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Unit-2:-

Matrix Algebra - II

Determinant of a matrix:-

problems:-

1. given a matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & -6 & x \end{bmatrix}$ & if $|A| = 48$ then find the value x.

A) $|A| = 1\begin{bmatrix} 3 & 4 \\ -6 & x \end{bmatrix} - 0\begin{bmatrix} 2 & 4 \\ 5 & x \end{bmatrix} + 0\begin{bmatrix} 2 & 3 \\ 5 & -6 \end{bmatrix}$

$48 = 1(3x+24) \Rightarrow 48 = 3x+24 \Rightarrow 48-24 = 3x$

$24 = 3x \Rightarrow x = \frac{\cancel{24}^{8}}{\cancel{3}} \Rightarrow \underline{x = 8}$

2. Find the Determinant of the following matrices.

$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

A) $|A| = a\begin{bmatrix} b & f \\ f & c \end{bmatrix} - h\begin{bmatrix} h & f \\ g & c \end{bmatrix} + g\begin{bmatrix} h & b \\ g & f \end{bmatrix}$

$= a(bc-f^2) - h(hc-gf) + g(hf-gb)$

$= abc - af^2 - h^2c + hgf + ghf - g^2b$

$|A| = \underline{abc - af^2 - ch^2 + 2hgf - bg^2}$

3. find the Determinant of $A = \begin{bmatrix} 2 & 3 & 20 \\ 3 & 5 & 34 \\ 1 & 2 & 17 \end{bmatrix}$

A) $|A| = 2\begin{bmatrix} 5 & 34 \\ 2 & 17 \end{bmatrix} - 3\begin{bmatrix} 3 & 34 \\ 1 & 17 \end{bmatrix} + 20\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

$= 2(85-68) - 3(51-34) + 20(6-5) = 2(17) - 3(17) + 20(1)$

$= 34 - 51 + 20 = -3$

3. $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$, 4. $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$\overset{2}{\cancel{15}} \times 4$, $\cancel{60}$, $\frac{60}{\overset{6}{\cancel{66}}}$

3A) $|A| = 4\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} - 1\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} + 1\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

$= 4(16-1) - 1(4-1) + 1(1-4) = 4(15) - 1(3) + 1(-3)$

$= 60 - 3 - 3 = 60 - 6 = 54.$

4A)

$|A| = a\begin{bmatrix} c & a \\ a & b \end{bmatrix} - b\begin{bmatrix} b & a \\ c & b \end{bmatrix} + c\begin{bmatrix} b & c \\ c & a \end{bmatrix}$

$= a(cb - a^2) - b(b^2 - ca) + c(ba - c^2)$

$= acb - a^3 - b^3 + abc + abc - c^3$

$|A| = \cancel{3abc} - a^3 - b^3 - c^3 + 3abc$. (or)

$|A| = -(a^3 + b^3 + c^3 - 3abc)$

5. If $A = \begin{bmatrix} 2 & -1 & 1 \\ x & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ is singular matrix then find the value of x.

A) $|A| = 0$

$|A| = 2\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} + 1\begin{bmatrix} x & 2 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} x & 0 \\ 1 & 2 \end{bmatrix} = 0$

$= 2(0-4) + 1(0-2) + 1(2x-0) = 0$

$= -8 - 2 + 2x = 0 \Rightarrow -10 + 2x = 0 \Rightarrow 2x = 10$

$x = \frac{\overset{5}{\cancel{10}}}{\cancel{2}}$ $x = 5$

Adjoint of a Square matrix :-

The transpose of the matrix (co-factor matrix) formed by replacing the elements of a square matrix with the corresponding co-factors is called the Adjoint of A. and it is denoted by "Adj A" (or) adj.A

i.e, Adj A = $(\text{co-factor matrix})^T$

NOTE:-

co-factor of an element = $(-1)^{\text{row no} + \text{column no.}} \times$ minor of the element

problems:-

1. find the Adj of the following matrices $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$.

A) Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

Calculation of co-factors:-

Co factor of an element = $(-1)^{\text{row no} + \text{Column no}} \times$ minor of the element

1st row:-

co factor of $1 = (-1)^{1+1} \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = (-1)^2 (4-2) = +2$

co factor of $2 = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = (-1)^3 (6-2) = -4$

cofactor of $1 = (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = (-1)^4 (3-2) = 1$

2nd row:-

co factor of $3 = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (-1)^3 (4-1) = -3$

cofactor of 2 $= (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (-1)^4 (2-1) = 1$

cofactor of 2 $= (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = (-1)^5 (1-2) = +1$

row 3 :-

cofactor of 1 $= (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = (-1)^4 (\cancel{2}4-2) = 2$

cofactor of 1 $= (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = (-1)^5 (2-3) = 1$

cofactor of 2 $= (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = (-1)^6 (2-6) = -4$

cofactor matrix :- of A

$$\begin{bmatrix} 2 & -4 & 1 \\ -3 & 1 & 1 \\ 2 & 1 & -4 \end{bmatrix}$$

Adj A = $(\text{cofactor matrix})^T$

$$= \begin{bmatrix} 2 & -3 & 2 \\ -4 & 1 & 1 \\ 1 & 1 & -4 \end{bmatrix}$$

2. $A = \begin{bmatrix} 0 & -3 & 1 \\ 4 & 1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$

A) Calculation of cofactors :-

$$\begin{array}{r} 28 \\ 15 \\ \hline 43 \end{array}$$

1st row :-

cofactor of 0 $= (-1)^{1+1} \begin{bmatrix} 1 & 5 \\ -1 & 7 \end{bmatrix} = 1(7+5) = 12$

$-3 = (-1)^{1+2} \begin{bmatrix} 4 & 5 \\ 3 & 7 \end{bmatrix} = -1(28-15) = \cancel{43}\ -13$

