วศ 5012206

วิศวกรรมแม่เหล็กไฟฟ้า

EN 5012206 Electromagnetic Engineering

วิศวกรรมการสื่อสารและสารสนเทศ

คณะเทคโนโลยีอุตสาหกรรม มหาวิทยาลัยราชภัฏเทพสตรี

วศ 5012206 วิศวกรรมแม่เหล็กไฟฟ้า 3(3-0-6)

EN 5012206 Electromagnetic Engineering

วิชาที่ต้องศึกษาก่อน : ไม่มี

การวิเคราะห์เวกเตอร์สนามไฟฟ้าสถิต กฎของคูลอมบ์ กฎของเกาส์ กฎของฟาราเดย์และกฎของแอมแปร์ ตัวนำและไดอิเล็กตริก ตัวเก็บประจุและความจุไฟฟ้า กระแสการพาและกระแสนำ ลาปลาซและปัวซอง ความต้านทาน สนามแม่เหล็กสถิต วัสดุแม่เหล็ก ตัวเหนี่ยวนำ ความเหนี่ยวนำและความเหนี่ยวนำร่วม สนามแม่เหล็กไฟฟ้าเปลี่ยนตามเวลา สมการของแมกซ์เวลล์ คลื่นระนาบ ทฤษฎีพอยท์ติ้ง โพลาไรเซชัน คลื่นแม่เหล็กไฟฟ้าในตัวกลางและเงื่อนไขขอบเขต เฟสแมตชิง แรงงานและกำลังงาน พลศาสตร์ไฟฟ้า การแพร่กระจายคลื่นและฟังก์ชันการถ่ายโอน การประยุกต์ใช้งานคลื่นแม่เหล็กไฟฟ้า

ระหว่างภาค 60 คะแนน

สอบเก็บคะแนนครั้งที่ 1 (บทที่ 1) [1]	10 คะแนน
สอบกลางภาค (บทที่ 2, 3, 4) [2]	30 คะแนน
สอบเก็บคะแนนครั้งที่ 2 (บทที่ 5) [3]	10 คะแนน
งานที่ได้รับมอบหมาย	10 คะแนน

ปลายภาค 40 คะแนน

ต่ำกว่า 50 คะแนน -> ไม่ผ่าน (F)

[1] สอบสัปดาห์ที่ 2

[2] สอบสัปดาห์ที่ 8

[3] สอบสัปดาห์ที่ 11

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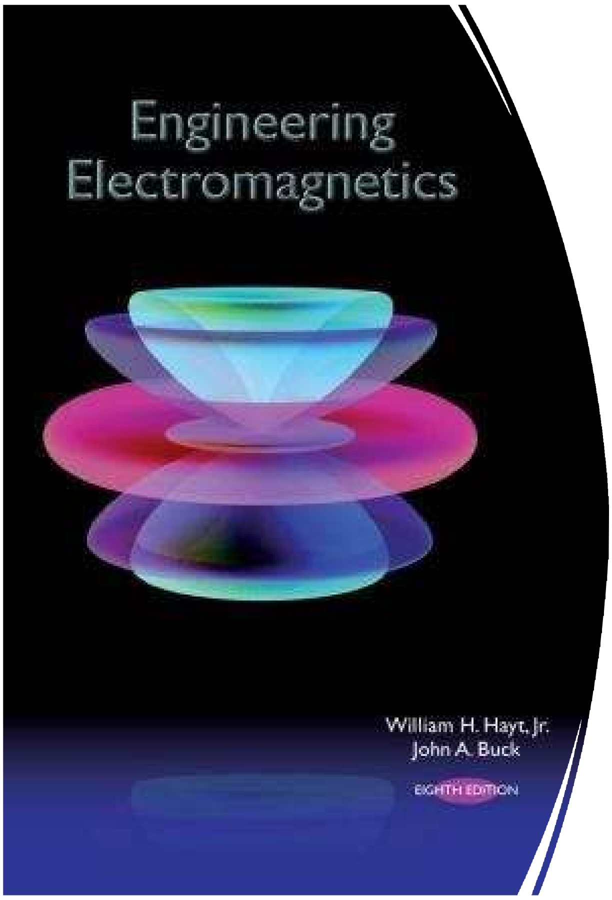


1. Vector Analysis

1.1 Scalars and Vectors
1.2 Vector Algebra
1.3 The Rectangular Coordinate System
1.4 Vector Components and Unit Vectors
1.5 The Vector Field
1.6 The Dot Product
1.7 The Cross Product
1.8 The Circular Cylindrical Coordinates
1.9 The Spherical Coordinate

1.1 Scalars and Vectors

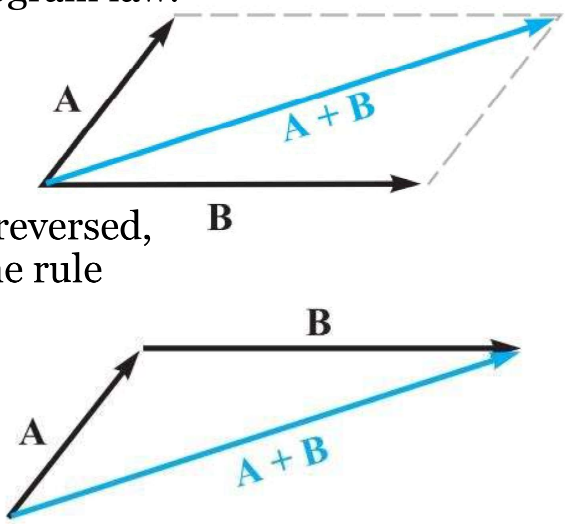
- The term scalar refers to a quantity whose value may be represented by a single (positive or negative) real number.
- If we speak of a body falling a distance L in a time t, or the temperature T at any point in a bowl of soup whose coordinates are x, y, and z, then L, t, T, x, y, and z are all scalars.
- Other scalar quantities are mass, density, pressure (but not force), volume, volume resistivity, and voltage.
- A vector quantity has both a magnitude and a direction in space.
- Force, velocity, acceleration, and a straight line from the positive to the negative terminal of a storage battery are examples of vectors.
- Each quantity is characterized by both a magnitude and a direction.

1.2 Vector Algebra

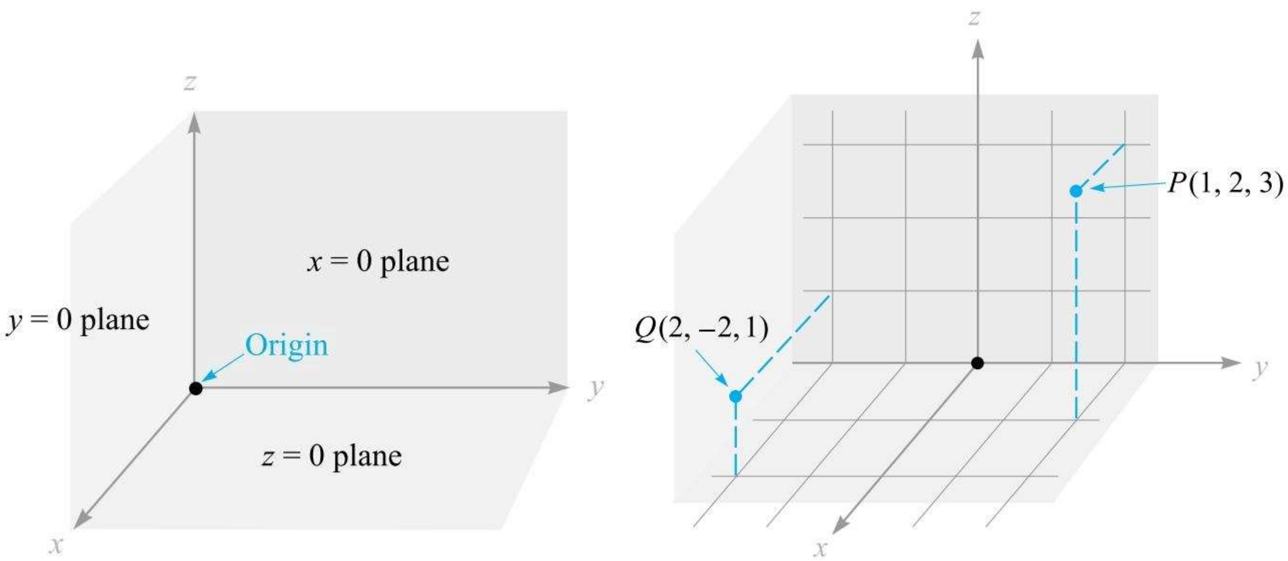
- The addition of vectors follows the parallelogram law.
- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
- $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition

- $\mathbf{A} = \mathbf{B}$ if $\mathbf{A} - \mathbf{B} = 0$



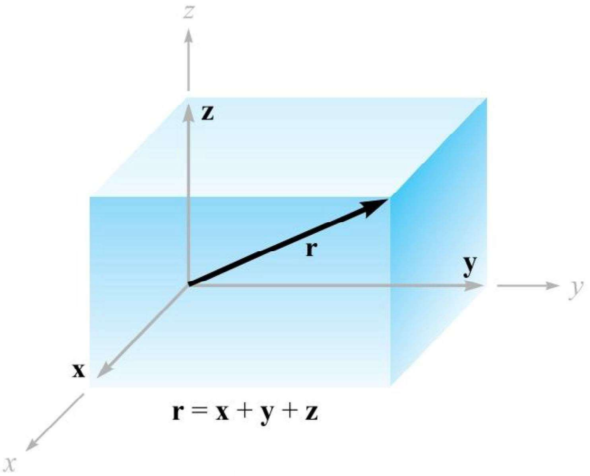
1.3 The Rectangular Coordinate System



(a) (b)

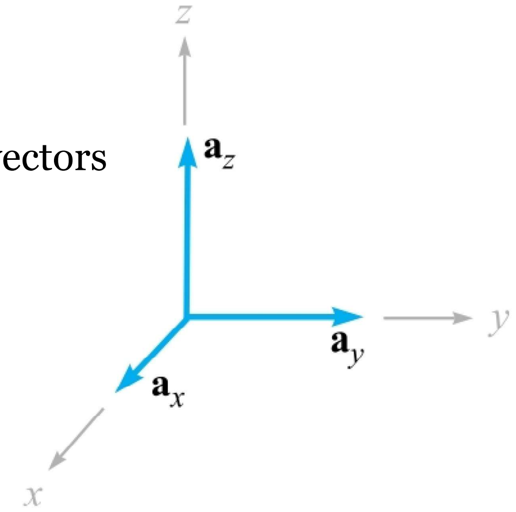
1.4 Vector Components and Unit Vectors

- If the component vectors of the vector **r** are **x**, **y**, and **z**, then **r** = **x** + **y** + **z**.
- The component vectors have magnitudes that depend on the given vector (such as r), but they each have a known and constant direction.

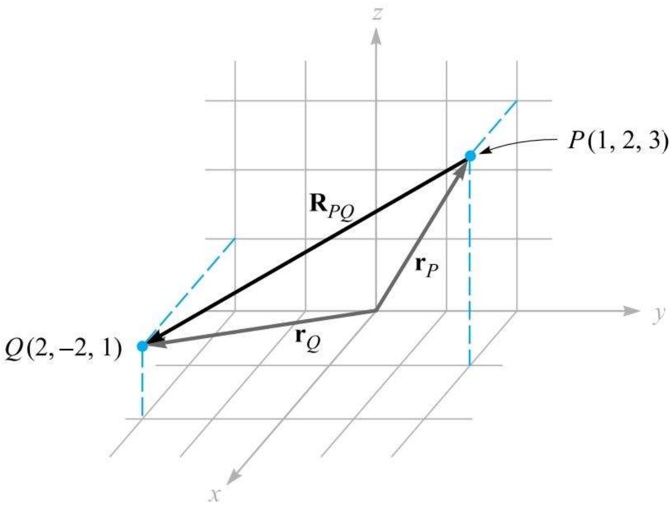


1.4 Vector Components and Unit Vectors

- This suggests the use of unit vectors having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values.
- We reserve the symbol **a** for a unit vector and identify its direction by an appropriate subscript. Thus $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the unit vectors in the rectangular coordinate system.



1.4 Vector Components and Unit Vectors



- A vector $\mathbf{r}_P$ pointing from the origin to point $P(1,2,3)$ is written $\mathbf{r}_P = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$.
- The vector from P to Q may be obtained by applying the rule of vector addition.
- The desired vector from $P(1,2,3)$ to $Q(2,-2,1)$ is therefore

$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

1.4 Vector Components and Unit Vectors

- Any vector $\mathbf{B}$ then may be described by $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The magnitude of $\mathbf{B}$ written $|\mathbf{B}|$ or simply B, is given by

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

- and a unit vector in the direction of the vector $\mathbf{B}$ is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

1.4 Vector Components and Unit Vectors

Given points $M(-1,2,1), N(3,-3,0)$, and $P(-2,-3,-4)$, find:
(a) $\mathbf{R}_{MN}$; (b) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (c) $|\mathbf{r}_M|$; (d) $\mathbf{a}_{MP}$; (e) $|2\mathbf{r}_P - 3\mathbf{r}_N|$.

