NON-STABILIZER QUANTUM CODES FROM ABELIAN SUBGROUPS OF THE ERROR GROUP

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This paper is motivated by the computer-generated nonadditive code described in Rains et al[1]. We describe a theory of non-stabilizer codes of which the nonadditive code of Rains et al is an example. Furthermore, we give a general strategy of constructing good nonstabilizer codes from good stabilizer codes and give some explicit constructions and asymptotically good nonstabilizer codes. Like in the case of stabilizer codes, we can design fairly efficient encoding and decoding procedures.

1 Introduction

Let A be a finite abelian group with operation denoted by $+$ and null element denoted by 0. We identify A with the alphabet of symbols transmitted on a classical communication channel. Consider the n-fold cartesian product A^n of copies of A. Elements of A^n are called *words* of *length n*. A commonly used group is $\{0, 1\}$ with addition modulo 2. Let $\hat{A}$ denote the character group of A, the multiplicative group of all homomorphisms from A into the multiplicative group of complex numbers of modulus unity. For $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T \in A^n$ we define its *weight* $w(\mathbf{a})$ to be $\#\{i \mid a_i \neq 0\}$. We say that a subgroup $\mathcal{C}_n$ of A^n is a *t-error correcting group code* if for every non-zero element $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ in $\mathcal{C}_n$, $w(\mathbf{x}) \geq 2t + 1$. In other words, if messages transmitted through a noisy channel are encoded into words from $\mathcal{C}_n$ and during transmission of a word errors at the output occur in at most t positions, then the message can be decoded without any error. There is a vast literature on the construction of t-error correcting group codes and the reader may find an introduction to this subject and pointers to literature in standard texts[2,3].

A broad class of quantum error correcting codes known as stabilizer codes was introduced by Gottesman[4] and Calderbank et al[5] (also see references[6,1,7]). Rains et al[1] obtained the first example of a non-stabilizer quantum code of distance 2. Subsequently, an entire family of non-stabilizer codes of distance 2 was constructed by Rains[8]. Later, Roychowdhury and Vatan[9] showed that there are non-stabilizer codes with different minimum distances and also proved an asymptotic result about the existence of t-error correcting non-stabilizer codes of good rate.

In this paper we develop a theory of *nonstabilizer* codes based on the Weyl commutation relations. The nonadditive code of Rains et al[1] is an instance of our theory and we derive it directly from the theory. Furthermore, we

outline a general strategy of constructing good nonstabilizer codes from good stabilizer codes and give some explicit constructions and asymptotically good nonstabilizer codes. For a rich family of nonstabilizer codes, we present elegant and efficient encoding circuits. We also present a simple effective decoding procedure for these nonstabilizer codes.

First we introduce some definitions. We choose and fix an M-dimensional complex Hilbert space $\mathcal{H}$ and consider the unit vectors of $\mathcal{H}$ as pure states of a finite level quantum system. If A is a finite abelian group with M elements and $\{e_x \mid x \in A\}$ is an orthonormal basis of $\mathcal{H}$ indexed by elements of A we express it in the Dirac notation as $|x\rangle = e_x$. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in A^n$ is a word of length n, we write

$$|\mathbf{x}\rangle = |x_1 x_2 \ldots x_n\rangle = e_{x_1} \otimes e_{x_2} \otimes \ldots \otimes e_{x_n}$$

where the right-hand side is a product vector in the n-fold tensor product $\mathcal{H}^{\otimes^n}$ of n copies of $\mathcal{H}$. Thus, with the chosen orthonormal basis, every word $\mathbf{x}$ in A^n is translated into a basis state $|\mathbf{x}\rangle$ of $\mathcal{H}^{\otimes^n}$.

A *quantum code* is a subspace $\mathcal{C}_n$ of $\mathcal{H}^{\otimes^n}$. Note that a pure state in $\mathcal{H}^{\otimes^n}$ described by a unit vector $|\psi\rangle$ in $\mathcal{H}^{\otimes^n}$ has density matrix $|\psi\rangle \langle\psi|$. A density matrix ρ in $\mathcal{H}^{\otimes^n}$ is a non-negative operator of unit trace. In quantum probability, a projection operator E in $\mathcal{H}^{\otimes^n}$ is interpreted as an event concerning the quantum system and a density matrix ρ as a state of the quantum system. The probability of the event E in the state ρ is given by $Tr\rho E$. Messages to be transmitted through a quantum channel are encoded into pure states in $\mathcal{H}^{\otimes^n}$. When a pure state $|\psi\rangle$, or equivalently, a density matrix $|\psi\rangle \langle\psi|$ is transmitted the channel output is hypothesized to be a state of the form

$$\rho = \sum_i L_i |\psi\rangle \langle\psi| L_i^\dagger \tag{1}$$

where the operators $\{L_i\}$ belong to a linear subspace $\mathcal{A}$ of the algebra of all operators on $\mathcal{H}^{\otimes^n}$. The operators $\{L_i\}$ may depend on ρ, but in order to ensure that ρ is a density matrix it is assumed that $\langle\psi| \sum_i L_i^\dagger L_i |\psi\rangle = 1$. By the spectral theorem ρ can be expressed as

$$\rho = \sum_j p_j |\psi_j\rangle \langle\psi_j|$$

where ψ_j is an orthonormal set in $\mathcal{H}^{\otimes^n}$ and $\{p_j\}$ is a probability distribution with $p_j > 0$ for each j. In other words, the output state ρ is not necessarily pure even though the input state is pure. The operators L_i are called *error operators* and the linear space $\mathcal{A}$ from which they come is called the *error space*.

Let P be the projection operator corresponding to a quantum code $\mathcal{C}_n$. The subspace $\mathcal{D}(P)$ of *error operators detected* by P is defined as

$$\mathcal{D}(P) = \{L \in \mathcal{B}(\mathcal{H}^{\otimes^n}) \mid PLP = c.P \text{ for some } c \in \mathbb{C}\}.$$

It is evident that there is a complex-valued functional $\phi : \mathcal{D}(P) \to \mathbb{C}$ so that we can write $PLP = \phi(L)P$ for all $L \in \mathcal{D}(P)$.

A finite family $\{M_j\} \subseteq \mathcal{B}(\mathcal{H}^{\otimes^n})$ constitutes a set of *decoding operators* for the code $\mathcal{C}_n$ and error space $\mathcal{A}$ if the following conditions are satisfied.

(a) $\sum_j M_j^\dagger M_j = I$.

