j \rangle$ . Therefore, the double sum can be rewritten as:

$$\sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{1}{n^3}$$

Because  $\langle j+k = n \rangle$  implies  $\langle k = n-j \rangle$ , and since both  $\langle j \rangle$  and  $\langle k \rangle$  are at least 1,  $\langle n \rangle$  starts from 2 (since the smallest  $\langle j \rangle$  and  $\langle k \rangle$  can be is 1, so  $\langle 1+1 = 2 \rangle$ ).

So, simplifying the inner sum, it's just  $\sum_{j=1}^{n-1} \frac{1}{n^3}$ . Since  $\frac{1}{n^3}$  is constant with respect to  $\langle j \rangle$ , the inner sum is just  $\frac{n-1}{n^3}$ .

Therefore, the entire double sum becomes:

$$\sum_{n=2}^{\infty} \frac{n-1}{n^3}$$

Now, I can split this into two separate sums:

$$\sum_{n=2}^{\infty} \frac{n}{n^3} - \sum_{n=2}^{\infty} \frac{1}{n^3}$$

I know that the original definitions of  $\langle p \rangle$  and  $\langle q \rangle$  start from  $\langle n=1 \rangle$ , so I can write these sums in terms of  $\langle p \rangle$  and  $\langle q \rangle$  by subtracting the  $\langle n=1 \rangle$  terms.

So,  $\sum_{n=2}^{\infty} \frac{n}{n^3} = p - \frac{1}{1^2} = p - 1$ , and similarly,  $\sum_{n=2}^{\infty} \frac{1}{n^3} = q - \frac{1}{1^3} = q - 1$ .

Therefore, substituting back into our expression:

$$(p - 1) - (q - 1) = p - q$$

So, the double sum simplifies to  $\langle p - q \rangle$ . I think that makes sense. So, the key steps were changing variables to  $\langle n = j+k \rangle$ , then rewriting the double sum as a single sum over  $\langle n \rangle$ , evaluating the inner sum, and then expressing the result in terms of  $\langle p \rangle$  and  $\langle q \rangle$ . The important part was recognizing that the inner sum simplifies nicely and then using the definitions of  $\langle p \rangle$  and  $\langle q \rangle$  to relate it back.

I don't see any mistakes in the steps, so I'm pretty confident the answer is  $\langle p - q \rangle$ .

**Final Answer**

The double sum can be written as  $\boxed{p - q}$ .

Remaining: None

LC-R1 Longest: 4287 Tokens

NRP Ratio: 100%

### NRP: 3283 Tokens

Okay, so I need to find a way to express the double sum  $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$  in terms of  $\langle p \rangle$  and  $\langle q \rangle$ , where  $\langle p \rangle = \sum_{k=1}^{\infty} \frac{1}{k^2}$  and  $\langle q \rangle = \sum_{k=1}^{\infty} \frac{1}{k^3}$ .

Hmm, let me think about how to approach this. First, I know that  $\langle p \rangle$  is the well-known Basel problem, which sums to  $\frac{\pi^2}{6}$ , but I don't think I need the exact value here. Similarly,  $\langle q \rangle$  is the sum of reciprocals of cubes, which converges to Apéry's constant, approximately 1.202..., but again, exact value isn't necessary here.

The double sum given is over all positive integers  $\langle j \rangle$  and  $\langle k \rangle$ , and the term inside is  $\frac{1}{(j+k)^3}$ . So, it's a double series where each term depends on the sum of two variables. This seems a bit tricky because it's a two-dimensional sum, unlike single sums which are straightforward.

Maybe I can change variables or find a way to express this double sum in terms of  $\langle p \rangle$  and  $\langle q \rangle$ . Let me consider switching the order of summation or using some symmetry.

Another idea: perhaps express  $\frac{1}{(j+k)^3}$  as an integral. Since  $\frac{1}{n^3}$  can be written as an integral, maybe I can use that to interchange the sum and the integral.

Recall that for positive integers  $\langle n \rangle$ , we have:

$$\frac{1}{n^3} = \frac{1}{2} \int_0^1 \int_0^1 (xy)^{n-1} dx dy$$

Wait, is that correct? Let me check. Actually, [omitted...] So, that gives us  $\langle p - q \rangle$ .

### Remaining: 492 Tokens

Hmm, that seems too straightforward.

Wait, but is that correct? [omitted...] Which is equal to:

$$\left( \sum_{m=1}^{\infty} \frac{1}{m^2} - 1 \right) - \left( \sum_{m=1}^{\infty} \frac{1}{m^3} - 1 \right) = (p - 1) - (q - 1) = p - q$$

So, is that correct? Hmm. Let me verify. [omitted...] which is indeed  $\langle p - q \rangle$ .

Therefore, the double sum is equal to  $\langle p - q \rangle$ .

So, the answer is  $\boxed{p - q}$ .

**Final Answer**

$\boxed{p - q}$

NRP Ratio: 87%