Topic 02: Functions - Limits, Continuity, Differentiability, Analyticity

MA201 Mathematics III

AC, MGPP, SN, SNB

IIT Guwahati

Topic 02: Learning Outcome

We learn

- Complex Functions and its visualization
- Limits of Functions
- Point at Infinity (∞), Extended Complex Plane and Riemann Sphere
- Limits involving ∞
- Continuity
- Properties of Continuous Functions
- Differentiation
- Properties of Differentiable Functions
- Cauchy Riemann Equations
- Analytic Functions
- Properties of Analytic Functions
- Harmonic Functions
- Finding Harmonic Conjugate

Complex Functions

Definition

A complex valued function f of a complex variable is a rule that assigns to each complex number z in a set $D \subseteq \mathbb{C}$ one and only complex value w. We write $w = f(z)$ and call w the image of z under f. The set D is called the domain of the definition of f and the set of all images $R = \{w = f(z) \; : \; z \in D\}$ is called the range of f.

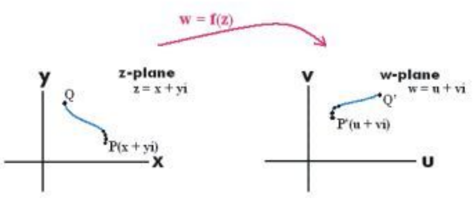
Usually, the real and imaginary parts of z are denoted by x and y, and those of the image point w are denoted by u and v respectively, so that $w = f(z) = u + i\,v$, where $u \equiv u(z) = u(x, \; y)$ and $v \equiv v(z) = v(x, \; y)$ are real valued functions of $z = x + iy$.

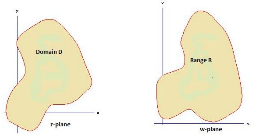
Example: Consider the function $f(z) = z^2$ for $z \in \mathbb{C}$. This function assigns to each complex number z in $\mathbb{C}$ one and only complex value $w = z^2$. The real and imaginary parts of $f(z)$ are given by

$$\Re(f(z)) = u(x, \; y) = x^2 - y^2 \qquad \Im(f(z)) = v(x, \; y) = 2xy \; .$$

Visualizing Complex Functions

In order to investigate a complex function $w = f(z)$, it is necessary to visualize it.
We view z and its image w as points in the complex plane, so that f becomes a transformation or mapping from D in the z-plane (xy-plane) on to the range R in the w-plane (uv-plane).





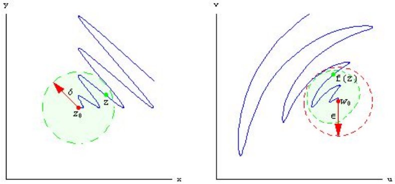
Limits of functions

Definition

Let $w = f(z)$ be a complex function of a complex variable z that is defined for all values of z in some neighborhood of z_0, except perhaps at the point z_0. We say that f has the limit w_0 as z approaches z_0 if **for each** positive number $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta \,.$$

We write it as $\lim_{z \to z_0} f(z) = w_0$.



- Geometrically, this says that for each ϵ-neighborhood $B_\epsilon(w_0) = \{w \in \mathbb{C} \ : \ |w - w_0| < \epsilon\}$ of the point w_0 in the w-plane, there exists a deleted or punctured δ-neighborhood $B^*_\delta(z_0) = \{z \in \mathbb{C} \ : \ 0 < |z - z_0| < \delta\}$ of z_0 in the z-plane such that $f(B^*_\delta(z_0)) \subset B_\epsilon(w_0)$.
- In case of functions $f : \mathbb{R} \to \mathbb{R}$, the variable x approaches the point x_0 in only two directions, either right or left. But, in the complex case, z can approach z_0 from any direction. That is, for the limit $\lim\limits_{z \to z_0} f(z)$ to exist, it is required that $f(z)$ *must* approach the same value no matter how z approaches z_0.

Example 1: If $f(z) = 2i/z$ then examine the existence of $\lim\limits_{z \to i} f(z)$.

Example 2: If $f(z) = \bar{z}$ then examine the existence of $\lim\limits_{z \to (1+2i)} f(z)$.

Example 3: If $f(z) = \Re(z)/|z|$ then examine the existence of $\lim\limits_{z \to 0} f(z)$.

Example 4: If $f(z) = \bar{z}/z$ then examine the existence of $\lim\limits_{z \to 0} f(z)$. Also examine the existence of $\lim\limits_{z \to z_0} f(z)$ if $z_0 \neq 0$.