(b) For any pure state $|\psi\rangle \in \mathcal{C}_n$ if the output state ρ corresponding to $|\psi\rangle$ is given by $\rho = \sum_{i=1}^m L_i |\psi\rangle \langle\psi| L_i^\dagger$, $L_i \in \mathcal{A}$ then

$$\sum_j M_j \rho M_j^\dagger = \sum_{i,j} M_j L_i |\psi\rangle \langle\psi| L_i^\dagger M_j^\dagger = |\psi\rangle \langle\psi|.$$

In this case we say that $\mathcal{C}_n$ is an $\mathcal{A}$-*error correcting quantum code*.

We have the following fundamental theorem of Knill and Laflamme[10] which characterizes the errors that a quantum code can correct. It essentially states that errors coming from a family $\mathcal{A}$ of operators can be corrected for a quantum code with projection P if and only if

$$\{L_1^\dagger L_2 \mid L_1, L_2, \in \mathcal{A}\} \subseteq \mathcal{D}(P).$$

Theorem 1.1 *Let $\mathcal{A}$ be a family of operators in $\mathcal{H}^{\otimes^n}$ and let $\mathcal{C}_n \subset \mathcal{H}^{\otimes^n}$ be a quantum code with an orthonormal basis $\psi_1, \psi_2, \ldots, \psi_d$. Let P be the projection corresponding to the code $\mathcal{C}_n$. Then $\mathcal{C}_n$ is an $\mathcal{A}$-error correcting quantum code if and only if*

$$\mathcal{A}^\dagger \mathcal{A} = \{L_1^\dagger L_2 \mid L_1, L_2 \in \mathcal{A}\} \subseteq \mathcal{D}(P).$$

Remark 1.2 *The proof of the above theorem is constructive and yields the decoding operators in terms of $\mathcal{A}$ and the basis $\psi_1, \ldots, \psi_d$ of $\mathcal{C}_n$.*

Now we specialize the choice of $\mathcal{A}$. Consider all unitary operators in $\mathcal{H}^{\otimes^n}$ of the form $U = U_1 \otimes U_2 \otimes \ldots \otimes U_n$ where each U_i is a unitary operator on $\mathcal{H}$ and all but t of the U_i's are equal to I. Such a U when operating on $\psi = \psi_1 \otimes \ldots \otimes \psi_n \in \mathcal{H}^{\otimes^n}$ produces $U|\psi\rangle$ which is an n-fold tensor product that differs from ψ in at most t places. Denote by $\mathcal{A}_t$ the linear span of all such unitary operators U. A quantum code $\mathcal{C}_n$ is called a *t-error correcting quantum code* if $\mathcal{C}_n$ is an $\mathcal{A}_t$-correcting quantum code.

2 Quantum Codes and Projections in a Group Algebra

Let $(A, +)$ be a finite abelian group of M elements with null element 0. By the fundamental theorem of finite abelian groups, A is isomorphic to $\bigoplus_{i=1}^k \mathbb{Z}_{n_i}$ via the isomorphism τ. For every m, let $\omega_m = e^{2\pi i/m}$. Define the *canonical bicharacter* of the group A as the following complex-valued function on $A \times A$.

$$\langle\langle a, b \rangle\rangle = \prod_{j=1}^k \omega_{n_j}^{x_j y_j}, \text{ where } \tau(a) = (x_1, \ldots, x_k) \text{ and } \tau(b) = (y_1, \ldots, y_k).$$

Notice that for all $a, b, c \in A$ we have $\langle\langle a, b \rangle\rangle = \langle\langle b, a \rangle\rangle$, $\langle\langle a + b, c \rangle\rangle = \langle\langle a, c \rangle\rangle \langle\langle b, c \rangle\rangle$, and $\langle\langle a, b \rangle\rangle = 1$ for all $b \in A$ if and only if $a = 0$. Denote by $\hat{A}$ the character group of A. For each fixed $a \in A$, the bicharacter $\langle\langle a, b \rangle\rangle$, as a function of b, is a distinct element χ_a of $\hat{A}$ and the correspondence $a \mapsto \chi_a$ is a group isomorphism between A and the multiplicative character group $\hat{A}$.

Denote by $\mathcal{H}$ the M-dimensional Hilbert space $L^2(A)$ of all complex-valued functions on A, spanned by $\{|x\rangle\}_{x \in A}$ (where the vector $|x\rangle$ denotes the indicator function 1_x of the singleton $\{x\}$). Define the unitary operators U_a and V_a on $\mathcal{H}$ for every $a \in A$ by

$$ U_a |x\rangle = |x + a\rangle, \qquad V_a |x\rangle = \langle\langle a, x \rangle\rangle |x\rangle $$

where $x \in A$. Then we have

$$ U_a U_b = U_{a+b} \quad V_a V_b = V_{a+b}, \text{ and } \langle\langle a, b \rangle\rangle U_a V_b = V_b U_a \;\; \forall \; a, b \in A. $$

These are the Weyl commutation relations between the unitary operators U_a and V_a on $\mathcal{H}$. The family of operators $\{U_a V_b \mid a, b \in A\}$ is irreducible.

The canonical bicharacter on A gives rise to the following bicharacter on A^n. For two elements $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in A^n, $\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle$ is defined as

$$ \langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle = \prod_{i=1}^{n} \langle\langle a_i, b_i \rangle\rangle. $$

Put $U_{\mathbf{a}} = U_{a_1} \otimes \ldots \otimes U_{a_n}$ and $V_{\mathbf{b}} = V_{b_1} \otimes \ldots \otimes V_{b_n}$. Then $\{U_{\mathbf{a}} V_{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in A^n\}$ is again an irreducible family of unitary operators such that $U_{\mathbf{a}} U_{\mathbf{b}} = U_{\mathbf{a}+\mathbf{b}}$ and $V_{\mathbf{a}} V_{\mathbf{b}} = V_{\mathbf{a}+\mathbf{b}}$, and they satisfy the Weyl commutation relations

$$ \langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle U_{\mathbf{a}} V_{\mathbf{b}} = V_{\mathbf{b}} U_{\mathbf{a}} \;\; \forall \; \mathbf{a}, \mathbf{b} \in A^n. $$

In the Hilbert space $\mathcal{B}(\mathcal{H}^{\otimes^n})$ of all linear operators on $\mathcal{H}^{\otimes^n}$ with the scalar product $\langle X \mid Y \rangle = Tr X^{\dagger} Y$ the set $\{M^{-n/2} U_{\mathbf{a}} V_{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in A^n\}$ is an orthonormal basis. In particular

$$ Tr\, U_{\mathbf{a}} V_{\mathbf{b}} = \begin{cases} 0 & \text{if } (\mathbf{a}, \mathbf{b}) \neq (0, 0), \\ M^n & \text{otherwise}, \end{cases} $$

The weight $wt(\mathbf{a}, \mathbf{b})$ of a pair $(\mathbf{a}, \mathbf{b}) \in A^n \times A^n$ is defined to be $\#\{i \mid 1 \leq i \leq n, (a_i, b_i) \neq (0, 0)\}$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. The irreducibility of $\{U_{\mathbf{a}} V_{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in A^n\}$ implies that $\{U_{\mathbf{a}} V_{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in A^n, wt(\mathbf{a}, \mathbf{b}) \leq t\}$ spans $\mathcal{A}_t$. As a result, the Knill-Laflamme theorem for $\mathcal{A}_t$-correcting quantum codes takes the following form which is easy to derive from Theorem 1.1.

