FORMULAS

RELATIONS

On a finite set A with $|A|=n$, $|A\times A|=n^2$

Total # of relations $= 2^{n^2}$ (Each pair in $A\times A$ can either be included or not)

Total # of REFLEXIVE relations $= 2^{n(n-1)} = 2^{n^2-n}$

Total - diagonal: $n^2 - n$ pairs

$\forall (a,a)\in R$

There are n such diagonal pairs and they MUST BE PRESENT.

→ Remaining n^2-n pairs can be freely included or excluded.

Total # of SYMMETRIC relations $= 2^n \cdot 2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}+n} = 2^{\frac{n(n+1)}{2}}$

$(a,b)\in R \Rightarrow (b,a)\in R$.

Diagonal pairs (a,a) – freely chosen to include/exclude: n choices $\to 2^n$

Off-diagonal pairs (a,b) with $a\neq b$, the pair and its symmetric counterpart (b,a) must be either both in or both be out.

- There are $\binom{n}{2} = \frac{n(n-1)}{2}$ such unordered pairs
- Each can be included or not $\to 2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}}$

Hence, # of sym. relations $= 2^n \cdot 2^{\frac{n(n-1)}{2}} = 2^{\frac{n(n+1)}{2}}$ (diagonal, # unordered pairs)

Total # of ANTISYMMETRIC relations $= 2^n \cdot 3^{\binom{n}{2}} = 2^n \cdot 3^{\frac{n(n-1)}{2}}$

Recall the condition: $(a,b)\in R$ and $(b,a)\in R \Rightarrow a=b$
or $(a,b)\in R$ and $(b,a)\notin R \Rightarrow a\neq b$

Construction

- For each diagonal pair (a,a), we can include it or not 2^n
- For each unordered off-diagonal pair $\{a,b\}$ with $a\neq b$, we have 3 choices
  1. include (a,b)
  2. include (b,a)
  3. include neither

  $3^{\frac{n(n-1)}{2}}$

BUT NOT BOTH !!

- There are $\frac{n(n-1)}{2}$ such unordered pairs.

THUS $\Rightarrow 2^n \cdot 3^{\frac{n(n-1)}{2}}$

Total # of TRANSITIVE relations: No closed formula – must test for closure

Total # of EQUIVALENCE relations = Equal to the # of partitions of the set (Bell number B_n) $\Rightarrow$

Bonus: Bell Number

n	B_n
1	1
2	2
3	5
4	15
5	52

FUNCTIONS

For 2 finite sets A & B where $|A|=m$, $|B|=n$ (domain) (codomain)

Total functions from $A\to B$ where $|A|=n$, $|B|=m$ $= n^m$ (Each element in A can map to any element in B)

Total # of INJECTIVE functions $= \frac{n!}{(n-m)!} = P(n,m)$ if $m\leq n$

An injective function requires distinct outputs for distinct inputs. So we must choose m different elements from B (only possible if $m\leq n$) and assign them uniquely.

$$\text{\# Injective functions} = \begin{cases} \frac{n!}{(n-m)!}, & \text{if } m\leq n \\ 0, & \text{if } m>n \end{cases}$$

Total # of SURJECTIVE functions $= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = n!\, S(m,n)$ (Inclusion-Exclusion)

A surjective function must hit every value in the codomain. This involves using inclusion-exclusion or Stirling numbers of the 2nd kind.

Surjective functions $= n!\cdot S(m,n)$

Total # of BIJECTIVE functions $= n!$ if $m=n$

A bijection is both injective & surjective, so it requires:
- $m=n$
- all elements used exactly once

So # BIJECTIVE FUNCTIONS $= \begin{cases} n!, & \text{if } m=n \\ 0, & \text{otherwise} \end{cases}$

Total # of PARTIAL functions $= (n+1)^m$ (assigning a "null" value option)

A partial function from $A\to B$ means the function is not necessarily defined for every element in A.

Each element in A can:
- not be mapped (undefined)
- be mapped to any one of the n values in B

So each of the m elements in A has $n+1$ choices

Partial Functions $= (n+1)^m$

Stirling Numbers

of ways to PARTITION a set of m elements into EXACTLY n, non-empty, unlabeled subsets

$S(m,n) = n\cdot \underbrace{S(m-1,n)}_{①} + \underbrace{S(m-1,n-1)}_{②}$

Meaning:
- $n\cdot S(m-1,n)$: Add the new element to one of the n existing subsets
- $S(m-1,n-1)$: Put the new element in a subset itself

BASE CASES:
- $S(m,1)=1$ (one subset: all elements together)
- $S(m,m)=1$ (each element in its own subset)
- $S(m,0)=0$ for $m>0$
- $S(0,0)=1$

Eg.
- $m=3$, $n=2$, $S(3,2)$
- Divide a 3-element set (e.g. $\{a,b,c\}$) into 2 non empty subsets

Valid Partitions
- $\{a,b\},\{c\}$
- $\{a,c\},\{b\}$
- $\{b,c\},\{a\}$

So, $S(3,2)=3$

Alternate Formula

$$S(m,n) = \underbrace{\frac{1}{n!}}_{①} \underbrace{\sum_{k=0}^{n}}_{②} \underbrace{(-1)^k}_{③} \underbrace{\binom{n}{k}}_{④} \underbrace{(n-k)^m}_{⑤}$$

- "Divide" ① $\frac{1}{n!}$ dividing by $n!$: to REMOVE ORDERING of subsets
- "Sum" ② $\sum_{k=0}^{n}$ Calculate a value for each k and add the results
- "Alternate" ③ $(-1)^k$ alternating signs (INCLUSION-EXCLUSION Logic)
- "Choose" ④ $\binom{n}{k}$ # of ways to choose k elements from n – helps count how many subsets to subtract/add
- "Power" ⑤ $(n-k)^m$ # of functions from a set of size m to a size $n-k$ – adjusts the count based on how many codomain values are excluded by the k selected in $\binom{n}{k}$.