$1 = (-1)^{1+3}\begin{bmatrix}4 & 1\\ 3 & -1\end{bmatrix} = (-1)^4(-4-3) = (-1)^4(-7) = -7$

$$\begin{array}{r} 28 \\ 15 \\ \hline 13 \end{array}$$

2nd row:-

$4 = (-1)^{2+1}\begin{bmatrix}-3 & 1\\ -1 & 7\end{bmatrix} = (-1)(-21+1) = 20$

$1 = (-1)^{2+2}\begin{bmatrix}0 & 1\\ 3 & 7\end{bmatrix} = (-1)^4(0-3) = -3$

$5 = (-1)^{2+3}\begin{bmatrix}0 & -3\\ 3 & -1\end{bmatrix} = (-1)^5(-0+9) = -9$

3rd row:-

$3 = (-1)^{3+1}\begin{bmatrix}-3 & 1\\ 1 & 5\end{bmatrix} = (-1)^4(-15-1) = 1(-16) = -16.$

$-1 = (-1)^{3+2}\begin{bmatrix}0 & 1\\ 4 & 5\end{bmatrix} = (-1)^5(0-4) = 4$

$7 = (-1)^{3+3}\begin{bmatrix}0 & -3\\ 4 & 1\end{bmatrix} = (-1)^6(0+12) = 12$

Cofactor Matrix $= \begin{bmatrix}12 & -13 & -7\\ 20 & -3 & -9\\ -16 & 4 & 12\end{bmatrix}$

Adj $A = (\text{cofactor matrix})^T = \begin{bmatrix}12 & 20 & -16\\ -13 & -3 & 4\\ -7 & -9 & 12\end{bmatrix}$

3. $A = \begin{bmatrix}2 & 3 & 2\\ 1 & 7 & 3\\ 0 & 1 & 7\end{bmatrix}$

$$\begin{array}{r} 49 \\ 3 \\ \hline 46 \end{array}$$

Calculation of co-factors:-

1st row:-

$2 = (-1)^{1+1}\begin{bmatrix}7 & 3\\ 1 & 7\end{bmatrix} = 1(49-3) = 1(46) = 46.$

$3 = (-1)^{1+2}\begin{bmatrix}1 & 3\\ 0 & 7\end{bmatrix} = -1(7-0) = -7$

$2 = (-1)^{1+3}\begin{bmatrix}1 & 7\\ 0 & 1\end{bmatrix} = +1(1-0) = 1$

2nd row:-

$1 = (-1)^{2+1}\begin{bmatrix}3 & 2\\ 1 & 7\end{bmatrix} = -1(21-2) = -19$

$7 = (-1)^{2+2}\begin{bmatrix}2 & 2\\ 0 & 7\end{bmatrix} = 1(14-0) = 14$

$3 = (-1)^{2+3}\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix} = -1(2-0) = -2$

3rd row:-

$0 = (-1)^{3+1}\begin{bmatrix}3 & 2\\ 7 & 3\end{bmatrix} = 1(9-14) = -5$

$1 = (-1)^{3+2}\begin{bmatrix}2 & 2\\ 1 & 3\end{bmatrix} = -1(6-2) = -4$

$7 = (-1)^{3+3}\begin{bmatrix}2 & 3\\ 1 & 7\end{bmatrix} = 1(14-3) = 11$

$$\text{cofactor matrix} = \begin{bmatrix}46 & -7 & 1\\ -19 & 14 & -2\\ -5 & -4 & 11\end{bmatrix}$$

$$\text{Adj } A = (\text{cofactor matrix})^T = \begin{bmatrix}46 & -19 & -5\\ -7 & 14 & -4\\ 1 & -2 & 11\end{bmatrix}$$

4. $A = \begin{bmatrix} 2 & 3 & -1 \\ -4 & 0 & 3 \\ 3 & -1 & 7 \end{bmatrix}$ find Adj A

$\begin{array}{r} 21 \\ 9 \\ \hline 30 \end{array}$ $\begin{array}{r} 21 \\ 7 \\ \hline 28 \end{array}$ $\begin{array}{r} 28 \\ 19 \\ \hline 37 \end{array}$

A) calculation of co-factors :-

1st row :-

$2 = (-1)^{1+1} \begin{bmatrix} 0 & 3 \\ -1 & 7 \end{bmatrix} = 1(0+3) = \cancel{10}\ 3$

$3 = (-1)^{1+2} \begin{bmatrix} -4 & 3 \\ 3 & 7 \end{bmatrix} = -1(-28-9) = \cancel{30}\ +37$

$-1 = (-1)^{1+3} \begin{bmatrix} -4 & 0 \\ 3 & -1 \end{bmatrix} = 1(4-0) = 4$

2nd row :-

$-4 = (-1)^{2+1} \begin{bmatrix} 3 & -1 \\ -1 & 7 \end{bmatrix} = -1(21-1) = -20$

$0 = (-1)^{2+2} \begin{bmatrix} 2 & -1 \\ 3 & 7 \end{bmatrix} = 1(\cancel{\ \ }) = +17$ $(21-3)$

$3 = (-1)^{2+3} \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} = -1(-2-9) = 11$

3rd row :-

$3 = (-1)^{3+1} \begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix} = 1(9-0) = 9$

$-1 = (-1)^{3+2} \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} = -1(6-4) = -2$

$7 = (-1)^{3+3} \begin{bmatrix} 2 & 3 \\ -4 & 0 \end{bmatrix} = 1(0+12) = 12$

co factors matrix $= \begin{bmatrix} 3 & +37 & 4 \\ -20 & +17 & 11 \\ 9 & -2 & 12 \end{bmatrix}$

Adj A $= (\text{cofactor matrix})^T = \begin{bmatrix} 3 & -20 & 9 \\ +37 & +17 & -2 \\ 4 & 11 & 12 \end{bmatrix}$

Inverse of a square matrix:-

If for a square matrix A their exists another matrix B such that "AB = BA = I", then the matrix A is said to be Invertible and B is called the Inverse of and it is denoted by A^{-1} (A Inverse). That is B = A' (i.e.)