Limit of $f(z)$ and Limit of $\Re(f(z))$ and $\Im(f(z))$

Theorem

Let $f(z) = u(x,\ y) + i\,v(x,\ y)$ be a complex function that is defined in some neighborhood of z_0, except perhaps at $z_0 = x_0 + i\,y_0$. Then

$$\lim_{z \to z_0} f(z) = w_0 = u_0 + i\,v_0$$

if and only if

$$\lim_{(x,\,y)\to(x_0,\,y_0)} u(x,\ y) = u_0 \qquad \textit{and} \qquad \lim_{(x,\,y)\to(x_0,\,y_0)} v(x,\ y) = v_0\ .$$

Example: Let $f(z) = z^2$. Then, $f(z) = u(x,\ y) + i\,v(x,\ y)$ where $u(x,\ y) = x^2 - y^2$ and $v(x,\ y) = 2xy$. Using above theorem, show that

$$\lim_{z \to (1+2i)} z^2 = -3 + 4\,i\ .$$

Limit of Functions and Algebraic Operations

Theorem

If $\lim_{z\to z_0} f(z) = A$ *and* $\lim_{z\to z_0} g(z) = B$ *then*

$$\lim_{z\to z_0} k\, f(z) \;=\; k\,A\,, \quad \textit{where } k \textit{ is a complex constant}\,,$$

$$\lim_{z\to z_0} (f(z) + g(z)) = A + B\,,$$

$$\lim_{z\to z_0} (f(z) - g(z)) = A - B\,,$$

$$\lim_{z\to z_0} f(z)g(z) = AB\,,$$

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \quad \textit{provided } \; B \neq 0\,.$$

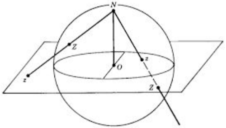
Point at Infinity ∞ and the Extended Complex Plane

It is convenient to include with the complex number system $\mathbb{C}$ one ideal element, called point at infinity, denoted by the symbol ∞. Then the set $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the extended complex plane and satisfies the following properties.

- For $z \in \mathbb{C}$,
$$z + \infty = \infty + z = z - \infty = \infty, \qquad \text{and} \qquad \frac{z}{\infty} = 0\,.$$
- For $z \in \mathbb{C} \setminus \{0\}$,
$$z \cdot \infty = \infty \cdot z = \infty, \qquad \text{and} \qquad \frac{z}{0} = \infty\,.$$
- $\infty \cdot \infty = \infty$.

Expressions such as $\infty + \infty$, $\infty - \infty$, $0 \cdot \infty$, ∞/∞ are not defined since they do not lead to meaningful results.

Riemann Sphere and Stereographic Projection



- Join the North Pole $N = (0, 0, 1)$ with the complex number $z = x + iy$ by a straight line L which pierce the sphere at Z.
- The mapping $z \mapsto Z$ gives one-to-one correspondence between $S \setminus \{N\}$ and $\mathbb{C}$.
- As $|z|$ approaches ∞ (along any direction in the plane), the corresponding point Z on S approaches N.
- Associate the North Pole N with the point at infinity ∞.
- $|z| > 1 \mapsto$ Upper hemisphere of S. $|z| < 1 \mapsto$ Lower hemisphere of S. $|z| = 1 \mapsto$ Equator of S.
- S is called the Riemann sphere. This bijection between S and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the Stereographic Projection.

Limits involving infinity

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. Let z_0 be a limit point of D. Then, $\lim_{z \to z_0} f(z) = \infty$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \qquad \Longrightarrow \qquad |f(z)| > 1/\epsilon \ .$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. Let ∞ be a limit point of D. Then, $\lim_{z \to \infty} f(z) = w_0$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|z| > 1/\delta \qquad \Longrightarrow \qquad |f(z) - w_0| < \epsilon \ .$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. Let ∞ be a limit point of D. Then, $\lim_{z \to \infty} f(z) = \infty$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|z| > 1/\delta \qquad \Longrightarrow \qquad |f(z)| > 1/\epsilon \ .$$

Results related Limits involving Infinity

1. $$\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0\,.$$

2. $$\lim_{z\to \infty} f(z) = w_0 \iff \lim_{z\to 0} f\left(1/z\right) = w_0\,.$$

3. $$\lim_{z\to \infty} f(z) = \infty \iff \lim_{z\to 0} \frac{1}{f(1/z)} = 0\,.$$

Exercises: Find (i) $\lim\limits_{z\to\infty} \dfrac{4z^2}{(z-1)^2}$, (ii) $\lim\limits_{z\to 1} \dfrac{1}{(z-1)^3}$, (iii) $\lim\limits_{z\to\infty} \dfrac{z^2+1}{z-1}$.

