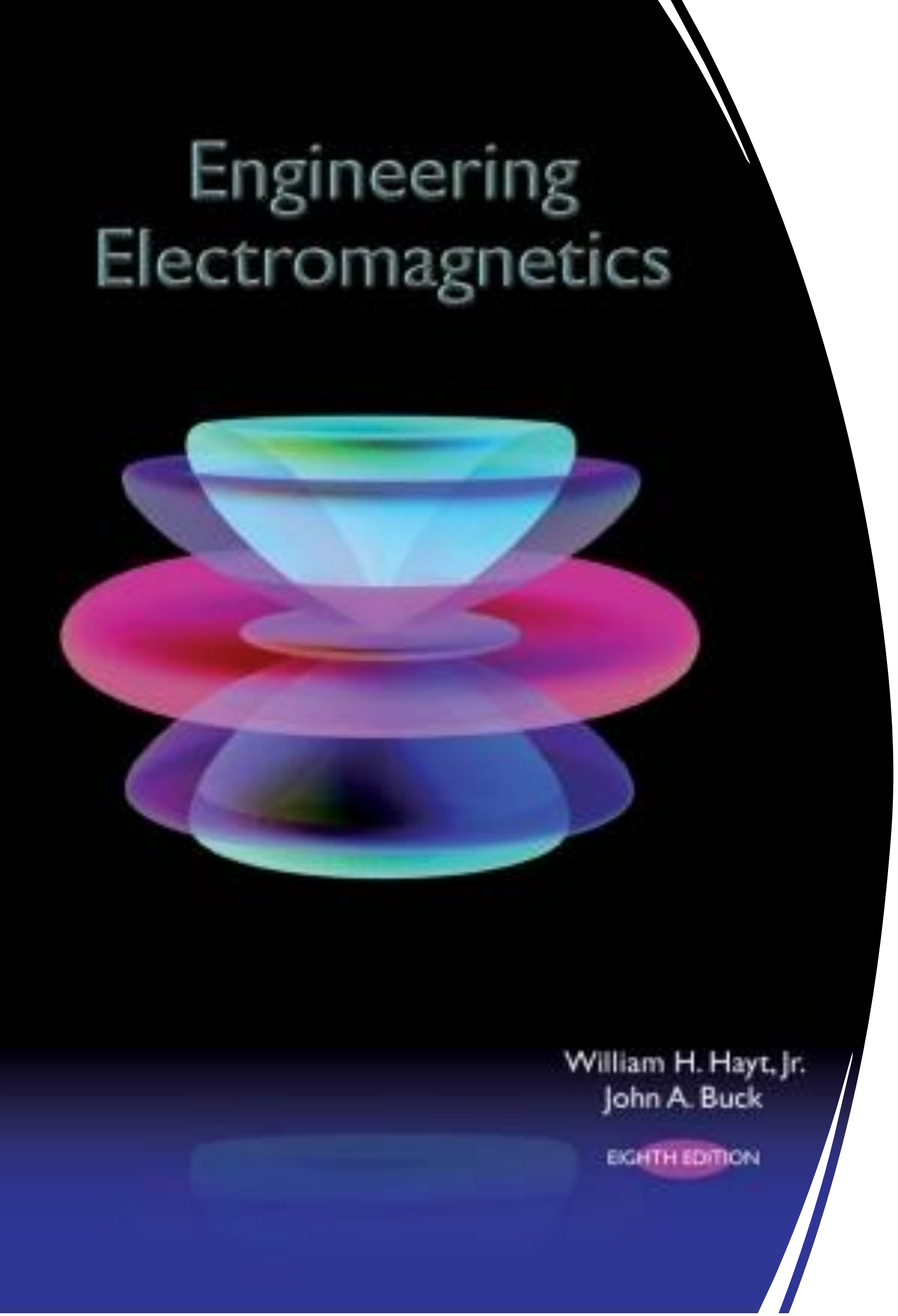


4. Energy and Potential

4.1 Energy Expended in Moving a Point Charge in an Electric Field

4.2 The Line Integral

4.3 Definition of Potential Difference and Potential

4.4 The Potential Field of a System of Charges: Conservative Property

4.5 Potential Gradient

4.1 Energy Expended in Moving a Point Charge in an Electric Field

- To move a charge Q a distance dL in an electric field E. The force on Q arising from the electric field is

$$\mathbf{F}_E = Q\mathbf{E}$$

- where the subscript reminds us that this force arises from the field. The component of this force in the direction $d\mathbf{L}$ which we must overcome is

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

where $\mathbf{a}_L$ = a unit vector in the direction of $d\mathbf{L}$.