Ans. $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$; $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$; 2.45; $-0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z$; 15.56

1.5 The Vector Field

- We have defined a vector field as a vector function of a position vector.
- In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question.
- We may write the velocity vector as $\mathbf{v} = v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z$, or $\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$; each of the components v_x, v_y, and v_z may be a function of the three variables x, y, and z.
- Further simplifying assumptions might be made if the velocity falls off with depth and changes very slowly as we move north, south, east, or west. A suitable expression could be $\mathbf{v} = 2e^{z/100}\mathbf{a}_x$. We have a velocity of 2 m/s at the surface and a velocity of 0.368×2, or 0.736 m/s, at a depth of 100 m$(z = -100)$. The velocity continues to decrease with depth, while maintaining a constant direction.

1.6 The Dot Product

- Given two vectors **A** and **B**, the dot product, or scalar product, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle between them. The expression **A** · **B** is read "A dot B."

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

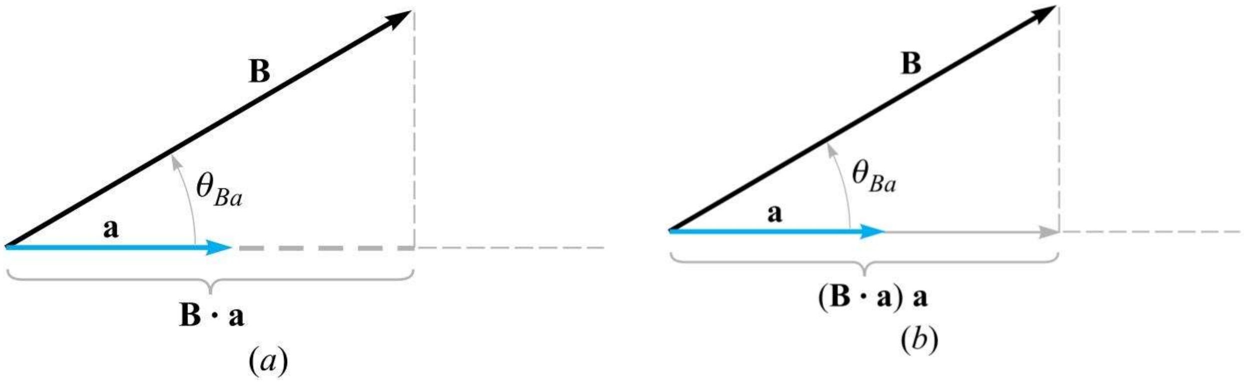
$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$$

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta_{AB}$$

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

1.6 The Dot Product

- One of the most important applications of the dot product is that of finding the component of a vector in a given direction.
- The geometrical term projection is also used with the dot product. Thus, **B** · **a** is the projection of **B** in the **a** direction.



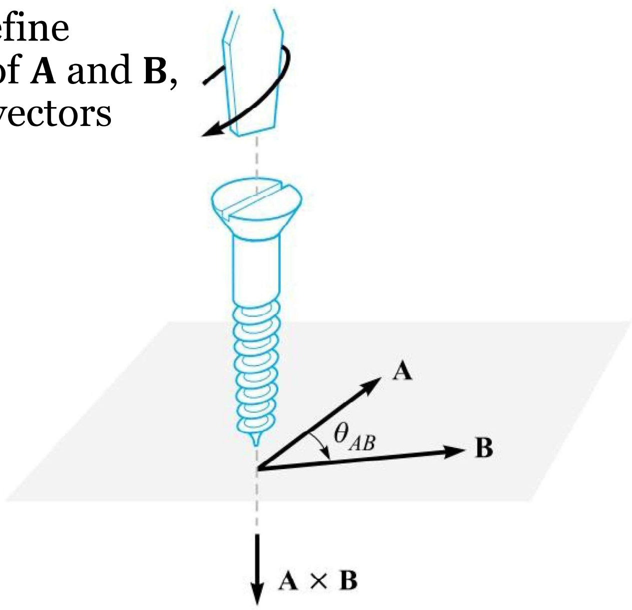
1.6 The Dot Product

The three vertices of a triangle are located at $A(6,-1,2), B(-2,3,-4)$, and $C(-3,1,5)$. Find: $(a)\mathbf{R}_{AB}$; $(b)\mathbf{R}_{AC}$; (c) the angle θ_{BAC} at vertex A; (d) the (vector) projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$.

Ans. $-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z; -9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z; 53.6°;$
$-5.94\mathbf{a}_x + 1.319\mathbf{a}_y + 1.979\mathbf{a}_z$

1.7 The Cross Product

- Given two vectors **A** and **B**, we now define the cross product, or vector product, of **A** and **B**, written with a cross between the two vectors as $\mathbf{A} \times \mathbf{B}$ and read "A cross **B**.“



$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}||\mathbf{B}| \sin\theta_{AB}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

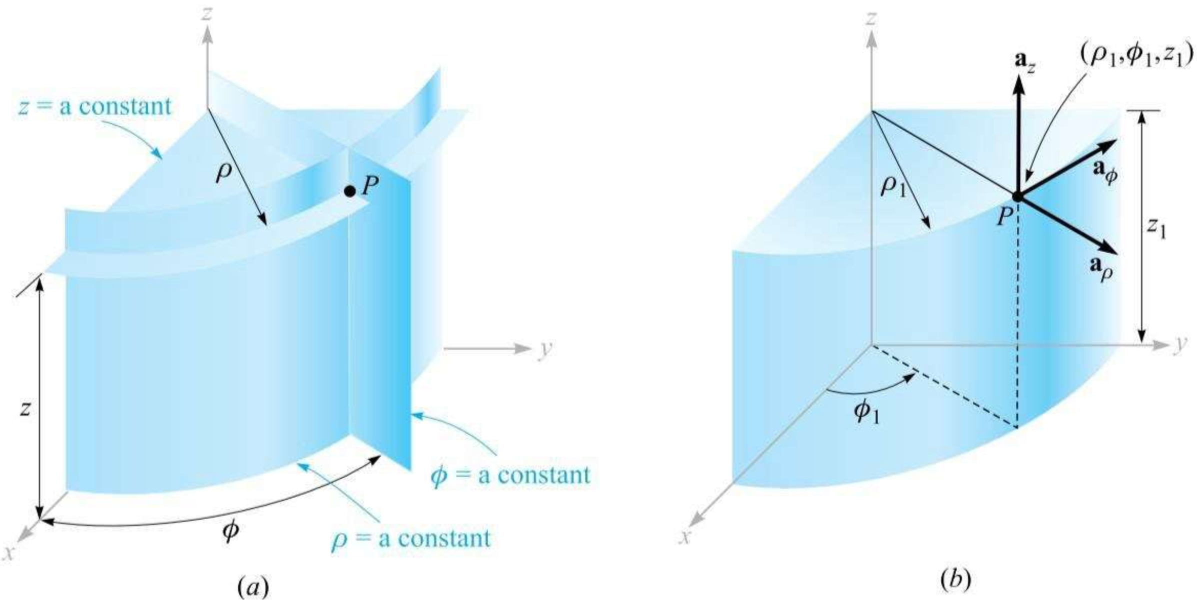
$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z$$

1.7 The Cross Product

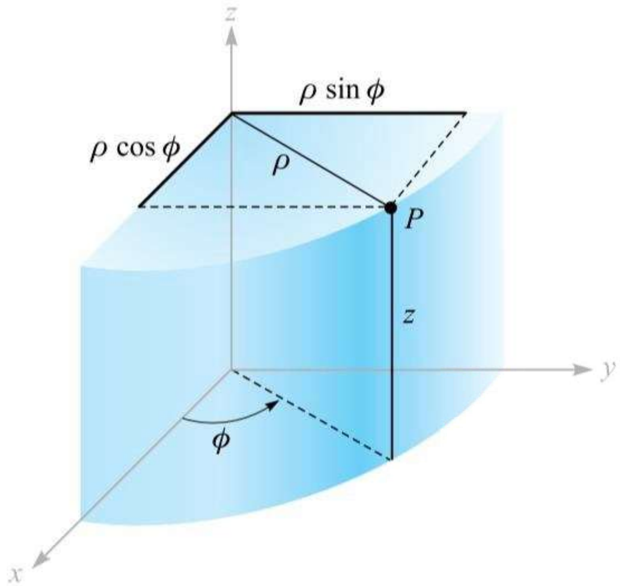
The three vertices of a triangle are located at $A(6,-1,2)$, $B(-2,3,-4)$, and $C(-3,1,5)$. Find: $(a)\mathbf{R}_{AB} \times \mathbf{R}_{AC}$; (b) the area of the triangle; (c) a unit vector perpendicular to the plane in which the triangle is located.

Ans. $24\mathbf{a}_x + 78\mathbf{a}_y + 20\mathbf{a}_z$; 42.0; $0.286\mathbf{a}_x + 0.928\mathbf{a}_y + 0.238\mathbf{a}_z$

1.8 The Circular Cylindrical Coordinates



1.8 The Circular Cylindrical Coordinates



$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$
$$\phi = \tan^{-1} \frac{y}{x}$$
$$z = z$$

1.8 The Circular Cylindrical Coordinates

$$\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \qquad \mathbf{A} = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$$

$$A_\rho = \mathbf{A} \cdot \mathbf{a}_\rho \qquad A_\rho = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_\rho = A_x \mathbf{a}_x \cdot \mathbf{a}_\rho + A_y \mathbf{a}_y \cdot \mathbf{a}_\rho$$
$$A_\phi = \mathbf{A} \cdot \mathbf{a}_\phi \qquad A_\phi = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_\phi = A_x \mathbf{a}_x \cdot \mathbf{a}_\phi + A_y \mathbf{a}_y \cdot \mathbf{a}_\phi$$
$$A_z = \mathbf{A} \cdot \mathbf{a}_z \qquad A_z = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_z = A_z \mathbf{a}_z \cdot \mathbf{a}_z = A_z$$

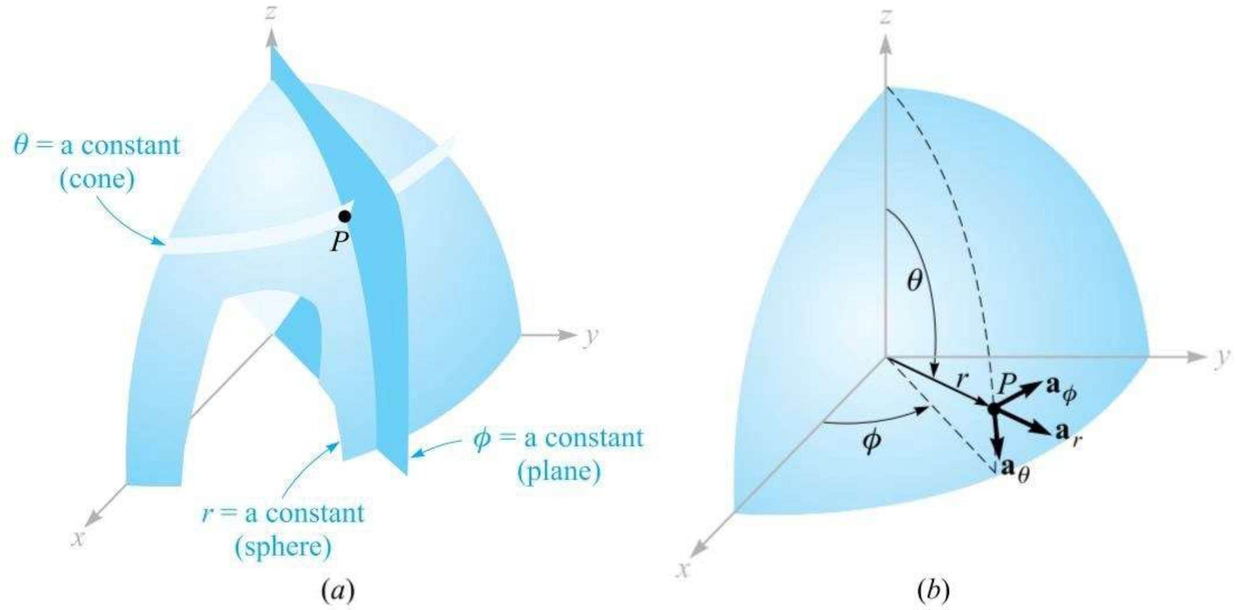
	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	1

1.8 The Circular Cylindrical Coordinates

(a) Give the rectangular coordinates of the point $C(\rho = 4.4, \phi = -115°, z = 2)$. (b) Give the cylindrical coordinates of the point $D(x = -3.1, y = 2.6, z = -3\ )$. (c) Specify the distance from C to D.

