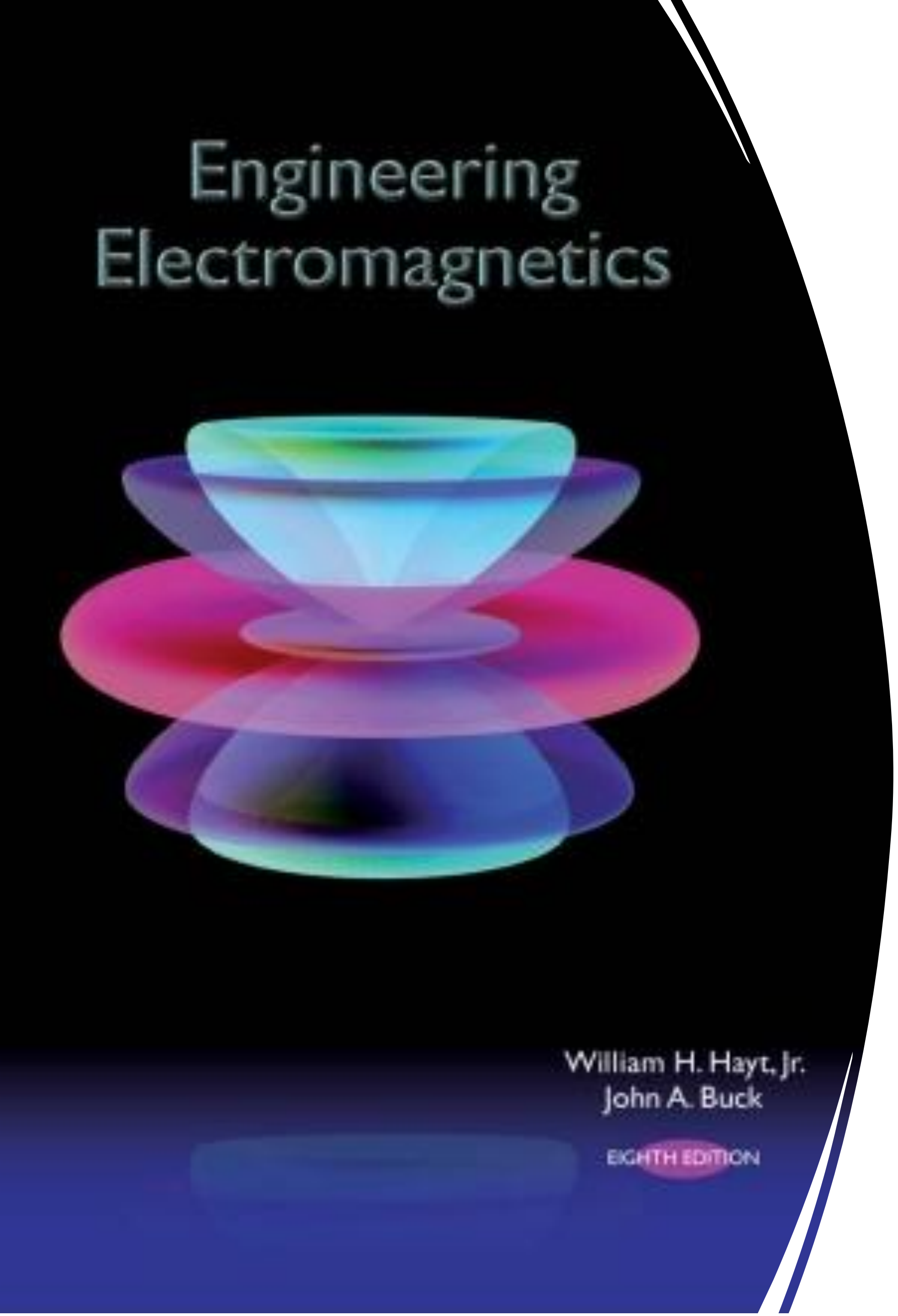

3. Electric Flux Density, Gauss's Law, and Divergence

3.1 Electric Flux Density

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3.5 Divergence and Maxwell’s First Equation

3.6 The Vector Operator ∇ and the Divergence Theorem

3.1 Electric Flux Density

- Faraday's experiment consisted essentially of the following steps:
  1) With the equipment dismantled, the inner sphere was given a known positive charge.
  2) The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
  3) The outer sphere was discharged by connecting it momentarily to ground.
  4) The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured
- Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres.

