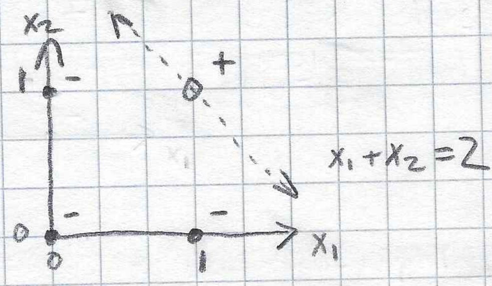
Assignment 2

Q 2 a)

i) And function

Consider the following graph encoding the input space and desired classifications:



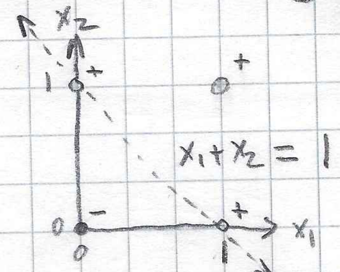
$$\Rightarrow \quad y = \begin{cases} 1, \text{ if } \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

The decision boundary $x_1 + x_2 = 2$ fits the data perfectly.

Thus, the and function can be encoded as a threshold perceptron with $w^T = [1, 1]$ and $w_0 = -2$

ii) Or function

Consider the following graph:



$$\Rightarrow \quad y = \begin{cases} 1 \text{ if } \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

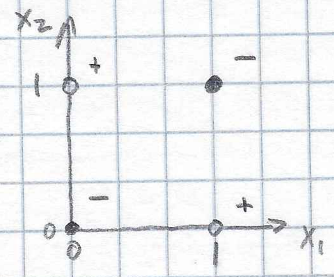
The decision boundary $x_1 + x_2 = 1$ fits the data perfectly.

Thus, the or function can be encoded as a threshold perceptron with

$w^T = [1, 1]$ and $w_0 = -1$

iii) Exclusive OR

Consider the following graph :



Assume we have a perceptron with weights w_1, w_2, w_0 that correctly classifies these points.

This implies :

$$w_1 + w_0 > 0 \quad (1)$$
$$w_2 + w_0 > 0 \quad (2)$$
$$w_1 + w_2 + w_0 < 0 \quad (3)$$
and
$$w_0 < 0 \quad (4)$$

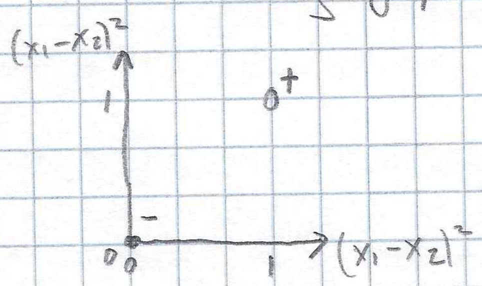
By (1)+(2) $\Rightarrow$ $w_1 + w_2 + 2w_0 > 0$ (5)

By (5), (3) $\Rightarrow$ $w_0 > 0$ (6).

(6), (4) form a contradiction.

Thus, the points are not linearly seperable and we must apply some mapping $\phi : X \rightarrow X$

Consider the mapping $\phi(x_1, x_2) = ((x_1 - x_2)^2, (x_1 - x_2)^2)$, and the following graph :

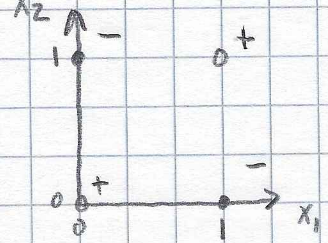


Now, with the transformed space, the data is clearly linearly seperable. We can apply the same decision boundary from the and function.

Thus, the mapping $\phi(x_1, x_2) = ((x_1 - x_2)^2, (x_1 - x_2)^2)$ and a threshold perceptron with $w^T = [1, 1]$ and $w_0 = -2$ correctly induces the exclusive or function.

iv) IFF function

Consider the following graph:

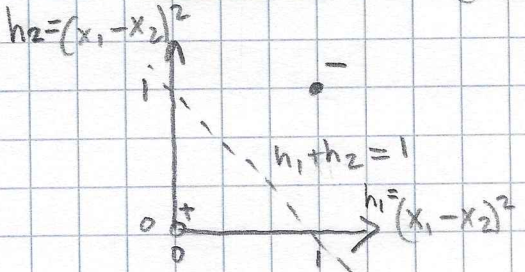


This function is the inverse of XOR function.

WLOG, since we could not linearly seperate the points induced by XOR, we cannot find a threshold perceptron that perfectly classifies IFF.

However, we can consider the same feature map used for XOR.

$\phi(x_1, x_2) = \left((x_1-x_2)^2, (x_1-x_2)^2\right)$ induces:



$$\Rightarrow \quad y = \begin{cases} 1 & \text{if } \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + 1 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the graph above, the threshold perceptron described induces the correct classifications on the transformed input space.

Thus $\phi(x_1, x_2) = \left((x_1-x_2)^2, (x_1-x_2)^2\right)$ and $w^T = [-1, -1]$ and $w_0 = 1$ create a threshold perceptron that encodes IFF.