- The force that we must apply is equal and opposite to the force associated with the field,

$$F_{\text{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

4.1 Energy Expended in Moving a Point Charge in an Electric Field

- The expenditure of energy is the product of the force and distance. That is, the differential work done by an external source moving charge Q is

$$dW = -Q\mathbf{E} \cdot d\mathbf{L}$$

- The work required to move the charge a finite distance must be determined by integrating,

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

4.1 Energy Expended in Moving a Point Charge in an Electric Field

Given the electric field $\mathbf{E} = \frac{1}{z^2}\left(8xyz\mathbf{a}_x + 4x^2z\mathbf{a}_y - 4x^2y\mathbf{a}_z\right)$V/m, find the differential amount of work done in moving a $6 - \mathrm{nC}$ charge a distance of 2μm, starting at $P(2, -2{,}3)$ and proceeding in the direction $\mathbf{a}_L = (a) -\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z$; (b) $\frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z$; (c) $\frac{3}{7}\mathbf{a}_x + \frac{6}{7}\mathbf{a}_y$.

Ans. −149.3fJ; 149.3fJ; 0

4.2 The Line Integral

- The work involved in moving a charge Q from B to A is then approximately

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

or, using vector notation,

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

- The result for the uniform field can be obtained from the integral expression

$$W = -Q \int_B^A \mathbf{E} \cdot d\mathbf{L}$$

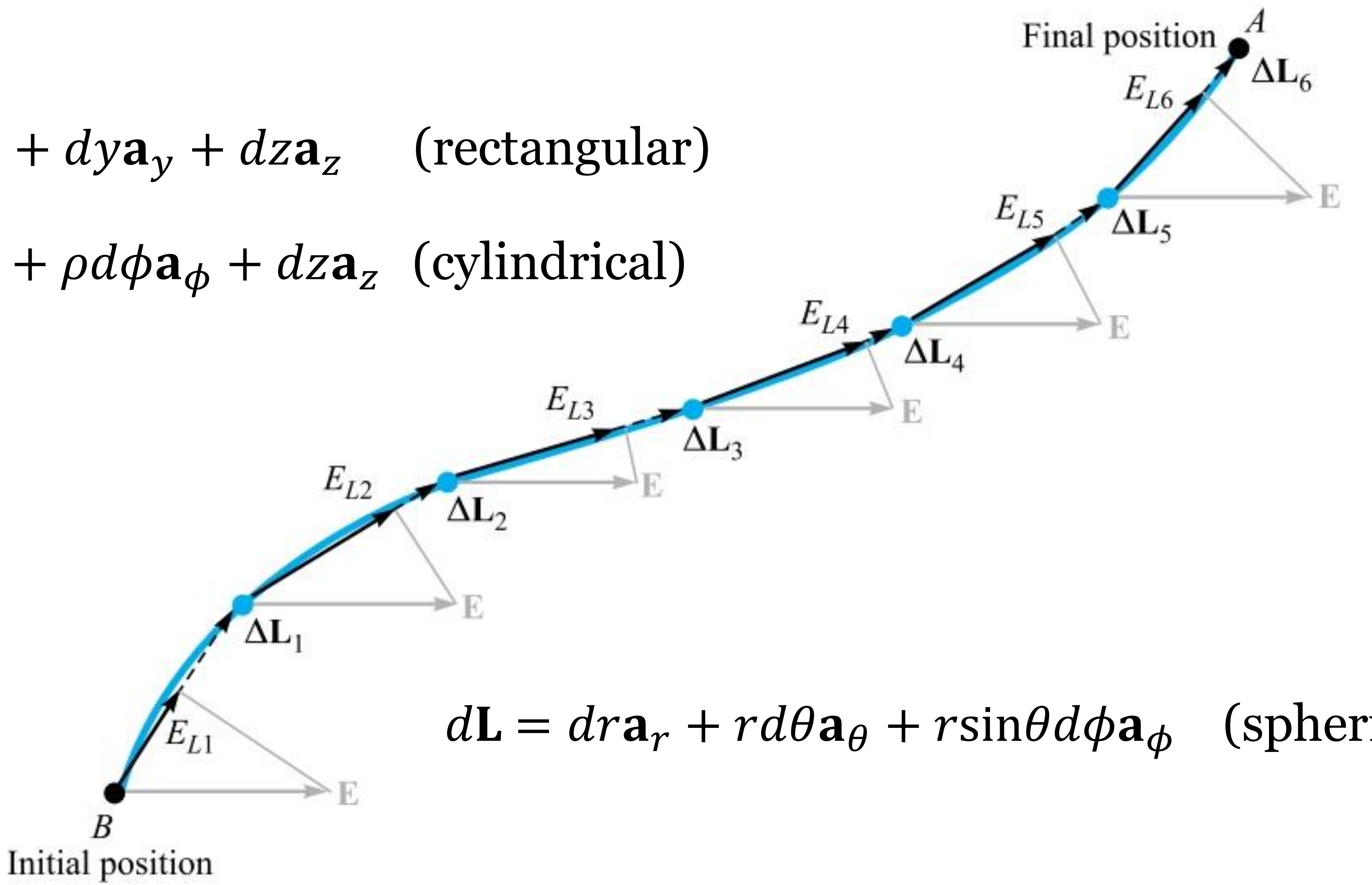
as applied to a uniform field

$$W = -Q\mathbf{E} \cdot \int_B^A d\mathbf{L}$$

4.2 The Line Integral

$$d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z \quad \text{(rectangular)}$$

$$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z \quad \text{(cylindrical)}$$



$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi \quad \text{(spherical)}$$

4.2 The Line Integral

We are given the nonuniform field