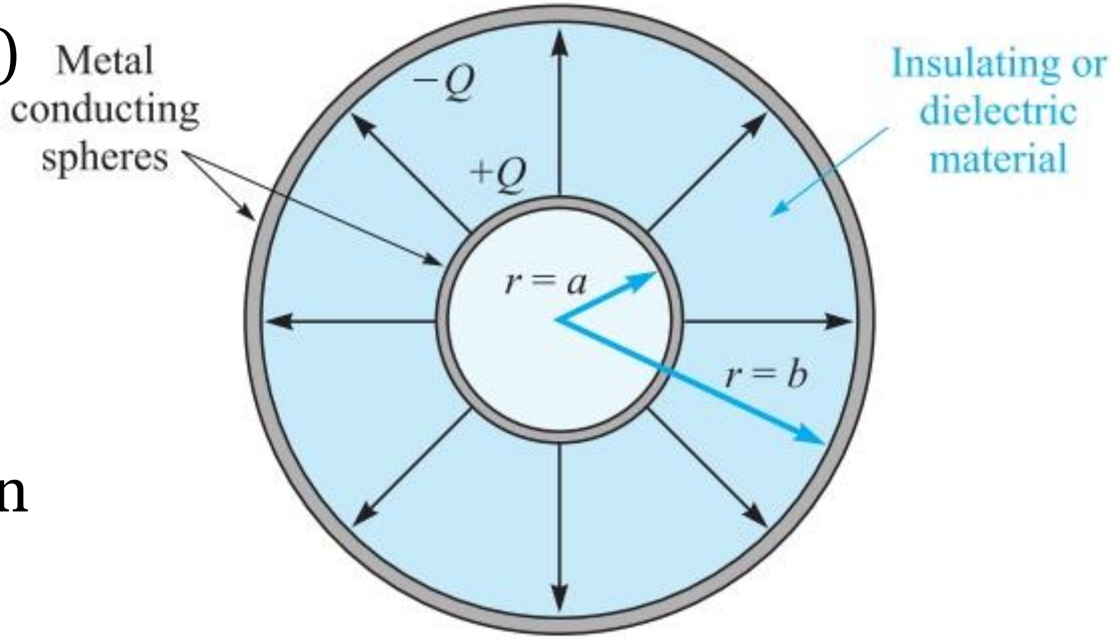
3.1 Electric Flux Density

- If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\Psi = Q$$

and the electric flux Ψ is measured in coulombs.

- The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields



$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} \qquad \text{(free space only)}$$

3.1 Electric Flux Density

Calculate $\mathbf{D}$ in rectangular coordinates at point $P(2,-3,6)$ produced by: (a) a point charge $Q_A = 55\text{mC}$ at $Q(-2,3,-6)$; (b) a uniform line charge $\rho_{LB} = 20\text{mC/m}$ on the x axis; (c) a uniform surface charge density $\rho_{SC} = 120\mu\text{C/m}^2$ on the plane $z = -5$ m.

Ans. $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z \mu\text{C/m}^2; -212\mathbf{a}_y + 424\mathbf{a}_z \mu\text{C/m}^2; 60\mathbf{a}_z \mu\text{C/m}^2$

3.2 Gauss's Law

- These generalizations of Faraday's experiment lead to the following statement, which is known as Gauss's law:

  The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

- The total flux passing through the closed surface is

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed} \\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

3.3 Application of Gauss's Law: Some Symmetrical Charge Distributions

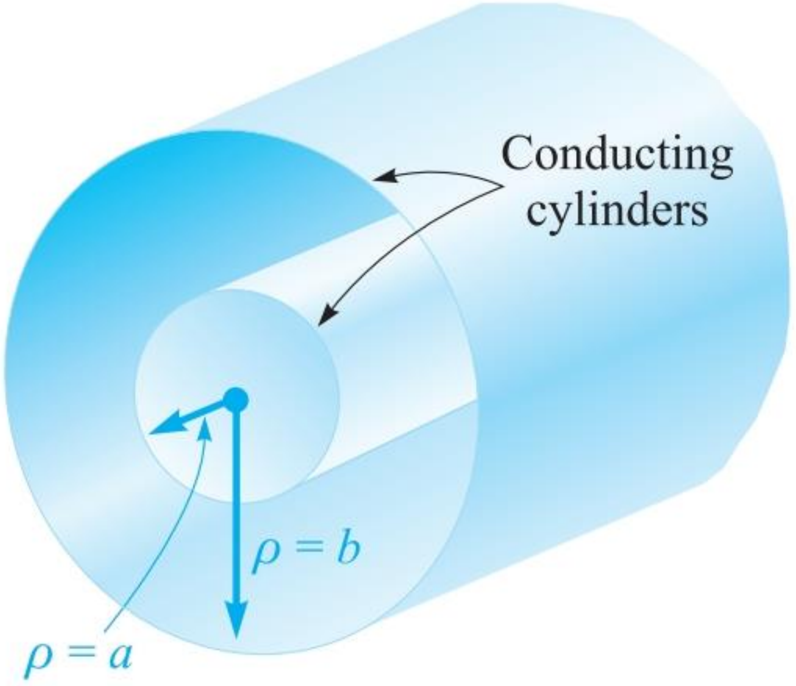
- We now consider how we may use Gauss's law

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral. The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1) $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2) On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

3.3 Application of Gauss's Law: Some Symmetrical Charge Distributions



- The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_S a d\phi dz = 2\pi a L \rho_S$$

$$D_S = \frac{a\rho_S}{\rho} \quad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \ (a < \rho < b)$$

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

3.4 Application of Gauss's Law: Differential Volume Element

- We are now going to apply the methods of Gauss's law to a slightly different type of problem-one that does not possess any symmetry at all.