Theorem 2.1 $\mathcal{C}_n \subset L^2(A)^{\otimes^n}$ *is a t-error correcting quantum code if and only if*

$$ \{U_{\mathbf{a}} V_{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in A^n, wt(\mathbf{a}, \mathbf{b}) \leq 2t\} \subseteq \mathcal{D}(P), $$

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where P is the projection corresponding to $\mathcal{C}_n$.

Let N be the least positive integer such that $Na = 0$ for all $a \in A$. Let $\omega = e^{2\pi i/N}$, We define the error group $\mathcal{E}$ as follows.

Definition 2.2 *The* error group $\mathcal{E}$ *is defined as*

$$\mathcal{E} = \{\omega^i U_{\mathbf{a}} V_{\mathbf{b}} \mid 0 \leq i \leq N - 1, \mathbf{a}, \mathbf{b} \in A^n\},$$

with the group operation defined by

$$\omega^i U_{\mathbf{a}} V_{\mathbf{b}} \omega^j U_{\mathbf{c}} V_{\mathbf{d}} = \omega^{i+j} \langle\langle \mathbf{b}, \mathbf{c} \rangle\rangle U_{\mathbf{a}+\mathbf{c}} V_{\mathbf{b}+\mathbf{d}}.$$

Let $\mathcal{S} \leq \mathcal{E}$ be a subgroup of the error group. In the theory of stabilizer codes we are interested in subspaces of $L^2(A)^{\otimes n}$ that are left invariant under the action of $\mathcal{S}$. It turns out that the invariant subspace is nontrivial if and only if $\mathcal{S}$ is abelian with the property that $\omega^i I \notin \mathcal{S}$ for all $1 \leq i \leq N - 1$.

Definition 2.3 *An abelian subgroup $\mathcal{S}$ of $\mathcal{E}$ is said to be a* Gottesman subgroup *of $\mathcal{E}$ if $\omega^i I \notin \mathcal{S}$ for all $1 \leq i \leq N - 1$. The* closure *of a Gottesman subgroup $\mathcal{S}$ is the abelian subgroup $\overline{\mathcal{S}}$ of $\mathcal{E}$ defined as*

$$\overline{\mathcal{S}} = \{\omega^i g \mid g \in \mathcal{S}, \ 0 \leq i \leq N - 1\}.$$

Remark 2.4 *For any Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$ the element $\omega^i U_{\mathbf{a}} V_{\mathbf{b}} \in \mathcal{S}$ for at most one $i : 0 \leq i \leq N - 1$.*

Let $\mathbb{C}[\mathcal{S}]$ be the group algebra of formal sums $\sum_{s \in \mathcal{S}} T_s s$, $T_s \in \mathbb{C}$ where the sum and product (convolution) are defined as follows

$$T + T' = \sum_{s \in \mathcal{E}} (T_s + T'_s)s,$$

$$T' * T'' = T, \text{where } T_s = \sum_{g \in \mathcal{E}} T'_g T''_{g^{-1}s}.$$

Since any Gottesman subgroup $\mathcal{S}$ is a set of linearly independent elements of $\mathcal{B}(\mathcal{H}^{\otimes n})$, the identity map is a natural injective linear embedding from $\mathbb{C}[\mathcal{S}]$ into $\mathcal{B}(\mathcal{H}^{\otimes n})$. In other words, $\mathbb{C}[\mathcal{S}]$ is a subalgebra of $\mathcal{B}(\mathcal{H}^{\otimes n})$ under operator addition and operator multiplication, where the convolution operation "$*$" in $\mathbb{C}[\mathcal{S}]$ coincides with operator multiplication.

Every subspace (i.e. quantum code) in $\mathcal{H}^{\otimes n}$ is defined by its corresponding projection operator in $\mathcal{B}(\mathcal{H}^{\otimes n})$. In this paper we are interested in the projection operators in $\mathbb{C}[\mathcal{S}]$ for a Gottesman subgroup $\mathcal{S}$ of $\mathcal{E}$. In the following easy proposition we characterize the elements in $\mathbb{C}[\mathcal{S}]$ which are projection operators in $\mathcal{H}^{\otimes n}$. Then our goal will be to seek for projections whose range is a good t-error correcting quantum code.

Proposition 2.5 *The element $T \in \mathbb{C}[\mathcal{S}]$ is a projection operator on $\mathcal{H}^{\otimes n}$ if and only if $\overline{T_s} = T_{s^{-1}}$ and $T * T = T$.*

We use the Fourier transform over $\mathbb{C}[\mathcal{S}]$ and the above proposition to describe projection operators in $\mathbb{C}[\mathcal{S}]$. We recall the Fourier transform and some of its properties.

Let $\mathcal{S}$ be an abelian group and let $\hat{\mathcal{S}}$ denote the character group of $\mathcal{S}$. For each $s \in \mathcal{S}$ we can associate the element $\sum \alpha_g g$ in the algebra $\mathbb{C}[\mathcal{S}]$ where, $\alpha_s = 1$ and $\alpha_g = 0$ for $g \neq s \in \mathcal{S}$. Similarly, to $\chi \in \hat{S}$ we associate the element $\sum \chi(g)g$ of $\mathbb{C}[\mathcal{S}]$. Fix an isomorphism $s \mapsto \chi_s$ between the groups $\mathcal{S}$ and $\hat{\mathcal{S}}$. The Fourier transform over $\mathbb{C}[\mathcal{S}]$ is now defined as follows.