Continuous functions

Definition

Let $f(z)$ be a complex function of a complex variable z that is defined for all values of z in some neighborhood of z_0. We say that f is continuous at z_0 if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \qquad \Longrightarrow \qquad |f(z) - f(z_0)| < \epsilon \ .$$

Equivalently, $f(z)$ is continuous at the point z_0 if $\lim_{z \to z_0} f(z)$ exists and is equal to $f(z_0)$.

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. We say that f is continuous in the set D if f is continuous at each point of D.

Geometrical Interpretation of Continuity

To be continuous at z_0, the function f should map Near by points of z_0 in to Near by points of $f(z_0)$.
Near by concept is written in terms of neighborhood.

The continuity of $f(z)$ at a point z_0 can be interpreted geometrically as for each ϵ-neighborhood $B_\epsilon(f(z_0)) = \{w \in \mathbb{C} \;:\; |w - f(z_0)| < \epsilon\}$ of the point $f(z_0)$ in the w-plane, there exists a δ-neighborhood $B_\delta(z_0) = \{z \in \mathbb{C} \;:\; |z - z_0| < \delta\}$ of z_0 in the z-plane such that the function $f(z)$ maps $B_\delta(z_0)$ inside $B_\epsilon(f(z_0))$.

Example

Let $f(z) = z^2$. Then,

$$\lim_{z\to(1+2i)} f(z) = \lim_{z\to(1+2i)} z^2 = (1+2i)^2 = -3+4\,i = f(1+2i)\ .$$

Therefore, the function $f(z)$ is continuous at the point $(1+2i)$.

Example

Let $f(z) = \Re(z)/|z|$ for $z \neq 0$ and $f(0) = 1$. The function $f(z)$ is **not** continuous at 0, since $\lim_{z\to 0} \dfrac{\Re(z)}{|z|}$ does not exist.

Example

Let $f(z) = \Re(z)/|1+z|$ for $z \neq 0$ and $f(0) = 1$. The function $f(z)$ is **not** continuous at 0, since $\lim_{z\to 0} \dfrac{\Re(z)}{|1+z|} = 0$ which is not equal to $f(0) = 1$.

Results on Continuity

Theorem

Let $f(z) = u(x,\ y) + i\ v(x,\ y)$ be defined in some neighborhood of $z_0 = x_0 + i\ y_0$. Then, f is continuous at z_0 if and only if $u(x, y)$ and $v(x, y)$ are continuous at $(x_0,\ y_0)$.

Theorem

Suppose that the functions f and g are continuous at z_0. Then, the following functions are continuous at z_0: (i) $f(z) + g(z)$, (ii) $f(z) - g(z)$, (iii) $f(z)g(z)$ and (iv) $\frac{f(z)}{g(z)}$ provided that $g(z_0) \neq 0$.

Theorem

Suppose that f is continuous at z_0 and $g(z)$ is continuous at $f(z_0)$. Then, the composition function $h = g \circ f = g(f(z))$ is continuous at z_0.

Results on Continuity (continuation...)

Theorem

Suppose that $f(z)$ is continuous at z_0. Then, $|f(z)|$ and $\overline{f(z)}$ are continuous at z_0.

Theorem

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. If D is a connected set and f is continuous in D then the set $f(D)$ is a connected set. That is, Continuous image of connected set is connected.

Theorem

Let $f : D \subset \mathbb{C} \to \mathbb{C}$. If D is a compact set and f is continuous in D then the set $f(D)$ is a compact set. That is, Continuous image of compact set is compact. Further $|f|$ attains its maximum and minimum values in D.

DIFFERENTIABLE FUNCTIONS

Differentiability

Definition

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ where D is an open set. Let $z_0 \in D$. If

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, then f is said to be differentiable at the point z_0, and the number

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

is called the derivative of f at z_0.

If we write $\Delta z = z - z_0$, then the above definition can be expressed in the form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

We can also use the Leibnitz notation for the derivative, $\dfrac{df}{dz}(z_0)$, or $\dfrac{df}{dz}|_{z=z_0}$.

Note: In the complex variable case there are infinitely many directions in which a variable can approach a point z_0. But, in the real case, there are only two directions, namely, left and right to approach. So the statement that a function of a complex variable has a derivative is **stronger** than the same statement about a function of a real variable. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is differentiable on $\mathbb{R} \setminus \{0\}$. But, if we consider the same function as a function of complex variable, that is, $f : \mathbb{C} \to \mathbb{R}$ given by $f(z) = |z|$, then it is nowhere differentiable in $\mathbb{C}$ (which will be proved later). Therefore, it has lost the differentiability on the set $\mathbb{R} \setminus \{0\}$ in the complex case.