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

- To evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

3.4 Application of Gauss's Law: Differential Volume Element

- Consider the first of these in detail. Because the surface element is very small, D is essentially constant (over this portion of the entire closed surface) and

$$\int_{\text{front}} \doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}}$$

$$\doteq \mathbf{D}_{\text{front}} \cdot \Delta y\, \Delta z\, \mathbf{a}_x$$

$$\doteq D_{x,\text{front}} \Delta y\, \Delta z$$

$$\int_{\text{back}} \doteq \mathbf{D}_{\text{back}} \cdot \Delta\mathbf{S}_{\text{back}}$$

$$\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y\, \Delta z\, \mathbf{a}_x)$$

$$\doteq -D_{x,\text{back}} \Delta y\, \Delta z$$

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x\, \Delta y\, \Delta z$$

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x\, \Delta y\, \Delta z$$

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x\, \Delta y\, \Delta z$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

3.4 Application of Gauss's Law: Differential Volume Element

In free space, let $\mathbf{D} = 8xyz^4\mathbf{a}_x + 4x^2z^4\mathbf{a}_y + 16x^2yz^3\mathbf{a}_z \text{pC/m}^2$. (a) Find the total electric flux passing through the rectangular surface $z = 2$, $0 < x < 2, 1 < y < 3$, in the $\mathbf{a}_z$ direction. (b) Find $\mathbf{E}$ at $P(2, -1, 3)$. (c) Find an approximate value for the total charge contained in an incremental sphere located at $P(2, -1, 3)$ and having a volume of $10^{-12}\ \text{m}^3$.

Ans. $1365\text{pC}; -146.4\mathbf{a}_x + 146.4\mathbf{a}_y - 195.2\mathbf{a}_z\ \text{V/m}; -2.38 \times 10^{-21}\text{C}$

3.5 Divergence and Maxwell's First Equation

- We will now obtain an exact relationship from

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta \nu$$

by allowing the volume element Δv to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta\nu \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta \nu} = \lim_{\Delta\nu \to 0} \frac{Q}{\Delta \nu} = \rho_\nu$$

3.5 Divergence and Maxwell's First Equation

- The divergence of the vector flux density **A** is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \qquad \text{(rectangular)}$$

$$\text{div } \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta\, D_\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}$$

3.5 Divergence and Maxwell's First Equation

Find div **D** at the origin if $\mathbf{D} = e^{-x}\sin y\mathbf{a}_x - e^{-x}\cos y\mathbf{a}_y + 2z\mathbf{a}_z$.

Ans. 2

3.5 Divergence and Maxwell's First Equation

In each of the following parts, find a numerical value for div**D** at the point specified: (a) $\mathbf{D} = (2xyz - y^2)\mathbf{a}_x + (x^2z - 2xy)\mathbf{a}_y + x^2y\mathbf{a}_z \mathrm{C/m^2}$ at $P_A(2,3,-1)$;
(b) $\mathbf{D} = 2\rho z^2 \sin^2\phi \mathbf{a}_\rho + \rho z^2 \sin 2\phi \mathbf{a}_\phi + 2\rho^2 z \sin^2\phi \mathbf{a}_z \mathrm{C/m^2}$ at $P_B(\rho = 2, \phi = 110°, z = -1)$;
$(c)\mathbf{D} = 2r\sin\theta\cos\phi \mathbf{a}_r + r\cos\theta\cos\phi \mathbf{a}_\theta - r\sin\phi \mathbf{a}_\phi \mathrm{C/m^2}$ at $P_C(r = 1.5, \theta = 30°, \phi = 50°)$.

Ans. $-10.00; 9.06; 1.29$

3.5 Divergence and Maxwell's First Equation

- Finally, we can combine

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}=\rho_v$$

and

$$\text{div}\,\mathbf{D}=\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)$$

and form the relation between electric flux density and charge density:

$$\text{div}\,\mathbf{D}=\rho_v$$

3.6 The Vector Operator ∇ and the Divergence Theorem

- we define the del operator ∇ as a vector operator,

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

- Consider $\nabla \cdot \mathbf{D}$, signifying

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot \left(D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z\right)$$

- We first consider the dot products of the unit vectors, discarding the six zero terms, and obtain the result that we recognize as the divergence of **D** :

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div } \mathbf{D}$$

3.6 The Vector Operator ∇ and the Divergence Theorem

- We have obtained it already and now have little more to do than point it out and name it, for starting from Gauss's law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

- The first and last expressions constitute the divergence theorem,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

- The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

3.6 The Vector Operator ∇ and the Divergence Theorem

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y \mathrm{C/m^2}$ and the rectangular parallelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3 .

Ans. 12 C

Problems

1. A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y \mathrm{C/m^2}$ (a) Apply Gauss's law to find the total flux leaving the closed surface of the cube. (b) Evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube. (c) Estimate the total charge enclosed within the cube.