~~A inverse that~~ A inverse $\boxed{(A^{-1}) = \frac{\text{Adj } A}{|A|} \; ; \; |A| \neq 0}$

problems:-

1. find the inverse of the following matrices. $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

A) $|A| = 1\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + (-2)\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

$= 1(3-0) - 2(-1-0) - 2(2-0) = 1(3) + 2 - 4$

$= 3 + 2 - 4 = 5 - 4 = 1 \qquad 1 \neq 0.$

Adj A =

calculation of co factors:-

1st row:-

$1 = (-1)^{1+1}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} = 1(3-0) = 3$

$2 = (-1)^{1+2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1(-1-0) = 1$

$-2 = (-1)^{1+3}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} = 1(2-0) = 2$

2nd row:-

$-1 = (-1)^{2+1}\begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix} = -1(2-4) = 2$

$3 = (-1)^{2+2}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = 1(1-0) = 1$

$$0 = (1)^{2+3} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = -1(-2-0) = 2$$

3rd row:-

$$0 = (1)^{3+1} \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix} = 1(0+6) = 6$$

$$-2 = (1)^{3+2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = -1(0-2) = 2$$

$$1 = (1)^{3+3} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = 1(3+2) = 5$$

$$\text{cofactor matrix} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3/1 & 2/1 & 6/1 \\ 1/1 & 1/1 & 2/1 \\ 2/1 & 2/1 & 5/1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

(Q) $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ find the inverse

A) $|A| = 2 \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} + 2 \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} + 4 \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

$= 2(-3-2) + 2(-2+2) + 4(2+3)$

$= 2(-5) + 2(0) + 4(5)$

$= -10 + 0 + 20 = 10 \quad |A| = 10 \neq 0$

A is non-singular matrix.

Adj A:-

1st row:-

$2 = (-1)^2 \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = +(-3-2) = -5$

$-2 = (-1)^{1+2} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = -(-2+2) = -0$

$4 = (-1)^{1+3} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = +(2+3) = 5$

2nd row:-

$2 = (-1)^{2+1} \begin{bmatrix} -2 & 4 \\ 1 & -1 \end{bmatrix} = -(2-4) = 2$

$3 = (-1)^{2+2} \begin{bmatrix} 2 & 4 \\ -1 & -1 \end{bmatrix} = +(-2+4) = 2$

$2 = (-1)^{2+3} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = -(2-2) = -0$

$\begin{array}{r} 12 \\ +4 \\ \hline 16 \end{array}$

3rd row:-

$-1 = (-1)^{3+1} \begin{bmatrix} -2 & 4 \\ 3 & 2 \end{bmatrix} = +(-4-12) = -16$

$1 = (-1)^{3+2} \begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix} = -(4-8) = 4$

$-1 = (-1)^{3+3}\begin{pmatrix} 2 & -2 \\ 2 & 3 \end{pmatrix} = +(6+4) = \underline{10}$

co-factor matrix = $\begin{bmatrix} -5 & 0 & 5 \\ 2 & 2 & 0 \\ -16 & 4 & 10 \end{bmatrix}$

Adj A = $\begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix}$

$A^{-1} = \dfrac{\text{Adj } A}{|A|}$

$= \dfrac{1}{10}\begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix} = \begin{bmatrix} -\cancel{5}/\cancel{10}_2 & \cancel{2}/\cancel{10}_5 & -\cancel{16}/\cancel{10}_5 \\ 0 & \cancel{2}/\cancel{10}_5 & \cancel{4}/\cancel{10}_5 \\ \cancel{5}/\cancel{10}_2 & 0 & \cancel{10}/\cancel{10} \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1/2 & 1/5 & -8/5 \\ 0 & 1/5 & 2/5 \\ 1/2 & 0 & 1 \end{bmatrix}$

(3) find the inverse of $A = \begin{bmatrix} 0 & 8 & 2 \\ 3 & 9 & 5 \\ 6 & 0 & 6 \end{bmatrix}$

A) $|A| = 0\begin{bmatrix} 9 & 5 \\ 0 & 6 \end{bmatrix} - 8\begin{bmatrix} 3 & 5 \\ 6 & 6 \end{bmatrix} + 2\begin{pmatrix} 3 & 9 \\ 6 & 0 \end{pmatrix}$

$= 0 - 8(18-30) + 2(0-54)$

$= 0 - 8(-12) + 2(-54) = +96 \cancel{-108} = -12$

$108 - 96 = 12$

$|A| = -12$

$-12 \neq 0$.

Adj A :-

1st row :-

$0 = (-1)^{1+1} \begin{bmatrix} 9 & 5 \\ 0 & 6 \end{bmatrix} = +(54-0) = 54$

$8 = (-1)^{1+2} \begin{bmatrix} 3 & 5 \\ 6 & 6 \end{bmatrix} = -(18-30) = 12$

$9 = (-1)^{1+3} \begin{bmatrix} 3 & 9 \\ 6 & 0 \end{bmatrix} = +(0-54) = -54$

$\begin{bmatrix} 0 & 8 & 2 \\ 3 & 9 & 5 \\ 6 & 0 & 6 \end{bmatrix}$

2nd row :-

$3 = (-1)^{2+1} \begin{bmatrix} 8 & 2 \\ 0 & 6 \end{bmatrix} = -(48-0) = -48$

$9 = (-1)^{2+2} \begin{bmatrix} 0 & 2 \\ 6 & 6 \end{bmatrix} = +(0-12) = -12$

$5 = (-1)^{2+3} \begin{bmatrix} 0 & 8 \\ 6 & 0 \end{bmatrix} = -(0-48) = +48$

3rd row :-

$6 = (-1)^{3+1} \begin{bmatrix} 8 & 2 \\ 9 & 5 \end{bmatrix} = +(40-18) = +22$

$0 = (-1)^{3+2} \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = -(0-6) = +6$

$6 = (-1)^{3+3} \begin{bmatrix} 0 & 8 \\ 3 & 9 \end{bmatrix} = +(0-24) = -24$

$\text{co factors} = \begin{bmatrix} 54 & 12 & -54 \\ -48 & -12 & +48 \\ +22 & +6 & -24 \end{bmatrix}$

$\text{Adj } A = \begin{matrix} 54 & -48 & 22 \\ 12 & -12 & +48 \\ 22 & -6 & -24 \end{matrix}$

$$\text{Adj } A = \begin{bmatrix} 54 & -48 & 22 \\ 12 & -12 & 6 \\ -54 & 48 & -24 \end{bmatrix}$$

$12 \times 8 = 96$

$12 \times 3 = 36$

$$12\overline{)22}(1.833$$
$$\begin{array}{r} 12 \\ \hline 100 \\ 96 \\ \hline 40 \\ 36 \\ \hline 40 \end{array}$$

$$12\overline{)60}(5$$
$$\begin{array}{r} 60 \\ \hline 0 \end{array}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} \Rightarrow \frac{1}{-12} \begin{bmatrix} 54 & -48 & 22 \\ 12 & -12 & 6 \\ -54 & 48 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{54}{-12} & \frac{-48}{-12} & \frac{22}{-12} \\ \frac{12}{-12} & \frac{-12}{-12} & \frac{6}{-12} \\ \frac{-54}{-12} & \frac{48}{-12} & \frac{-24}{-12} \end{bmatrix} = \begin{bmatrix} -4.5 & 4 & -1.833 \\ -1 & 1 & -0.5 \\ 4.5 & -4 & 2 \end{bmatrix}$$

(4) $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 2 & 3 & 1 \end{bmatrix}$ find the inverse of A.