Definition 2.6 (Fourier Transform) *The linear transformation that maps $s \in \mathbb{C}[\mathcal{S}]$ to $\chi_s \in \mathbb{C}[\mathcal{S}]$ is called the* Fourier transform *over $\mathbb{C}[\mathcal{S}]$. The Fourier transform $\hat{T}$ of T in $\mathbb{C}[\mathcal{S}]$ is given by the formula*

$$\hat{T}_u = \sum_{s \in \mathcal{S}} \chi_u(s) T_s.$$

The *inverse Fourier transform* is given by the following formula:

$$T_s = \frac{1}{\#\mathcal{S}} \sum_{u \in \mathcal{S}} \overline{\chi}_u(s) \hat{T}_u.$$

Let TT' denote the component-wise product of $T, T' \in \mathbb{C}[\mathcal{S}]$. I.e.

$$TT' = \sum_{s \in \mathcal{S}} T_s T'_s s.$$

We now recall some useful properties of the Fourier transform.

$$\widehat{T_1 * T_2} = \hat{T}_1 \hat{T}_2, \qquad T_1, T_2 \in \mathbb{C}[\mathcal{S}].$$
$$\widehat{gT} = \chi_g \hat{T}, \qquad g \in \mathcal{S}, \quad T \in \mathbb{C}[\mathcal{S}].$$

We can characterize projection operators in $\mathbb{C}[\mathcal{S}]$ using the Fourier transform.

Theorem 2.7 *An element $T \in \mathbb{C}[\mathcal{S}]$ is a projection (and hence a code) iff $\hat{T} = 1_B = \sum_{g \in B} g$ for some subset B of $\mathcal{S}$.*

Proof. T is a projection iff $T * T = T$ and $T^\dagger = T$. On taking Fourier transforms on both sides of the equation $T * T = T$ we get

$$\hat{T}\hat{T} = \hat{T}.$$

This implies $\hat{T}_s^2 = \hat{T}_s$ for every $s \in \mathcal{S}$. Thus, $\hat{T}_s \in \{0, 1\}$ for every $s \in \mathcal{S}$ which gives the desired result.

Conversely, note that if $\hat{T} = 1_B$ then by inverse Fourier transform we get

$$T = \frac{1}{\#\mathcal{S}} \sum_{s \in \mathcal{S}} \sum_{u \in B} \overline{\chi_u}(s) s.$$

From the above equation it is clear that the condition $T^\dagger = T$ is automatically satisfied. ∎

From Theorem 2.7 it is clear that the code defined by a projection T in $\mathbb{C}[\mathcal{S}]$ is completely specified by the subset B of $\mathcal{S}$. We will call B the *Fourier description* of the quantum code and denote the corresponding projection by $P(B)$.

Next, we give a formula for the dimension of a quantum code defined by a projection T in $\mathbb{C}[\mathcal{S}]$.

Lemma 2.8 *Let $\mathcal{C}_n \subseteq \mathcal{H}^{\otimes^n}$ be a quantum code defined by a projection T in $\mathbb{C}[\mathcal{S}]$ with Fourier description $B \subseteq \mathcal{S}$. Then $\mathcal{C}_n \neq 0$ if and only if $\omega^i I \notin \mathcal{S}$ for $i : 1 \leq i \leq N-1$ (i.e. $\mathcal{S}$ is a Gottesman subgroup), and if $\mathcal{C}_n \neq 0$ the dimension of the code $\mathcal{C}_n$ is given by*

$$\dim(\mathcal{C}_n) = \frac{\#A^n \#B}{\#\mathcal{S}}.$$

Proof. If B is the Fourier description of the code then the projection corresponding to the code is given by

$$P = \frac{1}{\#\mathcal{S}} \sum_{s \in \mathcal{S}} \sum_{u \in B} \overline{\chi_u}(s) s.$$

The dimension is given by $Tr(P)$. Observe that $Tr(U_\mathbf{a} V_\mathbf{b}) = 0$ if $U_\mathbf{a} V_\mathbf{b} \neq I$, and $Tr(\omega^i I) = \omega^i \#A^n$, for $\omega^i I \in \mathcal{S}$. Now, since ω is a nontrivial root of unity, $\sum_{\omega^i I \in \mathcal{S}} \omega^i \#A = 0$ if $\omega^i I \in \mathcal{S}$ for some i such that $1 \leq i \leq N-1$. Thus, we have $Tr(P) = 0$ if $\omega^i I \in \mathcal{S}$ for some $i : 1 \leq i \leq N-1$, and otherwise $Tr(P) = \frac{\#B}{\#\mathcal{S}} \#A^n$. This proves the lemma. ∎

Proposition 2.9 *Let $g = \omega^i U_\mathbf{a} V_\mathbf{b}$ and $h = \omega^j U_\mathbf{c} V_\mathbf{d}$ be elements in the error group $\mathcal{E}$. Then*

$$ghg^{-1}h^{-1} = \gamma(g,h)I$$

where $\gamma(g,h) = \langle\langle b,c \rangle\rangle \overline{\langle\langle a,d \rangle\rangle}$. In particular we have

$$\gamma(g_1 g_2, h) = \gamma(g_1, h)\gamma(g_2, h)$$

$$\gamma(g, h_1 h_2) = \gamma(g, h_1)\gamma(g, h_2)$$

and

$$\gamma(g,h) = \overline{\gamma(h,g)}$$

Proof. Straightforward from the Weyl commutation relations. ∎

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Remark 2.10 *When $h \in \mathcal{E}$ is fixed and s varies in $\mathcal{S}$ the map $s \mapsto \gamma(s, h)$ is a character of $\mathcal{S}$ which we will denote by γ_h.*

Let $\mathcal{C}_n \subseteq \mathcal{H}^{\otimes^n}$ be a code with Fourier description B. In the next theorem we derive a condition on B such that $\mathcal{C}_n$ is a t-error correcting quantum code. We introduce a convenient notation: For $g = \omega^i U_{\mathbf{a}} V_{\mathbf{b}}$ in the error group $\mathcal{E}$, let $wt(g)$ denote the number $wt(\mathbf{a}, \mathbf{b})$.

Theorem 2.11 *A quantum code $\mathcal{C}_n \subseteq \mathcal{H}^{\otimes^n}$ with Fourier description B is t-error correcting iff the following two conditions hold.*

1. *For each $g \in \mathcal{S}$ such that $wt(g) \leq 2t$*

$$\chi_g(u_1^{-1} u_2) = 1 \text{ for all } u_1, u_2 \in B.$$

   (i.e. every $u \in B$ is in the same coset of the kernel of χ_g.)

2. *For each $g \in \mathcal{E} \setminus \mathcal{S}$ such that $wt(g) \leq 2t$, and for every $u \in B^{-1}B$ we have*

$$\sum_{s \in \mathcal{S}} \gamma_g(s) \chi_u(s) = 0.$$

   (i.e. the character γ_g is different from $\chi_{u^{-1}}$ for every $u \in B^{-1}B$.)

Proof.

