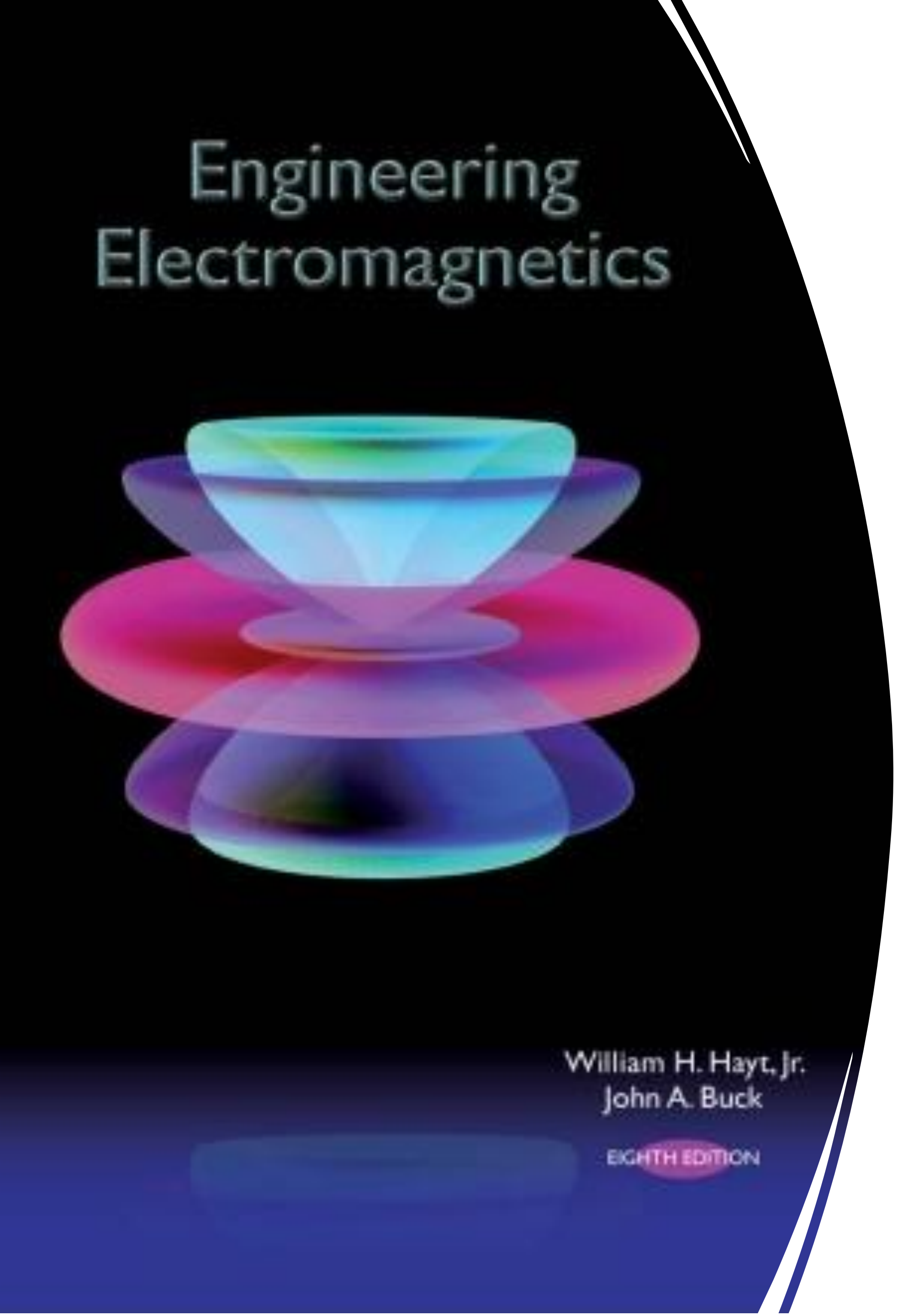


1. Vector Analysis

1.1 Scalars and Vectors

1.2 Vector Algebra

1.3 The Rectangular Coordinate System

1.4 Vector Components and Unit Vectors

1.5 The Vector Field

1.6 The Dot Product

1.7 The Cross Product

1.8 The Circular Cylindrical Coordinates

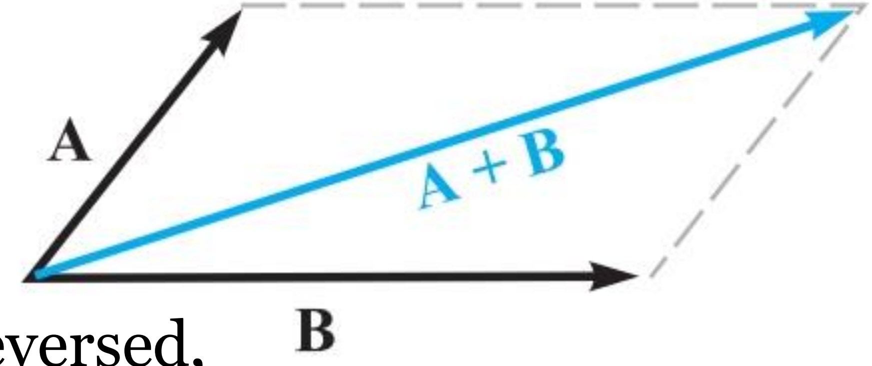
1.9 The Spherical Coordinate

1.1 Scalars and Vectors

- The term scalar refers to a quantity whose value may be represented by a single (positive or negative) real number.
- If we speak of a body falling a distance L in a time t, or the temperature T at any point in a bowl of soup whose coordinates are x, y, and z, then L, t, T, x, y, and z are all scalars.
- Other scalar quantities are mass, density, pressure (but not force), volume, volume resistivity, and voltage.
- A vector quantity has both a magnitude and a direction in space.
- Force, velocity, acceleration, and a straight line from the positive to the negative terminal of a storage battery are examples of vectors.
- Each quantity is characterized by both a magnitude and a direction.