(A)

$$|A| = 1\begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix} - 0\begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = 1(4-15) + 2(9-8)$$

$$= 1(-11) + 2(1) = -11 + 2 = -9 \neq 0$$

$\therefore$ A is a non-singular matrix.

The co-factors of elements:-

co-factors of a element $= (-1)^{\text{row no} + \text{column no}} \times$ minor of the element

1st row

$$1 = (-1)^{1+1} \begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix} = +(4-15) = -11$$

$$0 = (-1)^{1+2} \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix} = -(3-10) = -(-7) = 7$$

$$\begin{bmatrix}1&0&2\\3&4&5\\2&3&1\end{bmatrix}$$

$$2 = (-1)^{1+3}\begin{bmatrix}3&4\\2&3\end{bmatrix} = +(9-8) = 1$$

2nd row:-

$$3 = (-1)^{2+1}\begin{bmatrix}0&2\\3&1\end{bmatrix} = -(0-6) = 6$$

$$4 = (-1)^{2+2}\begin{bmatrix}1&2\\2&1\end{bmatrix} = +(1-4) = -3$$

$$5 = (-1)^{2+3}\begin{bmatrix}1&0\\2&3\end{bmatrix} = -(3-0) = -3$$

3rd row:-

$$2 = (-1)^{3+1}\begin{bmatrix}0&2\\4&5\end{bmatrix} = +(0-8) = -8$$

$$3 = (-1)^{3+2}\begin{bmatrix}1&2\\3&5\end{bmatrix} = -(5-6) = 1$$

$$1 = (-1)^{3+3}\begin{bmatrix}1&0\\3&4\end{bmatrix} = +(4-0) = 4$$

co-factor matrix $= \begin{bmatrix}-11&7&1\\6&-3&-3\\-8&1&4\end{bmatrix}$

$$\text{Adj } A = \begin{bmatrix}-11&6&-8\\7&-3&1\\1&-3&4\end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\cdot \text{Adj } A$$

$$= \frac{1}{-9}\begin{bmatrix}-11&6&-8\\7&-3&1\\1&-3&4\end{bmatrix} = \boxed{\begin{bmatrix}\frac{-11}{-9}&\frac{\cancel{6}^{2}}{-\cancel{9}3}&\frac{-8}{-9}\\ \frac{7}{-9}&\frac{-\cancel{3}1}{-\cancel{9}3}&\frac{1}{-9}\\ \frac{1}{-9}&\frac{-\cancel{3}1}{-\cancel{9}3}&\frac{-4}{9}\end{bmatrix} = \begin{bmatrix}-11/9&2/3&\frac{-8}{-9}\\ \frac{7}{-9}&-\frac{1}{3}&\frac{1}{-9}\\ \frac{1}{-9}&1/3&-4/9\end{bmatrix}}$$

(5) $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ -1 & 0 & 2 \end{bmatrix}$ find the inverse of A

A) $|A| = 1\begin{bmatrix} 6 & 8 \\ 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 2 & 8 \\ -1 & 2 \end{bmatrix} + 4\begin{bmatrix} 2 & 6 \\ -1 & 0 \end{bmatrix}$

$= 1(12-0) - 3(4+8) + 4(0+6) = 12 - 36 + 16 = 36 - 36 = 0$

$|A| = 0 \Rightarrow A$ is sigular matrix

A^{-1} is not possible

Solutions of a linear equations :-

Matrix inversion methods:-

Consider a system of linear equations be

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

Above equation can be written in matrix form ie, $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{X = A^{-1}B}$$

Problems:-

1. Solve the following equations by matrix inversion method.

i) $3x+y+2z=3,\ 2x-3y-z=-3,\ x+2y+z=4$

Above equation can be written in matix form

i.e $AX=B$

$$\begin{bmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ -3\\ 4\end{bmatrix}$$ where,

$$A=\begin{bmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad B=\begin{bmatrix}3\\ -3\\ 4\end{bmatrix}$$

$$\boxed{x=A^{-1}B}\rightarrow (1)$$

To find A^{-1} :-

$$|A|=3\begin{bmatrix}-3 & -1\\ 2 & 1\end{bmatrix}-1\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}+2\begin{bmatrix}2 & -3\\ 1 & 2\end{bmatrix}=3(-3+2)-1(2+1)+2(4+3)$$

$$=3(-1)-1(3)+2(7)=-3-3+14=\cancel{-18+14=-4}-6+14=8$$

$8\neq 0$

Adj A :-

Calculating co-factors:-

1st row:-

$$3=(-1)^{1+1}\begin{bmatrix}-3 & -1\\ 2 & 1\end{bmatrix}=+(-3+2)=-1$$

$$1=(-1)^{1+2}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}=-(2+1)=-3$$

$$2=(-1)^{1+3}\begin{bmatrix}2 & -3\\ 1 & 2\end{bmatrix}=+(4+3)=7$$

2nd row:-

$$2 = (-1)^{2+1}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -(1-4) = +3$$

$$-3 = (-1)^{2+2}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = +(3-2) = 1$$

$$-1 = (-1)^{2+3}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = -(6-1) = -5$$

