IME625A: PROJECT REPORT

TOPIC: BLACK-SCHOLES-MERTON MODEL FOR ANALYSIS OF OPTIONS

Course Title: Introduction to Stochastic Processes and their Applications

Submitted By:-

Piyush agarwal (190600)

Nirmal Agarwal (190557)

Sparsh Sihotiya (190860)

April 23, 2023

1 TERMINOLOGY

Let's begin by defining the key terms used throughout this report.

1.1 Options

In finance, an option is a contract that grants the buyer the right, but not the obligation, to purchase or sell an underlying asset at a specified price and time. The fundamental asset may be a stock, a commodity, a currency, or any other type of financial instrument. There are two primary options types: calls and puts.

In financial markets, options are frequently used for speculation, hedging, and risk management. Options may also be used to protect a portfolio against potential losses or to generate income through options trading strategies.

1.1.1 Call Option

A call option is a financial contract that grants the buyer the right, but not the obligation, to acquire the underlying asset at the strike price before the option's expiration date. The fundamental asset may be a stock, a commodity, a currency, or any other type of financial instrument.

When people purchase a call option, they are wagering that the underlying asset's price will rise above the strike price before the option expires. If the price of the underlying asset rises above the strike price, the buyer can exercise the option and purchase the asset at the lower strike price, allowing them to generate a profit by selling the asset at the higher market price.

1.1.2 Put Option

A put option is a financial contract that grants the buyer the right, but not the obligation, to sell the underlying asset at the strike price before the option's expiration date. The fundamental asset may be a stock, a commodity, a currency, or any other type of financial instrument.

When a person purchases a put option, they are betting that the underlying asset's price will decline below the strike price before the option expires. If the price of the underlying asset falls below the strike price, the buyer can exercise the option and sell the asset at the higher strike price, allowing them to profit by purchasing the asset at the lower market price.

1.2 Intrinsic Value

The intrinsic value of an option is the difference between the current market value of the underlying asset and the option's strike price. The intrinsic value of a call option is calculated as the difference between the market price of the underlying asset and the strike price, whereas the intrinsic value of a put option is the difference between the strike price and the market price of the underlying asset.

1.3 Premium

A premium is a price an investor or investor pays to acquire an option contract. It is the cost of purchasing the right, but not the obligation, to purchase or sell an underlying asset at a predetermined price, known as the strike price, before the option's expiration date.

Traders typically evaluate whether an option is overvalued or under-priced by comparing its premium to its intrinsic value (if any).

1.4 Time Value

In finance, time value is the component of an option's premium that reflects the time remaining until the option's expiration date. It represents the additional value an option buyer is willing to pay for the right, but not the obligation, to purchase or sell an underlying asset at a predetermined price during a specified time period.

The time value of an option is determined by subtracting its intrinsic value from its total premium. Intrinsic value is the difference between the underlying asset's strike and market prices, whereas time value represents the remainder of the premium.

2 FACTOR OF OPTIONS PRICING

It is time to shift our attention to the variables that influence option pricing. Six parameters primarily influence the process of determining option pricing.

2.1 Underlying Price

The current market price of the underlying asset is one of the most significant determinants of the option's price. Call options generally increase in value as the price of the underlying asset increases, whereas put options increase in value as the price of the underlying asset falls. Its impact on option types can be summarized in the table given below:

Underlying Price	Call Price	Put Price
↑	↑	↓
↓	↓	↑

2.2 Expected Volatility

Implied volatility measures how much the market expects the underlying asset's price fluctuates over time. All else being equivalent, options on assets with higher implied volatility will generally be more expensive than options on assets with lower implied volatility. The typical relationship between expected volatility and option price is as follows:

Underlying Price	Call Price
↑	↑

2.3 Strike Price

The strike price is the price at which an option may be exercised. The relationship between the strike price and the underlying asset price is crucial in determining the price of an option. For call options to be in the money, the strike price must be less than the price of the underlying asset, while for put options, the strike price must be greater than the underlying asset's price.