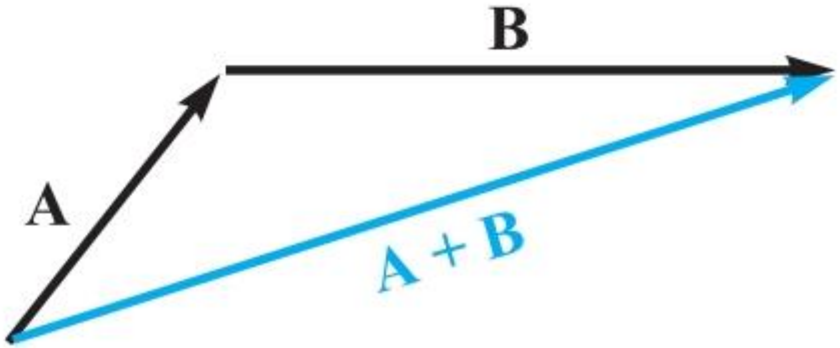
1.2 Vector Algebra

- The addition of vectors follows the parallelogram law.



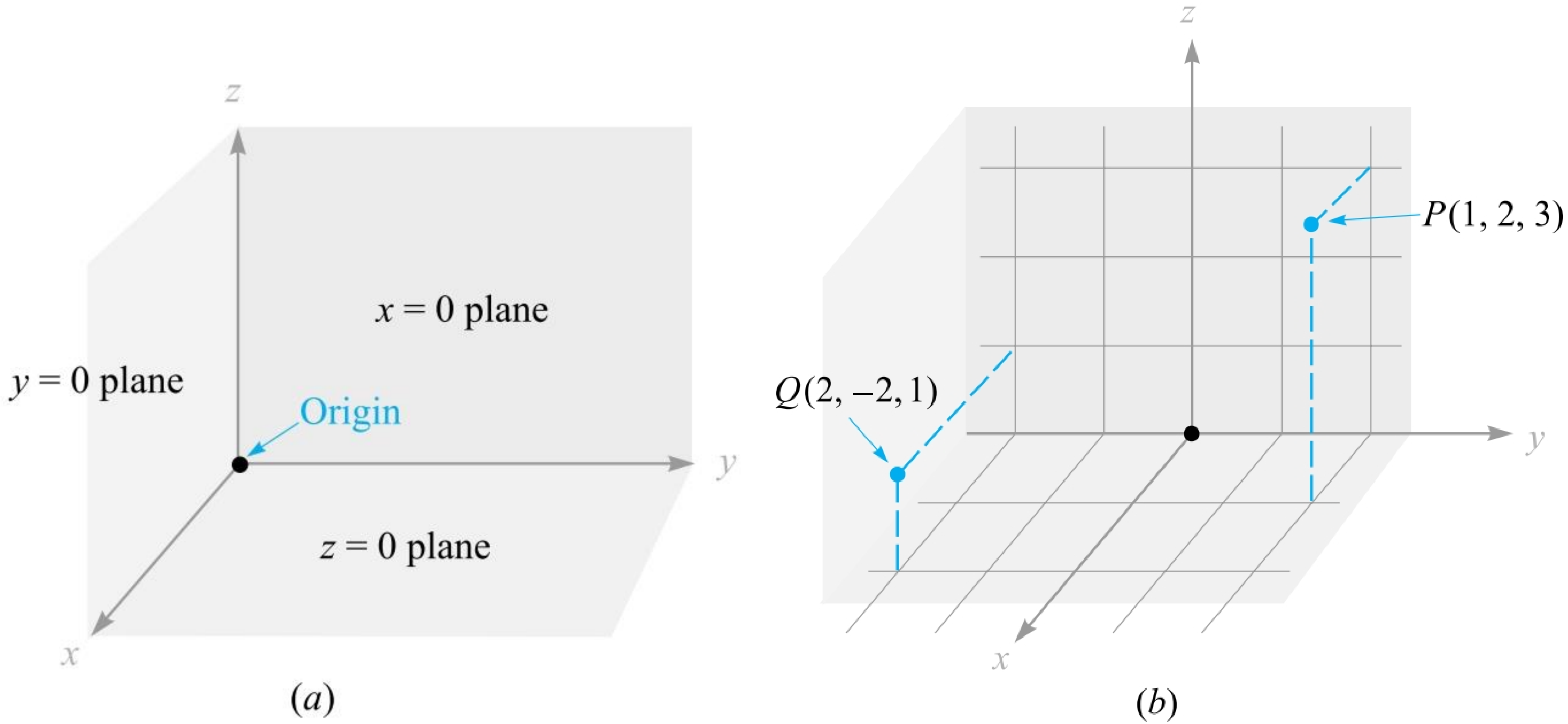
- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
- $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition



