Course: Mathematical Foundation for Electrical Stream- 2 **(23MA2BSMES)**

Mathematical Foundation for Computer Science Stream- 2 **(23MA2BSMCS)**

Unit 1: INTEGRAL CALCULUS

Fubini's Theorem (First form)

If $f(x,y)$ is continuous on the rectangular region $R: a \le x \le b, c \le y \le d$, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Fubini's Theorem (Second form)

Let $f(x,y)$ be the continuous function on the rectangular region R.

I. If R is defined by $a \le x \le b, g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then $\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$

II. If R is defined by $c \le y \le d, h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then $\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

The area of a closed, bounded region R in polar coordinate plane is $A = \iint_R r dr d\theta$

The volume of a closed, bounded region D in space is $V = \iiint_D dV$

DOUBLE INTEGRALS

I. Evaluate the following:

1. $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$. $\quad Ans: \frac{\pi^2}{4}$.
2. $\int_1^2 \int_3^4 (xy + e^y)dydx$. $\quad Ans: \frac{21+4(e^4-e^3)}{4}$.
3. $\int_3^4 \int_1^2 \frac{dydx}{(x+y)^2}$. $\quad Ans: \log\left(\frac{25}{24}\right)$.
4. $\int_1^2 \int_0^x \frac{dydx}{x^2+y^2}$. $\quad Ans: \frac{\pi}{4}\log(2)$.
5. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$ $\quad Ans: \frac{\pi}{4}\log(1+\sqrt{2})$.
6. $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)dydx$. $\quad Ans: \frac{3}{35}$.
7. $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}dxdy$ $\quad Ans: 8\ln(8)-16+e$.

II. Evaluate the following over the specified region:

1. $\iint_R xydxdy$, where R is the region bounded by the circle $x^2+y^2=a^2$ in the first quadrant.

   $Ans: \frac{a^4}{8}$

2. $\iint_A xy(x+y)dA$, where A is the area bounded by the parabola $y=x^2$ and the line $y=x$.

   $Ans: \frac{3}{56}$

3. $\iint_R xydxdy$, where R is the domain bounded by x-axis, ordinate $x=2a$ and the curve $x^2=4ay$. $Ans: \frac{a^4}{3}$

4. $\iint_D x^2dxdy$, where D is the domain in the first quadrant bounded by the hyperbola $xy=16$, and the lines $y=x, y=0$ and $x=8$. $Ans: 448$

III. Change the order of integration and hence evaluate the following:

1. $\int_0^a\int_y^a \frac{xdxdy}{x^2+y^2}$ $Ans: \frac{\pi a}{4}$.
2. $\int_0^a\int_{\sqrt{ax}}^a \frac{y^2dydx}{\sqrt{y^4-a^2x^2}}$ $Ans: \frac{\pi a^2}{6}$.
3. $\int_0^1\int_x^1 \frac{1}{1+y^4}dydx$ $Ans: \frac{\pi}{8}$
4. $\int_0^1\int_x^{\sqrt{x}} xydydx$ $Ans: \frac{1}{24}$.
5. $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dydx$ $Ans: 1$
6. $\int_0^\infty\int_0^x xe^{-x^2/y}dydx$ $Ans: \frac{1}{2}$.
7. $\int_0^3\int_0^{\sqrt{4-y}}(x+y)dxdy$ $Ans: 601/60$.
8. $\int_0^1\int_{e^x}^e \frac{dydx}{\log y}$ $Ans: e-1$.
9. $\int_0^a\int_{x/a}^{\sqrt{x/a}}(x^2+y^2)dxdy$ $Ans: \frac{a^3}{28}+\frac{a}{20}$.
10. $\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}dydx$ $Ans: \frac{\pi a^3}{6}$.
11. $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}}dydx$ $Ans: \frac{16}{3}a^2$.
12. $\int_0^1\int_{x^2}^{2-x} xydxdy$ $Ans: \frac{3}{8}$.
13. $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}dydx$ $Ans: 1-1/\sqrt{2}$.

IV. Evaluate the following by transforming into polar coordinates:

1. $\int_0^a\int_0^{\sqrt{a^2-x^2}}(x^2+y^2)dydx$ $Ans: \frac{\pi a^4}{8}$.

2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$ $Ans: \frac{\pi}{4}$.

3. $\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dydx$ $Ans: 9\pi$.

4. $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2} dxdy$ $Ans: \frac{\pi}{2}$(1-log 2).