Let $\mathcal{C}_n$ be a t-error correcting code and let $T = \sum_{s \in \mathcal{S}} T_s s$ be the corresponding projection. Recall that if B is the Fourier description for the code then

$$T_s = \frac{1}{\#\mathcal{S}} \sum_{u \in B} \overline{\chi}_u(s).$$

By Theorem 1.1, $\mathcal{C}_n$ is t-error correcting if and only if there is a scalar-valued function ϕ such that for every $g \in \mathcal{E}$ with $wt(g) \leq 2t$

$$TgT = \phi(g)T.$$

This is equivalent to the following condition.

$$\left(\sum_{s_1 \in \mathcal{S}} T_{s_1} s_1 \right) g \left(\sum_{s_2 \in \mathcal{S}} T_{s_2} s_2 \right) = \phi(g) \sum_{s \in \mathcal{S}} T_s s. \tag{2}$$

Case 1 $g \in \mathcal{S}$:

In this case Equation 2 yields $gT * T = \phi(g)T$. Taking Fourier transform on both sides we get

$$\chi_g \hat{T} \hat{T} = \phi(g) \hat{T}.$$

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Since $\hat{T} = 1_B$, we have $\chi_g(u) = \phi(g)$ for all $u \in B$. Thus, χ_g is constant on B for every $g \in \mathcal{S}$ such that $wt(g) \leq 2t$. This is true precisely when B is contained in some coset of the kernel of χ_g.

Case 2 $g \notin \mathcal{S}$:

In this case the Knill-Laflamme condition takes the following form

$$\sum_{s \in \mathcal{S}} \sum_{s_1 s_2 = s} T_{s_1} T_{s_2} \gamma(s_1, g) g s_1 s_2 = \phi(g) \sum_{s \in \mathcal{S}} T_s s.$$

Since the operators on the two sides of the above equation have disjoint support, each side of the equation vanishes. Consequently, $\phi(g) = 0$ and for all $s \in \mathcal{S}$

$$\sum_{s_1 s_2 = s} T_{s_1} T_{s_2} \gamma(s_1, g) = 0.$$

This yields

$$\sum_{s_1 s_2 = s} \left(\sum_{u_1 \in B} \overline{\chi}_{u_1}(s_1) \right) \left(\sum_{u_2 \in B} \overline{\chi}_{u_2}(s_2) \right) \gamma(s_1, g) = 0.$$

On simplification we get

$$\sum_{u_1, u_2 \in B} \sum_{s_1 s_2 = s} \gamma_g(s_1) \overline{\chi}_{u_1}(s_1) \overline{\chi}_{u_2}(s_2) = 0,$$

which gives

$$\sum_{u_1, u_2 \in B} \overline{\chi}_{u_2}(s) \sum_{s_1 \in \mathcal{S}} (\gamma_g \chi_{u_2} \overline{\chi}_{u_1})(s_1) = 0. \tag{3}$$

Note that the inner summation in equation 3 is summing up of a character of $\mathcal{S}$, namely, $\gamma_g \chi_{u_1^{-1}} \chi_{u_2}$, over the whole group $\mathcal{S}$. Therefore, the inner summation yields either 0 or $\#\mathcal{S}$. Hence, the necessary and sufficient condition for equation 3 to hold is

$$\sum_{s \in \mathcal{S}} \gamma_g(s) \chi_u(s) = 0 \qquad \forall u \in B^{-1} B.$$

$\blacksquare$

Remark 2.12 *If the Fourier description of a code, B, is a subgroup of $\mathcal{S}$ then the code is actually a stabilizer code with stabilizer group $B^\perp$, where $B^\perp$ is the annihilator of B in $\mathcal{S}$ defined by*

$$B^\perp = \{a \in \mathcal{S} : \forall b \in B \ \langle\langle a, b \rangle\rangle = 0\}.$$

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In particular if we set $B = \{I\}$, where I is the identity element, the code $\mathcal{C}_n$ with Fourier description B is the stabilizer quantum code:

$$\mathcal{C}_n = \{|\psi\rangle \mid s\,|\psi\rangle = |\psi\rangle \,\forall s \in \mathcal{S}\}.$$

Thus the stabilizer codes of[4] are a subclass of the class of codes defined in this paper.

At this point we recall some useful facts from the theory of stabilizer codes (see[11] for details).

Let $\mathcal{S}$ be a Gottesman subgroup of $\mathcal{E}$

$$\mathcal{C}(\mathcal{S}) = \{|\psi\rangle \in L^2(A)^{\otimes^n} \mid U\,|\psi\rangle = |\psi\rangle \;\; \forall \; U \in \mathcal{S}\}.$$

Let $Z(\mathcal{S})$ denote the centralizer of $\mathcal{S}$ in $\mathcal{E}$, that is,

$$Z(\mathcal{S}) = \{U \in \mathcal{E} \mid UU' = U'U \;\; \forall \; U' \in \mathcal{S}\}.$$

For Gottesman subgroup $\mathcal{S}$ let the closure be $\overline{\mathcal{S}}$ (see Definition 2.3).

Theorem 2.13 *Let $\mathcal{S}$ be a Gottesman subgroup of the error group $\mathcal{E}$ and let $\overline{\mathcal{S}}$ be its closure. Then $\mathcal{C}(\mathcal{S})$ is a t-error correcting quantum code if $wt(\mathbf{a}, \mathbf{b}) > 2t$ for each $\omega^i U_{\mathbf{a}} V_{\mathbf{b}} \in Z(\mathcal{S}) \setminus \overline{\mathcal{S}}$.*

The above theorem is a generalization of a basic result of Calderbank et al [5], where it is stated for the special case $A = \mathbb{F}_4$.

We introduce a useful notation for describing quantum stabilizer codes. Let $\mathcal{S}$ be a Gottesman subgroup of $\mathcal{E}$ with centralizer $Z(\mathcal{S})$. The *minimum distance* $d(\mathcal{S})$ is defined to be the minimum of

$$\{wt(\mathbf{a}, \mathbf{b}) \mid \omega^i U_{\mathbf{a}} V_{\mathbf{b}} \in Z(\mathcal{S}) \setminus \overline{\mathcal{S}}\}.$$

When A is the additive abelian group of the finite field $\mathbb{F}_q$ we define an $[[n, k, d]]_q$ quantum stabilizer code to be a q^k-dimensional subspace $\mathcal{C}(\mathcal{S})$ of $L^2(\mathbb{F}_q)^{\otimes^n}$, where $\mathcal{S}$ is a Gottesman subgroup of $\mathcal{E}$ with $d(\mathcal{S}) \geq d$ and cardinality q^{n-k}.

By Theorem 2.13 it follows that an $[[n, k, d]]_q$ quantum stabilizer code is a $\lfloor (d-1)/2 \rfloor$-error correcting quantum code.

