$A - \left[\frac{5}{..} \quad \frac{..}{..}\right] \qquad \det(A - \lambda I) = 0$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m_1 (\dot{x}_1 - \dot{\overline{l_1 \sin\theta_1}})^2 + \frac{1}{2} m_2 (\dot{x}_2 - \dot{\overline{l_2 \sin\theta_2}})^2 + \frac{1}{2} m_1 (\dot{\overline{l_1 - l_1\cos\theta_1}})^2 + \frac{1}{2} m_2 (\dot{\overline{l_2 - l_2\cos\theta_2}})^2$$

$$= \frac{1}{2}(M \quad)\dot{x}^2 + \frac{1}{2} m_1 \left((\dot{x}_1 - l_1\cos\theta_1 \dot{\theta}_1)^2 + (l_1 \sin\theta_1 \dot{\theta}_1)^2\right) + \frac{1}{2} m_2 \left((\dot{x} - l_2 \cos\theta_2 \dot{\theta}_2)^2 + (l_2 \sin\theta_2 \dot{\theta}_2)^2\right)$$

$$= \frac{1}{2}(M \quad)\dot{x}^2 + \frac{1}{2} m_1 (\dot{x}_1^2 - 2 l_1 \cos\theta_1 \dot{\theta}_1 \dot{x} + l_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + l_1^2 \sin^2\theta_1 \dot{\theta}_1^2) + \frac{1}{2} m_2 (\dot{x}^2 - 2 l_2 \cos\theta_2 \dot{\theta}_2 \dot{x} + l_2^2 \sin^2\theta_2 \dot{\theta}_2^2 + l_2^2 \cos^2\theta_2 \dot{\theta}_2^2)$$

$$T = \frac{1}{2}(M \quad)\dot{x}^2 + \frac{1}{2} m_1 (\dot{x}_1^2 - 2 l_1 \cos\theta_1 \dot{\theta}_1 \dot{x} + l_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (\dot{x}^2 - 2 l_2 \cos\theta_2 \dot{\theta}_2 \dot{x} + l_2^2 \dot{\theta}_2^2)$$

$$V = m_1 g l_1 (1 - \cos\theta_1) + m_2 g l_2 (1 - \cos\theta_2)$$

$$L = T - V$$

$$L = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_1 (-2 l_1 \cos\theta_1 \dot{\theta}_1 \dot{x} + l_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (-2 l_2 \cos\theta_2 \dot{\theta}_2 \dot{x} + l_2^2 \dot{\theta}_2^2) - m_1 g l_1 (1 - \cos\theta_1) - m_2 g l_2 (1 - \cos\theta_2)$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \text{Input}$$

x:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = F \; ; \; \frac{\partial L}{\partial x} = 0 \; ; \; \frac{\partial L}{\partial \dot{x}} = (M + m_1 + m_2)\dot{x} - m_1 l_1 \cos\theta_1 \dot{\theta}_1 - m_2 l_2 \cos\theta_2 \dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M + m_1 + m_2)\ddot{x} - m_1 l_1 (-\sin\theta_1 \dot{\theta}_1^2 + \cos\theta_1 \ddot{\theta}_1) - m_2 l_2 (-\sin\theta_2 \dot{\theta}_2^2 + \cos\theta_2 \ddot{\theta}_2)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = F \Rightarrow \boxed{(M + m_1 + m_2)\ddot{x} + m_1 l_1 \sin\theta_1 \dot{\theta}_1^2 - m_1 l_1 \cos\theta_1 \ddot{\theta}_1 + m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 - m_2 l_2 \cos\theta_2 \ddot{\theta}_2 - F = 0}$$

$\rightarrow$ Eq (1)

$\Rightarrow \theta_1$:

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 0 \; ; \; \frac{\partial L}{\partial \theta_1} = m_1 l_1 \sin\theta_1 \dot{\theta}_1 \dot{x} - m_1 g l_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = -m_1 l_1 \cos\theta_1 \dot{x} + m_1 l_1^2 \dot{\theta}_1 \; ; \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 (-\sin\theta_1 \dot{\theta}_1 \dot{x} + \cos\theta_1 \ddot{x}) = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \cos\theta_1 \ddot{x} + m_1 l_1 \sin\theta_1 \dot{\theta}_1 \dot{x}$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 0 \Rightarrow \cancel{m_1 l_1 \sin\theta_1 \dot{\theta}_1 \dot{x}} - m_1 g l_1 \sin\theta_1 - m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 \cos\theta_1 \ddot{x} - \cancel{m_1 l_1 \sin\theta_1 \dot{\theta}_1 \dot{x}}$$

$$\Rightarrow \boxed{m_1 l_1 \cos\theta_1 \ddot{x} - m_1 l_1^2 \ddot{\theta}_1 - m_1 g l_1 \sin\theta_1 = 0} \longrightarrow \text{Eq (2)}$$