3rd row:-

$$1 = (-1)^{3+1}\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = +(-1+6) = 5$$

$$2 = (-1)^{3+2}\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} = -(-3-4) = 7$$

$$1 = (-1)^{3+3}\begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix} = +(-9-2) = -11$$

$$\text{co-factor matrix} = \begin{bmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 5 & 7 & -11 \end{bmatrix}, \quad \text{Adj } A = \begin{bmatrix} -1 & +3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} = \begin{bmatrix} \frac{-1}{8} & \frac{3}{8} & \frac{5}{8} \\ \frac{-3}{8} & \frac{1}{8} & \frac{7}{8} \\ 7/8 & \frac{-5}{8} & \frac{-11}{8} \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} \Rightarrow \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

2. $x+y-2z=3$
$2x-y+z=0$
$3x+y-z=8$

A) Above equation can be written in matrix form i.e
$AX=B$

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}, \text{ where } A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \boxed{X = A^{-1}B} \rightarrow (1)$$

to find A^{-1}:-

$$|A| = 1(1-1) - 1(-2-3) - 2(2+3) = 1(0) - 1(-5) - 2(5) = +5 - 10$$
$= -5 \neq 0$ A is non singular matrix

Adj A:-
1st row:-

$$1 = (-1)^{1+1}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = +(1-1) = 0$$

$$1 = (-1)^{1+2}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = -(-2-3) = +5$$

$$-2 = (-1)^{1+3}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = +(2+3) = 5$$

2nd row:-

$$2 = (-1)^{2+1}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = -(-1+2) = -1$$

$$-1 = (-1)^{2+2}\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} = +(-1+6) = 5$$

$$1 = (-1)^{2+3}\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = -(1-3) = 2$$

3rd row :-

$3 = (-1)^{3+1}\begin{bmatrix}1 & -2\\ -1 & 1\end{bmatrix} = +(1-2) = -1$

$1 = (-1)^{3+2}\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix} = -(1+4) = -5$

$-1 = (-1)^{3+3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix} = +(1-2) = -3$

co factors matrix :-

$$\begin{bmatrix}0 & 5 & 5\\ -1 & 5 & 2\\ -1 & -5 & -3\end{bmatrix} \Rightarrow \text{Adj}A = \begin{bmatrix}0 & -1 & -1\\ 5 & 5 & -5\\ 5 & 2 & -3\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\cdot \text{Adj} \Rightarrow \frac{1}{-5}\begin{bmatrix}0 & -1 & -1\\ 5 & 5 & -5\\ 5 & 2 & -3\end{bmatrix}$$

-40
$+15$
25

24
15
9

$X = A^{-1}B$

$$X = \frac{1}{-5}\begin{bmatrix}0 & -1 & -1\\ 5 & 5 & -5\\ 5 & 2 & -3\end{bmatrix}\begin{bmatrix}3\\ 0\\ 8\end{bmatrix} = \begin{bmatrix}0-0-8\\ 15+0-40\\ 15+0-24\end{bmatrix} = \begin{bmatrix}-8\\ -25\\ -9\end{bmatrix}$$

$$x = \frac{1}{-5}\begin{bmatrix}-8\\ -25\\ -9\end{bmatrix} = \begin{bmatrix}-8/-5\\ -25/-5\\ -9/-5\end{bmatrix} = \begin{bmatrix}1.6\\ 5\\ 1.8\end{bmatrix}$$

Check :-

$= 3x + y - z$

$= 3(1.6) + 5 - 1.8$

$= 4.8 + 5 - 1.8$

$= 3 + 5 = \underline{\underline{8}}$

1.6×3
4.8

4.8
1.8
3.0

(3). $x+y+2z=4$

$2x-y+3z=9$

$3x-y-z=2$

A) The following equation can be written as matrix form i.e $AX=B$

$$A=\begin{bmatrix}1 & 1 & 2\\ 2 & -1 & 3\\ 3 & -1 & -1\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad B=\begin{bmatrix}4\\ 9\\ 2\end{bmatrix}$$

$$\boxed{x=A^{-1}B}\rightarrow (1)$$

To find A^{-1}:-

$|A|=1(1+3)-1(-2-9)+2(-2+3)=1(4)-1(-11)+2(1)$

$=4+11+2=\cancel{17}\neq 0$ ∴ A is non-singular matrix

Adj A:-

1st row:-

$1=(-1)^{1+1}\begin{bmatrix}-1 & 3\\ -1 & -1\end{bmatrix}=+(1+3)=4$

$1=(-1)^{1+2}\begin{bmatrix}2 & 3\\ 3 & -1\end{bmatrix}=-(-2-9)=11$

$2=(-1)^{1+3}\begin{bmatrix}2 & -1\\ 3 & -1\end{bmatrix}=+(-2+3)=1$

2nd row:-

$2=(-1)^{2+1}\begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix}=-(-1+2)=-1$

$-1=(-1)^{\cancel{3}2+2}\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}=+(-1-6)=-7$

$3 = (-1)^{2+3} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = -(-1-3) = 4$

3rd row:-

$3 = (-1)^{3+1} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = +(3+2) = 5$

$-1 = (-1)^{3+2} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = -(3-4) = 1$

$-1 = (-1)^{3+3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = +(-1-2) = -3$

co factors matrix = $\begin{bmatrix} 4 & 11 & 1 \\ -1 & -7 & 4 \\ 5 & 1 & -3 \end{bmatrix}$, Adj A = $\begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 4 \\ 1 & 4 & -3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj$

$= \frac{1}{17} \begin{bmatrix} 4 & -1 & 1 \\ 11 & -7 & 4 \\ 5 & 1 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{A1} Adj$

$= \frac{1}{17} \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$,

$X = A^{-1} B$

$= \frac{1}{17} \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 16-9+10 \\ 44-63+2 \\ 20+9-6 \end{bmatrix}$

$= \frac{1}{17} \begin{bmatrix} 17 \\ -17 \\ 34 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Check :- $x + y + 2(z) = 1 - 1 + 2(2) = \underline{\underline{4}}$

③ $2x - y + 3z = 9$
$x + y + z = 6$
$x - y + z = 2$.

Solve the following equations in Inversion method.