3 Nonstabilizer codes over finite fields

We focus our attention to the case when the abelian group A is the field $\mathbb{F}_q$. Recall that the additive group $\mathbb{F}_q^n$ is a vector space over $\mathbb{F}_q$. If ω is a nontrivial character then the characters of the additive group $\mathbb{F}_q^n$ constitute the set $\{\omega_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{F}_q^n\}$ where $\omega_{\mathbf{a}}(\mathbf{b}) = \omega(\mathbf{a}.\mathbf{b})$, $\mathbf{a}.\mathbf{b} = \sum a_i b_i$.

The set of operators $\{\omega_p^i U_{\mathbf{a}} V_{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n, 0 \leq i \leq p-1\}$, ω_p is a pth root of unity, where p is the characteristic of the field $\mathbb{F}_q$, forms an irreducible representation of associated error group. It is shown in[11] that any Gottesman subgroup $\mathcal{S}$ is of the following form

$$\{\omega(\rho(\mathbf{a}))U_{L\mathbf{a}}V_{M\mathbf{a}} \mid \mathbf{a} \in \mathbb{F}_q^r\},$$

where L and M are $n \times r$ matrices over $\mathbb{F}_q$ such that $L^T M$ is symmetric and $\rho(.)$ satisfies the condition that

$$\rho(\mathbf{v}_1 + \mathbf{v}_2) - \rho(\mathbf{v}_1) - \rho(\mathbf{v}_2) = \mathbf{v}_2{}^T L^T M \mathbf{v}_1.$$

Our goal in this section is to seek for nonstabilizer codes with the help of Theorem 2.11.

Definition 3.1 *A Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$ is said to be d-pure if $wt(g) \geq d$ for every $g \in Z(\mathcal{S})$ that is not a scalar multiple of the identity operator.*

By the theory of stabilizer codes, it follows that the stabilizer code $\mathcal{C}(\mathcal{S}) = \{\psi \in L^2(A)^{\otimes^n} \mid U\psi = \psi \ \forall \ U \in \mathcal{S}\}$ corresponding to a d-pure Gottesman subgroup $\mathcal{S}$, is a $\lfloor (d-1)/2 \rfloor$-error correcting quantum code.

More precisely, our aim is to start with the stabilizer code $\mathcal{C}(\mathcal{S})$ of distance d, and use Theorem 2.11 to construct nonstabilizer codes of the same distance but larger dimension.

Observe that if $\mathcal{S}$ is a $2t$-pure Gottesman subgroup of $\mathcal{E}$, the first condition in Theorem 2.11 is vacuously true. Thus, we only need to ensure that the second condition in Theorem 2.11 is satisfied. For a d-pure Gottesman code we define the *forbidden set* as follows

Definition 3.2 *Let $\mathcal{S}$ be a d-pure Gottesman subgroup of the error group $\mathcal{E}$. We define the d-forbidden subset of $\mathcal{S}$, denoted by $\mathcal{F}_d(\mathcal{S})$, to be the subset*

$$\mathcal{F}_d(\mathcal{S}) = \left\{ u \in \mathcal{S} : u \neq I, \ and \ \exists g \in \mathcal{E} \setminus \mathcal{S} \ wt(g) < d \ and \ \sum_{s \in S}(\gamma_g \chi_u)(s) = \#\mathcal{S} \right\}.$$

We have the following theorem that is an easy consequence of Theorem 2.11.

Theorem 3.3 *Let $\mathcal{S}$ be a d-pure Gottesman subgroup of the error group $\mathcal{E}$. Then $B \subseteq \mathcal{S}$ is the Fourier description of a distance d code if and only if $B^{-1}B \cap \mathcal{F}_d(\mathcal{S})$ is empty.*

Let $s_{\mathbf{a}}$ denote $\omega_p(\rho(a))U_{L\mathbf{a}}V_{M\mathbf{a}} \in \mathcal{S}$. Observe that $\tau : s_{\mathbf{a}} \mapsto \mathbf{a}$ is a group isomorphism from $\mathcal{S}$ to $\mathbb{F}_q^r$, and $\chi_{s_{\mathbf{a}}} \mapsto \omega_{\mathbf{a}}$ is an isomorphism from $\widehat{\mathcal{S}}$ to $\widehat{\mathbb{F}_q^r}$.

Let $g = \omega^i U_{\mathbf{x}} V_{\mathbf{y}} \notin \mathcal{S}$ with $wt(g) \leq 2t$. By applying the Weyl commutation relations we get $\gamma_g(s_{\mathbf{a}}) = \omega(\mathbf{a}^T M^T \mathbf{x} - \mathbf{a}^T L^T \mathbf{y}) = \omega_{\mathbf{a}}(M^T \mathbf{x} - L^T \mathbf{y})$. We have the following theorem.

Theorem 3.4 *Let $\mathcal{S} = \{\omega(\rho(\mathbf{a}))U_{L\mathbf{a}}V_{M\mathbf{a}} : a \in \mathbb{F}_q^r\}$ be a d-pure Gottesman subgroup of the error group $\mathcal{E}$ over $\mathbb{F}_q^n$. Then the d-forbidden subset of $\mathcal{S}$ is given by*

$$\mathcal{F}_d(\mathcal{S}) = \{s_{\mathbf{u}} : \exists \mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n \ wt(\mathbf{x}, \mathbf{y}) < d \ and \ \mathbf{u} = L^T \mathbf{y} - M^T \mathbf{x}\}.$$

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Proof.

$$\sum_{s \in \mathcal{S}} \gamma_g(s) \chi_u(s) = \sum_{\mathbf{a} \in \mathbb{F}_q^r} \omega_a(\mathbf{u} + M^T \mathbf{x} - L^T \mathbf{y})$$

Note that the right hand side of the equation is the sum over all characters of $\mathbb{F}_q^r$ and hence is nonzero iff $\mathbf{u} + M^T \mathbf{x} - L^T \mathbf{y} = 0$. Hence

$$\mathcal{F}_d(\mathcal{S}) = \{s_{\mathbf{u}} : \exists \mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n \ wt(\mathbf{x}, \mathbf{y}) < d \text{ and } \mathbf{u} = L^T \mathbf{y} - M^T \mathbf{x}\}$$

∎

Remark 3.5 *In the above setting we will call the set*

$$F_d(\mathcal{S}) = \{\mathbf{u} : \mathbf{u} \neq 0, \exists \mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n \ wt(\mathbf{x}, \mathbf{y}) < d \text{ and } \mathbf{u} = L^T \mathbf{y} - M^T \mathbf{x}\}$$

the forbidden set. Note that $\mathcal{F}_d(\mathcal{S}) = \{s_{\mathbf{u}} : \mathbf{u} \in F_d(\mathcal{S})\}$.