2.4 Time Until Expiration

The remaining time until an option's expiration date is also a significant factor in determining its price. As the time remaining until expiration decreases, so does the option's time value. The underlying asset's price has less time to move in the option buyer's favor.

2.5 Interest Rate

In addition to impacting the cost of financing and holding the fundamental asset, the level of interest rates can also affect the pricing of options. As interest rates rise, call options will become more costly, and put options will become less expensive.

Interest Rate	Call Price	Put Price
↑	↑	↓
↓	↓	↑

2.6 Dividends

The fact that the fundamental asset pays dividends can also affect the pricing of options. When the underlying asset pays dividends, call options become less costly, and put options become more expensive as the value of holding the underlying asset decreases due to the dividend payments.

Dividends	Call Price	Put Price
↑	↓	↑
↓	↑	↓

3 Brownian Motion

3.1 Introduction

Brownian motion can be described as a mathematical model that describes the evolution of a system over time in a probabilistic manner. It is an example of continuous time, continuous state space Markov process, a stochastic process which is characterized by the continuous random movement of particles in a fluid. This movement is described by a stochastic differential equation that captures the random fluctuations in the motion of the particles. It is also a commonly used model in finance, where it is used to describe the random fluctuations of stock prices and other financial assets over time.

3.2 Properties

- It is normally distributed with zero mean and non-zero variance.
- It has normally distributed increments.
- It follows Markov property which states that the state in which the stochastic process is present only depends on the most recent known past state.

- It follows Martingale Property which states that conditional expectation of future is equal to the present value, given the information about past events

3.3 Geometric Brownian Motion

Geometric Brownian Motion (GBM) is a stochastic process that is commonly used in financial modeling to describe the behavior of stock prices, currency exchange rates, and other financial assets over time.

The GBM process is defined by the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where $S(t)$ is the price of the asset at time t, μ is the expected rate of return on the asset, σ is the volatility of the asset, $dW(t)$ is a Wiener process or Brownian motion, and dt is an infinitesimal time interval.

4 Black-Scholes-Merton Model

4.1 ITO's Lemma

The price of a stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any derivative is a function of the stochastic variables underlying the derivative and time. An important result which is known as Itoˆ's lemma.
Suppose that the value of a variable x follows the Ito's process:

$$dx = a(x,t)dt + b(x,t)dz$$

where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2. Ito's lemma shows that a function f of x and t follows the process

$$df = (\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2)dt + \frac{\partial f}{\partial x}bdz$$

4.2 Lognormal Property Of Stock Prices

Black-Scholes-Model assumes that the percentage changes in the stock price in a short period of time are normally distributed.

μ : Expected return on stock per year.

σ : Volatility of the stock price per year.

The mean of the return in time Δt is $\mu \Delta t$ and the standard deviation of the return is $\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma^2 \Delta t)$$

4.3 Assumptions

The assumptions we use to derive the Black–Scholes–Merton differential equation are as follows:

- The stock price follows the geometric brownian motion process with μ and σ constant.
- The short selling of securities with full use of proceeds is permitted.
- There are no transactions costs or taxes. All securities are perfectly divisible.
- There are no dividends during the life of the derivative.
- There are no riskless arbitrage opportunities.
- Security trading is continuous.
- The risk-free rate of interest, r, is constant and the same for all maturities.

Now, Some of these assumptions can be relaxed. For example, σ and r can be known functions of t. We can even allow interest rates to be stochastic provided that the stock price distribution at maturity of the option is still lognormal.

4.4 Black Scholes Formula

The most famous solutions to the above differential equation are the Black–Scholes–Merton formulas for the prices of European call and put options.