Ans. $C(x = -1.860, y = -3.99, z = 2); D(\rho = 4.05, \phi = 140.0°, z = -3); 8.36$

1.9 The Spherical Coordinate



1.9 The Spherical Coordinate

$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2} \quad (r \geq 0)$$

$$y = r\sin\theta\sin\phi \qquad \theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \quad (0^\circ \leq \theta \leq 180^\circ)$$

$$z = r\cos\theta \qquad \phi = \tan^{-1}\frac{y}{x}$$

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

1.9 The Spherical Coordinate

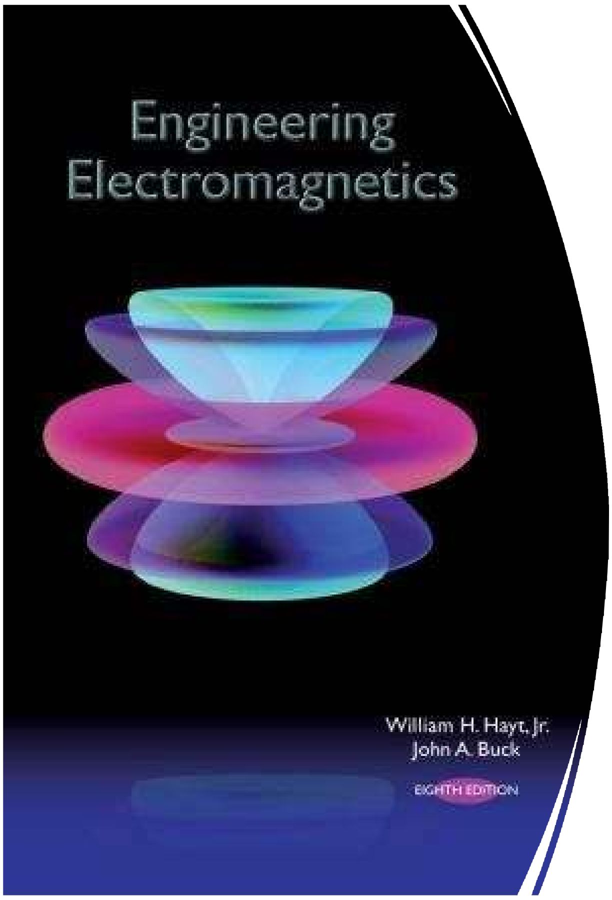
Given the two points, $C(-3,2,1)$ and $D(r=5, \theta=20^\circ, \phi=-70^\circ)$, find: (a) the spherical coordinates of C; (b) the rectangular coordinates of D; (c) the distance from C to D.

Ans. $C(r=3.74, \theta=74.5^\circ, \phi=146.3^\circ)$;

$D(x=0.585, y=-1.607, z=4.70)$; 6.29

Problems

1. Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find: (a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$; (b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$; $(c)|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$.
2. Given the points $M(0.1, -0.2, -0.1), N(-0.2,0.1,0.3)$, and $P(0.4,0,0.1)$, find (a) the vector $\mathbf{R}_{MN}$; (b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$; (c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$; (d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$.
3. Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of (a) the dot product; (b) the cross product.
4. Express in cylindrical components: (a) the vector from $C(3,2,-7)$ to $D(-1,-4,2)$; (b) a unit vector at D directed toward C; (c) a unit vector at D directed toward the origin.



2. Coulomb's Law and Electric Field Intensity

2.1 The Experimental Law of Coulomb

- Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance

$$F = k\frac{Q_1 Q_2}{R^2} \qquad k = \frac{1}{4\pi\epsilon_0} \qquad \epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi} 10^{-9} \text{ F/m}$$

- The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in SI units as 1.602×10^{-19}C; hence a negative charge of one coulomb represents about 6×10^{18} electrons.
- Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is 9×10^{9} N, or about one million tons.

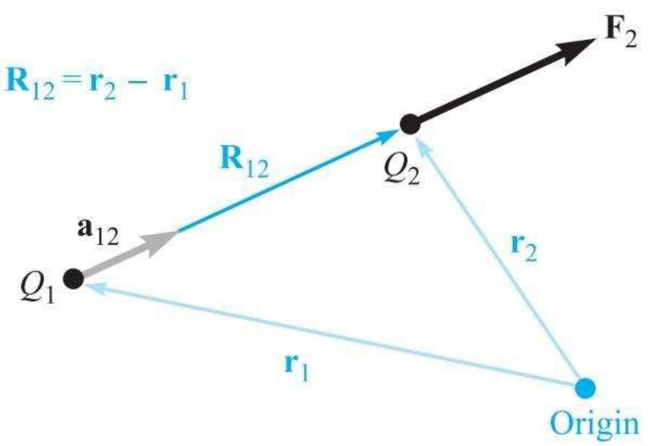
2.1 The Experimental Law of Coulomb

- The force acts along the line joining the two charges and is repulsive if the charges are alike in sign or attractive if they are of opposite sign.
- Let the vector $\mathbf{r}_1$ locate Q_1, whereas $\mathbf{r}_2$ locates Q_2. Then the vector $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ represents the directed line segment from Q_1 to Q_2.
- The vector $\mathbf{F}_2$ is the force on Q_2 and is shown for the case where Q_1 and Q_2 have the same sign. The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}$$

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

2.1 The Experimental Law of Coulomb



- If Q_1 and Q_2 have like signs, the vector force F_2 on Q_2 is in the same direction as the vector R_{12}.

2.1 The Experimental Law of Coulomb

A charge $Q_A = -20\mu\text{C}$ is located at $A(-6,4,7)$, and a charge $Q_B = 50\mu\text{C}$ is at $B(5,8,-2)$ in free space. If distances are given in meters, find: (a) $\mathbf{R}_{AB}$; (b) R_{AB}. Determine the vector force exerted on Q_A by Q_B if $\epsilon_0 =$ (c) $10^{-9}/(36\pi)\text{F/m}$; $(d) 8.854 \times 10^{-12}$ F/m.

Ans. $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; 14.76 m; $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$mN; $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$mN

2.2 Electric Field Intensity

- A force field that is associated with charge, Q_1. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

- Writing this force as a force per unit charge gives the electric field intensity, $\mathbf{E}_1$ arising from Q_1 :

$$\mathbf{E}_1 = \frac{\mathbf{F}_t}{Q_1} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

- The electric field of a single point charge becomes:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R$$

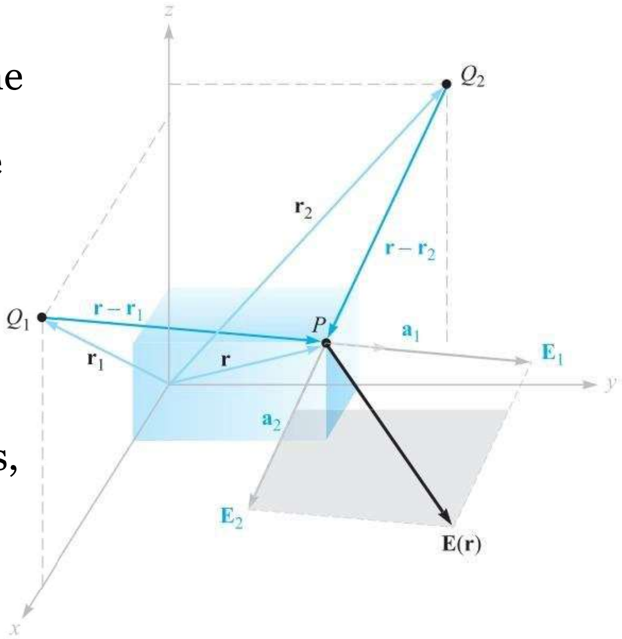
2.2 Electric Field Intensity

- Because the coulomb forces are linear, the electric field intensity arising from two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

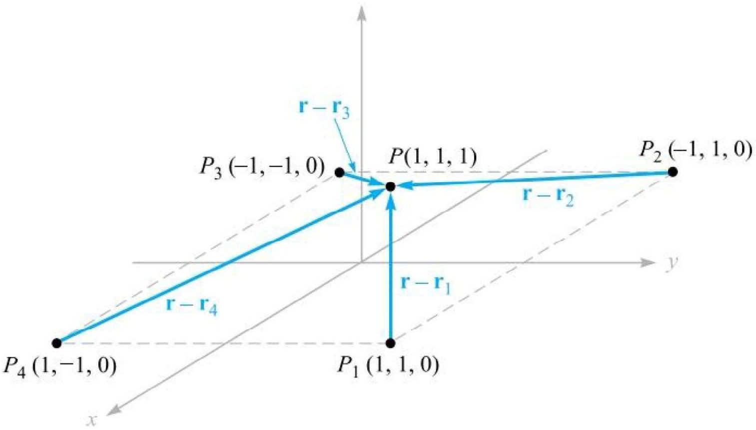
- If we add more charges at other positions, the field due to n point charges is

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$



2.2 Electric Field Intensity

Find $\mathbf{E}$ at $P(1,1,1)$ caused by four identical 3-nC charges located at $P_1(1,1,0)$, $P_2(-1,1,0)$, $P_3(-1,-1,0)$, and $P_4(1,-1,0)$



Ans. $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m

2.3 Field Arising from a Continuous Volume Charge Distribution

- We denote volume charge density by ρ_v, having the units of coulombs per cubic meter $(\mathrm{C/m^3})$.The small amount of charge ΔQ in a small volume Δv is

$$\Delta Q = \rho_v \Delta v$$

- and we may define ρ_v mathematically by using a limiting process,

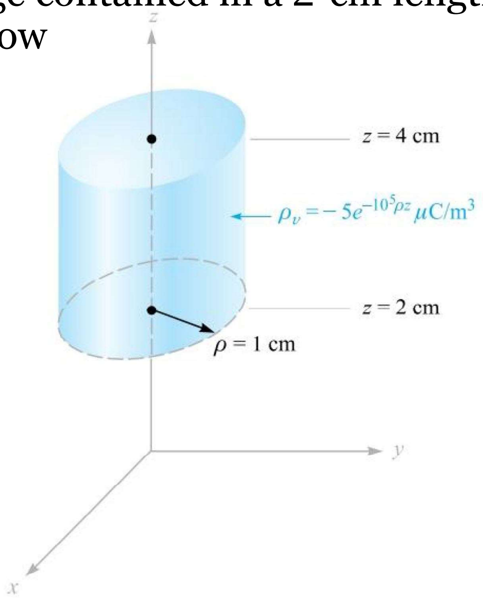
$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

- The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{\mathrm{vol}} \rho_v dv$$

2.3 Field Arising from a Continuous Volume Charge Distribution

Find the total charge contained in a 2-cm length of the electron beam shown in figure below



Ans. 0.0785pC

2.3 Field Arising from a Continuous Volume Charge Distribution

Calculate the total charge within each of the indicated volumes:

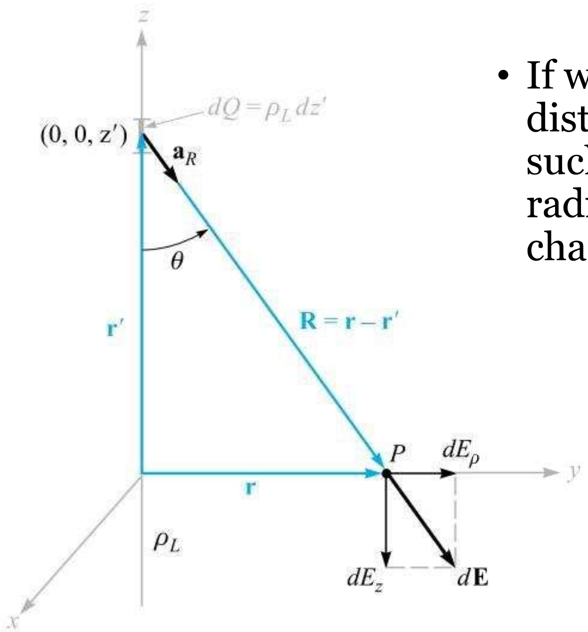
(a) $0.1 \le |x|, |y|, |z| \le 0.2$: $\rho_v = \frac{1}{x^3 y^3 z^3}$;

(b) $0 \le \rho \le 0.1, 0 \le \phi \le \pi, 2 \le z \le 4$; $\rho_v = \rho^2 z^2 \sin 0.6\phi$;

(c) universe: $\rho_v = e^{-2r}/r^2$.

Ans. 0; 1.018mC; 6.28C

2.4 Field of a Line Charge



- If we now consider a filamentlike distribution of volume charge density, such as a charged conductor of very small radius, we find it convenient to treat the charge as a line charge of density ρ_L C/m.