$\Rightarrow \theta_2$:

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = 0 \; ; \; \frac{\partial L}{\partial \theta_2} = m_2 l_2 \sin\theta_2 \dot{\theta}_2 \dot{x} - m_2 g l_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = -m_2 l_2 \cos\theta_2 \dot{x} + m_2 l_2^2 \dot{\theta}_2 \; ; \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 (-\sin\theta_2 \dot{\theta}_2 \dot{x} + \cos\theta_2 \ddot{x}) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 \sin\theta_2 \dot{\theta}_2 \dot{x} - m_2 l_2 \cos\theta_2 \ddot{x}$$

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = 0 \; ; \; \cancel{m_2 l_2 \sin\theta_2 \dot{\theta}_2 \dot{x}} - m_2 g l_2 \sin\theta_2 - m_2 l_2^2 \ddot{\theta}_2 - \cancel{m_2 l_2 \sin\theta_2 \dot{\theta}_2 \dot{x}} + m_2 l_2 \cos\theta_2 \ddot{x} = 0$$

$$\boxed{\Rightarrow m_2 l_2 \cos\theta_2 \ddot{x} - m_2 l_2^2 \ddot{\theta}_2 - m_2 g l_2 \sin\theta_2 = 0} \longrightarrow \text{Eq (3)}$$

Eq 1 : $(M+m_1+m_2)\ddot{x}+m_1 l_1(\sin\theta_1\dot{\theta}_1^2-\cos\theta_1\ddot{\theta}_1)+m_2 l_2(\sin\theta_2\dot{\theta}_2^2-\cos\theta_2\ddot{\theta}_2)-F=0$

Eq 2 : $m_1 l_1\cos\theta_1\ddot{x}-m_1 l_1^2\ddot{\theta}_1-m_1 g l_1\sin\theta_1=0 \Rightarrow \cos\theta_1\ddot{x}-\ddot{\theta}_1-g\sin\theta_1=0$

Eq 3 : $m_2 l_2\cos\theta_2\ddot{x}-m_2 l_2^2\ddot{\theta}_2-m_2 g l_2\sin\theta_2=0 \Rightarrow \cos\theta_2\ddot{x}-\ddot{\theta}_2-g\sin\theta_2=0$

(2)

$$\cos\theta_1 \ddot{x} = l_1\ddot{\theta}_1 + g\sin\theta_1 \Rightarrow \ddot{\theta}_1 = \frac{\cos\theta_1 \ddot{x}}{l_1} - \frac{g\sin\theta_1}{l_1}$$

$$\cos\theta_2 \ddot{x} - l_2\ddot{\theta}_2 - g\sin\theta_2 = 0 \Rightarrow \ddot{\theta}_2 = \frac{\cos\theta_2 \ddot{x}}{l_2} - \frac{g\sin\theta_2}{l_2}$$

Finding equations for $\ddot{x}$, $\ddot{\theta}_1$ & $\ddot{\theta}_2$

$$(M+m_1+m_2)\ddot{x} + m_1 l_1 \sin\theta_1 \dot{\theta}_1^2 - m_1 l_1 \cos\theta_1\left(\frac{\cos\theta_1 \ddot{x} - g\sin\theta_1}{l_1}\right) + m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 - m_2 l_2 \cos\theta_2\left(\frac{\cos\theta_2 \ddot{x} - g\sin\theta_2}{l_2}\right) - F = 0$$

$$(M+m_1+m_2)\ddot{x} + m_1 l_1\sin\theta_1 \dot{\theta}_1^2 - m_1\cos^2\theta_1 \ddot{x} + m_1 g\sin\theta_1\cos\theta_1 + m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 - m_2\cos^2\theta_2 \ddot{x} + m_2 g\sin\theta_2\cos\theta_2 - F = 0$$

$$(M+\underbrace{m_1 - m_1\cos^2\theta_1}_{m_1(1-\cos^2\theta_1)} + m_2 - m_2\cos^2\theta_2)\ddot{x} + m_1\sin\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) + m_2\sin\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2) - F = 0$$

$$(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)\ddot{x} = F - m_1\sin\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2)$$

$$\boxed{\ddot{x} = \frac{F - m_1\sin\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2)}{M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2}} \quad - (1)$$

$$l_1\ddot{\theta}_1 = \cos\theta_1\left(\frac{F - m_1\sin\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2)}{M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2}\right) - g\sin\theta_1$$

$$\ddot{\theta}_1 = \frac{F\cos\theta_1 - m_1\sin\theta_1\cos\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2\cos\theta_1(l_2\dot{\theta}_2^2 + g\cos\theta_2) - Mg\sin\theta_1 - m_1 g\sin^3\theta_1 - m_2 g\sin\theta_1\sin^2\theta_2}{l_1(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)}$$