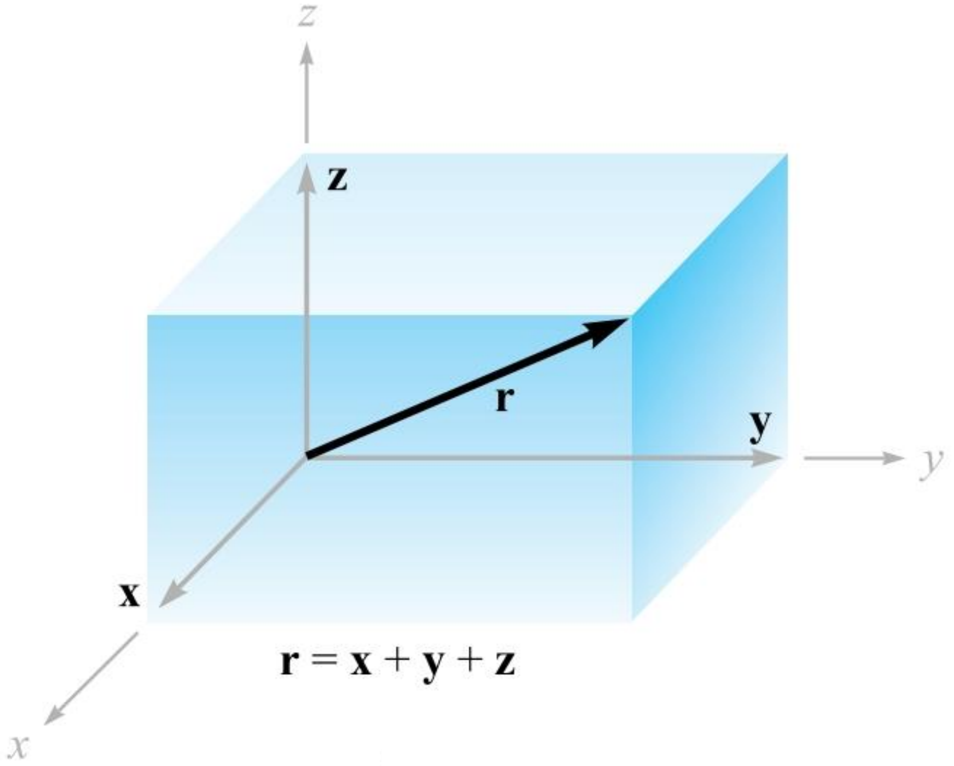
- $\mathbf{A} = \mathbf{B}$ if $\mathbf{A} - \mathbf{B} = 0$

1.3 The Rectangular Coordinate System



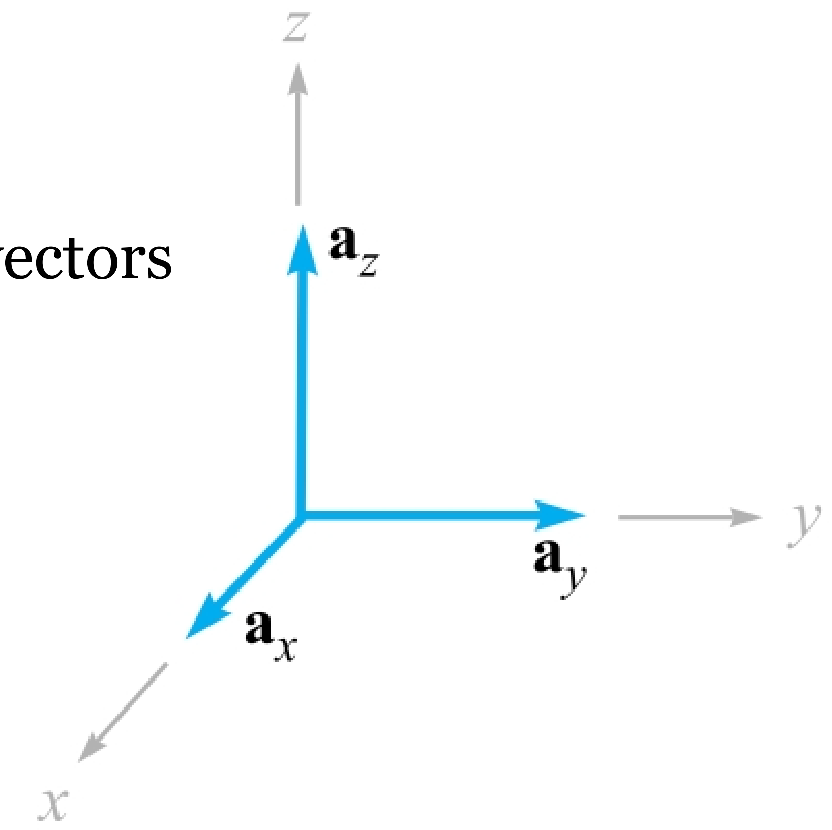
1.4 Vector Components and Unit Vectors

- If the component vectors of the vector $\mathbf{r}$ are $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, then $\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$.
- The component vectors have magnitudes that depend on the given vector (such as r), but they each have a known and constant direction.