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho$$

2.4 Field of a Line Charge

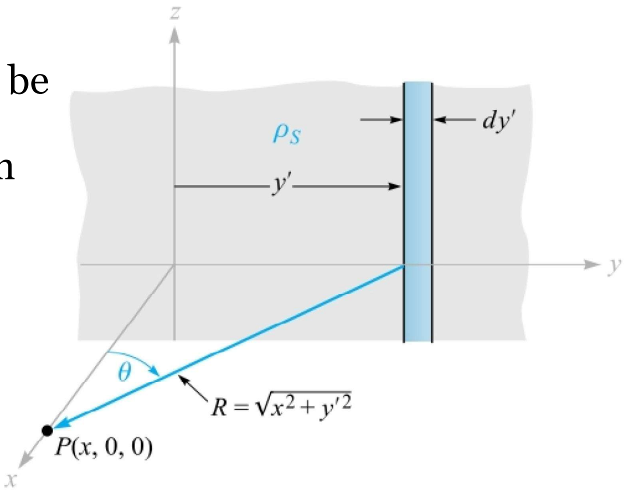
Infinite uniform line charges of 5nC/m lie along the (positive and negative) x and y axes in free space. Find $\mathbf{E}$ at: $(a) P_A(0,0,4)$; $(b) P_B(0,3,4)$.

Ans. $45\mathbf{a}_z$ V/m; $10.8\mathbf{a}_y + 36.9\mathbf{a}_z$ V/m

2.5 Field of a Sheet of Charge

- Another basic charge configuration is the infinite sheet of charge having a uniform density of ρ_S C/m^2.
- Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor.

$$\mathbf{E} = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_N$$



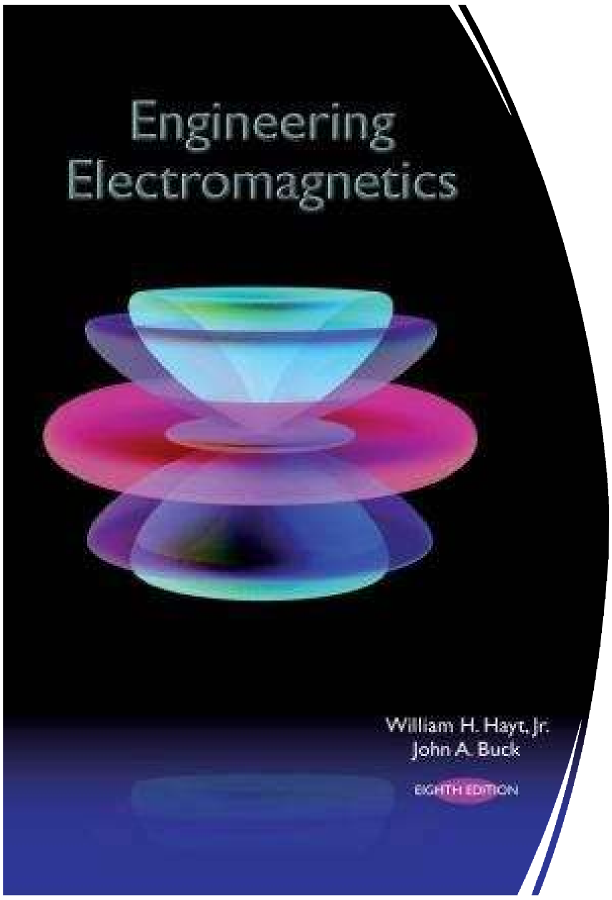
2.5 Field of a Sheet of Charge

Three infinite uniform sheets of charge are located in free space as follows: 3nC/m^2 at $z = -4, 6\text{nC/m}^2$ at $z = 1$, and -8nC/m^2 at $z = 4$. Find $\mathbf{E}$ at the point: $(a) P_A(2,5,-5)$; (b) $P_B(4,2,-3)$; (c) $P_C(-1,-5,2)$; (d) $P_D(-2,4,5)$.

Ans. $45\mathbf{a}_z$ V/m; $10.8\mathbf{a}_y + 36.9\mathbf{a}_z$ V/m

Problems

1. Point charges of 50nC each are located at $A(1,0,0), B(-1,0,0), C(0,1,0)$, and $D(0,-1,0)$ in free space. Find the total force on the charge at A.
2. Let a point charge $Q_1 = 25\text{nC}$ be located at $P_1(4,-2,7)$ and a charge $Q_2 = 60\text{nC}$ be at $P_2(-3,4,-2)$. (a) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1,2,3)$. (b) At what point on the y axis is $E_x = 0$?
3. Find $\mathbf{E}$ at the origin if the following charge distributions are present in free space: point charge, 12nC, at $P(2,0,6)$; uniform line charge density, 3nC/m, at $x = -2, y = 3$; uniform surface charge density, 0.2nC/m^2 at $x = 2$.



3. Electric Flux Density, Gauss's Law, and Divergence

3.1 Electric Flux Density

- Faraday's experiment consisted essentially of the following steps:
  1) With the equipment dismantled, the inner sphere was given a known positive charge.
  2) The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
  3) The outer sphere was discharged by connecting it momentarily to ground.
  4) The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured
- Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres.

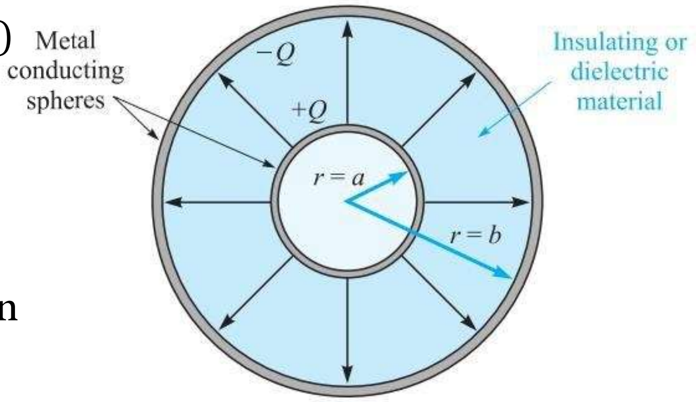
3.1 Electric Flux Density

- If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\Psi = Q$$

and the electric flux Ψ is measured in coulombs.

- The electric flux density $\mathbf{D}$ is a vector field and is a member of the "flux density" class of vector fields



$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} \qquad \text{(free space only)}$$

3.1 Electric Flux Density

Calculate $\mathbf{D}$ in rectangular coordinates at point $P(2,-3,6)$ produced by: (a) a point charge $Q_A = 55\text{mC}$ at $Q(-2,3,-6)$; (b) a uniform line charge $\rho_{LB} = 20\text{mC/m}$ on the x axis; (c) a uniform surface charge density $\rho_{SC} = 120\mu\text{C/m}^2$ on the plane $z = -5$ m.

Ans. $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z \mu\text{C/m}^2; -212\mathbf{a}_y + 424\mathbf{a}_z \mu\text{C/m}^2; 60\mathbf{a}_z \mu\text{C/m}^2$

3.2 Gauss's Law

- These generalizations of Faraday's experiment lead to the following statement, which is known as Gauss's law:

  The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

- The total flux passing through the closed surface is

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

3.3 Application of Gauss's Law: Some Symmetrical Charge Distributions

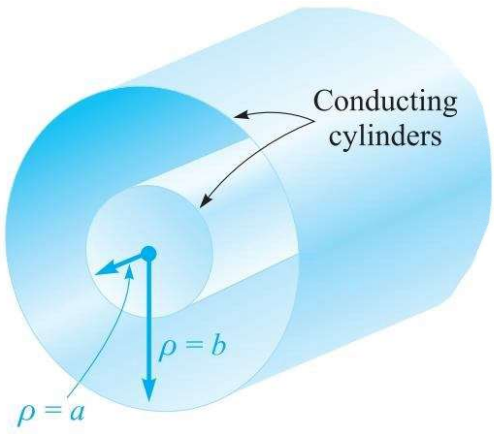
- We now consider how we may use Gauss's law

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral. The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1) $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2) On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

3.3 Application of Gauss's Law: Some Symmetrical Charge Distributions



- The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_S a d\phi dz = 2\pi a L \rho_S$$

$$D_S = \frac{a\rho_S}{\rho} \quad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \ (a < \rho < b)$$

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

3.4 Application of Gauss's Law: Differential Volume Element

- We are now going to apply the methods of Gauss's law to a slightly different type of problem-one that does not possess any symmetry at all.

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

- To evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

3.4 Application of Gauss's Law: Differential Volume Element

- Consider the first of these in detail. Because the surface element is very small, D is essentially constant (over this portion of the entire closed surface) and

$$\int_{\text{front}} \doteq \mathbf{D}_{\text{front}} \cdot \Delta \mathbf{S}_{\text{front}}$$
$$\doteq \mathbf{D}_{\text{front}} \cdot \Delta y \, \Delta z \, \mathbf{a}_x$$
$$\doteq D_{\text{x,front}} \Delta y \, \Delta z$$

$$\int_{\text{back}} \doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}}$$
$$\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \mathbf{a}_x)$$
$$\doteq -D_{x,\text{back}} \Delta y \, \Delta z$$

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

3.4 Application of Gauss's Law: Differential Volume Element

In free space, let $\mathbf{D} = 8xyz^4\mathbf{a}_x + 4x^2z^4\mathbf{a}_y + 16x^2yz^3\mathbf{a}_z \text{pC/m}^2$. (a) Find the total electric flux passing through the rectangular surface $z = 2$, $0 < x < 2, 1 < y < 3$, in the $\mathbf{a}_z$ direction. (b) Find $\mathbf{E}$ at $P(2, -1, 3)$. (c) Find an approximate value for the total charge contained in an incremental sphere located at $P(2, -1, 3)$ and having a volume of 10^{-12} m^3.

Ans. $1365\text{pC}; -146.4\mathbf{a}_x + 146.4\mathbf{a}_y - 195.2\mathbf{a}_z$ V/m; -2.38×10^{-21}C

3.5 Divergence and Maxwell's First Equation

- We will now obtain an exact relationship from

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

by allowing the volume element Δv to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$$

3.5 Divergence and Maxwell's First Equation

- The divergence of the vector flux density **A** is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \qquad \text{(rectangular)}$$

$$\text{div } \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, D_\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}$$

3.5 Divergence and Maxwell's First Equation

Find div $\mathbf{D}$ at the origin if $\mathbf{D} = e^{-x}\sin y\mathbf{a}_x - e^{-x}\cos y\mathbf{a}_y + 2z\mathbf{a}_z$.

Ans. 2

3.5 Divergence and Maxwell's First Equation

In each of the following parts, find a numerical value for div$\mathbf{D}$ at the point specified: (a) $\mathbf{D} = (2xyz - y^2)\mathbf{a}_x + (x^2z - 2xy)\mathbf{a}_y + x^2y\mathbf{a}_z \text{C/m}^2$ at $P_A(2,3,-1)$;

(b) $\mathbf{D} = 2\rho z^2\sin^2\phi\mathbf{a}_\rho + \rho z^2\sin 2\phi\mathbf{a}_\phi + 2\rho^2 z\sin^2\phi\mathbf{a}_z \text{C/m}^2$ at $P_B(\rho = 2, \phi = 110°, z = -1)$;

$(c)\mathbf{D} = 2r\sin\theta\cos\phi\mathbf{a}_r + r\cos\theta\cos\phi\mathbf{a}_\theta - r\sin\phi\mathbf{a}_\phi \text{C/m}^2$ at $P_C(r = 1.5, \theta = 30°, \phi = 50°)$.

Ans. -10.00; 9.06; 1.29

3.5 Divergence and Maxwell's First Equation

- Finally, we can combine

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$$

and

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)$$

and form the relation between electric flux density and charge density:

$$\text{div } \mathbf{D} = \rho_v$$

3.6 The Vector Operator ∇ and the Divergence Theorem

- we define the del operator ∇ as a vector operator,

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

- Consider $\nabla \cdot \mathbf{D}$, signifying

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot \left(D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z\right)$$

- We first consider the dot products of the unit vectors, discarding the six zero terms, and obtain the result that we recognize as the divergence of **D** :

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div } \mathbf{D}$$

3.6 The Vector Operator ∇ and the Divergence Theorem

- We have obtained it already and now have little more to do than point it out and name it, for starting from Gauss's law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

- The first and last expressions constitute the divergence theorem,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

- The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

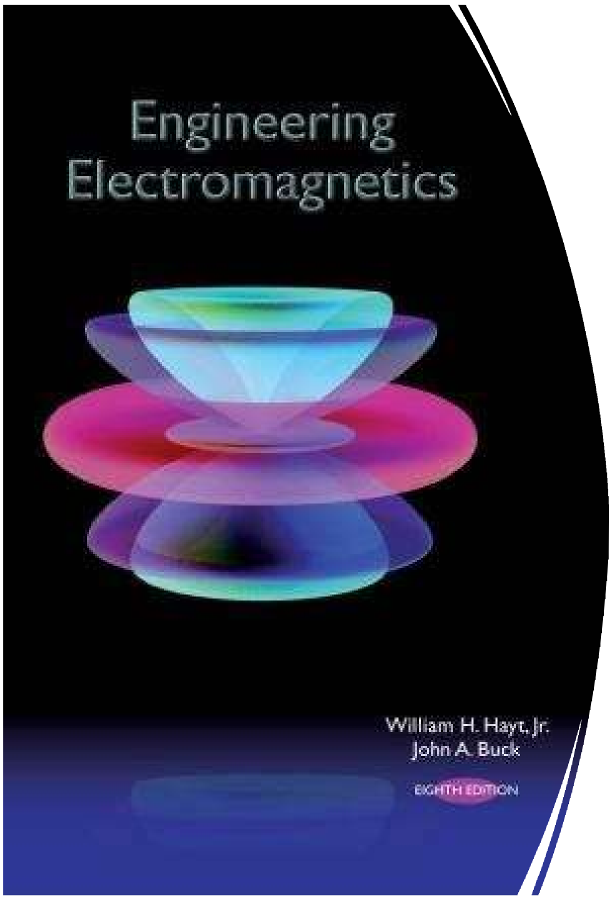
3.6 The Vector Operator ∇ and the Divergence Theorem

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y \mathrm{C/m^2}$ and the rectangular parallelepiped formed by the planes $x = 0$ and $1, y = 0$ and 2, and $z = 0$ and 3 .