This Black Scholes Formula is used to find the price of a European option given its price of the underlying asset, time till expiration of the option, volatility, and the risk-free interest rate.

$$c = SN(d_1) - N(d_2)Ke^{-rt}$$

$$p = Ke^{-rt}N(-d_2) - SN(-d_1)$$

where

- $d_1 = \frac{\ln(\frac{S}{K})+(r+\frac{s^2}{2})t}{\sigma\sqrt{t}}$
- $d_2 = d_1 - \sigma\sqrt{t}$
- S : current stock price
- C : Call Premium
- t : time until expiration
- K : option strike price
- r : risk-free interest rate
- N : cumulative standard normal distribution
- σ : standard deviation

4.5 Deriving Black-Scholes-Merton Differential Equation

Suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t.

$$\Delta S = \mu S \Delta t + \sigma S \Delta z$$

using Ito's Lemma

$$\Delta f = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\triangle t + \frac{\partial f}{\partial S}\sigma S \triangle z$$

The holder of this portfolio is short one derivative and long an amount $\frac{\partial f}{\partial S}$ of shares. Define Π as the value of the portfolio. By definition

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change $\triangle\Pi$ in the value of the portfolio in the time interval $\triangle t$ is given by

$$\triangle\Pi = -\triangle f + \frac{\partial f}{\partial S}\triangle S$$

Also,

$$\triangle\Pi = r\Pi\triangle t$$

Using the above equations and solving them, we get the Black–Scholes–Merton differential equation as

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf$$

Numerous solutions exist, corresponding to the various derivatives that can be defined with S as the fundamental variable. The specific derivative obtained when the equation is solved is determined by the boundary conditions used. These indicate the values of the derivative at the limits of possible S and t values.

In the case of a European call option, the key boundary condition is

$$f = max(S - K, 0)\ when\ t = T$$

In the case of a European put option, it is

$$f = max(K - S, 0)\ when\ t = T$$

5 Analysis

Implied volatility is a crucial component of the Black-Scholes model, which is used to calculate the theoretical price of European-style options. It represents the expected future volatility of the underlying asset, as implied by the market price of the option. In other words, implied volatility is the level of volatility that would make the Black-Scholes model output the same price as the actual market price of the option. Implied volatility can be thought of as a measure of uncertainty or risk in the market, and it is influenced by a variety of factors, such as changes in interest rates, market sentiment, and geopolitical events. Accurately predicting implied volatility is essential for traders and investors who use options as a hedging or speculative too.

To estimate implied volatility using the Black-Scholes model, traders and investors can use a process called **"back-solving"**. This involves inputting the current market price of the option, along with other known variables such as the strike price, time to expiration, and current underlying asset price, into the Black-Scholes formula and solving for the unknown variable of implied volatility. While this can be a time-consuming process, Estimating implied volatility using the Black-Scholes model is a valuable tool for traders and investors looking to make informed decisions about options trading strategies.

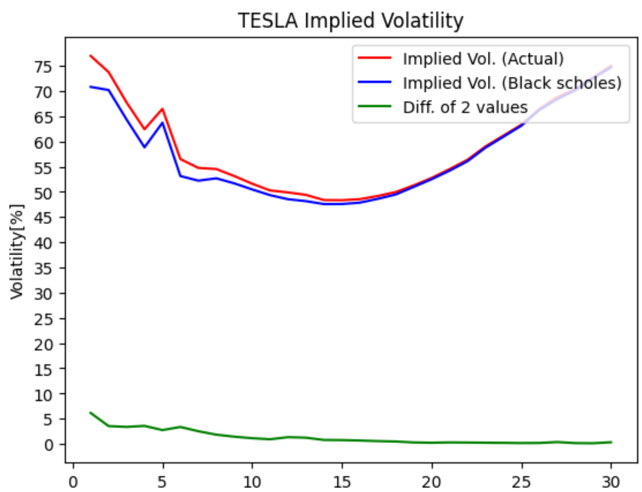
We utilized the well-known **Newton-Raphson method** to estimate the Implied Volatility in the back-solving process. Our approach involved approximating the fair value of the option price and updating the volatility simultaneously. For our analysis, we focused on the closest expiry call options for Tesla and Google. We retrieved the latest option chain data using the **yfinance library** and filtered out options with a strike price close to the underlying asset. Given that our analysis was based on the most recent data, certain parameters were taken into consideration.