$$= \frac{F\cos\theta_1 - m_1\sin\theta_1(l_1\cos\theta_1\dot{\theta}_1^2 + g\cos^2\theta_1 + g\sin^2\theta_1) - m_2\sin\theta_2\cos\theta_1(l_2\dot{\theta}_2^2 + g\cos\theta_2) - g\sin\theta_1(M + m_2\sin^2\theta_2)}{l_1(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)}$$

$$\boxed{\ddot{\theta}_1 = \frac{F\cos\theta_1 - m_1\sin\theta_1(l_1\cos\theta_1\dot{\theta}_1^2 + g) - m_2\sin\theta_2\cos\theta_1(l_2\dot{\theta}_2^2 + g\cos\theta_2) - g\sin\theta_1(M + m_2\sin^2\theta_2)}{l_1(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)}} \quad (2)$$

$$\ddot{\theta}_2 = \frac{F\cos\theta_2 - m_1\sin\theta_1\cos\theta_2(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2\cos\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2) - Mg\sin\theta_2 - m_1 g\sin\theta_2\sin^2\theta_1 - m_2 g\sin^3\theta_2}{l_2(M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2)}$$

$$\boxed{\ddot{\theta}_2 = \frac{F\cos\theta_2 - m_1\sin\theta_1\cos\theta_2(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\cos\theta_2\dot{\theta}_2^2 + g) - g\sin\theta_2(M + m_1\sin^2\theta_1)}{l_2(M + m_1\sin^2\theta_1 - m_2\sin^2\theta_2)}} \quad (3)$$

Now we write the nonlinear system as

$$\dot{x} = A_{nl}\, x(t)$$

and continue to linearize

3

$$\mathcal{M}\Big|_{\substack{x=0\\ \theta_1=\theta_2=0}} = M$$

$$\mathcal{M} = M + m_1\sin^2\theta_1 + m_2\sin^2\theta_2, \quad \dot{\mathcal{M}} = \begin{pmatrix} 2m_1\sin\theta_1\cos\theta_1 \\ + 2m_2\sin\theta_2\cos\theta_2 \end{pmatrix} = 2(m_1\sin\theta_1\cos\theta_1 + m_2\sin\theta_2\cos\theta_2)$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dfrac{-m_1\sin\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2) + F}{\mathcal{M}} \\ \dot{\theta}_1 \\ \dfrac{-m_1\sin\theta_1(l_1\cos\theta_1\dot{\theta}_1^2 + g) - m_2\sin\theta_2\cos\theta_1(l_2\dot{\theta}_2^2 + g\cos\theta_2) - g\sin\theta_1(M + m_2\sin^2\theta_2) + F\cos\theta_1}{l_1\mathcal{M}} \\ \dot{\theta}_2 \\ \dfrac{-m_1\sin\theta_1\cos\theta_2(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\cos\theta_2\dot{\theta}_2^2 + g) - g\sin\theta_2(M + m_1\sin^2\theta_1) + F\cos\theta_2}{l_2\mathcal{M}} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

Let $s_1 = \sin\theta_1$ & $c_1 = \cos\theta_1$

$$A = \frac{\partial f}{\partial x}\Big|_{\substack{x = \bar{x} \\ u = \bar{u}}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial f}{\partial \theta_1}(23) & \frac{\partial f}{\partial \dot{\theta}_1}(24) & \frac{\partial f}{\partial \theta_2}(25) & \frac{\partial f}{\partial \dot{\theta}_2}(26) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial f}{\partial \theta_1}(43) & \frac{\partial f}{\partial \dot{\theta}_1}(44) & \frac{\partial f}{\partial \theta_2}(45) & \frac{\partial f}{\partial \dot{\theta}_2}(46) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\partial f}{\partial \theta_1}(63) & \frac{\partial f}{\partial \dot{\theta}_1}(64) & \frac{\partial f}{\partial \theta_2}(65) & \frac{\partial f}{\partial \dot{\theta}_2}(66) \end{bmatrix} \qquad B = \frac{\partial f}{\partial F} = \begin{bmatrix} 0 \\ \frac{1}{\mathcal{M}} \\ 0 \\ \frac{\cos\theta_1}{l_1\mathcal{M}} \\ 0 \\ \frac{\cos\theta_2}{l_2\mathcal{M}} \end{bmatrix}$$

$$\frac{\partial f}{\partial \theta_1(23)} = \frac{\mathcal{M}\left(-m_1\cos\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_1\sin\theta_1(-g\sin\theta_1)\right) - 2m_1\sin\theta_1\cos\theta_1\left(-m_1\sin\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2) + F\right)}{\mathcal{M}^2}$$

$$\frac{\partial f}{\partial \theta_1(23)}\Big|_{\substack{x=0\\ \theta_1=0 \\ \theta_2=0}} = \frac{M(-m_1 g)}{M^2} = \frac{-m_1 g}{M}$$