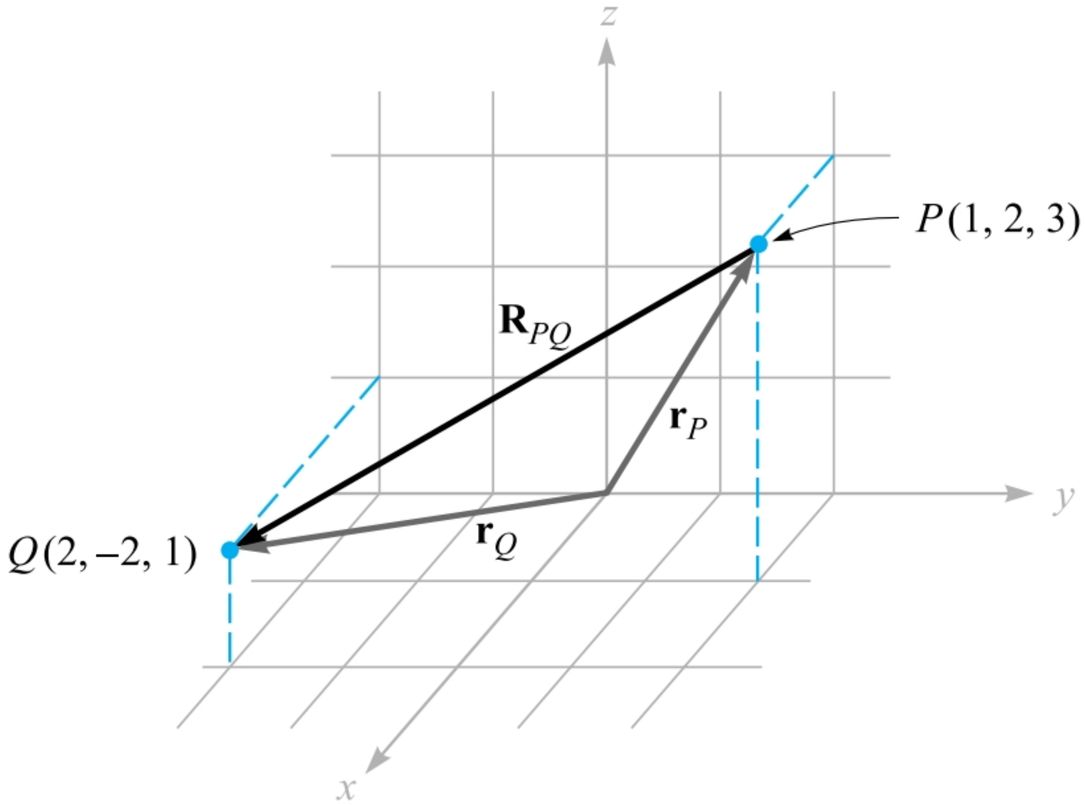


1.4 Vector Components and Unit Vectors

- This suggests the use of unit vectors having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values.
- We reserve the symbol **a** for a unit vector and identify its direction by an appropriate subscript. Thus $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the unit vectors in the rectangular coordinate system.



1.4 Vector Components and Unit Vectors



- A vector $\mathbf{r}_P$ pointing from the origin to point $P(1,2,3)$ is written $\mathbf{r}_P = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$.
- The vector from P to Q may be obtained by applying the rule of vector addition.
- The desired vector from $P(1,2,3)$ to $Q(2,-2,1)$ is therefore

$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

1.4 Vector Components and Unit Vectors

- Any vector **B** then may be described by $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The magnitude of **B** written $|\mathbf{B}|$ or simply B, is given by

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

- and a unit vector in the direction of the vector **B** is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

1.4 Vector Components and Unit Vectors

Given points $M(-1,2,1), N(3,-3,0)$, and $P(-2,-3,-4)$, find:
(a) $\mathbf{R}_{MN}$; (b) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (c) $|\mathbf{r}_M|$; (d) $\mathbf{a}_{MP}$; (e) $|2\mathbf{r}_P - 3\mathbf{r}_N|$.

Ans. $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z; 3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z; 2.45; -0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z; 15.56$

1.5 The Vector Field

- We have defined a vector field as a vector function of a position vector.
- In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question.
- We may write the velocity vector as $\mathbf{v} = v_x \mathbf{a}_x + v_y \mathbf{a}_y + v_z \mathbf{a}_z$, or $\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$; each of the components v_x, v_y, and v_z may be a function of the three variables x, y, and z.
- Further simplifying assumptions might be made if the velocity falls off with depth and changes very slowly as we move north, south, east, or west. A suitable expression could be $\mathbf{v} = 2e^{z/100}\mathbf{a}_x$. We have a velocity of 2 m/s at the surface and a velocity of 0.368×2, or 0.736 m/s, at a depth of 100 m$(z = -100)$. The velocity continues to decrease with depth, while maintaining a constant direction.

1.6 The Dot Product

- Given two vectors **A** and **B**, the dot product, or scalar product, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle between them. The expression $\mathbf{A} \cdot \mathbf{B}$ is read "A dot B."