Ans. 12 C

Problems

1. A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2 y\mathbf{a}_x + 3x^2 y^2 \mathbf{a}_y \, \mathrm{C/m^2}$ (a) Apply Gauss's law to find the total flux leaving the closed surface of the cube. (b) Evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube. (c) Estimate the total charge enclosed within the cube.



4. Energy and Potential

4.1 Energy Expended in Moving a Point Charge in an Electric Field

- To move a charge Q a distance dL in an electric field E. The force on Q arising from the electric field is

$$\mathbf{F}_E = Q\mathbf{E}$$

- where the subscript reminds us that this force arises from the field. The component of this force in the direction $d\mathbf{L}$ which we must overcome is

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

where $\mathbf{a}_L$ = a unit vector in the direction of $d\mathbf{L}$.

- The force that we must apply is equal and opposite to the force associated with the field,

$$F_{\text{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

4.1 Energy Expended in Moving a Point Charge in an Electric Field

- The expenditure of energy is the product of the force and distance. That is, the differential work done by an external source moving charge Q is

$$dW = -Q\mathbf{E} \cdot d\mathbf{L}$$

- The work required to move the charge a finite distance must be determined by integrating,

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

4.1 Energy Expended in Moving a Point Charge in an Electric Field

Given the electric field $\mathbf{E} = \frac{1}{z^2}\left(8xyz\mathbf{a}_x + 4x^2z\mathbf{a}_y - 4x^2y\mathbf{a}_z\right)\mathrm{V/m}$, find the differential amount of work done in moving a $6-\mathrm{nC}$ charge a distance of $2\mu\mathrm{m}$, starting at $P(2,-2,3)$ and proceeding in the direction $\mathbf{a}_L = (a) -\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z$; (b) $\frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z$; (c) $\frac{3}{7}\mathbf{a}_x + \frac{6}{7}\mathbf{a}_y$.

Ans. $-149.3\mathrm{fJ}; 149.3\mathrm{fJ}; 0$

4.2 The Line Integral

- The work involved in moving a charge Q from B to A is then approximately

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

or, using vector notation,

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

- The result for the uniform field can be obtained from the integral expression

$$W = -Q\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

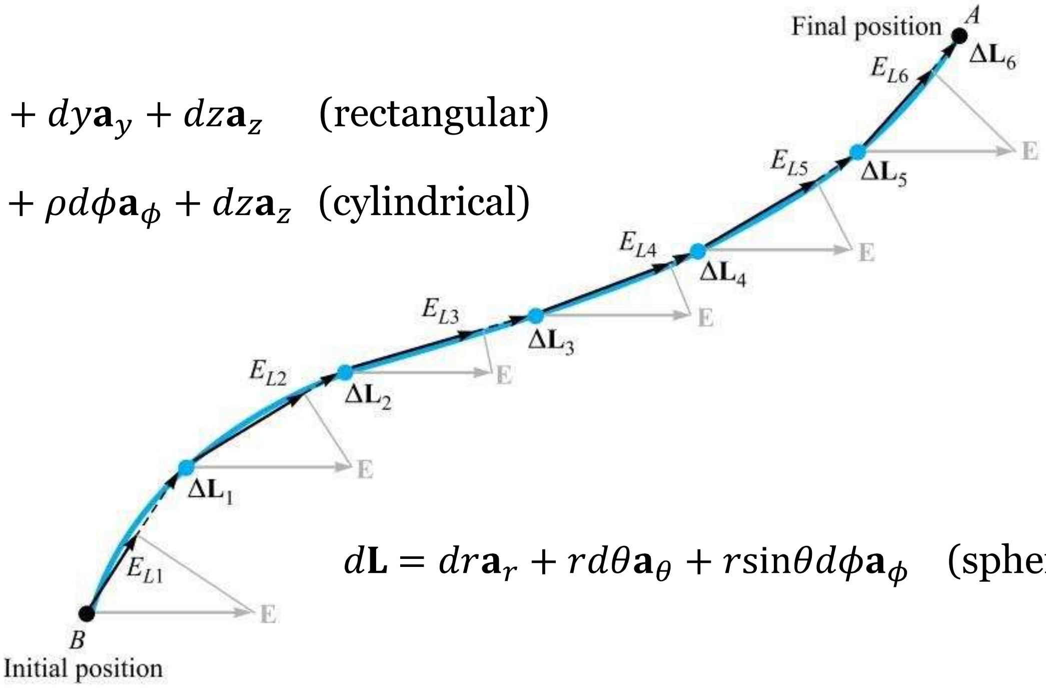
as applied to a uniform field

$$W = -Q\mathbf{E} \cdot \int_B^A d\mathbf{L}$$

4.2 The Line Integral

$$d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z \quad \text{(rectangular)}$$

$$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z \quad \text{(cylindrical)}$$



$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi \quad \text{(spherical)}$$

4.2 The Line Integral

We are given the nonuniform field

$$\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$$

and we are asked to determine the work expended in carrying 2C from $B(1,0,1)$ to $A(0.8,0.6,1)$ along the shorter arc of the circle

$$x^2 + y^2 = 1 \qquad z = 1$$

Ans. -0.96 J

4.2 The Line Integral

Calculate the work done in moving a 4-C charge from $B(1,0,0)$ to $A(0,2,0)$ along the path $y = 2 - 2x, z = 0$ in the field $\mathbf{E} =$ (a) $5\mathbf{a}_x$ V/m; (b) $5x\mathbf{a}_x$ V/m; (c) $5x\mathbf{a}_x + 5y\mathbf{a}_y$ V/m.

Ans. 20 J; 10 J; -30 J

4.3 Definition of Potential Difference and Potential

- We are now ready to define a new concept from the expression for the work done by an external source in moving a charge Q from one point to another in an electric field $\mathbf{E}$, "Potential difference and work."

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

- In much the same way as we defined the electric field intensity as the force on a unit test charge, we now define potential difference V as the work done (by an external source) in moving a unit positive charge from one point to another in an electric field,

$$\text{Potential difference } = V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

4.3 Definition of Potential Difference and Potential

- Potential difference is measured in joules per coulomb, for which the volt is defined as a more common unit, abbreviated as V. Hence the potential difference between points A and B is

$$V_{AB} = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}\mathbf{V}$$

and V_{AB} is positive if work is done in carrying the positive charge from B to A.

- If the potential at point A is V_A and that at B is V_B, then

$$V_{AB} = V_A - V_B$$

4.3 Definition of Potential Difference and Potential

An electric field is expressed in rectangular coordinates by $\mathbf{E} = 6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z$ V/m. Find: (a) V_{MN} if points M and N are specified by $M(2,6,-1)$ and $N(-3,-3,2)$; (b) V_M if $V = 0$ at $Q(4,-2,-35)$; (c) V_N if $V = 2$ at $P(1,2,-4)$.

Ans. -139.0 V; -120.0 V; 19.0 V

4.4 The Potential Field of a System of Charges: Conservative Property

- In other words, the expression for potential (zero reference at infinity),

$$V_A = -\int_{\infty}^{A} \mathbf{E} \cdot d\mathbf{L}$$

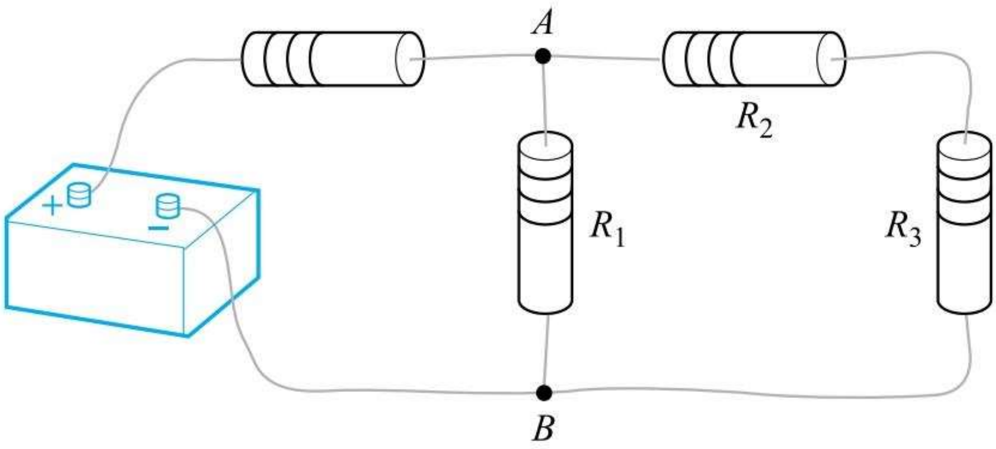
  or potential difference,

$$V_{AB} = V_A - V_B = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}$$

- This result is often stated concisely by recognizing that no work is done in carrying the unit charge around any closed path, or

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

4.4 The Potential Field of a System of Charges: Conservative Property



- A simple dc-circuit problem that must be solved by applying $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ in the form of Kirchhoff's voltage law.

4.5 Potential Gradient

- We already have the general line-integral relationship between these quantities,

$$V = -\int \mathbf{E} \cdot d\mathbf{L}$$

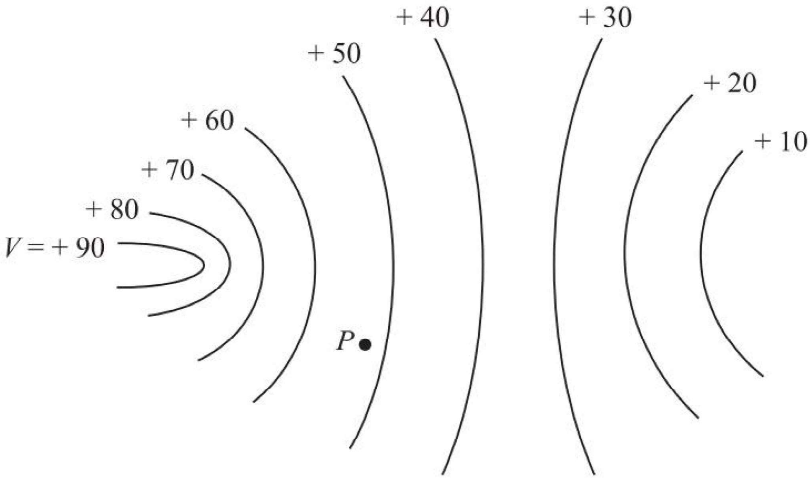
- This little exercise shows us two characteristics of the relationship between $\mathbf{E}$ and V at any point:

  1) The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.

  2)This maximum value is obtained when the direction of the distance increment is opposite to $\mathbf{E}$ or, in other words, the direction of $\mathbf{E}$ is opposite to the direction in which the potential is increasing the most rapidly.

4.5 Potential Gradient

- A potential field is shown by its equipotential surfaces. At any point the E field is normal to the equipotential surface passing through that point and is directed toward the more negative surfaces.