Example

By using the definition of derivative, let us compute $f'(z)$ at an arbitrary point $z_0 \in \mathbb{C}$ for the function $f(z) = z$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1 \; .$$

Therefore, the derivative of $f(z) = z$ is $f'(z) = 1$ for any $z \in \mathbb{C}$.

Example

By using the definition of derivative, let us compute $f'(z)$ at an arbitrary point $z_0 \in \mathbb{C}$ for the function $f(z) = z^2$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0 \; .$$

Therefore, the derivative of $f(z) = z^2$ is $f'(z) = 2z$ for any $z \in \mathbb{C}$.

Example

Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \bar{z}$ for $z \in \mathbb{C}$.
Examine the differentiability of $f(z)$ at each point z of $\mathbb{C}$.
Answer: $f(z) = \bar{z}$ is not differentiable at any point of $\mathbb{C}$ (No where differentiable in $\mathbb{C}$).
Details are worked out on the board.

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x, -y)$ for $(x, y) \in \mathbb{R}^2$.
Examine the Frechet differentiability of f on $\mathbb{R}^2$ (Differentiability in $\mathbb{R}^2$ from multivariable calculus)
Compare with the above example.
Details are worked out on the board.

Example

On $\mathbb{C}$, examine the differentiability of $f(z) = |z|^2$ for $z \in \mathbb{C}$.
Answer: The $f(z) = |z|^2$ is not differentiable in $\mathbb{C} \setminus \{0\}$. Further it is differentiable at $z = 0$.
That is, $|z|^2$ is differentiable only at the point 0 in $\mathbb{C}$.
Comparison: The function $f(x) = |x|^2$ for $x \in \mathbb{R}$ is (real) differentiable at each point of $\mathbb{R}$.
Details are worked out on the board.

Example

On $\mathbb{C}$, examine the differentiability of $f(z) = |z|$ for $z \in \mathbb{C}$.
Answer: Now, we show $f(z) = |z|$ is not differentiable at $z_0 = 0$.
Later we will show that $f(z) = |z|$ is not differentiable at $z_0 \neq 0$ (using CR equations)
Thus, $|z|$ is nowhere differentiable in $\mathbb{C}$.
Comparison: The function $f(x) = |x|$ for $x \in \mathbb{R}$ is (real) differentiable at each point of $\mathbb{R} \setminus \{0\}$.
Details are worked out on the board.

Results and Properties

1. If $f(z)$ is differentiable at z_0, then f is continuous at z_0.
2. If $f(z) \equiv c$ is a constant function, then $f'(z) = 0$.

Theorem

Let $f(z)$ and $g(z)$ be two differentiable functions. Then,

1. *Sum:* $\dfrac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$
2. *Product:* $\dfrac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$
3. *Quotient:* $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ *provided* $g(z) \neq 0$
4. *Composition:* $\dfrac{d}{dz}[f(g(z))] = f'(g(z))\, g'(z)$

Total Derivative of f and Partial Derivatives of Component Functions

Let $f(z) = u(x, y) + iv(x, y)$ be a complex function defined on an open set $G \subseteq \mathbb{C}$. Then, the function $u(x, y)$ and $v(x, y)$ are functions from the set $G \subseteq \mathbb{R}^2$ to $\mathbb{R}$. Suppose that $f(z)$ is differentiable at a point $z_0 = x_0 + iy_0 \in G$. Is there any relation between $f'(z_0)$ and the partial derivatives of $u(x, y)$ and $v(x, y)$ at the point (x_0, y_0)?

The answer to the above question was discovered independently by the French mathematician A. L. Cauchy (1789-1857) and the German mathematician G. F. B. Riemann (1826-1866).

Cauchy

Riemann

Cauchy-Riemann Equations

Theorem

Necessary Condition for Differentiability:

Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be differentiable at the point $z_0 = x_0 + iy_0$. Then, the first order partial derivatives of $u(x, y)$ and $v(x, y)$ exist at the point $z_0 = (x_0, y_0)$ and satisfy the following equations

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0)\ .$$

The above equations are called the Cauchy-Riemann Equations or briefly CR equations.

Proof: Now, $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists and equals $f'(z_0)$ (whichever way z approaches z_0).
Let us make z to approach z_0 along two specific paths (i) Horizontally and (ii) Vertically.