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

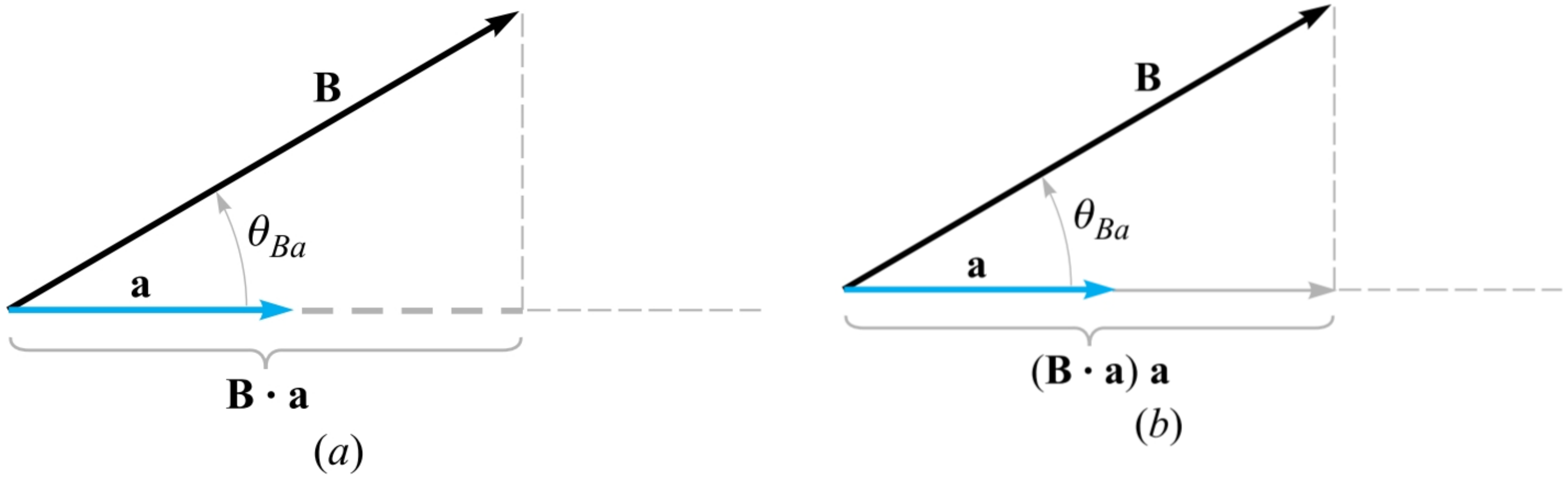
$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$$

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta_{AB}$$

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

1.6 The Dot Product

- One of the most important applications of the dot product is that of finding the component of a vector in a given direction.
- The geometrical term projection is also used with the dot product. Thus, $\mathbf{B} \cdot \mathbf{a}$ is the projection of $\mathbf{B}$ in the $\mathbf{a}$ direction.



1.6 The Dot Product

The three vertices of a triangle are located at $A(6,-1,2)$, $B(-2,3,-4)$, and $C(-3,1,5)$. Find: $(a)\mathbf{R}_{AB}$; $(b)\mathbf{R}_{AC}$; (c) the angle θ_{BAC} at vertex A; (d) the (vector) projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$.

Ans. $-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z$; $-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$; $53.6°$;
$-5.94\mathbf{a}_x + 1.319\mathbf{a}_y + 1.979\mathbf{a}_z$

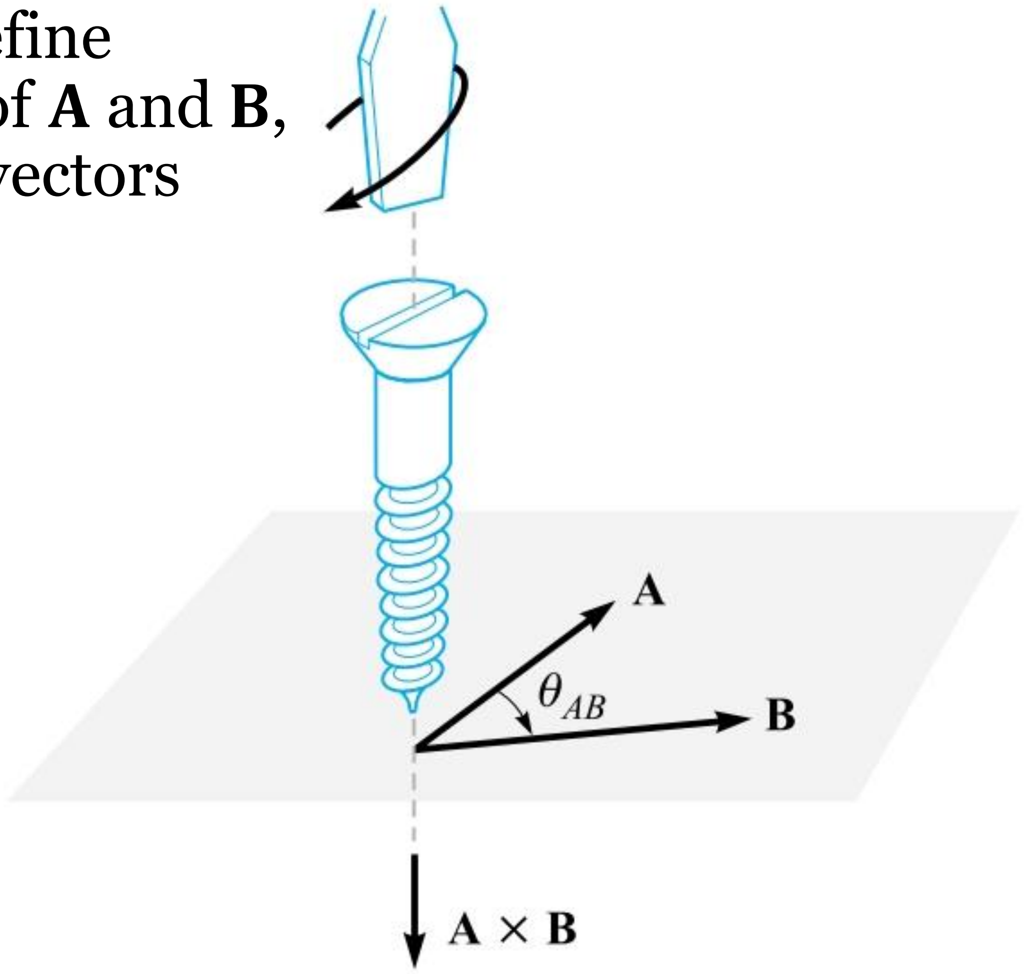
1.7 The Cross Product

- Given two vectors **A** and **B**, we now define the cross product, or vector product, of **A** and **B**, written with a cross between the two vectors as **A** × **B** and read "A cross **B**.“