4.5 Potential Gradient

- We now may write the relationship between V and $\mathbf{E}$ as

$$\mathbf{E} = -\text{grad } V$$

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

- The vector operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

$$\mathbf{E} = -\nabla V$$

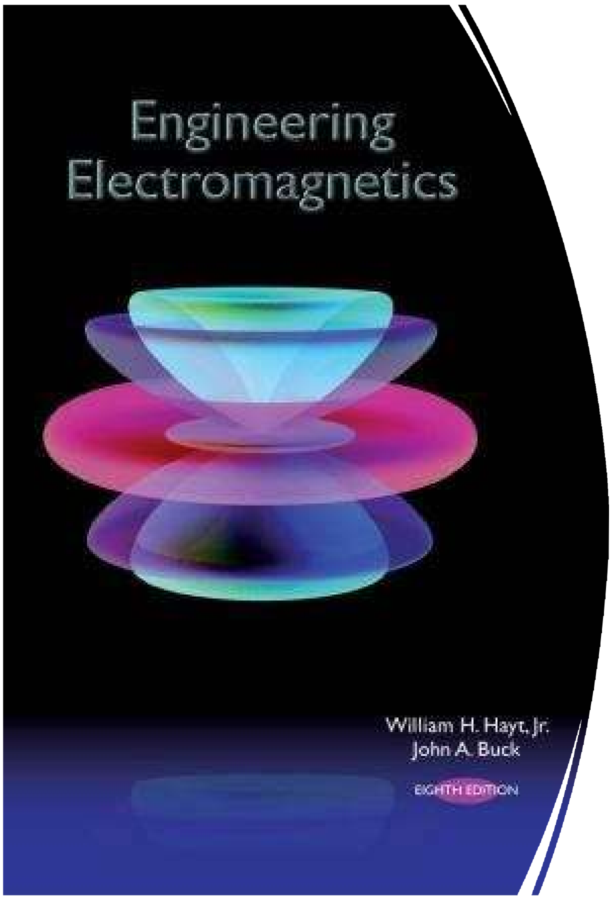
4.5 Potential Gradient

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4,3,6)$, we wish to find several numerical values at point P : the potential V, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Ans. $66\ \text{V}; 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z\ \text{V/m}; 57.9\ \text{V/m};$
$0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z\ \text{V/m};$
$-35.4xy\mathbf{a}_x - 17.71x^2\mathbf{a}_y + 44.3\mathbf{a}_z\text{p C/m}^3;$
$-35.4y\text{p C/m}^3; -106.2\text{pC/m}^3$

Problems

1. Given the electric field $\mathbf{E} = (y+1)\mathbf{a}_x + (x-1)\mathbf{a}_y + 2\mathbf{a}_z$ find the potential difference between the points $(a)(2,-2,-1)$ and $(0,0,0)$; $(b)(3,2,-1)$ and $(-2,-3,4)$.
2. Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)\text{V}$ in free space. Evaluate each of the following quantities at $P(3,2,-1)$
$(a)V; (b)|V|; (c)\mathbf{E}; (d)|\mathbf{E}|; (e)\mathbf{a}_N; (f)$ D.



5. The Steady Magnetic Field

5.1 Biot-Savart Law
5.2 Ampere’s Circuital Law
5.3 Curl
5.4 Stokes’ Theorem
5.5 Magnetic Flux and Magnetic Flux Density

5.1 Biot-Savart Law

- The Biot-Savart law may be written concisely using vector notation as

$$d\mathbf{H} = \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{Id\mathbf{L} \times \mathbf{R}}{4\pi R^3}$$

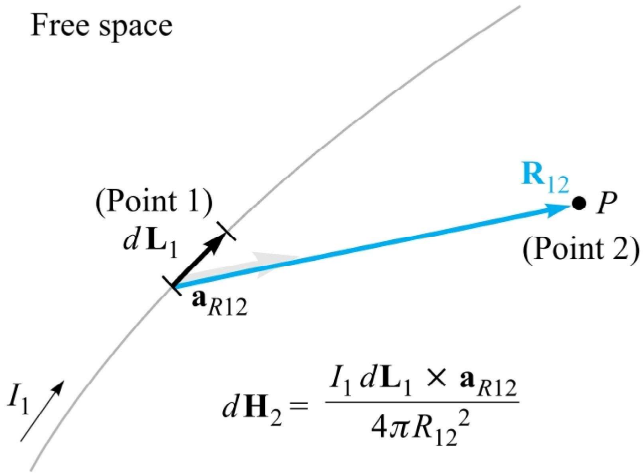
The units of the magnetic field intensity **H** are evidently amperes per meter (A/m).

- If we locate the current element at point 1 and describe the point P at which the field is to be determined as point 2, then

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

5.1 Biot-Savart Law

- The law of Biot-Savart expresses the magnetic field intensity $d\mathrm{H}_2$ produced by a differential current element $l_1 d\ \mathrm{L}_1$. The direction of $d\mathrm{H}_2$ is into the page.



5.1 Biot-Savart Law

- It follows that only the integral form of the Biot-Savart law can be verified experimentally,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

5.2 Ampere's Circuital Law

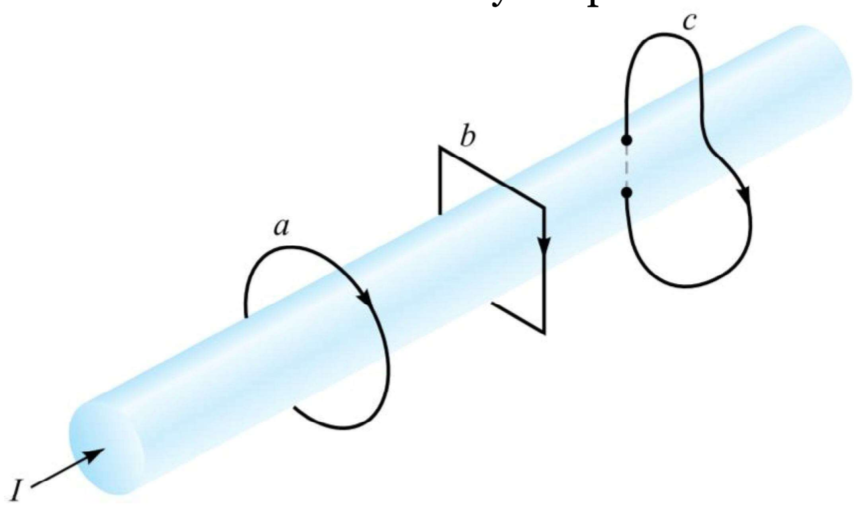
- Ampère's circuital law states that the line integral of **H** about any closed path is exactly equal to the direct current enclosed by that path,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

- We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

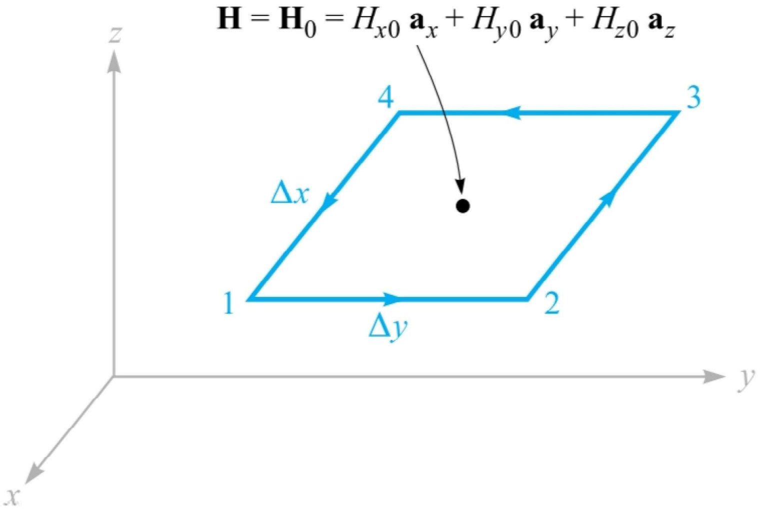
5.2 Ampere's Circuital Law

- A conductor has a total current l. The line integral of H about the closed paths a and b is equal to I, and the integral around path c is less than I, since the entire current is not enclosed by the path.



5.2 Ampere’s Circuital Law

- An incremental closed path in rectangular coordinates is selected for the application of Ampère's circuital law to determine the spatial rate of change of H.



5.2 Ampere’s Circuital Law

Evaluate total current I for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \leq x \leq 5, -1 \leq y \leq 1, z = 0$.

Ans. -126 A

5.3 Curl

- After beginning with Ampere's circuital law equating the closed line integral of H to the current enclosed.
- We see that a component of the current density is given by the limit of the quotient of the closed line integral of H about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero.

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}$$

5.3 Curl

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

This result may be written in the form of a determinant,

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

- and may also be written in terms of the vector operator,

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H}$$

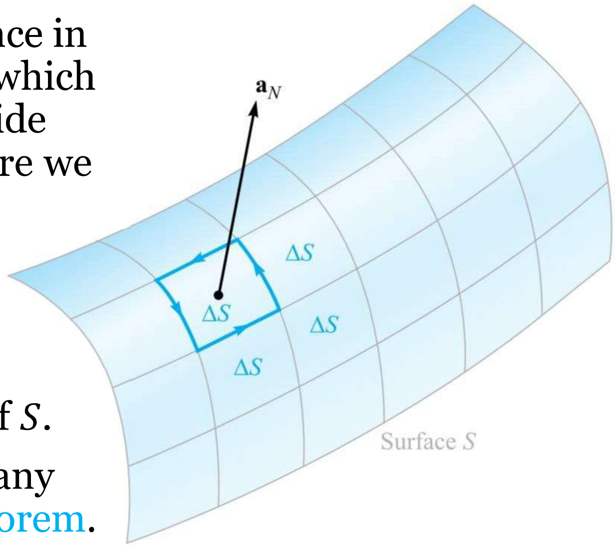
5.4 Stokes' Theorem

- because every interior wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S. Therefore we have

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

where $d\mathbf{L}$ is taken only on the perimeter of S.

- The equation is an identity, holding for any vector field, and is known as Stokes' theorem.



5.4 Stokes' Theorem

Evaluate total current I for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5, -1 \le y \le 1, z = 0$.

Ans. -126 A

5.5 Magnetic Flux and Magnetic Flux Density

- In free space, let us define the magnetic flux density **B** as

$$\mathbf{B} = \mu_0 \mathbf{H} \quad \text{(free space only)}$$

- where **B** is measured in webers per square meter ($\mathrm{Wb/m^2}$)
- In a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or 1 $\mathrm{Wb/m^2}$ is the same as 10,000G. The constant μ_0 is not dimensionless and has the defined value for free space, in henrys per meter (H/m), of

$$\mu_0 = 4\pi \times 10^{-7} \mathrm{H/m}$$

- The name given to μ_0 is the permeability of free space.

5.5 Magnetic Flux and Magnetic Flux Density

- Let us represent magnetic flux by Φ and define Φ as the flux passing through any designated area,

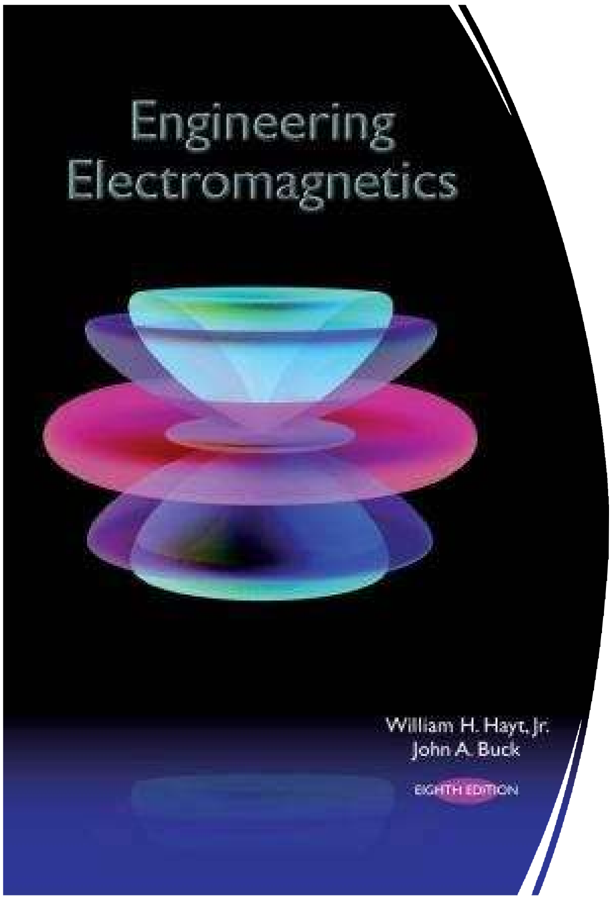
$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

- In the example of the infinitely long straight filament carrying a direct current I, the **H** field formed concentric circles about the filament. Because $\mathbf{B} = \mu_0 \mathbf{H}$, the **B** field is of the same form. The magnetic flux lines are closed and do not terminate on a "magnetic charge".
- For this reason, Gauss's law for the magnetic field is

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

Problems

1. The magnetic field intensity is given in a certain region of space as $\mathbf{H} = [(x + 2y)/z^2]\mathbf{a}_y + (2/z)\mathbf{a}_z$ A/m. (a) Find $\nabla \times \mathbf{H}$. (b) Find $\mathbf{J}$. (c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4, 1 \leq x \leq 2, 3 \leq z \leq 5$, in the $\mathbf{a}_z$ direction. (d) Show that the same result is obtained using the other side of Stokes' theorem.
2. Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space. (a) Show that $\nabla \cdot \mathbf{A} = 0$. (b) At $P(2, -1, 3)$, find $\mathbf{A}, \mathbf{B}, \mathbf{H}$, and $\mathbf{J}$.