Continuation of Proof

- Horizontally: $z = x + iy_0$ approaches $z_0 = x_0 + iy_0$ as x approaches x_0. So,

$$f'(z_0) = \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0},$$

  i.e., $f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0).$ (i)

- Vertically: $z = x_0 + iy$ approaches $z_0 = x_0 + iy_0$ as y approaches y_0. So,

$$f'(z_0) = \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)},$$

  i.e., $f'(z_0) = -i\, u_y(x_0, y_0) + v_y(x_0, y_0).$ (ii)

From (i) and (ii) we have at $z_0 = (x_0, y_0)$, $u_x = v_y, v_x = -u_y$.

From the proof of the previous theorem, one can observe that $f'(z_0)$ can be written in terms of the partial derivatives of u and v as follows:

$$\begin{aligned} f'(z_0) &= u_x(x_0,\ y_0) + \ i\, v_x(x_0,\ y_0)\ . \\ \textit{Also } f'(z_0) &= v_y(x_0,\ y_0) - \ i\, u_y(x_0,\ y_0)\ . \end{aligned}$$

Example

We know that $f(z) = z^2 = x^2 - y^2 + i2xy$ is differentiable at each point $z \in \mathbb{C}$. Then, $u(x,\ y) = x^2 - y^2$, $v(x,\ y) = 2xy$ so that $f(z) = u + iv$. The first order partial derivatives of u and v at a point z are $u_x = 2x$, $u_y = -2y$, $v_x = 2y$ and $v_y = 2x$. Therefore, it follows that at each point z

$$u_x = 2x = v_y \qquad \text{and} \qquad u_y = -2y = -v_x\ .$$

Thus, $f(z)$ satisfies the Cauchy-Riemann equations at each point $z \in \mathbb{C}$. Observe that $f'(z) = u_x + i\, v_x = 2x + i\, 2y = 2z = v_y - i\, u_y$.

Not satisfying Cauchy-Riemann equations $\implies$ Not differentiable

The idea of "**Not satisfying Cauchy-Riemann equations $\implies$ Not differentiable**" can be used as one of the methods to show the non-differentiability of complex functions.

Example

Let $f(z) = \bar{z} = x - i\,y$ for $z \in \mathbb{C}$. Set $u(x,\ y) = x$ and $v(x,\ y) = -y$. This gives that $u_x = 1$, $u_y = 0$, $v_x = 0$ and $v_y = -1$. Now, $1 = u_x \neq v_y = -1$ and $0 = u_y = -v_x = 0$ at any point $z \in \mathbb{C}$. Therefore, the function $f(z)$ does **NOT** satisfy the Cauchy-Riemann equations at any point $z \in \mathbb{C}$ and consequently, it is **not** differentiable at any point $z \in \mathbb{C}$.

Now, a serious question arising in our mind is that Will the satisfaction of the Cauchy-Riemann equations make the function differentiable? The answer is non-affirmative. The next example shows that the mere satisfaction of the Cauchy-Riemann equations is not a sufficient criterion to guarantee the differentiability of a function.

Function satisfying CR equations, but not differentiable

Let $f(z) = \bar{z}^2/z$ for $z \neq 0$ and $f(0) = 0$. First we test the differentiability of $f(z)$ at the point $z = 0$.

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{(x,\,y)\to(0,\,0)} \frac{\left(\frac{x^3-3xy^2}{x^2+y^2} + i\,\frac{y^3-3x^2y}{x^2+y^2}\right) - 0}{x + iy - 0}$$

Let z approach 0 along the x-axis. Then, we have

$$\lim_{(x,\,0)\to(0,\,0)} \frac{x - 0}{x - 0} = 1\ .$$

Let z approach 0 along the line $y = x$. This gives

$$\lim_{(x,\,x)\to(0,\,0)} \frac{-x - ix}{x + ix} = -1\ .$$

Since the limits are distinct, we conclude that f is not differentiable at the origin.

Continuation of previous slide

$$f(x+iy) = \left(\frac{x^3-3xy^2}{x^2+y^2} + i\,\frac{y^3-3x^2y}{x^2+y^2}\right) \qquad \text{and} \qquad f(0)=0\,.$$

Now, we verify the Cauchy-Riemann equations at the point $z=0$.
To calculate the partial derivatives of u and v at $(0,\ 0)$, we use the definitions (Why!).

$$u_x(0,\ 0) = \lim_{x\to 0}\frac{u(0+x,\ 0)-u(0,\ 0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1\,.$$

$$v_y(0,\ 0) = \lim_{y\to 0}\frac{v(0,\ 0+y)-v(0,\ 0)}{y} = \lim_{y\to 0}\frac{y-0}{y} = 1\,.$$

In a similar fashion, one can show that

$$u_y(0,\ 0)=0, \qquad \text{and} \qquad v_x(0,\ 0)=0\,.$$

Hence the function satisfies the Cauchy-Riemann equations at the point $z=0$.

