

## Question

A spin-half particle is in a linear superposition  $0.5|\uparrow\rangle + \sqrt{3}/2|\downarrow\rangle$  of its spin-up and spin-down states. If  $|\uparrow\rangle$  and  $|\downarrow\rangle$  are the eigenstates of  $\sigma_z$ , then what is the expectation value up to one decimal place, of the operator  $10\sigma_z + 5\sigma_x$ ? Here, symbols have their usual meanings (A) -1.4 (B) -0.7 (C) 1.65 (D) 0.85

Ground Truth Answer: B)

Ours Longest: 2766 Tokens

### NRP: 2294 Tokens

Okay, so I have this quantum mechanics problem here, and I need to find the expectation value of the operator  $10\sigma_z + 5\sigma_x$  for a given spin- $1/2$  particle state. Hmm, let me try to break this down step by step. First, the state of the particle is given as a linear combination of the spin-up and spin-down states:  $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$ . I remember that  $|\uparrow\rangle$  and  $|\downarrow\rangle$  are eigenstates of the Pauli matrix  $\sigma_z$ . So,  $\sigma_z$  acting on  $|\uparrow\rangle$  gives  $+1$  times  $|\uparrow\rangle$ , and  $\sigma_z$  acting on  $|\downarrow\rangle$  gives  $-1$  times  $|\downarrow\rangle$ , right? Now, the operator we're dealing with is  $10\sigma_z + 5\sigma_x$ . I need to find the expectation value of this operator, which is  $\langle\psi|(10\sigma_z + 5\sigma_x)|\psi\rangle$ . Since expectation value is linear, I can separate this into  $10\langle\psi|\sigma_z|\psi\rangle + 5\langle\psi|\sigma_x|\psi\rangle$ . That should make things easier. Let me handle each term separately. Starting with the first term,  $10\langle\psi|\sigma_z|\psi\rangle$ . I know that  $\sigma_z$  is diagonal in the  $\sigma_z$  basis, which is the same as the  $|\uparrow\rangle$  and  $|\downarrow\rangle$  basis. So, the expectation value of  $\sigma_z$  is just the weighted average of the eigenvalues, with the weights given by the square of the amplitudes in the state  $|\psi\rangle$ . The state  $|\psi\rangle$  is  $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$ . So, the coefficient for  $|\uparrow\rangle$  is  $0.5$ , and for  $|\downarrow\rangle$  is  $\sqrt{3}/2$ . The square of these gives the probabilities. Therefore, the expectation value of  $\sigma_z$  is  $(0.5)^2(+1) + (\sqrt{3}/2)^2(-1)$ . Calculating that:  $(0.25)(1) + (3/4)(-1) = 0.25 - 0.75 = -0.5$ . So,  $10$  times that would be  $10(-0.5) = -5$ . Okay, that's the first part. Now, the second term is  $5\langle\psi|\sigma_x|\psi\rangle$ . Hmm,  $\sigma_x$  is the Pauli X matrix, which is not diagonal in the  $\sigma_z$  basis. I remember that  $\sigma_x$  has eigenstates  $|+\rangle$  and  $|-\rangle$ , which are superpositions of  $|\uparrow\rangle$  and  $|\downarrow\rangle$ . Maybe I need to express  $|\psi\rangle$  in the  $\sigma_x$  basis to compute this expectation value, or perhaps there's another way. Alternatively, I recall that  $\sigma_x$  can be written in matrix form as  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . So, if I can write  $|\psi\rangle$  as a column vector, then  $\sigma_x$  acting on  $|\psi\rangle$  can be computed, and then the inner product  $\langle\psi|\sigma_x|\psi\rangle$  can be found. Let's represent  $|\psi\rangle$  as a vector. Since  $|\uparrow\rangle$  is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $|\downarrow\rangle$  is  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , then  $|\psi\rangle$  is  $0.5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (\sqrt{3}/2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$ . So, the vector is  $\begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$ . Now,  $\sigma_x$  is  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . So,  $\sigma_x|\psi\rangle$  is  $\begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$ . Wait, that's the same as  $|\psi\rangle$ ? Hmm, so  $\sigma_x|\psi\rangle = |\psi\rangle$ ? [omitted] Since  $a$  and  $b$  are real in this case (the coefficients are real), this becomes  $2ab$ . So, for our state,  $a = 0.5$ ,  $b = \sqrt{3}/2$ . Therefore,  $\langle\sigma_x\rangle = 2(0.5)(\sqrt{3}/2) = \sqrt{3}/2 \approx 0.866$ . Therefore,  $5\langle\sigma_x\rangle = 5(\sqrt{3}/2) \approx 5 \cdot 0.866 \approx 4.33$ . So, adding the two terms together:  $-5 + 4.33 \approx -0.67$ . So, approximately  $-0.7$  when rounded to one decimal place. Ah, okay, so that makes sense now. My initial mistake was probably in the way I computed  $\sigma_x|\psi\rangle$  and then the inner product. But using the formula for expectation value in terms of coefficients, it's much simpler. So, the expectation value of  $\sigma_x$  is  $2ab$ , which is  $2(0.5)(\sqrt{3}/2) = \sqrt{3}/2 \approx 0.866$ . Therefore,  $5$  times that is approximately  $4.33$ , and  $10$  times the expectation value of  $\sigma_z$  is  $-5$ . So, adding them gives approximately  $-0.67$ , which rounds to  $-0.7$ . So, the answer should be option B)  $-0.7$ .