6. Magnetic Forces, Materials, and inductance

6.1 Force on a Moving Charge

- In an electric field, the definition of the electric field intensity shows us that the force on a charged particle is

$$\mathbf{F} = Q\mathbf{E}$$

- A charged particle in motion in a magnetic field of flux density **B** is

$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$$

- The force on a moving particle arising from combined electric and magnetic fields is obtained easily by superposition

$$F = Q(E + v \times B)$$

- This equation is known as the Lorentz force equation

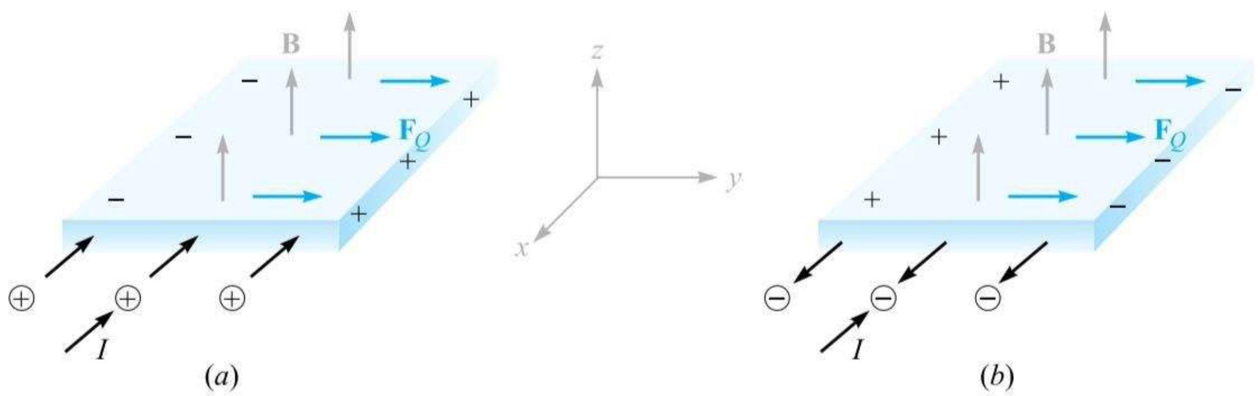
6.1 Force on a Moving Charge

The point charge $Q = 18\text{nC}$ has a velocity of 5×10^6 m/s in the direction $\mathbf{a}_v = 0.60\mathbf{a}_x + 0.75\mathbf{a}_y + 0.30\mathbf{a}_z$. Calculate the magnitude of the force exerte on the charge by the field: $(a)\mathbf{B} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z\text{mT}$;
(b) $\mathbf{E} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z\text{kV/m}$; (c)B and E acting together.

Ans. $660\mu\text{N}; 140\mu\text{N}; 670\mu\text{N}$

6.2 Force on a Differential Current Element

- Equal currents directed into the material are provided by positive charges moving inward in (a) and negative charges moving outward in (b). The two cases can be distinguished by oppositely directed Hall voltages, as shown.



6.3 Force and Torque on a Closed Circuit

- Now assume that two forces, $\mathbf{F}_1$ at P_1 and $\mathbf{F}_2$ at P_2, having lever arms $\mathbf{R}_1$ and $\mathbf{R}_2$ extending from a common origin O, as shown in Figure 8.5b, are applied to an object of fixed shape and that the object does not undergo any translation. Then the torque about the origin is

$$\mathbf{T} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2$$

where $\mathbf{F}_1 + \mathbf{F}_2 = 0$

- and therefore

$$\mathbf{T} = (\mathbf{R}_1 - \mathbf{R}_2) \times \mathbf{F}_1 = \mathbf{R}_{21} \times \mathbf{F}_1$$

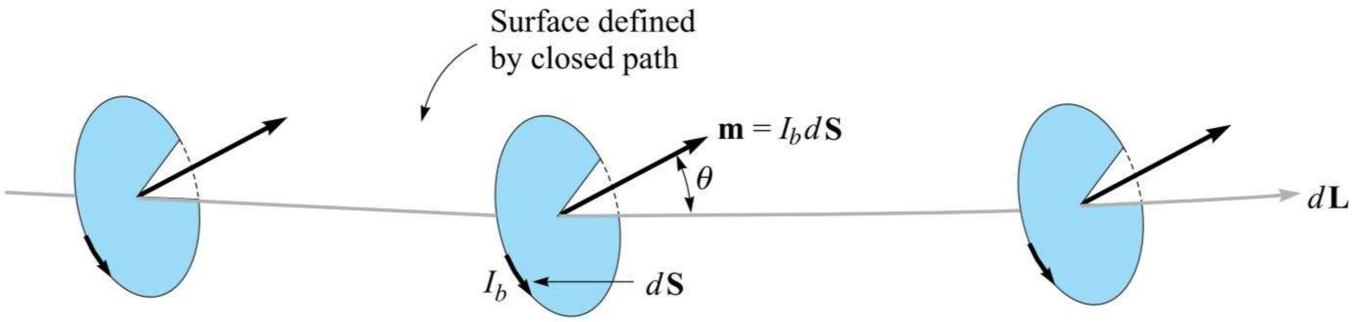
6.4 The Nature of Magnetic Materials

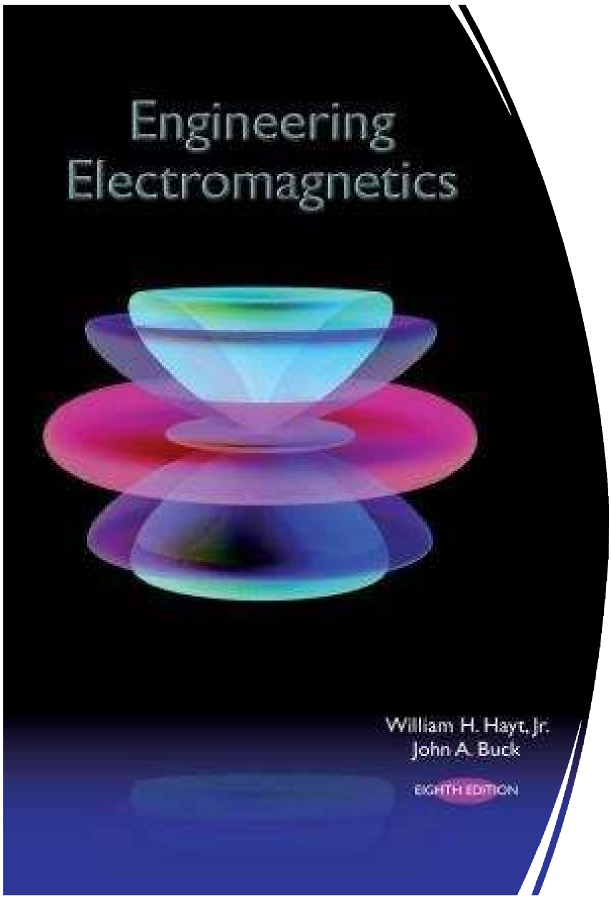
- Characteristics of magnetic materials

Classification	Magnetic Moments	B Values	Comments
Diamagnetic	$\mathbf{m}_{orb} + \mathbf{m}_{spin} = 0$	$B_{int} < B_{appl}$	$B_{int} \doteq B_{appl}$
Paramagnetic	$\mathbf{m}_{orb} + \mathbf{m}_{spin} =$ small	$B_{int} > B_{appl}$	$B_{int} \doteq B_{appl}$
Ferromagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} \gg B_{appl}$	Domains
Antiferromagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} \doteq B_{appl}$	Adjacent moments oppose
Ferrimagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} > B_{appl}$	Unequal adjacent moments oppose; low σ
Superparamagnetic	$\lvert\mathbf{m}_{spin}\rvert \gg \lvert\mathbf{m}_{orb}\rvert$	$B_{int} > B_{appl}$	Nonmagnetic matrix; recording tapes

6.5 Magnetization and Permeability

- A section d L of a closed path along which magnetic dipoles have been partially aligned by some external magnetic field. The alignment has caused the bound current crossing the surface defined by the closed path to increase by $nI_b dS \cdot d$ LA.





7. Time-Varying Fields and Maxwell's Equations

7.1 Faraday's Law

- An electromotive force (emf) is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields, and we shall define it in this section. Faraday's law is customarily stated as

$$\text{emf} = -\frac{d\Phi}{dt} \text{ V}$$

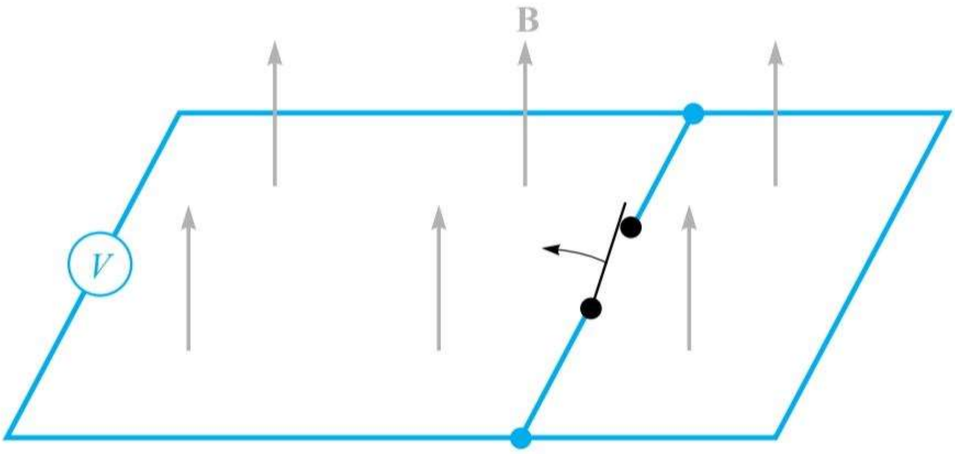
- If the closed path is that taken by an N-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and let

$$\text{emf} = -N\frac{d\Phi}{dt}$$

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

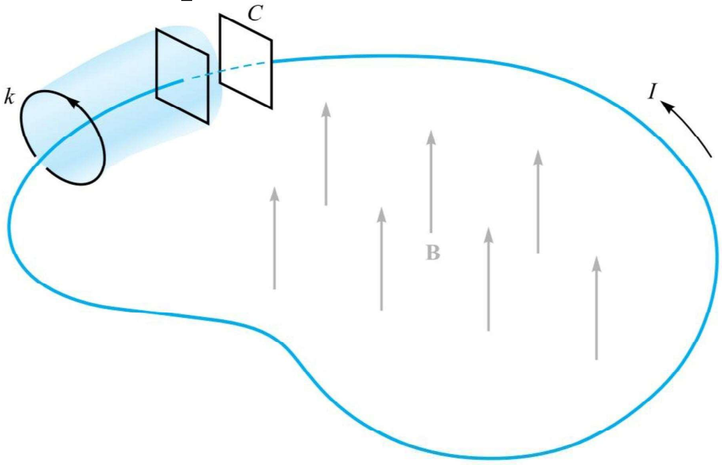
7.1 Faraday's Law

- An apparent increase in flux linkages does not lead to an induced voltage when one part of a circuit is simply substituted for another by opening the switch. No indication will be observed on the voltmeter.



7.2 Displacement Current

- A filamentary conductor forms a loop connecting the two plates of a parallel-plate capacitor. A time-varying magnetic field inside the closed path produces an emf of $V_0 \cos \omega t$ around the closed path. The conduction current / is equal to the displacement current between the capacitor plates.



7.3 Maxwell's Equations in Point Form

- non-time-varying

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot \mathbf{B} = 0$$

- time-varying

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

7.3 Maxwell's Equations in Point Form

- The auxiliary equations relating **D** and **E**,

$$\mathbf{D} = \epsilon \mathbf{E}$$

- The auxiliary equations relating **B** and **H**,

$$\mathbf{B} = \mu \mathbf{H}$$

- defining conduction current density,

$$\mathbf{J} = \sigma \mathbf{E}$$

- and defining convection current density in terms of the volume charge density ρ_ν,

$$\mathbf{J} = \rho_\nu \mathbf{v}$$

7.4 Maxwell's Equations in Integral Form

- non-time-varying

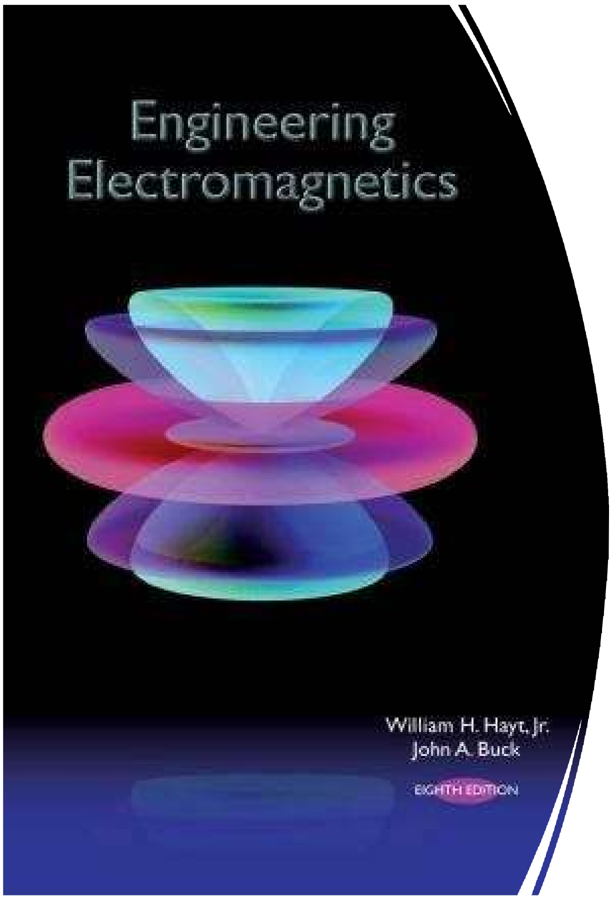
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v dv$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