$$\frac{\partial f}{\partial \dot{\theta}_1(24)} = \frac{\mathcal{M}(-2m_1\sin\theta_1 l_1\dot{\theta}_1) - 0 \times (\text{numerator})}{\mathcal{M}^2} \Rightarrow \frac{\partial f}{\partial \dot{\theta}_1(24)}\Big|_{\substack{x=0\\ \theta_1=\theta_2=0}} = 0$$

$$\frac{\partial f}{\partial \theta_2(25)} = \frac{\mathcal{M}\left(-m_2\cos\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2) - m_2\sin\theta_2(-g\sin\theta_2)\right) - 2m_2\sin\theta_2\cos\theta_2\left(-m_1\sin\theta_1(l_1\dot{\theta}_1^2 + g\cos\theta_1) - m_2\sin\theta_2(l_2\dot{\theta}_2^2 + g\cos\theta_2) + F\right)}{\mathcal{M}^2}$$

$$\frac{\partial f}{\partial \theta_2(25)}\Big|_{\substack{x=0\\ \theta_1=\theta_2=0}} = \frac{M(-m_2 g)}{M^2} = \frac{-m_2 g}{M}$$

$$\frac{\partial f}{\partial \dot{\theta}_2(26)} = \frac{\mathcal{M}(-2m_2\sin\theta_2 l_2\dot{\theta}_2) - 0 \times (\text{numerator})}{\mathcal{M}^2} \Rightarrow \frac{\partial f}{\partial \dot{\theta}_2(26)}\Big|_{\substack{x=0\\ \theta_1=\theta_2=0}} = 0$$

$$\frac{\partial f}{\partial \theta_{1(43)}} = \frac{l_1 \mathcal{M}\Big(-m_1\cos\theta_1(l_1\cos\theta_1\dot{\theta}_1^2+g) - m_1\sin\theta_1(-\sin\theta_1 l_1\dot{\theta}_1^2) + m_2\sin\theta_2\sin\theta_1(l_2\dot{\theta}_2^2+g\cos\theta_2) - m_2\sin\theta_2\cos\theta_1(0) - g\cos\theta_1(M+m_2\sin^2\theta_2) - g\sin\theta_1(0) - F\sin\theta_1\Big) - 2l_1 m_1\sin\theta_1\cos\theta_1\,(\text{numerator})}{l_1^2\mathcal{M}^2}$$

$$\left.\frac{\partial f}{\partial \theta_{1(43)}}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = \frac{l_1 M(-m_1 g - gM) - 0}{l_1^2 M^2} = \frac{-g(m_1+M)}{l_1 M}$$

$$\frac{\partial f}{\partial \dot{\theta}_{1(44)}} = \frac{l_1\mathcal{M}(-2m_1\sin\theta_1 l_1\cos\theta_1\dot{\theta}_1) - 0\times(\text{numerator})}{l_1^2\mathcal{M}^2} \Rightarrow \left.\frac{\partial f}{\partial \dot{\theta}_1}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = \frac{l_1 M(0)}{l_1^2 M^2} = 0$$

$$\frac{\partial f}{\partial \theta_{2(45)}} = \frac{l_1\mathcal{M}\Big(-m_2\cos\theta_2\cos\theta_1(l_2\dot{\theta}_2^2+g\cos\theta_2) - m_2\sin\theta_2\cos\theta_1(-g\sin\theta_2) - 2g\sin\theta_1 m_2\sin\theta_2\cos\theta_2\Big) - 2m_2\sin\theta_2\cos\theta_2 l_1(\text{numerator})}{l_1^2\mathcal{M}^2}$$

$$\left.\frac{\partial f}{\partial \theta_{2(45)}}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = \frac{l_1 M(-m_2 g) - 0\times(\text{numerator})}{l_1^2 M^2} = \frac{-m_2 g}{l_1 M}$$

$$\frac{\partial f}{\partial \dot{\theta}_{2(46)}} = \frac{l_1\mathcal{M}(-2m_2\sin\theta_2\cos\theta_1 l_2\dot{\theta}_2) - 0\times(\text{num})}{l_1^2\mathcal{M}^2} \Rightarrow \left.\frac{\partial f}{\partial \dot{\theta}_{2(46)}}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = 0$$

$$\frac{\partial f}{\partial \theta_{1(63)}} = \frac{l_2\mathcal{M}\Big(-m_1\cos\theta_1\cos\theta_2(l_1\dot{\theta}_1^2+g\cos\theta_1) - m_1\sin\theta_1\cos\theta_2(-g\sin\theta_1) - 2g\sin\theta_2 m_1\sin\theta_1\cos\theta_1\Big) - 2m_1\sin\theta_1\cos\theta_1 l_2\times(\text{numerator})}{l_2^2\mathcal{M}^2}$$