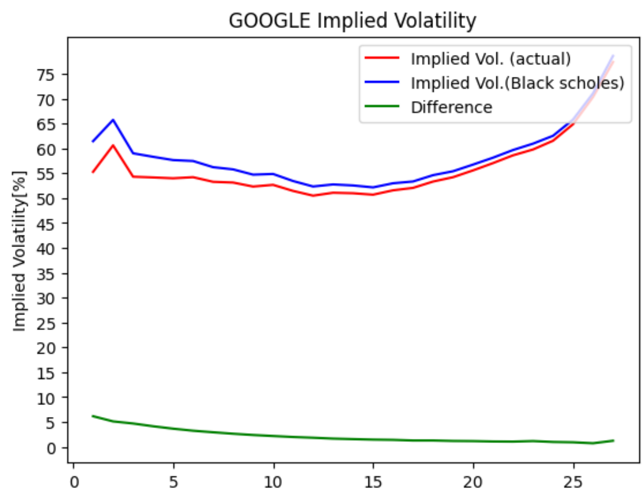
- Expiration date- 29/04/23
- Option type- CALL
- Interest Rate- 4.77% (US 1 year Treasury bond rate)
- Tesla underlying price- 165.08 (as of 23/04/23)
- Google underlying price- 105 (as of 23/04/23)
- Maximum iteration for Newton-Raphson method- 1000

Based on our analysis, it appears that the implied volatility derived from our algorithm is in close proximity to the actual implied volatility. This finding highlights the practical application of the Black-Scholes formula in the analysis of options within the context of real-world financial markets.

Attached, you will find a visual representation of the actual and calculated implied volatility. Additionally, the code snippet provided displays the fundamental functions used in the calculation process.

For the notebook : **click here**






```python
from scipy.stats import norm
import datetime
import numpy as np
import matplotlib.pyplot as plt
# The objective of this section is to determine the implied volatility for a given option.
# To accomplish this, we employ the Newton-Raphson method, which begins with an initial estimate of 0.5 for the sigma value.
# We then aim to approximate the current price of the underlying security.

def find_vol(target_price, call_put, S, K, T, r):
    NO_OF_ITERATIONS = 1000
    EPSILON = 0.00001

    constant = 0.5
    # Running the loop for 1000 times.
    for i in range(0, NO_OF_ITERATIONS):
        ## our F(x) for newton raphson is Target_price- price_from_black_scholes
        price = bs_price(call_put, S, K, T, r, constant) ##calculating price using Black scholes
        vega = bs_vega(call_put, S, K, T, r, constant) ## vega is rate of change of option price with repect to volatility
        delta = target_price - price
        if (abs(delta) < EPSILON):    ## stopping condition
            return constant
        constant = constant + delta/vega # f(x) / f'(x) ## equation of newton raphson

    # value wasn't found, return best guess so far
    return constant

n = norm.pdf
N = norm.cdf

def bs_price(cp_flag,S,K,T,r,v,q=0.0): ## standart function to calculate price using black scholes
    d1 = (np.log(S/K)+(r+v*v/2.)*T)/(v*np.sqrt(T))
    d2 = d1-v*np.sqrt(T)
    if cp_flag == 'call':
        price = S*np.exp(-q*T)*N(d1)-K*np.exp(-r*T)*N(d2)
    else:
        price = K*np.exp(-r*T)*N(-d2)-S*np.exp(-q*T)*N(-d1)
    return price

def bs_vega(cp_flag,S,K,T,r,v,q=0.0): ## calculating vega
    d1 = (np.log(S/K)+(r+v*v/2.)*T)/(v*np.sqrt(T))
    return S * np.sqrt(T)*n(d1)
```

References

[1] Hull, John, 1946 (2012). *Options, futures, and other derivatives*, Boston :Prentice Hall, 2012

[2] We have used ChatGPT and Google Search for the basic terminologies.