5. $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log_e (x^2+y^2) dxdy$, where $(a > 0)$ $Ans: \frac{\pi a^2}{8}(2\log a - 1)$

6. $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}} dxdy$ $Ans: \frac{a^3}{3}\log(\sqrt{2}+1)$

V. Area of the region using double integral

1. Find the area lying inside the cardioid $r = a(1+\cos\ \theta)$ and outside the circle $r = a$.
$Ans: a^2(2+\pi/4)$

2. Find the area of Lemniscate $r^2 = a^2\cos\ 2\theta$. $Ans: a^2$

3. Find the area of the cardioid$r = a(1+\cos\ \theta)$. $Ans: \frac{3\pi a^2}{2}$

4. Find the area which is inside the circle $r = 3a\cos\ \theta$ and outside the cardioid
$r = a(1+\cos\ \theta)$ $Ans: \pi a^2$

5. Find the area lying inside the circle $r = a\sin\ \theta$ and outside the cardioid $r = a(1-\cos\ \theta)$.
$Ans: a^2(1-\frac{\pi}{4})$

TRIPLE INTEGRALS

VI. Evaluate the following:

1. $\int_0^1 \int_0^2 \int_1^2 x^2 yz dxdydz$ $Ans: \frac{7}{3}$

2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2+z^2) dzdydx$ $Ans: \frac{8}{3}abc(a^2+b^2+c^2)$.

3. $\int_0^2 \int_1^z \int_0^{yz} xyz dxdydz$ $Ans: \frac{7}{2}$

4. $\int_0^6 \int_0^{6-x} \int_1^{6-x-z} dydzdx$ $Ans: 18$

5. $\int_0^4 \int_0^{\frac{1}{2}} \int_0^{x^2} \frac{1}{\sqrt{x^2-y^2}} dydxdz$ $Ans: 0.5113$.

6. $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} xdxdydz$. $Ans: \frac{\pi a^4}{16}$

7. $\int_0^1 \int_0^x \int_0^{x+y} (x+y+z) dzdydx$ $Ans: \frac{7}{8}$.

8. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dzdydx$ $Ans: \frac{1}{48}$

9. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$ $Ans: \frac{\pi^2}{8}$

10. $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dydxdz$ $Ans: 0$

11. $\int_{0}^{1}\int_{0}^{1}\int_{0}^{2-x^2-y^2} xye^{z}dzdxdy$ $Ans: \frac{1}{4}(e-2).$

12. $\int_{0}^{4}\int_{0}^{2\sqrt{z}}\int_{0}^{\sqrt{4z-x^2}} dydxdz$ $Ans: 8\pi$

VII. Evaluate the following over the region R bounded by the planes

$x=0, y=0, z=0$ and $x+y+z=1$.

1. $\iiint_R (x+y+z)dxdydz.$ $Ans: 1/8$
2. $\iiint_R xyzdxdydz$ $Ans: \frac{1}{720}.$
3. $\iiint_R \frac{dxdydz}{(1+x+y+z)^3}$ $Ans: \frac{1}{2}(\log 2-\frac{5}{8})$

VIII. Volume of the region using triple integral

1. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the co-ordinate planes. $Ans: \frac{abc}{6}$
2. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ $Ans: \frac{4}{3}\pi abc$
3. Find the volume of the sphere $x^2+y^2+z^2=a^2.$ $Ans: \frac{4\pi a^3}{3}$

BETA AND GAMMA FUNCTIONS

Beta function is defined as

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx, \quad n>0.$$

Gamma function is defined as

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m,n>0.$$

Properties:

1. $\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}dx, \quad n>0.$
2. $\Gamma(n+1)=n\Gamma(n), n\in\mathbb{R}$
3. $\Gamma(n+1)=n!, n\notin\mathbb{Z}^-$
4. $\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin(n\pi)}, \quad 0<n<1.$
5. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$
6. $\beta(m,n)=\beta(n,m)$
7. $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
8. $\beta(m,n)=2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}dx, \quad m,n>0$

IX. Express the following in terms of Gamma functions:

1. $\int_0^\infty e^{-kx}x^{p-1}dx, k>0$ $Ans: \frac{\Gamma(p)}{k^p}.$

2. $\int_0^\infty \frac{x^4}{4^x} dx.$ $Ans: \frac{24}{(\log 4)^5}.$

3. $\int_0^\infty 3^{-4x^2} dx.$ $Ans: \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

4. $\int_0^1 x^m [\log_e x]^n dx$, where n is an integer and $m > -1$. $Ans: (-1)^n n!/(m+1)^{n+1}$.

5. $\int_0^1 x^{q-1} \left[\log_e \left(\frac{1}{x}\right)\right]^{p-1} dx \ (p > 0, q > 0).$ $Ans: \frac{\Gamma(p)}{q^p}$

6. $\int_0^\infty e^{-t^2} t^{2n-1} dt.$ $Ans: \frac{1}{2}\Gamma(n)$

X. Express the following in terms of Gamma functions:

1. $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta.$ $Ans: \frac{\pi}{\sqrt{2}}$

2. $\int_0^{\pi/2} (\sqrt{\tan\theta} + \sqrt{\sec\theta}) d\theta.$ $Ans: \frac{\pi}{\sqrt{2}} + \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\Gamma(3/4)}$

3. $\int_0^1 x^m (1 - x^n)^p dx.$ $Ans: \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$

4. $\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$ $Ans: \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{3}{4}\right)}$

XI. Prove the following:

1. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, given $\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$ and hence deduce the value of $\int_0^\infty \frac{dy}{1+y^4}$.

2. $\int_0^\infty x e^{-x^8} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}.$

3. $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$

4. $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$

5. $\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n} = \frac{\beta(m,n+1)}{n}.$