A) Above equation can be written as matrix form i.e $AX = B$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\boxed{X = A^{-1}B} \rightarrow ①$

To find A^{-1}

$|A| = 2(1+1) + 1(1-1) + 3(-1-1)$
$= 2(2) + 1(0) + 3(-2) = 4 + 0 - 6 = -2 \neq 0$ A is non singular matrix.

Adj A:-

calculating cofactors:-

$2 = (-1)^{1+1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = +(1+1) = 2$

$-1 = (-1)^{1+2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = -(1-1) = 0$

$3 = (-1)^{1+3} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = +(-1-1) = -2$

2nd row:-

$1 = (-1)^{2+1} \begin{bmatrix} -1 & 3 \\ -1 & 1 \end{bmatrix} = -(-1+3) = -2$

$1 = (-1)^{2+2} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = +(2-3) = -1$

$1 = (-1)^{2+3} \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} = -(-2+1) = 1$

3rd row:-

$1 = (-1)^{3+1} \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} = +(-1-3) = -4$

$-1 = (-1)^{3+2} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = -(2-3) = 1$

$1 = (-1)^{3+3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = +(2+1) = 3$

$\frac{1}{\cancel{2}} \times \cancel{6}^{3} = 3$

$\frac{-1}{\cancel{2}} \times \cancel{2}$

$\frac{-3}{\cancel{2}} \times \cancel{2}$

co factor matrix $= \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$, $\text{Adj}\, A = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot \text{Adj}$

$A^{-1} = \frac{1}{-2} \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{-2} & \frac{-2}{-2} & \frac{-4}{-2} \\ \frac{0}{-2} & \frac{-1}{-2} & \frac{1}{-2} \\ \frac{-2}{-2} & \frac{1}{-2} & \frac{3}{-2} \end{bmatrix} = \begin{bmatrix} -1 & 1 & +2 \\ 0 & 1/2 & -1/2 \\ 1 & 1/2 & -3/2 \end{bmatrix}$

$X = A^{-1} B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & \frac{-1}{2} \\ 1 & 1/2 & -3/2 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -9+6+4 \\ 0+3-1 \\ 9-3-3 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

check :- $x - y + z = 2$

$= 1 - 2 + 3$

$= -1 + 3$

$= 2$

Cramer's Rule :- (or) Determinant method :-

Consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Above equation can be written in matrix form i.e, $AX=B$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$|A| = \Delta = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} ; \Delta \neq 0$$

$$\Delta_1 = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix} \qquad x = \frac{\Delta_1}{\Delta}$$

$$\Delta_2 = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix} \qquad y = \frac{\Delta_2}{\Delta}$$

$$\Delta_3 = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} \qquad z = \frac{\Delta_3}{\Delta}$$

Problems :-

1. Solve the following system of equations using Cramer's rule.

$x + 2y - x = -3,\ 3x + y + z = 4,\ x - y + 2z = 6$

A) Above equation can be written in matrix form i.e $AX = B$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Delta = |A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1(2+1) - 2(3-2) + 1(-3-1)$$
$$= 1(3) - 2(1) + 1(-4) = 3 - 2 - 4$$
$$= -3 \neq 0$$ A is non-singular matrix

$$\Delta_1 = \begin{vmatrix} -3 & 2 & -1 \\ 4 & 1 & 1 \\ 6 & -1 & 2 \end{vmatrix} \Rightarrow -3(2+1) - 2(8-6) - 1(-4-6)$$
$$= -3(3) - 2(2) - 1(-10) = -9 - 4 + 10 = -13 + 10$$
$$= -3$$

$$\begin{array}{r} 18 \\ 4 \\ \hline 14 \end{array} \qquad \begin{array}{r} 28 \\ 22 \\ \hline 16 \end{array}$$

$$\Delta_2 = \begin{vmatrix} 1 & -3 & -1 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{vmatrix} \Rightarrow 1(8-6) + 3(6-1) - 1(18-4)$$
$$= 1(2) + 3(5) - 1(14) = 2 + 15 - 14$$
$$= 3$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 1 & -1 & 6 \end{vmatrix} \Rightarrow 1(6+4) - 2(18-4) + 3(-3-1)$$
$$= 10 - 28 + 12 = -6$$

By cramer's rule, $x = \frac{\Delta_1}{\Delta} = \frac{-3}{-3} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{3}{-3} = -1$

$z = \frac{\Delta_3}{\Delta} = \frac{-6}{-3} = 2$, $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Check :-

$x+2y-z = 1+2(-1)-2 = 1-2-2 = 1-4 = -3$

(2). $3x+y+2z = 3$

$2x-3y-z = -3$

$x+2y+z = 4$

A) Above equation can be written in matrix form, $AX=B$

$$\begin{bmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}3\\ -3\\ 4\end{bmatrix},\ \text{where } A = \begin{bmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix}, B = \begin{bmatrix}3\\ -3\\ 4\end{bmatrix}; X = \begin{bmatrix}3\\ -3\\ 4\end{bmatrix}$$

$$\Delta = |A| = \begin{vmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{vmatrix} = 3(-3+2)-1(2+1)+2(4+3)$$

$= 3(-1)-1(3)+2(7) = -3-3+14 = -6+14$

$= 8 \neq 0$ A is non-singular matrix.

$$\Delta_1 = \begin{vmatrix}3 & 1 & 2\\ -3 & -3 & -1\\ 4 & 2 & 1\end{vmatrix} = 3(-3+2)-1(-3+4)+2(-6+12)$$

$= 3(-1)-1(1)+2(6) = -3-1+12$

$= -4+12 = 8$

$$\Delta_2 = \begin{vmatrix}3 & 3 & 2\\ 2 & -3 & -1\\ 1 & 4 & 1\end{vmatrix} = 3(-3+4)-3(2+1)+2(8+3)$$

$= 3(1)-3(3)+2(11) = 3-9+22$

$= 16.$

$$\Delta_3 = \begin{vmatrix}3 & 1 & 3\\ 2 & -3 & -3\\ 1 & 2 & 4\end{vmatrix} = 3(-12+6)-1(8+3)+3(4+3)$$

$= 3(-6)-1(11)+3(7) = \cancel{-3-3+14 = -6+14}$

$= -18-11+21 = -29+21 = -8$

By cramer's rule $x = \frac{\Delta_1}{\Delta} = \frac{8}{8} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{16}{8} = 2$,

$z = \frac{-8}{8} = -1$

$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Check :-

$3x + y + 2z = 3(1) + 2 + 2(-1) = 3 + 2 - 2 = 5 - 2$

$= \underline{\underline{3}}$

(3) $x - 2y - z = -1$

$x - y - z = 0$

$2x + 3y + 2z = 5$.