$$\left.\frac{\partial f}{\partial \theta_{1(63)}}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = \frac{l_2 M(-m_1 g) - 0}{l_2^2 M^2} = \frac{-m_1 g}{l_2 M}$$

$$\frac{\partial f}{\partial \dot{\theta}_{1(64)}} = \frac{l_2\mathcal{M}(-2m_1\sin\theta_1\cos\theta_2 l_1\dot{\theta}_1) - 0\times(\text{num})}{l_2^2\mathcal{M}^2} \Rightarrow \left.\frac{\partial f}{\partial \dot{\theta}_{1(64)}}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = 0$$

$$\frac{\partial f}{\partial \theta_{2(65)}} = \frac{l_2\mathcal{M}\Big(m_1\sin\theta_1\sin\theta_2(l_1\dot{\theta}_1^2+g\cos\theta_1) - m_2\cos\theta_2(l_2\cos\theta_2\dot{\theta}_2^2+g) - m_2\sin\theta_2(-l_2\sin\theta_2\dot{\theta}_2^2) - g\cos\theta_2(M+m_1\sin^2\theta_1) - F\sin\theta_2\Big) - 2l_2 m_2\sin\theta_2\cos\theta_2\times(\text{num})}{l_2^2\mathcal{M}^2}$$

$$\left.\frac{\partial f}{\partial \theta_{2(65)}}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = \frac{l_2 M(-m_2 g - gM) - 0}{l_2^2 M^2} = \frac{-g(m_2+M)}{l_2 M}$$

$$\frac{\partial f}{\partial \dot{\theta}_{2(66)}} = \frac{l_2\mathcal{M}(-2m_2\sin\theta_2 l_2\cos\theta_2\dot{\theta}_2) - 0\times(\text{num})}{l_2^2\mathcal{M}^2} \Rightarrow \left.\frac{\partial f}{\partial \dot{\theta}_{2(66)}}\right|_{\substack{x=0\\ \theta_1=\theta_2=0}} = 0$$

(5)

$$\mathcal{M} = M + m_1 \sin^2\theta_1 + m_2 \sin^2\theta_2$$

$$B = \frac{\partial f}{\partial F} = \begin{bmatrix} 0 \\ \frac{1}{\mathcal{M}} \\ 0 \\ \frac{\cos\theta_1}{l_1 \mathcal{M}} \\ 0 \\ \frac{\cos\theta_2}{l_2 \mathcal{M}} \end{bmatrix} \Rightarrow \frac{\partial f}{\partial F}\bigg|_{\substack{x=0 \\ \theta_1=\theta_2=0}} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{l_1 M} \\ 0 \\ \frac{1}{l_2 M} \end{bmatrix} = \bar{B} \rightarrow \text{linearized}$$

$$A_{\text{linearized}} = \bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(m_1+M)}{l_1 M} & 0 & \frac{-m_2 g}{l_1 M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-g m_1}{l_2 M} & 0 & \frac{-g(m_2+M)}{l_2 M} & 0 \end{bmatrix}$$

State space representn of linearized system

$$\dot{\delta}_x = \bar{A}\,\delta_x + \bar{B}\,\delta_u$$

C) Obtain conditions on M, m_1, m_2, l_1, l_2 for linearized system to be controllable.

Soln

$$\text{rank}\begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} = n$$

This controllability rank test needs to be satisfied for linearized system to be controllable

(6)

$$AB = \begin{bmatrix} \frac{1}{M} \\ 0 \\ \frac{1}{l_1 M} \\ 0 \\ \frac{1}{l_2 M} \\ 0 \end{bmatrix}$$

$$A^2B = AAB = \begin{bmatrix} 0 \\ -\frac{m_1 g}{l_1 M^2} - \frac{m_2 g}{l_2 M^2} = -\left(\frac{l_2 m_1 g + l_1 m_2 g}{l_1 l_2 M^2}\right) \\ 0 \\ -\frac{g(m_1+M)}{l_1^2 M^2} - \frac{m_2 g}{l_1 l_2 M^2} = \frac{-g\left(l_2(m_1+M) + l_1 m_2\right)}{l_1^2 l_2 M^2} \\ 0 \\ -\frac{g m_1}{l_1 l_2 M^2} - \frac{g(m_2+M)}{l_2^2 M^2} = \frac{-g\left(l_2 m_1 + l_1(m_2+M)\right)}{l_1 l_2^2 M^2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-g(l_2 m_1 + l_1 m_2)}{l_1 l_2 M^2} \\ 0 \\ \frac{-g(l_2(m_1+M)+l_1 m_2)}{l_1^2 l_2 M^2} \\ 0 \\ \frac{-g(l_2 m_1 + l_1(m_2+M))}{l_1 l_2^2 M^2} \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 \\ \frac{-g(l_2 m_1 + l_1 m_2)}{l_1 l_2 M^2} \\ 0 \\ \frac{-g(l_2(m_1+M)+l_1 m_2)}{l_1^2 l_2 M^2} \\ 0 \\ \frac{-g(l_2 m_1 + l_1(m_2+M))}{l_1 l_2^2 M^2} \end{bmatrix} \qquad A^3B = AA^2B = \begin{bmatrix} \frac{-g(l_2 m_1 + l_1 m_2)}{l_1 l_2 M^2} \\ 0 \\ \frac{-g(l_2(m_1+M)+l_1 m_2)}{l_1^2 l_2 M^2} \\ 0 \\ \frac{-g(l_2 m_1 + l_1(m_2+M))}{l_1 l_2^2 M^2} \\ 0 \end{bmatrix}$$