Sufficient Conditions for Differentiability

Theorem

Sufficient conditions for differentiability: Let $f(z) = u(x, y) + i\, v(x, y)$ be defined in some neighborhood of the point $z_0 = x_0 + i\, y_0$. Suppose that

- the first order partial derivatives u_x, u_y, v_x and v_y exist in a neighborhood of $z_0 = (x_0, y_0)$,
- u_x, u_y, v_x and v_y are continuous at the point (x_0, y_0),
- the Cauchy Riemann equations $u_x = v_y$, $u_y = -v_x$ hold at the point z_0.

Then, the function f is differentiable at z_0 and the derivative
$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$.

Example

Let $f(x+iy) = e^{-x}\cos y - \ i\, e^{-x}\sin y$ for $z = x+iy \in \mathbb{C}$. Then,

- $$u_x = -e^{-x}\cos y\ ,$$
  $$u_y = -e^{-x}\sin y\ ,$$
  $$v_x = e^{-x}\sin y\ ,$$
  $$v_y = -e^{-x}\cos y$$

  are continuous in $\mathbb{C}$, and
- For any $z = x+iy \in \mathbb{C}$, f satisfies the CR equations:

  $$u_x = -e^{-x}\cos y = v_y \qquad \text{and} \qquad u_y = -e^{-x}\sin y = -v_x\ .$$

Therefore, by the previous theorem (sufficient conditions for differentiability), we conclude that $f(z)$ is differentiable in $\mathbb{C}$ and $f'(z) = u_x + \ i\, v_x = -e^{-x}\cos y + \ i\, e^{-x}\sin y$ at each point of $\mathbb{C}$.

CR equations in Polar Form

Let $f(z) = f(re^{i\theta}) = u(r,\ \theta) + i\ v(r,\ \theta)$ be differentiable in D.
The polar form of the Cauchy-Riemann equations of f is given by

$$u_r(r,\ \theta) = \frac{1}{r} v_\theta(r,\ \theta) \qquad \text{and} \qquad v_r(r,\ \theta) = \frac{-1}{r} u_\theta(r,\ \theta)\ .$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned} u_r &= \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x\cos\theta + u_y\sin\theta\ , \\ u_\theta &= \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_x\ r\sin\theta\ +\ u_y\ r\cos\theta\ . \end{aligned}$$

Similarly,

$$v_r = v_x\cos\theta + v_y\sin\theta, \qquad v_\theta = -v_x\ r\ \sin\theta\ +\ v_y\ r\ \cos\theta\ .$$

Applying CR equations: $u_x = v_y, u_y = -v_x$, we get

$$v_r = -u_y\cos\theta + u_x\sin\theta, \qquad v_\theta = u_y r\sin\theta + u_x r\cos\theta\ .$$

CR Equations in Complex Form

The Cauchy-Riemann equations in complex form is given by

$$\frac{\partial f}{\partial \bar{z}} = 0 \,.$$

Proof:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u(x,y) \;+\; i\; v(x,y)) = \frac{\partial u}{\partial \bar{z}} \;+\; i\;\frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) \;+\; i\;\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(\frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\left(\frac{i}{2}\right)\right) \;+\; i\;\left(\frac{\partial v}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial v}{\partial y}\left(\frac{i}{2}\right)\right)$$

$$= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \;+\; i\;\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \;=\; 0$$

Note: A differentiable function $f(z)$ can not contain any terms involving $\bar{z}$ explicitly!

ANALYTIC FUNCTIONS

Analytic Functions (Holomorphic Functions)

Definition

Let $f(z)$ be a function defined on an open set $S \subseteq \mathbb{C}$. Then the function $f(z)$ is said to be analytic in the open set S if $f(z)$ is differentiable at each point of S.

Examples: The functions $f(z) = z$ and $g(z) = z^2$ are analytic in $\mathbb{C}$. The functions $f(z) = \overline{z}$ and $g(z) = |z|^2$ are no where analytic in $\mathbb{C}$.

Definition

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and let $z_0 \in D$. Then, the function f is said to be analytic at the point z_0 if there exists an open neighborhood $N(z_0) \subset D$ of z_0 such that f is differentiable at each point of $N(z_0)$.

Further f is said to be analytic in D if f is analytic at each point of D.

Note: The other terminologies for analytic are holomorphic or regular.

Think: Suppose f is analytic at a point z_0. Does it imply that f is analytic in an open set containing z_0?