- time-varying

$$\oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\oint \mathbf{H} \cdot d\mathbf{L} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$



8. The Uniform Plane Wave

8.1 Wave Propagation in Free Space

- When considering electromagnetic waves in free space, we note that the medium is sourceless ($\rho_v = \mathbf{J} = 0$). Under these conditions, Maxwell's equations may be written in terms of **E** and **H** only as

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

8.1 Wave Propagation in Free Space

- Transverse electromagnetic (TEM)

$\mathbf{E} = E_x \mathbf{a}_x$, or that the electric field is polarized in the x direction. The curl of $\mathbf{E}$ reduces to a single term:

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t}\mathbf{a}_y$$

The directions of $\mathbf{E}$ and $\mathbf{H}$ and the direction of travel are mutually orthogonal. Using the y-directed magnetic field, and the fact that it varies only in z

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \frac{\partial E_x}{\partial t}\mathbf{a}_x$$

8.1 Wave Propagation in Free Space

- Transverse electromagnetic (TEM)

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \frac{\partial^2 H_y}{\partial t \partial z} \qquad \frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad \frac{\partial^2 H_y}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 H_y}{\partial t^2}$$

8.1 Wave Propagation in Free Space

- We further identify the propagation velocity:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s} = c$$

- The intrinsic impedance of free space is defined as

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \doteq 120\pi \qquad \Omega$$

8.1 Wave Propagation in Free Space

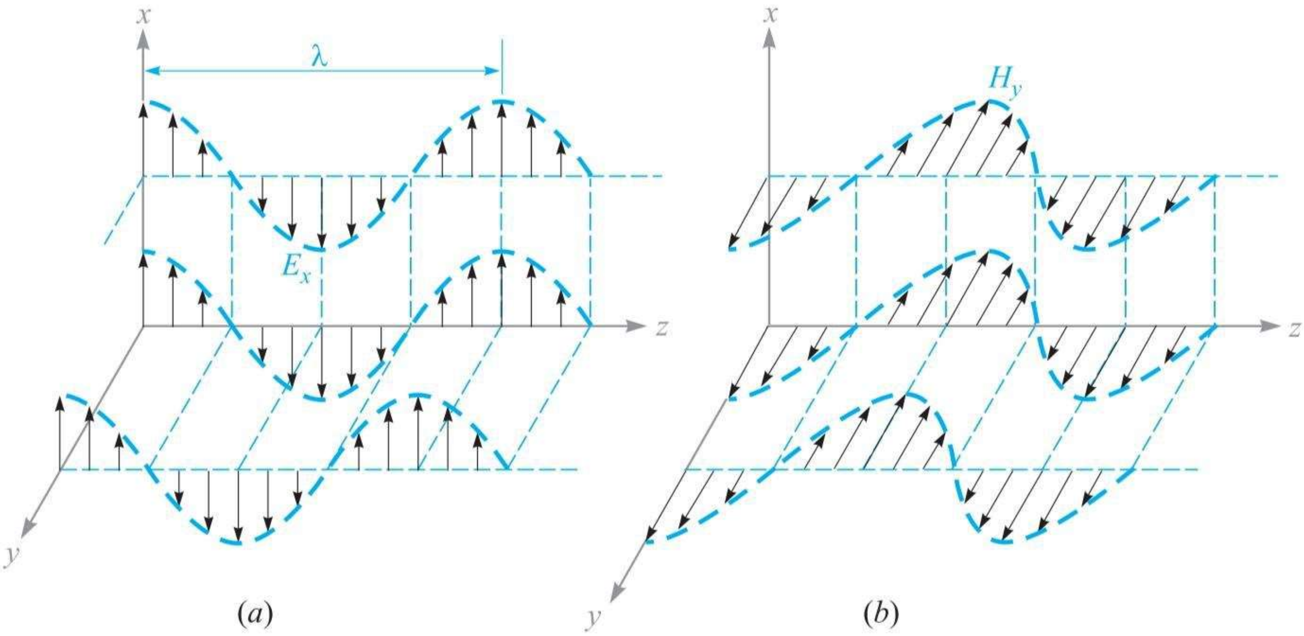
- The equations of the forward- and backward-propagating:

$$E_x(z,t) = f_1(t - z/v) + f_2(t + z/v)$$

where again f_1 and f_2 can be any function whose argument is of the form $t \pm z/v$.

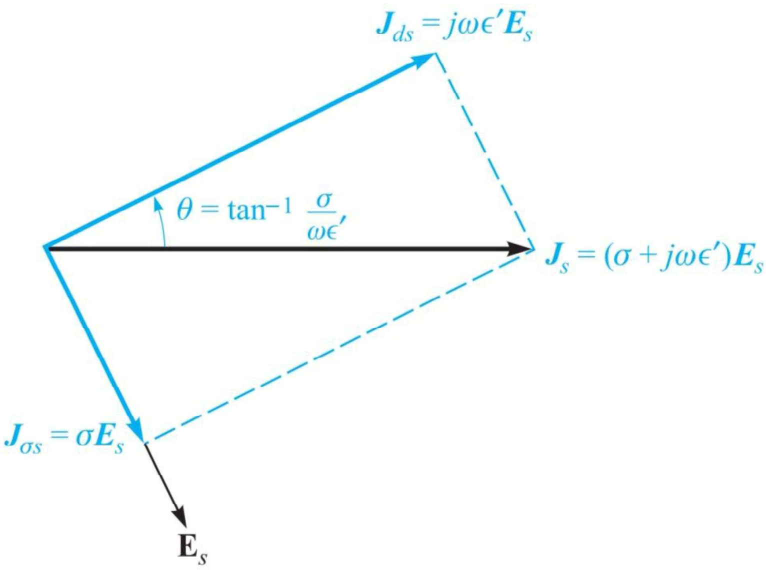
$$\begin{aligned} E_x(z,t) &= \mathcal{E}_x(z,t) + \mathcal{E}'_x(z,t) \\ &= |E_{x0}| \cos[\omega(t - z/v_p) + \phi_1] + |E'_{x0}| \cos[\omega(t + z/v_p) + \phi_2] \\ &= \underbrace{|E_{x0}| \cos[\omega t - k_0 z + \phi_1]}_{\text{forward } z \text{ travel}} + \underbrace{|E'_{x0}| \cos[\omega t + k_0 z + \phi_2]}_{\text{backward } z \text{ travel}} \end{aligned}$$

8.1 Wave Propagation in Free Space



8.2 Wave Propagation in Dielectrics

- The time-phase relationship between J_{ds}, $J_{\sigma S}$, J_S, and E_s. The tangent of θ is equal to $\sigma/\omega\epsilon'$, and $90° - \theta$ is the common power-factor angle, or the angle by which J_S leads E_s.



8.3 Poynting's Theorem and Wave Power

- The total power flowing out of the volume is

$$\oint_{\text{area}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \qquad \text{W}$$

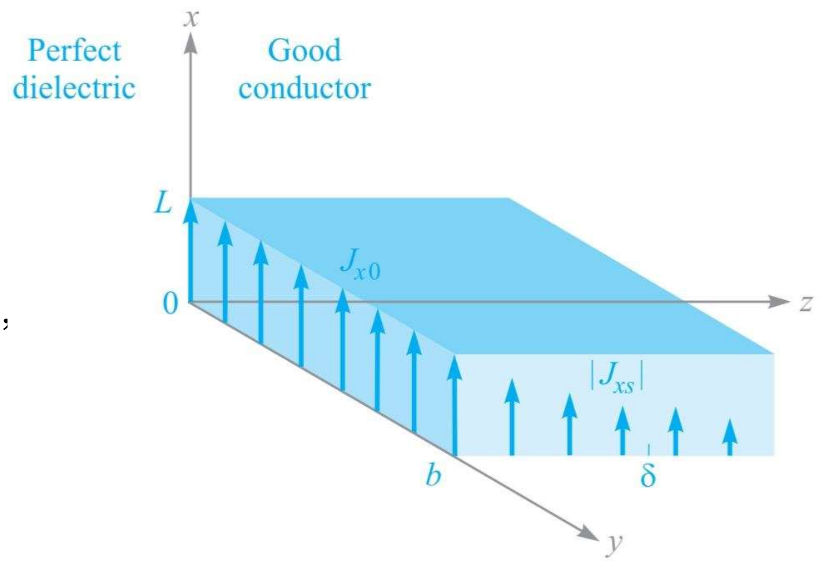
where the integral is over the closed surface surrounding the volume. The cross product $\mathbf{E} \times \mathbf{H}$ is known as the Poynting vector, $\mathbf{S}$,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad \text{W/m}^2$$

which is interpreted as an instantaneous power density, measured in watts per square meter (W/m^2). The direction of the vector $\mathbf{S}$ indicates the direction of the instantaneous power flow at a point, and many of us think of the Poynting vector as a "pointing" vector.

8.4 Propagation in Good Conductors: Skin Effect

- The current density $J_x = J_{x0} e^{-z/\delta} e^{-jz/\delta}$ decreases in magnitude as the wave propagates into the conductor. The average power loss in the region $0 < x < L, 0 < y < b, z > 0$, is $\delta b L J_{x0}^2 / 4\sigma$ watts.



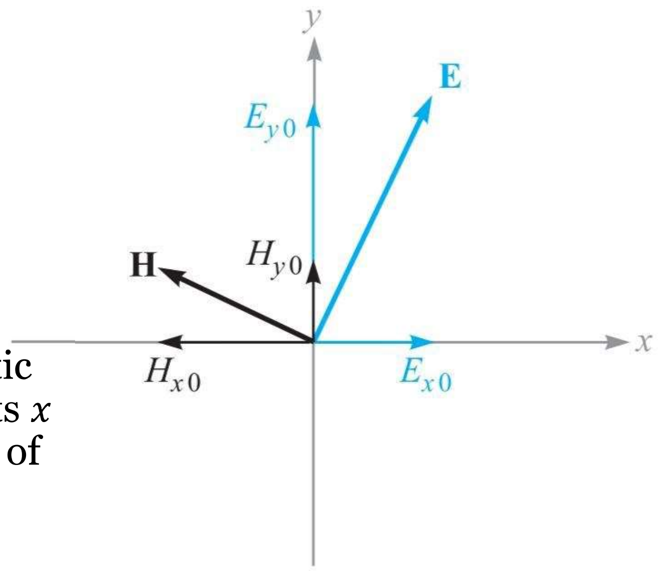
8.5 Wave Polarization

- Electric and magnetic field configuration for a general linearly polarized plane wave propagating in the forward z direction

$$\mathbf{E}_s = (E_{x0}\mathbf{a}_x + E_{y0}\mathbf{a}_y)e^{-\alpha z}e^{-j\beta z}$$

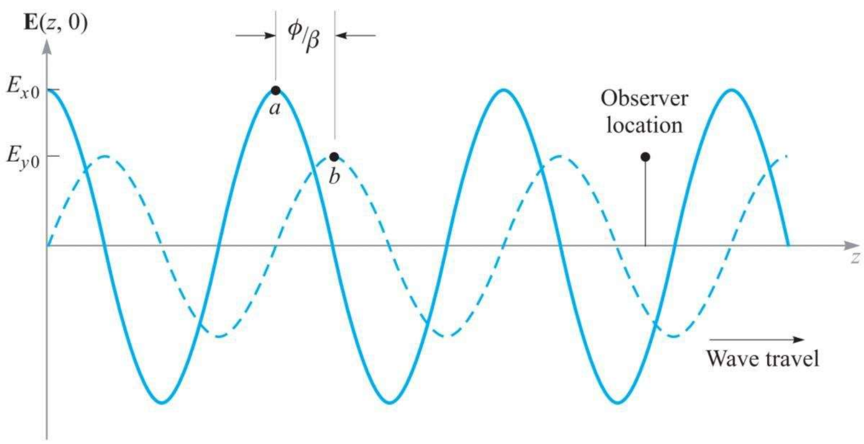
where E_{x0} and E_{y0} are constant amplitudes along x and y. The magnetic field is readily found by determining its x and y components directly from those of $\mathbf{E}_s$. Specifically, $\mathbf{H}_s$ for the wave is

$$\mathbf{H}_s = [H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y]e^{-\alpha z}e^{-j\beta z}$$



8.5 Wave Polarization

- Plots of the electric field component magnitudes as functions of z. Note that the y component lags behind the x component in z. As time increases from zero, both waves travel to the right. Thus, to an observer at a fixed location, the y component leads in time.



8.5 Wave Polarization

- Representation of a right circularly polarized wave. The electric field vector (in white) will rotate toward the y axis as the entire wave moves through the xy plane in the direction of k. This counterclockwise rotation (when looking toward the wave source) satisfies the temporal right-handed rotation convention as described in the text. The wave, however, appears as a left-handed screw, and for this reason it is called left circular polarization in the other convention.