$$A^4B = AA^3B = \begin{bmatrix} 0 \\ \frac{g^2 m_1(l_2(m_1+M)+l_1 m_2)}{l_1^2 l_2 M^3} + \frac{g^2 m_2(l_2 m_1 + l_1(m_2+M))}{l_1 l_2^2 M^3} = \frac{g^2\left(l_2 m_1(l_2(m_1+M)+l_1 m_2) + l_1 m_2(l_2 m_1 + l_1(m_2+\ldots\right)}{l_1^2 l_2^2 M^3} \\ 0 \\ \frac{g^2(m_1+M)(l_2(m_1+M)+l_1 m_2)}{l_1^3 l_2 M^3} + \frac{g^2 m_2(l_2 m_1 + l_1(m_2+M))}{l_1^2 l_2^2 M^3} = \frac{g^2\left(l_2(m_1+M)(l_2(m_1+M)+l_1 m_2) + l_1 m_2(l_2 m_1 + l_1(m_2+M)\ldots\right)}{l_1^3 l_2^2 M^3} \\ 0 \\ \frac{g^2 m_1(l_2(m_1+M)+l_1 m_2)}{l_1^2 l_2^2 M^3} + \frac{g^2(m_2+M)(l_2 m_1 + l_1(m_2+M))}{l_1 l_2^3 M^3} = \frac{g^2\left(l_2 m_1(l_2(m_1+M)+l_1 m_2) + l_1(m_2+M)(l_2 m_1 + l_1(m_2+M\ldots\right)}{l_1^2 l_2^3 M^3} \end{bmatrix}$$

$$A^4B = \begin{bmatrix} 0 \\ \frac{g^2\left(l_2 m_1(l_2(m_1+M)+l_1 m_2) + l_1 m_2(l_2 m_1 + l_1(m_2+M))\right)}{l_1^2 l_2^2 M^3} \\ 0 \\ \frac{g^2\left(l_2(m_1+M)(l_2(m_1+M)+l_1 m_2) + l_1 m_2(l_2 m_1 + l_1(m_2+M))\right)}{l_1^3 l_2^2 M^3} \\ 0 \\ g^2\left(l_2 m_1(l_2(m_1+M)+l_1 m_2) + l_1(m_2+M)(l_2 m_1 + l_1(m_2+M))\right) \ldots \end{bmatrix} \qquad A^5B = \begin{bmatrix} \frac{g^2\left(l_2 m_1(l_2(m_1+M)+l_1 m_2) + l_1 m_2(l_2 m_1 + l_1(m_2+\ldots\right)}{l_1^2 l_2^2 M^3} \\ 0 \\ \frac{g^2\left(l_2(m_1+M)(l_2(m_1+M)+l_1 m_2) + l_1 m_2(l_2 m_1 + l_1(m_2+\ldots\right)}{l_1^3 l_2^2 M^3} \\ 0 \\ \frac{g^2\left(l_2 m_1(l_2(m_1+M)+l_1 m_2) + l_1(m_2+M)(l_2 m_1 + l_1(m_2+\ldots\right)}{l_1^2 l_2^3 M^3} \\ 0 \end{bmatrix}$$