Divide, Sum, Alternate, Choose, Count with Power

DON'T DO THAT FOR SURJECTIVE FNs!

$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$

COMBINATORICS

Stars & Bars

$x_1+x_2+\dots+x_k = n$ $(x_1,x_2,x_3,x_4\geq 0)$

$\left(\!\binom{n+1}{k-1}\!\right) = \left(\!\binom{k}{n}\!\right) = \binom{n+k-1}{k-1}$ (for non-negative integers)

multiset coefficient

$\mathbb{Z}^+$ only

$x_i' = x_i - 1$

$x_1' + x_2' + \dots x_k' = n-k \Rightarrow \binom{n-1}{k-1}$

Example Problem (non-negative integers) ①

How many non-negative integer solutions are there to

$x_1+x_2+x_3=4$, $x_i\geq 0$!!!

$n=4$ STARS

$k=3$ variables → need $k-1=2$ bars (x_1,x_2,x_3)

- $****||\to x_1=4, x_2=0, x_3=0$
- $***|*|\to x_1=3, x_2=1, x_3=0$
- $**|**|\to x_1=2, x_2=2, x_3=0$
- $*|***|\to x_1=1, x_2=3, x_3=0$
- $|****|\to x_1=0, x_2=4, x_3=0$
- ⋮
- $||****\to x_1=0, x_2=0, x_3=4$

$\binom{4+3-1}{3-1} = \binom{6}{2} = 15$ such arrangements.

Example Problem (+ve Z)

$x_1+x_2+x_3=4$ $x_i\geq 1$ ← if these, use primes!

$x_i \geq 1$

Since each $x_i\geq 1$, let

$x_i' = x_i - 1 \Rightarrow x_i'\geq 0$ ⇐ make everything "go to 0"

then calculate the $x_1'+x_2'+x_3'=4$

Formula $\Rightarrow \binom{4+3-1}{3-1} = \binom{6}{2} = 15$

Using Caps

□ R R R R R R

Select 3 from 7
- no restrictions $\Rightarrow \binom{7}{3}$
- with corner $\Rightarrow \binom{7-2}{3} = \binom{5}{3}$
- with 2 corner $\Rightarrow \binom{7-2(2)}{3} = \binom{3}{3}$

SEQUENCES

Arithmetic Sequence $\Rightarrow a_n = a_1 + (n-1)d$
(check for constant difference)

Geometric Sequence $\Rightarrow S_n = \frac{a_1(1-r^n)}{1-r}$ (Sum of n terms)
(check for a constant ratio)

r is the COMMON RATIO $\Rightarrow r = \frac{a_n}{a_{n-1}}$

Quadratic Sequence $\Rightarrow An^2+Bn+C_n$
(check for 2nd differences)

GRAPH THEORY

For a graph with n vertices

# of edges on COMPLETE graphs $= \frac{n(n-1)}{2} = \binom{n}{2}$

Handshaking Lemma $= \sum \deg(v) = 2\cdot e$, e = number of edges

# of SIMPLE GRAPHS (undirected) $= 2^{\binom{n}{2}}$

# of DIRECTED GRAPHS $= 2^{n(n-1)}$ (no self loops)
$= 2^{n^2}$ (self loops)

# of edges in a COMPLETE GRAPH K_n $= \binom{n}{2} = \frac{n(n-1)}{2}$ (undirected)
$= n(n-1)$ (directed, (no self-loops))

# of HAMILTONIAN CIRCUITS in K_n $= \frac{(n-1)!}{2}$ (for undirected graphs)

Reason: Start at one vertex, permute the others, divide by 2 for the reverse duplicates.

GRAPH THEORY

1. Complete Graphs

Let K_n be the complete graph with n vertices

- # edges $= \binom{n}{2} = \frac{n(n-1)}{2}$
- # triangles $= \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$
- deg. of each vertex $= n-1$
- Total degree (all vertices) $= n(n-1)$

2. General Graphs

Let $G=(V,E)$ with $|V|=n$, $|E|=m$

- Max edges (simple graph) $= \frac{n(n-1)}{2}$
- Total degree of all vertices $= 2m$ (Handshaking Lemma)
- Max edges (complete bipartite $K_{a,b}$) $= ab$

3. Paths, Walks, and Cycles

Number of walks of length k from $u\to v$: Use $A^k[u][v]$, where A is the adj matrix

Why?
- matrix multiplication counts paths: number of walks of length k from i to j
  - Entry (i,j) in A^k gives number of walks of length k from i to j and recursive summation over intermediate steps.
  - (comes from linear algebra and recursive summation over intermediate steps.

Number of cycles in K_n: For length r: $\frac{1}{2r}\binom{n}{r}\cdot(r-1)!$

- $\binom{n}{r}$: choose vertices to form the cycle
- $(r-1)!$: # ways to arrange them in a cycle (fix 1 vertex to avoid overcounting rotations)
- Divide by 2 because each cycle is counted in forward and reverse.

4. Connected Graphs

- Minimum edges in connected graph: $n-1$
- max edges in a simple connected graph: $\frac{n(n-1)}{2}$ (same as complete graph)