4 Bounds on the dimension of codes

We now give upper and lower bounds on the dimension of nonstabilizer codes built from pure Gottesman subgroups of $\mathcal{E}$. Let the encoding space be $L^2(A)^{\otimes^n}$, and $N(n, q, d)$ denote the number $\sum_{i=0}^{d} \binom{n}{i}(q^2 - 1)^i$. We have the following upper bound on the dimension of the code.

Theorem 4.1 *Let $\mathcal{C}_n \subseteq L^2(A)^{\otimes^n}$ be a d-error correcting quantum code such that its corresponding projection P has support in a $2d + 1$-pure Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$. Let B be the Fourier description of $\mathcal{C}_n$. Then the dimension of the code $\mathcal{C}_n$ satisfies the inequality*

$$\dim(\mathcal{C}_n) \leq \frac{\#A^n}{N(n, \#A, d)}.$$

Proof. Since $\mathcal{S}$ is $2d+1$-pure, $\{g \in \mathcal{E} \mid wt(g) \leq 2d\} \subseteq \mathcal{E} \backslash \mathcal{S}$. By Theorem 2.11 we have:

$$PgP = 0$$

for all $g \in \mathcal{E}$ such that $wt(g) \leq 2d$. Let P_g denote the projection $g^{-1}Pg$. The range of P_g has dimension $\dim(\mathcal{C}_n)$ for every $g \in \mathcal{E}$. Furthermore, for all $g_1, g_2 \in \mathcal{E}$ such that $wt(g_1) \leq d$ and $wt(g_2) \leq d$, we have

$$P_{g_1} P_{g_2} = g_1^{-1} P g_1 g_2^{-1} P g_2 = 0,$$

since $wt(g_1 g_2) \leq 2d$ implies $Pg_1 g_2 P = 0$. Thus, $\{P_g \mid g \in \mathcal{E}, wt(g) \leq d\}$ is a collection of mutually orthogonal projections in $L^2(A)^{\otimes^n}$. Furthermore, the

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range of each P_g is $\dim(\mathcal{C}_n)$. Since there are $N(n, \#A, d)$ elements g in $\mathcal{E}$ with $wt(g) \leq d$, it follows by adding dimensions that

$$N(n, \#A, d) \ \dim(\mathcal{C}_n) \leq \dim L^2(A)^{\otimes^n} = \#A^n.$$

$\blacksquare$

We now obtain a lower bound for the dimension of codes satisfying the conditions of Theorem 3.4.

Theorem 4.2 *Let $A = \mathbb{F}_q^n$, and $\mathcal{S} = \{\omega(\rho(\mathbf{a}))U_{L\mathbf{a}}V_{M\mathbf{a}}|\mathbf{a} \in \mathbb{F}_q^r\}$ be a $2d+1$-pure Gottesman subgroup of the error group $\mathcal{E}$ for the encoding space $L^2(A)^{\otimes^n}$. Then there is a d-error correcting code $\mathcal{C}_n$ such that its corresponding projection has support in $\mathcal{S}$ and*

$$\dim(C_n) \geq \frac{q^n}{N(n, q, 2d)}.$$

Proof. For the $2d+1$-pure Gottesman subgroup $\mathcal{S} = \{\omega(\rho(\mathbf{a}))U_{L\mathbf{a}}V_{M\mathbf{a}}|\mathbf{a} \in \mathbb{F}_q^r\}$, let X be the corresponding forbidden set. By Theorem 3.4, X is the image of the set $\{(\mathbf{x}, \mathbf{y})|\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n, wt(\mathbf{x}, \mathbf{y}) \leq 2d\}$ under the map $(\mathbf{x}, \mathbf{y}) \mapsto L^T\mathbf{y} - M^T\mathbf{x}$. There are $N(n, q, 2d)$ pairs $(\mathbf{x}, \mathbf{y})$, such that $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ and $wt(\mathbf{x}, \mathbf{y}) \leq 2d$. It follow that $\#X \leq N(n, q, 2d)$. We prove the existence of the code $\mathcal{C}_n$ by constructing its Fourier description $B \subseteq \mathcal{S}$ using the following "greedy" procedure to pick elements from $\mathcal{S}$:

1. initially, let B be the empty set and let $T = A$.

2. Pick any $\mathbf{u} \in T$ and include in B.

3. Remove from T all elements $\mathbf{v}$ such that difference $\mathbf{u} - \mathbf{v}$ is in X, where $\mathbf{u}$ is the element picked in the previous step.

4. If T is empty, stop. Otherwise, return to Step 2.

Consider the Fourier description B constructed at the end of the above procedure. Note that the procedure will eliminate at most $\#X$ elements from T every time we include a new element in B. Thus, the number of elements picked into B will be at least $\left\lfloor \frac{\#A}{\#X} \right\rfloor \geq \frac{\#A}{N(n,q,2d)}$. Applying Lemma 2.8 yields the desired lower bound on $\dim(C_n)$. $\blacksquare$

We can now easily argue about the existence of asymptotically good non-stabilizer codes. The following theorem is a paraphrase of a result we proved in[11] about the existence of d-pure maximal Gottesman subgroups of the error group $\mathcal{E}$ for large d, for encoding space $L^2(A)^{\otimes^n}$ for large n ($A = \mathbb{F}_2$).

We first need the following technical definition.

Definition 4.3 *An $n \times n$ matrix R over $\mathbb{F}_2$ is said to be α-good if the following conditions are true.*

 (i) *The sum of every $\lfloor \alpha n \rfloor$ columns of R has weight at least αn.*

 (ii) *The sum of every $\lfloor \alpha n \rfloor$ rows of R has weight at least αn.*

 (iii) *The sum of every $\lfloor \alpha n \rfloor$ columns of R has weight at most $(1 - \alpha)n$.*

 (iv) *The sum of every $\lfloor \alpha n \rfloor$ rows of R has weight at most $(1 - \alpha)n$.*

It is shown in[11] that there is a constant $\alpha > 0$ and a corresponding positive integer n_α such that, for a randomly chosen R

$$\Pr[R \text{ is } \alpha\text{-good }] > 0.$$

Theorem 4.4 *For $0 < \alpha < 1$, suppose R is an $n \times n$ α-good matrix over $\mathbb{F}_2$. Let L be the following $2n \times 2n$ symmetric matrix over $\mathbb{F}_2$:*

$$\begin{pmatrix} 0 & R \\ R^T & 0 \end{pmatrix}$$

If we write $L = D + D^T$, where D is the upper triangular matrix with zeros on the principal diagonal, then $\mathcal{S} = \{\tilde{\omega}(\mathbf{a}^T D \mathbf{a}) U_{\mathbf{a}} V_{L\mathbf{a} - \mathbf{b}} \mid \mathbf{a} \in C, \mathbf{b} \in C^\perp\}$, is an $\lfloor \alpha n \rfloor$-pure maximal Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$.