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}||\mathbf{B}| \sin\theta_{AB}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z$$

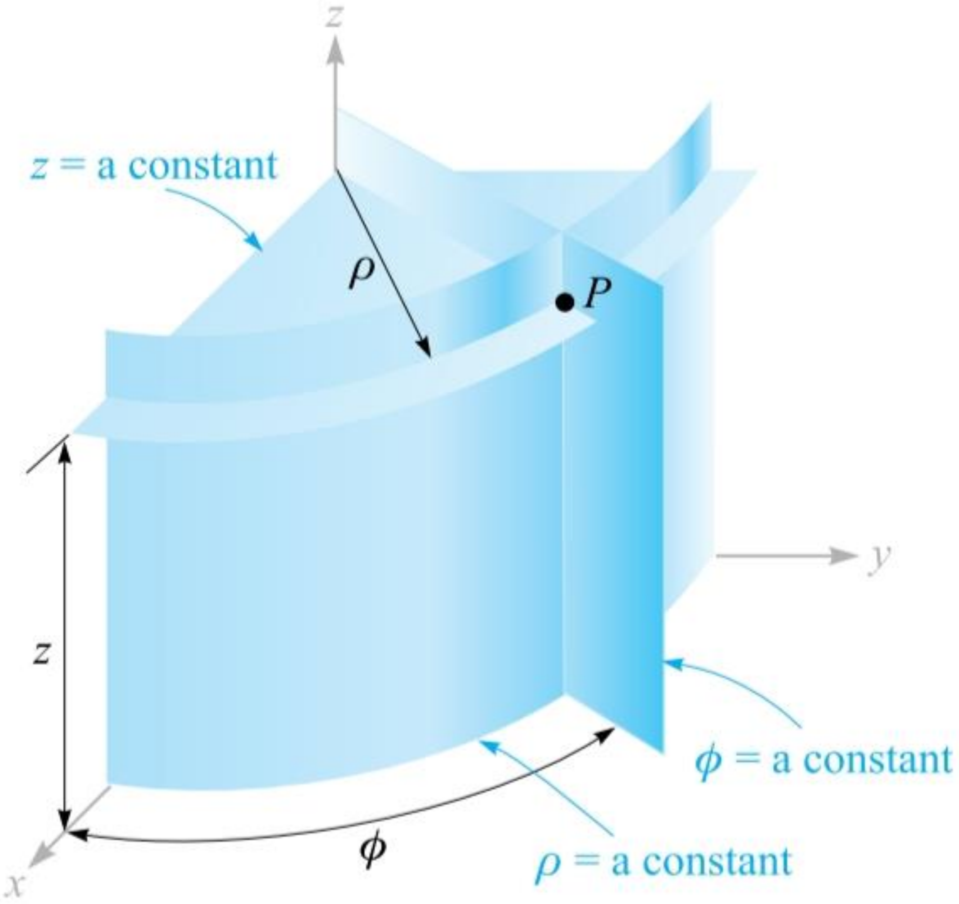


1.7 The Cross Product

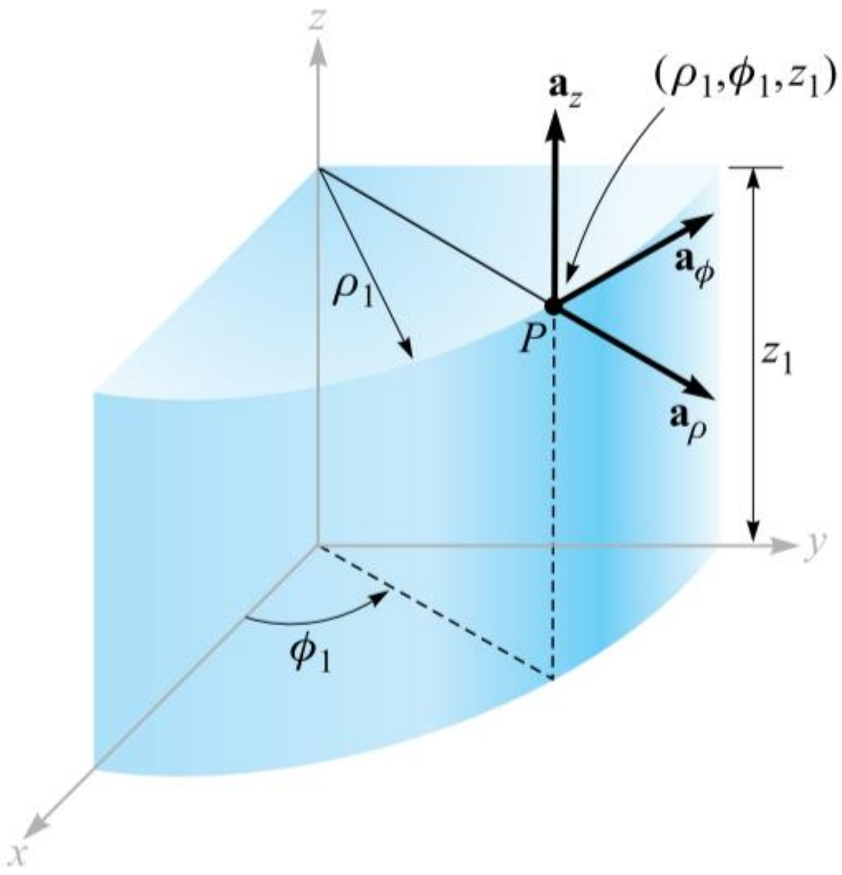
The three vertices of a triangle are located at $A(6,-1,2)$, $B(-2,3,-4)$, and $C(-3,1,5)$. Find: $(a)\mathbf{R}_{AB} \times \mathbf{R}_{AC}$; (b) the area of the triangle; (c) a unit vector perpendicular to the plane in which the triangle is located.