”Analytic” is a property ”defined over open sets”

- We emphasize that analyticity is a property defined over open sets, while differentiability could conceivable hold at one point only.
- That is, we can have a function which is differentiable at exactly one point in $\mathbb{C}$. But we cannot construct a function which is analytic at exactly one point in $\mathbb{C}$.
- If we say $f(z)$ is analytic in a set S which is not open in $\mathbb{C}$, then it actually means that $f(z)$ is analytic in an open set D which contains S.

Results on Analyticity

Theorem

If $f(z)$ is analytic in an open set D then $f(z)$ is differentiable in D.

Note: Converse of above theorem is not true. Example: $|z|^2$ at $z = 0$.

Theorem

Necessary condition for analyticity:

Let $f(z)$ be analytic in an open set D of $\mathbb{C}$. Then $f(z)$ satisfies the Cauchy-Riemann equations at each point of D.

Theorem

Sufficient conditions for analyticity: Let $f(z) = u(x,\ y) + i\ v(x,\ y)$ be defined in an open set D. If the first order partial derivatives of u and v exist, continuous and satisfy the Cauchy-Riemann equations at all points of D, then f is analytic in D.

In case of analytic function f in an open set D, the previous result becomes necessary and sufficient conditions.

A function $f(z) = u(x,\ y) + i\ v(x,\ y)$ is analytic in an open set $D \subseteq \mathbb{C}$

if and only if

the first order partial derivatives of u and v exist, continuous and satisfy the Cauchy-Riemann equations at all points of D.

Results on Analyticity (continuation)

Theorem

Suppose that $f(z)$ and $g(z)$ are analytic in an open set D of $\mathbb{C}$. Then the functions $f+g$, $f-g$, fg are analytic in D. If $g(z) \neq 0$ for all $z \in D$ then the function f/g is analytic in D.

Theorem

If f is analytic in an open set D and g is analytic in an open set containing $f(D)$, then the composite function $h(z) = g(f(z))$ is analytic in D.

Theorem

Let $f(z)$ be analytic in an open set D of $\mathbb{C}$. Then the derivatives of all orders of $f(z)$ exist in D and they are analytic in D. That is, $f^{(n)}(z)$ for all $n \in \mathbb{N}$ exist and analytic in D.

Proof: Will be proved later.

Results on Analyticity (continuation)

Theorem

If $f(z)$ is analytic in an open and connected set D in $\mathbb{C}$ and if $f'(z) = 0$ everywhere in D, then $f(z)$ is constant in D.

Proof: Worked out on the board.

Results on Analyticity (continuation)

Theorem

Let $f(z) = u(x, y) + i\, v(x, y)$ be an analytic function in a domain D of $\mathbb{C}$. If any one of the following conditions hold in the domain D, then the function $f(z)$ is constant in D:

1. $f'(z) = 0$ for all $z \in D$.
2. $u(x, y)$ is constant in D.
3. $v(x, y)$ is constant in D.
4. $f(z)$ is real valued for all $z \in D$.
5. $f(z)$ is pure imaginary valued for all $z \in D$.
6. $|f(z)|$ is constant in D.
7. $\operatorname{Arg}(f(z))$ is constant in D.
8. $\overline{f(z)}$ is also analytic in D.
9. $|f(z)|$ is also analytic in D.

Harmonic Functions

Laplace equation:

Let $\phi(x,\ y)$ be a real valued function of the two real variables x and y. The partial differential equation

$$\phi_{xx}(x,\ y) + \phi_{yy}(x,\ y) = 0$$

is known as Laplace equation and is sometimes referred to as the potential equation.

Definition

Harmonic Functions:

A real valued function $\phi(x,\ y)$ is said to be harmonic in a domain D if all its second order partial derivatives are continuous in D and if at each point of D, ϕ satisfies the Laplace equation

$$\phi_{xx}(x,\ y) + \phi_{yy}(x,\ y) = 0\ .$$

Real and Imaginary Part of Analytic Functions are Harmonic Functions

Theorem

If $f(z) = u(x, y) + i\, v(x, y)$ is analytic in a domain D, then each of the functions $u(x, y)$ and $v(x, y)$ is harmonic in D.

Proof: Worked out on the board.

Examples:

The functions $x^2 - y^2$, $2xy$, $e^x \cos y$, $e^x \sin y$ are harmonic functions in $\mathbb{C}$.

The function $u(x, y) = \log(x^2 + y^2)^{\frac{1}{2}}$ is harmonic on $G = \mathbb{C} \setminus \{0\}$.