$$\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$$

and we are asked to determine the work expended in carrying 2C from $B(1,0,1)$ to $A(0.8,0.6,1)$ along the shorter arc of the circle

$$x^2 + y^2 = 1 \qquad z = 1$$

Ans. -0.96 J

4.2 The Line Integral

Calculate the work done in moving a 4-C charge from $B(1,0,0)$ to $A(0,2,0)$ along the path $y = 2 - 2x, z = 0$ in the field $\mathbf{E} =$ (a) $5\mathbf{a}_x$ V/m; (b) $5x\mathbf{a}_x$ V/m; (c) $5x\mathbf{a}_x + 5y\mathbf{a}_y$ V/m.

Ans. 20 J; 10 J; -30 J

4.3 Definition of Potential Difference and Potential

- We are now ready to define a new concept from the expression for the work done by an external source in moving a charge Q from one point to another in an electric field **E**, "Potential difference and work."

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

- In much the same way as we defined the electric field intensity as the force on a unit test charge, we now define potential difference V as the work done (by an external source) in moving a unit positive charge from one point to another in an electric field,

$$\text{Potential difference } = V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

4.3 Definition of Potential Difference and Potential

- Potential difference is measured in joules per coulomb, for which the volt is defined as a more common unit, abbreviated as V. Hence the potential difference between points A and B is

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L}\mathbf{V}$$

and V_{AB} is positive if work is done in carrying the positive charge from B to A.

- If the potential at point A is V_A and that at B is V_B, then

$$V_{AB} = V_A - V_B$$

4.3 Definition of Potential Difference and Potential

An electric field is expressed in rectangular coordinates by $\mathbf{E} = 6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z$ V/m. Find: (a) V_{MN} if points M and N are specified by $M(2,6,-1)$ and $N(-3,-3,2)$; (b) V_M if $V = 0$ at $Q(4,-2,-35)$; (c) V_N if $V = 2$ at $P(1,2,-4)$.