Ans. $24\mathbf{a}_x + 78\mathbf{a}_y + 20\mathbf{a}_z; 42.0; 0.286\mathbf{a}_x + 0.928\mathbf{a}_y + 0.238\mathbf{a}_z$

1.8 The Circular Cylindrical Coordinates

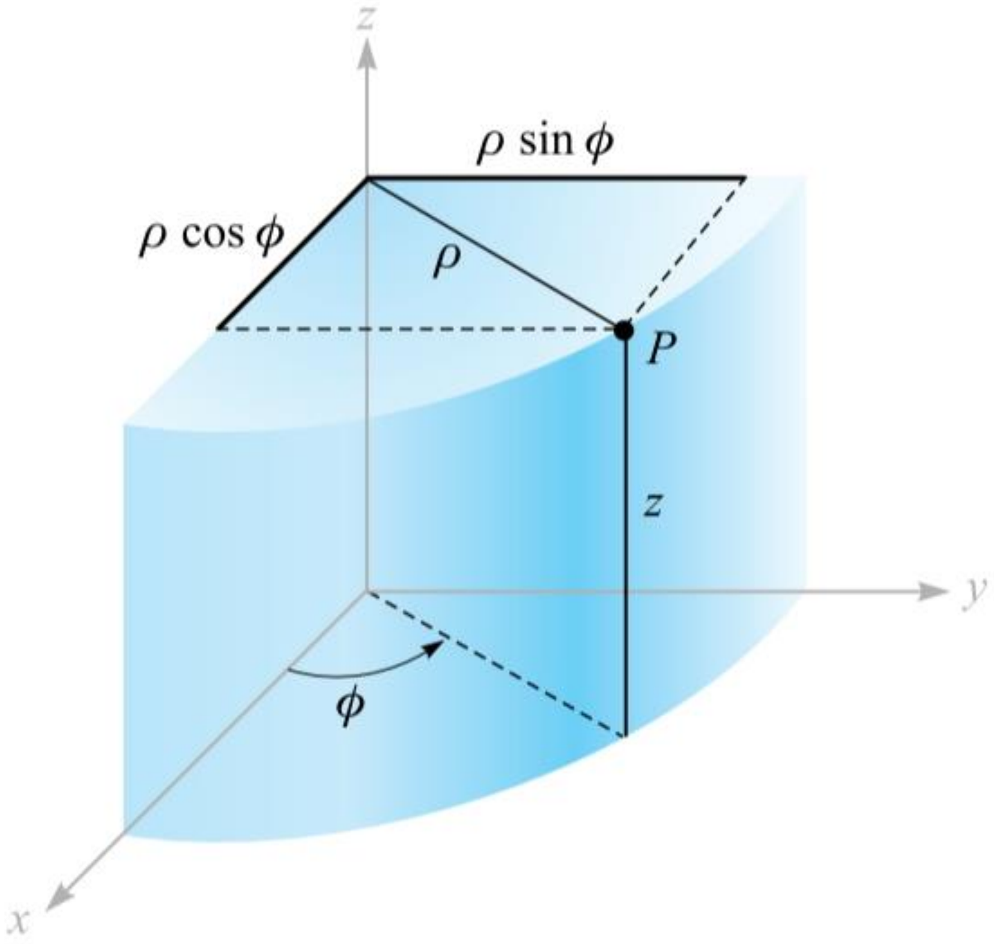


(a)



(b)

1.8 The Circular Cylindrical Coordinates



$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$
$$\phi = \tan^{-1} \frac{y}{x}$$
$$z = z$$

1.8 The Circular Cylindrical Coordinates

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \qquad \mathbf{A} = A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho \qquad A_\rho = \left(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z\right)\cdot\mathbf{a}_\rho = A_x\mathbf{a}_x\cdot\mathbf{a}_\rho + A_y\mathbf{a}_y\cdot\mathbf{a}_\rho$$

$$A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi \qquad A_\phi = \left(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z\right)\cdot\mathbf{a}_\phi = A_x\mathbf{a}_x\cdot\mathbf{a}_\phi + A_y\mathbf{a}_y\cdot\mathbf{a}_\phi$$

$$A_z = \mathbf{A}\cdot\mathbf{a}_z \qquad A_z = \left(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z\right)\cdot\mathbf{a}_z = A_z\mathbf{a}_z\cdot\mathbf{a}_z = A_z$$