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$$\text{rank}\begin{bmatrix} 0 & \frac{1}{M} & 0 & \frac{-g(l_2m_1+l_1m_2)}{l_1l_2M^2} & 0 & \frac{g^2(l_2m_1(l_2(m_1+M)+l_1m_2)+l_1m_2(l_2m_1+l_1(m_2+M)))}{l_1^2l_2^2M^3} \\ \frac{1}{M} & 0 & \frac{-g(l_2m_1+l_1m_2)}{l_1l_2M^2} & 0 & \frac{g^2(l_2m_1(l_2(m_1+M)+l_1m_2)+l_1m_2(l_2m_1+l_1(m_2+M)))}{l_1^2l_2^2M^3} & 0 \\ 0 & \frac{1}{l_1M} & 0 & \frac{-g(l_2(m_1+M)+l_1m_2)}{l_1^2l_2M^2} & 0 & \frac{g^2(l_2(m_1+M)(l_2(m_1+M)+l_1m_2)+l_1m_2(l_2m_1+l_1(m_2+M)))}{l_1^3l_2^2M^3} \\ \frac{1}{l_1M} & 0 & \frac{-g(l_2(m_1+M)+l_1m_2)}{l_1^2l_2M^2} & 0 & \frac{g^2(l_2(m_1+M)(l_2(m_1+M)+l_1m_2)+l_1m_2(l_2m_1+l_1(m_2+M)))}{l_1^3l_2^2M^3} & 0 \\ 0 & \frac{1}{l_2M} & 0 & \frac{-g(l_2m_1+l_1(m_2+M))}{l_1l_2^2M^2} & 0 & \frac{g^2(l_2m_1(l_1(m_1+M)+l_1m_2)+l_1(m_2+M)(l_2m_1+l_1(m_2+M)))}{l_1^2l_2^3M^3} \\ \frac{1}{l_2M} & 0 & \frac{-g(l_2m_1+l_1(m_2+M))}{l_1l_2^2M^2} & 0 & \frac{g^2(l_2m_1(l_2(m_1+M)+l_1m_2)+l_1(m_2+M)(l_2m_1+l_1(m_2+M)))}{l_1^2l_2^3M^3} & 0 \end{bmatrix}$$

$C_3 \quad C_4 \quad C_5 \quad C_6$

$$C_3 = \begin{bmatrix} 0 \\ \frac{-g(l_2m_1+l_1m_2)}{l_1l_2M^2} \\ 0 \\ \frac{-g(l_2(m_1+M)+l_1m_2)}{l_1^2l_2M^2} \\ 0 \\ \frac{-g(l_2m_1+l_1(m_2+M))}{l_1l_2^2M^2} \end{bmatrix} \qquad C_5 = \begin{bmatrix} 0 \\ \frac{g^2(l_2m_1(l_2(m_1+M)+l_1m_2)+l_1m_2(l_2m_1+l_1(m_2+M)))}{l_1^2l_2^2M^3} \\ 0 \\ \frac{g^2(l_2(m_1+M)(l_2(m_1+M)+l_1m_2)+l_1m_2(l_2m_1+l_1(m_2+M)))}{l_1^3l_2^2M^3} \\ 0 \\ \frac{g^2(l_2m_1(l_2(m_1+M)+l_1m_2)+l_1(m_2+M)(l_2m_1+l_1(m_2+M)))}{l_1^2l_2^2M^3} \end{bmatrix}$$

For system to be controllable the rank needs to be full, i.e. $n = 6$

we see for C_5 to be dependent on C_3,

$$C_5 = \frac{-g}{l_1l_2M} C_3, \text{ only if.}$$

$$\text{①} \quad l_2(m_1+M)+l_1m_2 = 1.$$
$$- \text{②} \quad l_2m_1+l_1(m_2+M) = 1.$$

$$l_2(m_1+M-m_1)+l_1(m_2-m_2-M) = 0$$
$$l_2M - l_1M = 0$$
$$M(l_2-l_1) = 0$$
$$M=0 \quad \text{or} \quad l_2 = l_1$$

Can't be true since $M > 0$, $\quad l_2 \neq l_1 \Rightarrow$ to not be controll.

So, $M > 0$, $\quad 0 < l_1 < l_2$
$m_1 > 0$ $\quad$ or $0 < l_2 < l_1$
$m_2 > 0$

Conditions

$M > 0,\ m_1 > 0,\ m_2 > 0,\ l_1 > 0,\ l_2 > 0$

$l_1 \neq l_2 \Rightarrow l_1 > l_2$ or $l_2 > l_1$

$\therefore$ As long as M, m_1, m_2 are positive and l_1, l_2 are positive but not equal to each other the linearized system is controllable

(8)

D) For $M = 1000\,kg$, $m_1 = m_2 = 100\,kg$, $l_1 = 20\,m$, $l_2 = 10\,m$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-100g}{1000} & 0 & \frac{-100g}{1000} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(1100)}{20,000} & 0 & \frac{-g(100)}{20,000} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-g\,100}{10,000} & 0 & \frac{-g(1100)}{10,000} & 0 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-g}{10} & 0 & \frac{-g}{10} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-11g}{200} & 0 & \frac{-g}{200} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-g}{100} & 0 & \frac{-11g}{100} & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{20,000} \\ 0 \\ \frac{1}{10,000} \end{bmatrix}, \quad \bar{A}\bar{B} = \begin{bmatrix} \frac{1}{1000} \\ 0 \\ \frac{1}{20,000} \\ 0 \\ \frac{1}{10,000} \\ 0 \end{bmatrix}, \quad \bar{A}^2\bar{B} = \begin{bmatrix} 0 \\ \frac{-g}{10}\left(\frac{1}{20,000} + \frac{1}{10,000}\right) = \frac{-g}{10}\left(\frac{1+2}{20,000}\right) = \frac{-3g}{20,000} \\ 0 \\ \frac{-g}{200}\left(\frac{11}{20,000} + \frac{1}{10,000}\right) = \frac{-g}{200}\left(\frac{11+2}{20,000}\right) = \frac{-13g}{4,000,000} \\ 0 \\ \frac{-g}{100}\left(\frac{1}{20,000} + \frac{11}{10,000}\right) = \frac{-g}{100}\left(\frac{1+22}{20,000}\right) = \frac{-23g}{2,000,000} \end{bmatrix}$$