(A) Above equation can be written in matrix form $AX = B$:

$\begin{bmatrix} 1 & -2 & -1 \\ 1 & -1 & -1 \\ 2 & 3 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$, $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\Delta = |A| = \begin{bmatrix} 1 & -2 & -1 \\ 1 & -1 & -1 \\ 2 & 3 & 2 \end{bmatrix} = 1(-2+3) + 2(2+2) - 1(3+2) = 1(1) + 2(4) - 1(5)$

$= 1 + 8 - 5 = 4 \neq 0$ A is non singular matrix.

$\Delta_1 = \begin{bmatrix} -1 & -2 & -1 \\ 0 & -1 & -1 \\ 5 & 3 & 2 \end{bmatrix} = -1(-2+3) + 2(0+5) - 1(0+5) = -1(1) + 2(5) - 1(5)$

$= -1 + 10 - 5 = 4$

$\Delta_2 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 2 & 5 & 2 \end{bmatrix} = 1(0+5) + 1(2+2) - 1(5-0)$

$= 1(+5) + 1(4) - 1(5)$

$= +5 + 4 - 5 = 4$

$\Delta_3 = \begin{bmatrix} 1 & -2 & -1 \\ 1 & -1 & 0 \\ 2 & 3 & 5 \end{bmatrix} = 1(-5-0) + 2(5-0) - 1(3+2)$

$= 1(-5) + 2(5) - 1(5) = -5 + 10 - 5 = 0$.

By cramer's rule $x = \frac{\Delta_1}{\Delta} = \frac{4}{4} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{4}{4} = 1$

$z = \frac{\Delta_3}{\Delta} = \frac{0}{4} = 0.$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

verification:-

$x - 2y - z = 1 - 2(1) - 0 = 1 - 2 - 0$
$= \underline{\underline{-1}}$

(4) $3x + y + 2z = 3$
$2x - 3y - z = -3$
$x + 2y + z = 4.$

A) The Above equation can be written as Matrix form i.e $AX = B$.

where $A = \begin{bmatrix} 3 & +1 & +2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

$\Delta = |A| = 3 \begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} \; 3(-3+2) - 1(2+1) + 2(4+3)$

$= 3(-1) - 1(3) + 2(7) = -3 - 3 + 14 = +8 \neq 0$

A is nonsingular matrix.

$\Delta_1 = \begin{bmatrix} 3 & 1 & +2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{bmatrix} = 3(-3+2) - 1(-3+4) + 2(-6+12)$
$= 3(-1) - 1(1) + 2(6) = -3 - 1 + 12$
$= -4 + 12 = 8$

$\Delta_2 = \begin{bmatrix} 3 & +3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{bmatrix} = 3(-3+4) - 3(2+1) + 2(8+3)$
$= 3(1) - 3(3) + 2(11)$
$= 3 - 9 + 22 = 16$

$$\Delta_3 = \begin{bmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{bmatrix} = 3(-12+6) - 1(8+3) + 3(4+3)$$
$$= 3(-6) - 1(11) + 3(7)$$
$$= -18 - 11 + 21 = -8$$

$$\begin{array}{r} -29 \\ +21 \\ \hline 8 \end{array} \quad \begin{array}{r} 11 \\ 6 \\ \hline -17 \end{array} \quad \begin{array}{r} 18 \\ 11 \\ \hline 29 \end{array} \quad \begin{array}{r} 21 \\ 17 \\ \hline 4 \end{array}$$

By cramer's rule $x = \frac{\Delta_1}{\Delta} = \frac{\cancel{8}}{\cancel{8}} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{\cancel{16}^{2}}{\cancel{8}} = 2$

$z = \frac{-\cancel{8}}{\cancel{8}} = -1$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

<u>verification</u> :-

$x + 2y + z = 1 + 2(2) + (-1)$
$= 1 + 4 - 1 = \underline{4}$

(5) $x + y + z = 8$
$3x + 5y - 7z = 14$
$x - y + 2z = 6.$

A) The equation can be written as matrix form i.e $AX = B$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & -7 \\ 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 14 \\ 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\Delta = |A| = 1(10-7) - 1(6+7) + 1(-3-5) = 1(3) - 1(13) - (8)$
$= 3 - 13 - 8 = 3 - 21 = -18 \neq 0$ A is non singular matrix

$$\Delta_1 = \begin{bmatrix} 8 & 1 & 1 \\ 14 & 5 & -7 \\ 6 & -1 & 2 \end{bmatrix} = 8(10-7) - 1(28+42) + 1(-14-30)$$
$$= 8(3) - 1(70) + 1(-44) = 24 - 70 - 44$$
$$= -90.$$

$$\Delta_2 = \begin{bmatrix} 1 & 8 & 1 \\ 3 & 14 & -7 \\ 1 & 6 & 2 \end{bmatrix} = 1(28+42) - 8(6+7) + 1(18-14)$$
$$= 1(70) - 8(13) + 1(4) = 70 - 104 + 4$$
$$= -30.$$

$$\Delta_3 = \begin{bmatrix} 1 & 1 & 8 \\ 3 & 5 & 14 \\ 1 & -1 & 6 \end{bmatrix} = 1(30+14) - 1(18-14) + 8(-3-5)$$

$$= 1(44) - 1(4) + 8(-8)$$

$$= 44 - 4 - 64 = -24$$

By cramer's rule $x = \frac{\Delta_1}{\Delta} = \frac{-90}{-18} = 5$, $y = \frac{\Delta_2}{\Delta} = \frac{-30}{-18} = \frac{5}{3}$

$z = \frac{-24}{-18} = \frac{4}{3}$

verification :-

$x + y + z = 5 + \frac{5}{3} + \frac{4}{3} = \frac{15+5+4}{3}$

$= \frac{\cancel{24}^{8}}{\cancel{3}} = \underline{\underline{8}}$

Rank of a Matrix :-

If A is a Square matrix, the rank of the matrix is said to be r

1. 3, if A is non singular matrix
2. If 2, A is a singular and atleast one of its 2X2 sub matrices is non singular.
3. If 1, If every 2X2 submatrices is singular.