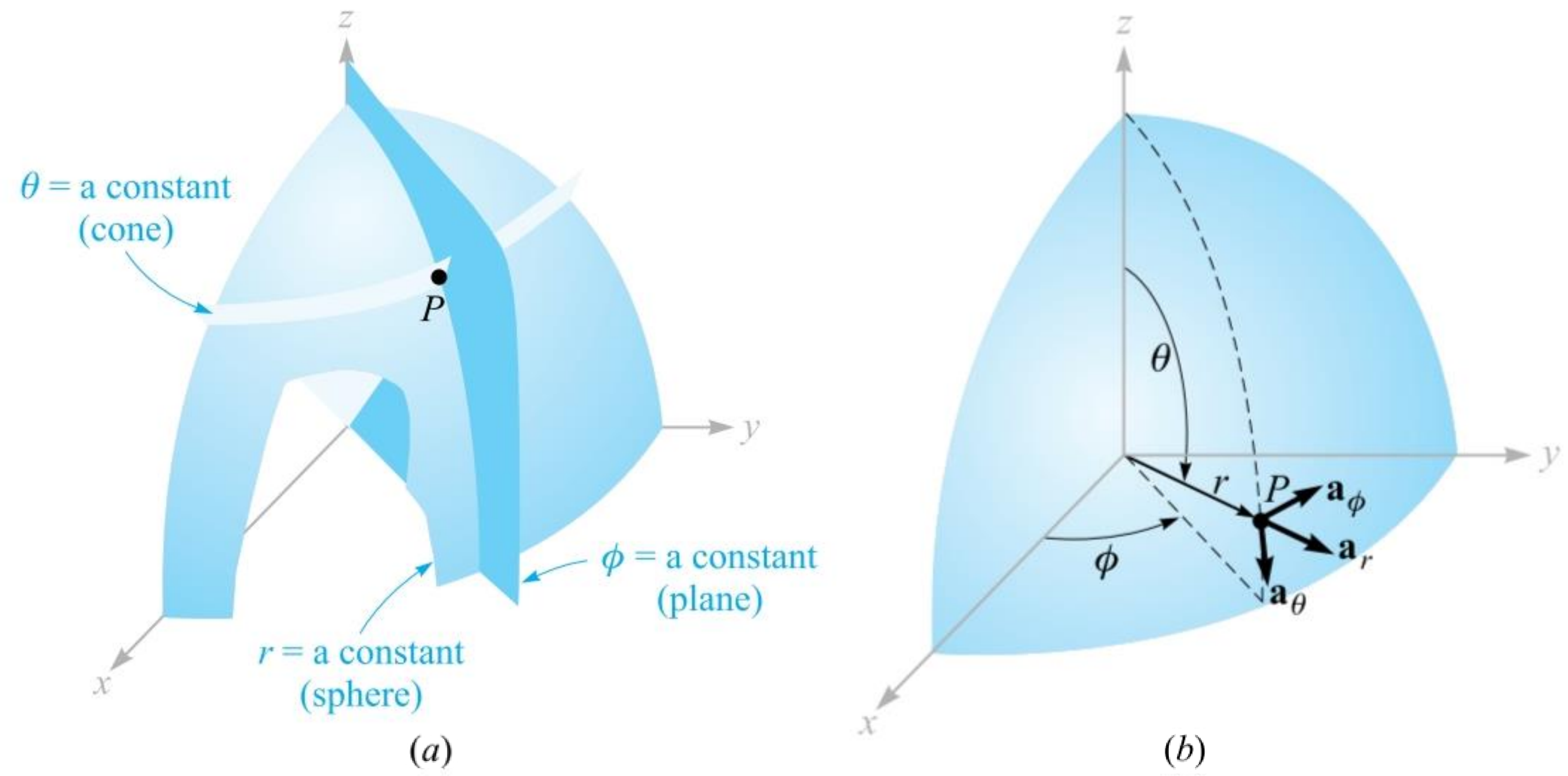
	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	1

1.8 The Circular Cylindrical Coordinates

(a) Give the rectangular coordinates of the point $C(\rho = 4.4, \phi = -115°, z = 2)$. (b) Give the cylindrical coordinates of the point $D(x = -3.1, y = 2.6, z = -3\ )$. (c) Specify the distance from C to D.

Ans. $C(x = -1.860, y = -3.99, z = 2); D(\rho = 4.05, \phi = 140.0°, z = -3); 8.36$

1.9 The Spherical Coordinate



1.9 The Spherical Coordinate

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad (r \geq 0)$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad (0^\circ \leq \theta \leq 180^\circ)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

1.9 The Spherical Coordinate

Given the two points, $C(-3,2,1)$ and $D(r = 5, \theta = 20°, \phi = -70°)$, find: (a) the spherical coordinates of C; (b) the rectangular coordinates of D; (c) the distance from C to D.

Ans. $C(r = 3.74, \theta = 74.5°, \phi = 146.3°)$;

$D(x = 0.585, y = -1.607, z = 4.70)$; 6.29

Problems

1. Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find: (a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$; (b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$; $(c) |\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$.
2. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find (a) the vector $\mathbf{R}_{MN}$; (b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$; (c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$; (d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$.
3. Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of (a) the dot product; (b) the cross product.
4. Express in cylindrical components: (a) the vector from $C(3, 2, -7)$ to $D(-1, -4, 2)$; (b) a unit vector at D directed toward C; (c) a unit vector at D directed toward the origin.