$$\bar{A}^2\bar{B} = \begin{bmatrix} 0 \\ \frac{-3g}{20,000} \\ 0 \\ \frac{-13g}{4,000,000} \\ 0 \\ \frac{-23g}{2,000,000} \end{bmatrix} \Rightarrow \bar{A}^3\bar{B} = \begin{bmatrix} \frac{-3g}{20,000} \\ 0 \\ \frac{-13g}{4,000,000} \\ 0 \\ \frac{-23g}{2,000,000} \\ 0 \end{bmatrix} \Rightarrow \bar{A}^4\bar{B} = \begin{bmatrix} 0 \\ \frac{+g^2}{100}\left(\frac{13}{400,000} + \frac{\overset{46}{23}}{20,000}\right) = \frac{g^2}{100}\left(\frac{59}{400,000}\right) = \frac{59g^2}{40,000,000} \\ 0 \\ \frac{g^2}{200}\left(\frac{\overset{143}{13\times 11}}{4,000,000} + \frac{\overset{46}{23}}{2,000,000}\right) = \frac{g^2(189)}{200\times 4,000,000} = \frac{189g^2}{800,000,000} \\ 0 \\ \frac{g^2}{100}\left(\frac{13}{4,000,000} + \frac{\overset{253\times 2}{23\times 11}}{2,000,000}\right) = \frac{g^2(519)}{100\times 4,000,000} = \frac{519g^2}{400,000,000} \end{bmatrix}$$

$$\bar{A}^4\bar{B} = \begin{bmatrix} 0 \\ \frac{59g^2}{40,000,000} \\ 0 \\ \frac{189g^2}{800,000,000} \\ 0 \\ \frac{519g^2}{400,000,000} \end{bmatrix} \Rightarrow \bar{A}^5\bar{B} = \begin{bmatrix} \frac{59g^2}{40,000,000} \\ 0 \\ \frac{189g^2}{800,000,000} \\ 0 \\ \frac{519g^2}{400,000,000} \\ 0 \end{bmatrix}$$

⑨

$$\text{Rank}\begin{bmatrix} 0 & \frac{1}{1000} & 0 & \frac{-3g}{20,000} & 0 & \frac{59g^2}{40,000,000} \\ \frac{1}{1000} & 0 & \frac{-3g}{20,000} & 0 & \frac{59g^2}{40,000,000} & 0 \\ 0 & \frac{1}{20,000} & 0 & \frac{-13g}{4,000,000} & 0 & \frac{189g^2}{800,000,000} \\ \frac{1}{20,000} & 0 & \frac{-13g}{4,000,000} & 0 & \frac{184g^2}{800,000,000} & 0 \\ 0 & \frac{1}{10,000} & 0 & \frac{-23g}{2,000,000} & 0 & \frac{519g^2}{400,000,000} \\ \frac{1}{10,000} & 0 & \frac{-23g}{2,000,000} & 0 & \frac{514g^2}{400,000,000} & 0 \end{bmatrix} = m$$

Full rank so the linearized system is controllable

when $M = 1000\,kg$, $m_1 = m_2 = 100\,kg$, $l_1 = 20\,m$ & $l_2 = 10\,m$

⇒ Obtain an LQR controller

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-9}{10} & 0 & \frac{-9}{10} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-11g}{200} & 0 & \frac{-9}{200} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-9}{100} & 0 & \frac{-11g}{100} & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{20,000} \\ 0 \\ \frac{1}{10,000} \end{bmatrix} \quad \text{let } Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Riccati eqn: $A^TP + PA - PBR^{-1}B^TP = -Q$, since Input is just "F", $R = 1$

$K = R^{-1}B^TP$

$$A^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-9}{10} & 0 & 0 & \frac{-11g}{200} & 0 & \frac{-9}{100} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{-9}{10} & 0 & \frac{-9}{200} & 0 & \frac{-11g}{100} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{12} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ P_{13} & P_{23} & P_{33} & P_{34} & P_{35} & P_{36} \\ P_{14} & P_{24} & P_{34} & P_{44} & P_{45} & P_{46} \\ P_{15} & P_{25} & P_{35} & P_{45} & P_{55} & P_{56} \\ P_{16} & P_{26} & P_{36} & P_{46} & P_{56} & P_{66} \end{bmatrix}$$

$A^TP + PA + PBK = -Q$