Remaining: None

NRP Ratio: 100%

ThinkPrune Longest: 8098 Tokens

### NRP: 1535 Tokens

Okay, so I have this quantum mechanics problem here about a spin-half particle. Hmm, let's see. The particle is in a linear superposition given by  $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$ . And I need to find the expectation value of the operator  $10\sigma_z + 5\sigma_x$ . The answer choices are A) -1.4, B) -0.7, C) 1.65, D) 0.85. Alright, first, let me recall what the expectation value of an operator is. For a quantum state  $|\psi\rangle$ , the expectation value  $\langle O \rangle$  is given by  $\langle\psi|O|\psi\rangle$ . So, I need to compute  $\langle\psi|(10\sigma_z + 5\sigma_x)|\psi\rangle$ . Since expectation value is linear, I can separate this into  $10\langle\psi|\sigma_z|\psi\rangle + 5\langle\psi|\sigma_x|\psi\rangle$ . So, I need to find the expectation values of  $\sigma_z$  and  $\sigma_x$  separately and then combine them. Let me write down the given state  $|\psi\rangle$ . It is  $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$ . Let me note that  $|\uparrow\rangle$  and  $|\downarrow\rangle$  are the eigenstates of  $\sigma_z$ , which is the Pauli z-matrix. So,  $\sigma_z|\uparrow\rangle = +|\uparrow\rangle$  and  $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$ . First, compute  $\langle\sigma_z\rangle$ . That is,  $\langle\psi|\sigma_z|\psi\rangle$ . Since  $\sigma_z$  is diagonal in the  $\sigma_z$  basis, which is the same as the basis in which  $|\uparrow\rangle$  and  $|\downarrow\rangle$  are defined, this should be straightforward. [omitted...] Looking at the options, B is  $-0.7$ , which would be the answer. But wait, let me verify the calculations again because sometimes I might have made a mistake. First,  $\langle\sigma_z\rangle$ : probabilities  $0.25$  and  $0.75$ , so  $0.25 - 0.75 = -0.5$ . That seems right. Then, for  $\langle\sigma_x\rangle$ : the expectation value. Another way to compute  $\langle\sigma_x\rangle$  is to note that  $\sigma_x = |\rightarrow\rangle\langle\leftarrow| - |\leftarrow\rangle\langle\rightarrow|$ , but maybe that's more complicated. Alternatively, for a general state  $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ ,  $\langle\sigma_x\rangle$  is  $|a|^2 - |b|^2$ ? Wait, no, that can't be. Wait,  $\sigma_x$  is  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , so when you compute  $\langle\psi|\sigma_x|\psi\rangle$ , it's  $|a|^2\langle\downarrow|\sigma_x|\uparrow\rangle + |b|^2\langle\uparrow|\sigma_x|\downarrow\rangle + \text{cross terms}$ . Wait, let's compute it again step by step.  $|\psi\rangle = \begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$ . Then,  $\sigma_x|\psi\rangle = \begin{bmatrix} \sqrt{3}/2 \\ 0.5 \end{bmatrix}$ . Then,  $\langle\psi|\sigma_x|\psi\rangle = (0.5)(\sqrt{3}/2) + (\sqrt{3}/2)(0.5) = \sqrt{3}/2 \approx 0.866$ . So,  $5\langle\sigma_x\rangle = 5(\sqrt{3}/2) \approx 4.33$ , and  $10\langle\sigma_z\rangle = -5$ , so total expectation value is  $-5 + 4.33 \approx -0.67$ , which is approximately  $-0.7$  when rounded to one decimal. Therefore, the answer should be B)  $-0.7$ .

### Remaining: 5990 Tokens

[omitted...] Wait, but let me think again. Is there another way to compute this? Maybe using the properties of spin states. [omitted...] So, in conclusion,  $\langle\sigma_x\rangle$  is approximately  $0.866$ , which is  $\sqrt{3}/2$ . Therefore,  $5\langle\sigma_x\rangle$  is  $5(\sqrt{3}/2) \approx 5 \cdot 0.866 \approx 4.33$ . Then,  $10\langle\sigma_z\rangle = -5$ , as computed earlier. Therefore, the total expectation value is  $-5 + 4.33 \approx -0.67$ , which is approximately  $-0.7$  when rounded to one decimal place. Therefore, the answer is B)  $-0.7$ .

NRP Ratio: 20%