Harmonic Conjugate Function

Definition

Let $u(x,\ y)$ be a harmonic function in a domain D. If there exists a harmonic function $v(x,\ y)$ in D such that $f(z) = u(x,\ y) + \ i\ v(x,\ y)$ is analytic in D then we say that the function $v(x,y) = \Im(f(z))$ is a harmonic conjugate of $u(x,y) = \Re(f(z))$.

Example: The function $v(x,\ y) = 2xy$ is a harmonic conjugate of $u(x,\ y) = x^2 - y^2$ in $\mathbb{C}$.

Note:

In the definition of ‘harmonic conjugate’, note that it is not said that u is a *harmonic conjugate* of v.

Sometimes the phrase ‘harmonic conjugate’ is briefly called as ‘conjugate’. But it has nothing to do with the conjugate $\bar{z}$ of a complex number z.

Existence of Harmonic Conjugate Function

One can raise the question that

Does harmonic conjugate v always exist for a given harmonic function u in a domain D? The answer is 'No'.

The function $u(x, y) = \log(x^2 + y^2)^{\frac{1}{2}}$ is harmonic in $G = \mathbb{C} \setminus \{0\}$ and it has no harmonic conjugate in G.

Refining the above question as:

Under what condition harmonic conjugate v exists for a given harmonic function u in a domain D?

The answer to this question is: there are some domains for which every harmonic function has a harmonic conjugate. The exact result is given below.

Theorem

Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \subseteq \mathbb{C} \to \mathbb{R}$ is a harmonic function in G then u has a harmonic conjugate in G.

The above theorem is true if G is a simply connected domain.

Finding Harmonic Conjugate Functions

Let $u(x, y) = x^2 - y^2$ for $z = x + iy \in \mathbb{C}$.
Find the harmonic conjugate functions of $u(x, y)$.
Further write the function $f(x + iy) = u(x, y) + \, i\, v(x, y)$ in terms of z.
Details: Worked out on the board.

Explaining idea/technique in the lecture class for the following.

Let $f(z) = u(x, y) + \, i\, v(x, y)$ be analytic in a domain D.

- Given the expression for the function $u(x, y) = \Re(f(z))$ in D, how to find the function $v(x, y) = \Im(f(z))$ in D? (see previous example)
- Given the expression for the function $v(x, y) = \Im(f(z))$ in D, how to find the function $u(x, y) = \Re(f(z))$ in D?
- Given the expression for the function $u(x, y) + v(x, y)$ in D, how to find the functions $u(x, y) = \Re(f(z))$ and $v(x, y) = \Im(f(z))$ in D?
  Hint: Consider $g(z) = (1 + i)f(z)$.
- Given the expression for the function $u(x, y) - v(x, y)$ in D, how to find the functions $u(x, y) = \Re(f(z))$ and $v(x, y) = \Im(f(z))$ in D?

Properties

1. If u is harmonic and v is a harmonic conjugate of u then $-u$ is a harmonic conjugate of v.
2. Suppose that u is a harmonic function in a domain D. Then
   (i) if v_1 and v_2 are harmonic conjugates of u in D, then v_1 and v_2 must differ by a real constant.
   (ii) if v is a harmonic conjugate of u, the v is also a harmonic conjugate of $u + c$ where c is any real constant.
3. If u is harmonic and v is a harmonic conjugate of u then $u^2 - v^2$ and uv are both harmonic.
4. If u, v, and $u^2 + v^2$ are harmonic in a domain D then u and v must be constant.
5. The function $f(z) = u(x, y) + i\, v(x, y)$ is analytic in a domain D iff $v(x, y)$ is a harmonic conjugate of $u(x, y)$ in D.
6. If $f = u + iv$ is analytic in a domain D and if v is a harmonic conjugate of u and u is a harmonic conjugate of v then f is a constant function in D.

Laplace Equation: Polar Form and Complex Form

The polar form of the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ is given by

$$r^2\, \phi_{rr}(r, \theta) \;+\; r\, \phi_r(r, \theta) \;+\; \phi_{\theta\theta}(r, \theta) = 0\ .$$

The complex form of the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ is given by

$$\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial \bar{z}}\right) = 0\ .$$

Singular Points/ Singularities

Definition

A point z_0 is said to be a singular point of a function $f(z)$ if $f(z)$ is NOT analytic at z_0 and every neighborhood $N(z_0)$ of the point z_0 contains at least one point at which $f(z)$ is analytic.

We say it as the function $f(z)$ has a singularity at $z = z_0$.

Examples:

The point $z = 1$ is a singular point of $f(z) = \frac{1}{(z-1)(z-2)}$.

The polynomials has no singular points.

Definition

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be an entire function if $f(z)$ is analytic at all points of the complex plane $\mathbb{C}$.

Example: Polynomials are entire functions.