Ans. -139.0 V; -120.0 V; 19.0 V

4.4 The Potential Field of a System of Charges: Conservative Property

- In other words, the expression for potential (zero reference at infinity),

$$V_A = -\int_{\infty}^{A} \mathbf{E} \cdot d\mathbf{L}$$

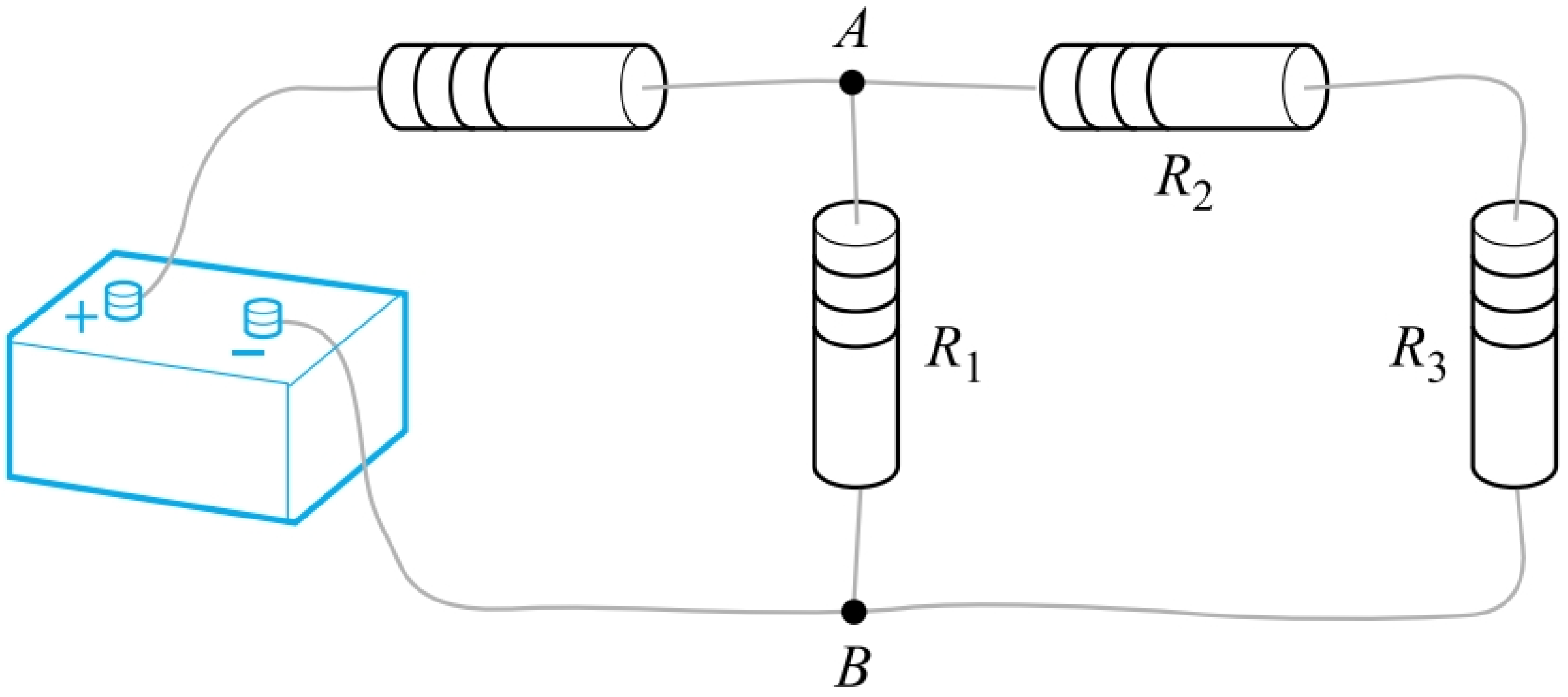
or potential difference,

$$V_{AB} = V_A - V_B = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}$$

- This result is often stated concisely by recognizing that no work is done in carrying the unit charge around any closed path, or

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

4.4 The Potential Field of a System of Charges: Conservative Property



- A simple dc-circuit problem that must be solved by applying $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ in the form of Kirchhoff's voltage law.

4.5 Potential Gradient

- We already have the general line-integral relationship between these quantities,