The Rank of capital A is denoted by $\rho(A)$ or there are 4 types to find the Rank.

1. Determinant method
2. Echelon ~~method~~ form
3. PAQ form
4. Normal form.

1. **Determinant method:-**

① Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

A) Given matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}_{3\times 3}$

$$|A| = 1(-1-2) - 1(1-6) + 1(1+3)$$
$$= 1(-3) - 1(-5) + 1(4)$$
$$= -3+5+4 = 6 \neq 0$$

A is a non-singular matrix, so $\rho(A) = r = \underline{\underline{3}}$.

② Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 2 & 4 & 6 \end{bmatrix}$

A) Det of $|A| = 1(36-36) - 2(18-18) + 3(12-12)$
$$= 1(0) - 2(0) + 3(0) = \underline{\underline{0 = 0}}$$

A is a singular matrix $|A|_{3\times3} = 0$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = (6-6) = 0$$

$$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} = (18-18) = 0$$

All sub-matrices are singular

$\therefore \rho(A) = \underline{\underline{1}}$

③ $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}$ find Rank..

A) $|A| = 1(-35+14) - 1(-14+2) + 1(14-5)$
$$= 1(-21) - 1(-12) + 1(9) = -21+12+9 = -21+21 = 0$$

A is a singular matrix.

$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} = 5-2 = 3 \neq 0$

The sub matrix of order 2×2 is non-singular so

The Rank of $A = 2$

4. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ find the rank.

A)

$|A| = 1(24-25) - 2(18-20) + 3(15-16)$

$= 1(-1) - 2(-2) + 3(-1)$

$= -1+4-3 = 0$

A is singular matrix $|A| = 0$.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4-6 = -2 \neq 0.$

The sub matrix of order 2×2 is non singular

so $\rho(A) = 2$

5. $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$ find the rank

$33 \times 6 = 198$

$198 - 17 = 181$

$|A| = -1(18-1) - 0(9+5) + 6(3+30)$

$= -1(17) - 0(14) + 6(33)$

$= -17 + 198 = 181 \neq 0.$

so rank of $A = 3$

Echelon form:-

A matrix is said to be in ~~echlon~~ form if it has the following properties

1. Zero rows, if any ~~are~~ below any non zero row.

2. The no. of zeroes before the first non-zero element in a row is less than the no. of such zeroes in the next row

3. The Rank of a matrix = no. of non zero rows

Problems:-

① find the rank of the following matrices by using Echelon form.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

A) Given matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

R_2:-
$2 - 2(1)$
$-3 - 2(1)$ $\quad 4 - 2$
$4 - 2(1)$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in Echelon form

Rank of A $\rho(A)$ = number of non-zeros rows $= \underline{\underline{2}}$

(2) $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

A) given matrix $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix} \begin{matrix} - R1 \\ - R2 \\ - R3 \end{matrix}$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - 5R_1$

$= \begin{bmatrix} -1 & 0 & 6 \\ 0 & 6 & 19 \\ 0 & 1 & -27 \end{bmatrix}$

$R_3 \to 6R_3 - R_2$

$= \begin{bmatrix} -1 & 0 & 6 \\ 0 & 6 & 19 \\ 0 & 0 & -181 \end{bmatrix}$

$1 - 5(6)$

which is in Echelon form.

$\rho(A) = \underline{\underline{3}}$

(3) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$-2R_2$ $-3R_2$ 0

$-2R_3$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -3 & -6 \end{bmatrix}$

$3 - 3(1)$
$= 0$

$4 - 3(2)$
$4 - 6$

$5 - 3(3)$
$5 - 9$

$4 - 4(1)$

$5 - 4(2)$
$5 - 8$

$5 - 4(4)$
$5 - 16$

$6 - 4(3)$
$6 - 12$

$$= \begin{matrix} 1 & \cancel{0} & 3 \\ 0 & -2 & -4 \\ 0 & & \end{matrix}$$

$R_3 \to \cancel{0} - 4R_3 - 4$

4.

$$= 2(-3) - 3(-2)$$
$$-6 + 6$$

$R_3 \to 2R_3 - 3R_2$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 2(-6) - 3(-4)$$
$$= -12 + 12$$
$$= 0.$$

= which is in Echelon form $\rho(A) = \underline{\underline{2}}$

f(A) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$ $R_2 \div$ $-1+1$

$1-1(1)$ | $3-3(1)$
$-1-1(1)$ | $1-3(1)$
$2-1(1)$ | $2-3(1)$
$2-1$

$R_2 \to R_2 - 1(R_1)$
$R_3 \to R_3 - \underline{3(R_1)}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$-2 + -2$

$R_3 \to R_3 \mp R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

= which is in echelon form $\rho(A) = \underline{\underline{3}}$

5) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}$

$= R_2 \to R_2 - 2(R_1)$

$R_3 \to R_3 - R_1$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 6 & -8 \end{bmatrix}$

$R_3 \to 6R_2 - R_3(3)$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$2 - 1^{\times 2} = 0$

$2 - 2(1)$
$2 - 2$ $\quad -2 - 2(1)$

$5 - 2(1)$ $\quad 1 - 1$
$5 - 2$

$7 - 1$
$-7 - 1$

$6 \times 3 - 6 \times 3$

$6R_2 - R_3(3)$

$6(3) - 6(3)$
$18 - 18$

$6(-4) + 8(3)$
$24 - 24$

$6(-4) - 0(-8)(3)$
$-24 - (+24)$
$-24 + 24$
$= 0$

which is in Echelon form $\rho(A) = 2$

Determinant :-

$|A| = 1(-35+14) - 1(-14+2) + 1(14-5)$

$= 1(-21) - 1(-12) + 1(9)$

$= -21 + 12 + 9 = -21 + 21 = 0.$ $\quad |A| = 0.$

$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} = 5 - 2 = 3 \neq 0.$

-35
$+14$
21

14
2
12
19
21