

Fast Growth, Slow Growth: The Rate of Growth of Continuous Processes

Video companion

1 Introduction

“Exponential rate of growth” can be a *discrete* exponential rate of growth or a *continuous* exponential rate of growth

Discrete rate of growth

$$\$1.00(1 + r)^t$$

How much money would grow in discrete intervals of time t

If $r = 100\%$ /year and $t = 1$, then would have \$2.00 after one year,
After 2 years, would have \$4.00
After 3 years, would have \$8.00...

2 Continuous exponential growth

Euler’s constant e

100% interest per year (discrete)

50% interest for 6 months, then interest on interest for another 6 months.

Interval	Factor	Repeats	Result
1 year	$1 + 1$	1	$(2)^1 = 2$
6 months	1.5	2	$(1.5)^2 = 2.25$
3 months	1.25	4	$(1.25)^4 = 2.44$

As time intervals decrease, does result increase in an unlimited way?

No...

Interval	Factor	Repeats	Result
1 month	1.08	12	$(1.08)^{12} = 2.613$
1 week	1.019	52	$(1.019)^{52} = 2.693$
1 day	1.002739	365	$(1.002739)^{365} = 2.7146$
1 hour	1.000114	8760	$(1.000114)^{8760} = 2.71813$
1 minute	1.0000019	525,600	$(1.0000019)^{525,600} = 2.71828$
1 second	1.0000000317	31,536,000	$(1.0000000317)^{31,536,000} = 2.71828$

$e = 2.71828$, Euler's constant

Problem A baby elephant weighing 200 kg grows at a continuously compounded rate of 5%/year. How much does it weigh in 3 years?

$$(200 \text{ kg})e^{(0.05)(3)} = 232.4 \text{ kg}$$

3 Continuous rate of return

“Log to the base e of x ” is given by the symbol $\ln(x)$, where \ln stands for *natural logarithm*.

Problem Rabbit population increases in mass at a rate of 200% per year. Population starts at 10 kg. If they increase at a continuously compounded rate, how many years is it until they weigh as much as the Earth (5.972×10^{24} kg)?

$$\begin{aligned} 5.972 \times 10^{24} \text{ kg} &= (10 \text{ kg})e^{2t} \\ 5.972 \times 10^{23} &= e^{2t} \\ \ln(5.972 \times 10^{23}) &= \ln(e^{2t}) = 2t \\ \frac{\ln(5.972 \times 10^{23})}{2} &= t = 27.37 \text{ years} \end{aligned}$$