$$V = -\int \mathbf{E} \cdot d\mathbf{L}$$

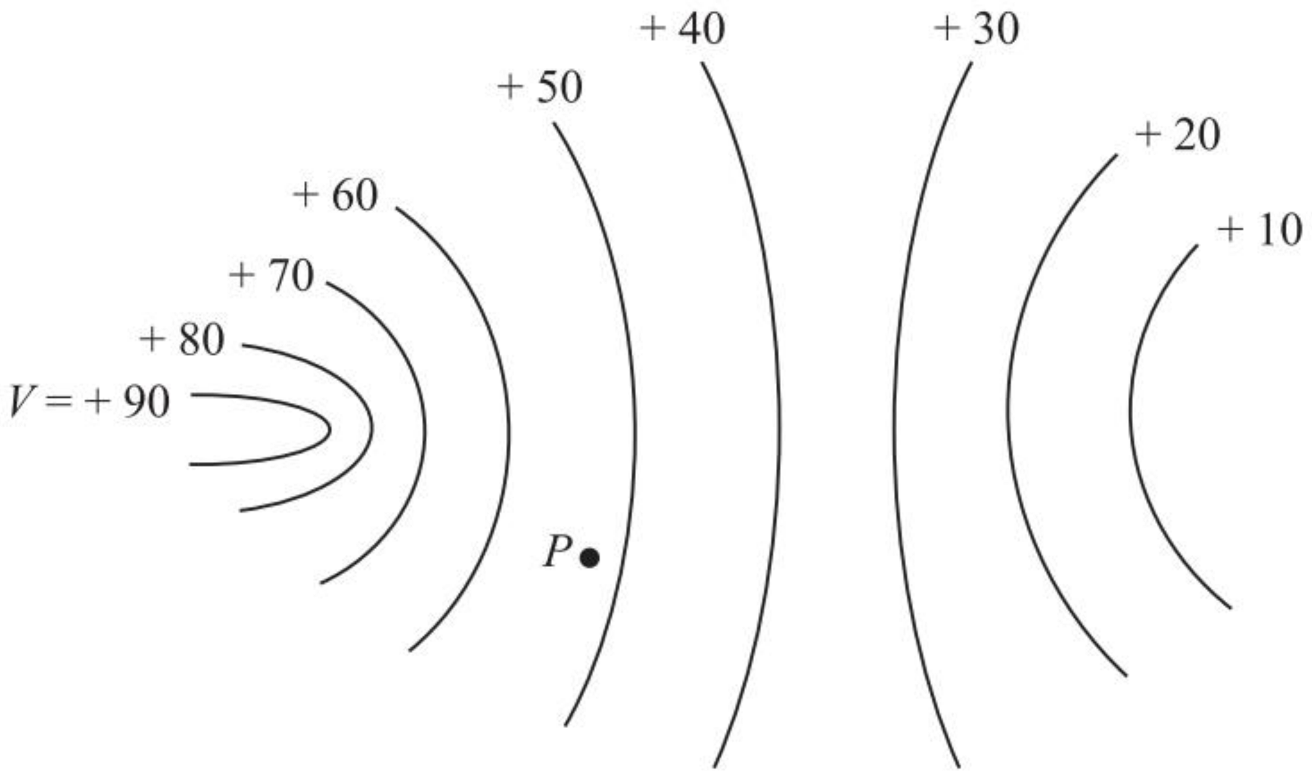
- This little exercise shows us two characteristics of the relationship between **E** and V at any point:

  1) The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.

  2)This maximum value is obtained when the direction of the distance increment is opposite to **E** or, in other words, the direction of **E** is opposite to the direction in which the potential is increasing the most rapidly.

4.5 Potential Gradient

- A potential field is shown by its equipotential surfaces. At any point the E field is normal to the equipotential surface passing through that point and is directed toward the more negative surfaces.



4.5 Potential Gradient

- We now may write the relationship between V and $\mathbf{E}$ as

$$\mathbf{E} = -\text{grad}\, V$$

$$\text{grad}\, V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

- The vector operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

$$\mathbf{E} = -\nabla V$$

4.5 Potential Gradient

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4,3,6)$, we wish to find several numerical values at point P : the potential V, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density ρ_v.

Ans. $66\ \mathrm{V}; 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z\ \mathrm{V/m}; 57.9\ \mathrm{V/m};$
$0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z\ \mathrm{V/m};$
$-35.4xy\mathbf{a}_x - 17.71x^2\mathbf{a}_y + 44.3\mathbf{a}_z\mathrm{p\,C/m^3};$
$-35.4y\mathrm{p\,C/m^3}; -106.2\mathrm{pC/m^3}$

Problems

1. Given the electric field $\mathbf{E} = (y+1)\mathbf{a}_x + (x-1)\mathbf{a}_y + 2\mathbf{a}_z$ find the potential difference between the points $(a)(2,-2,-1)$ and $(0,0,0)$; $(b)(3,2,-1)$ and $(-2,-3,4)$.
2. Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$V in free space. Evaluate each of the following quantities at $P(3,2,-1)$
$(a)V$; $(b)|V|$; $(c)\mathbf{E}$; $(d)|\mathbf{E}|$; $(e)\mathbf{a}_N$; (f) D.