Now, applying Theorem 4.2 we immediately get the following family of asymptotically good nonstabilizer codes.

Corollary 4.5 *For $0 < \alpha < 1$, suppose R is an $n \times n$ α-good matrix over $\mathbb{F}_2$ and $\mathcal{S}$ is the $\lfloor \alpha n \rfloor$-pure maximal Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$ (defined in the above theorem). Then there is an $\lfloor (\alpha n - 1)/2 \rfloor$-error correcting quantum code of dimension $\frac{2^n}{N(n,2,\lfloor \alpha n \rfloor)}$, whose projection has support in $\mathcal{S}$.*

5 Explicit construction of non-stabilizer codes

We now give an explicit construction of a family of distance 2 code. Recall that any abelian group of the error group is of the form

$$\mathcal{S} = \{\omega(\rho(\mathbf{a})) U_{L\mathbf{a}} V_{M\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^r\}$$

where L and M are $r \times n$ matrices over $\mathbb{F}_q$ such that $L^T M$ is symmetric and ρ satisfies the condition

$$\rho(\mathbf{a}_1 + \mathbf{a}_2) - \rho(\mathbf{a}_1) - \rho(\mathbf{a}_2) = \mathbf{a}_1^T L^T M \mathbf{a}_2.$$

Given an odd integer $n = 2m + 1$, we give the explicit construction of a $((n, 1 + n(q-1), 2))_q$ code. Note that if $q = 2$ and $n = 5$ we get a $((5, 6, 2))_2$ code. In[1] a $((5, 6, 2))_2$ code is given which is generated by a computer search.

They have also shown that for distance 2 this is the best possible code. We also show that there is a code of dimension greater than $\left\lceil \frac{q^n}{n(q^2-1)} \right\rceil$.

Let $\mathbf{x} \in \mathbb{F}_q^n$ is all zeros except at positions $m+1$ and $m+2$ where it is 1. Define the matrices S and L as follows

$$
S = \begin{pmatrix} \mathbf{x}^T \\ \sigma\mathbf{x}^T \\ \vdots \\ \sigma^i\mathbf{x}^T \\ \vdots \\ \sigma^{n-1}\mathbf{x}^T \end{pmatrix} \quad L = \left(\begin{array}{c|c} I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -1 \ldots -1 & 0 \end{array} \right).
$$

where σ is the cyclic shift on n elements and I_{n-1} is the $n-1 \times n-1$ identity matrix. Let J be the $n \times n$ matrix, all of whose entries are $1 \in \mathbb{F}_q$ and let $M = SL + J$. Note that $JL = 0$ and hence $M^T L = L^T SL$ is symmetric. As a result the pair L, M gives rise to a Gottesman subgroup

$$\mathcal{S} = \{\omega(\rho(a))U_{L\mathbf{a}}V_{M\mathbf{a}} : \mathbf{a} \in \mathbb{F}_q^n\}.$$

Let $\mathbf{e}_i, 0 \leq i \leq n-1$ be the standard basis for $\mathbb{F}_q^n$. $\mathbf{e}_i$ is the vector with a 1 in the ith position and $\mathbf{e}_0$ is the vector with a 1 at the nth position. Let $\underline{1} = (1,1,\ldots,1)$ then we have the following observation.

Observation 5.1

$$S\mathbf{e}_j = \mathbf{e}_{j+m} + \mathbf{e}_{j+m+1} \ (\text{ index addition mod } n).$$

$$
L^T\mathbf{e}_j = \begin{cases} \mathbf{e}_0 - \underline{1} & \text{if } j = 0, \\ \mathbf{e}_j & \text{otherwise}. \end{cases}
$$

$$
M^T\mathbf{e}_j = \begin{cases} \mathbf{e}_{j+m} + \mathbf{e}_{j+m+1} + \underline{1} & \text{if } j+m \not\equiv 0 \text{ and } j+m+1 \not\equiv 0 \ (\text{mod } n), \\ \mathbf{e}_0 + \mathbf{e}_1 & j+m \equiv 0 \ (\text{mod } n), \\ \mathbf{e}_0 + \mathbf{e}_{n-1} & j+m+1 \equiv 0 \ (\text{mod } n). \end{cases}
$$

Note that for L and M defined as above $\mathcal{S}$ will be a maximal abelian subgroup (because its cardinality is q^n) and is 2-pure. Furthermore, the 2-forbidden set is given by

$$F_2 = \{L^T\mathbf{y} - M^T\mathbf{x} : wt(\mathbf{x}, \mathbf{y}) = 1\} = \{aM^T\mathbf{e}_j + bL^T\mathbf{e}_j : (a,b) \neq (0,0); a, b \in \mathbb{F}_q 1 \leq j \leq n\}.$$

We can now derive the following asymptotic result.

Theorem 5.2 *Let $n = 2m + 1$ be an odd integer. There exists a $((n, \left\lceil \frac{q^n}{n(q^2-1)} \right\rceil, 2))_q$ quantum code.*

Proof. Let the matrices S, L, and M be as defined in the above observation and F_2 be the corresponding 2-forbidden set. Then $\#F_2 = n(q^2 - 1)$. Now, applying a greedy algorithm similar to the one in the proof of Theorem 4.2 we can construct a $\left(\left(n, \left\lceil \frac{q^n}{n(q^2-1)} \right\rceil, 2 \right) \right)_q$ quantum code. ∎

Consider the subset B of $\mathcal{S}$ defined as

$$B = \{\underline{0}\} \cup \left\{ \alpha \mathbf{e}_0 : \alpha \in \mathbb{F}_q^* \right\} \cup \left\{ \mathbf{e}_0 + \alpha \left(\sum_{i=1}^{n-1} \mathbf{e}_i \right) - \mathbf{e}_j : \alpha \in \mathbb{F}_q^*; \ j = 1, 2, \ldots n - 1 \right\}.$$

Notice that $\#B = 1 + n(q - 1)$.

Theorem 5.3 *The set B as defined above is the Fourier description of a $((n, 1 + n(q - 1), 2))_q$ code.*

Proof. Let $D = (B - B) \setminus \{\underline{0}\}$. Since F_2 does not contain the zero vector it is sufficient to prove that $F_2 \cap D$ is empty. Let $\mathbf{u}_i = \left(\sum_{j=1}^{n-1} \mathbf{e}_j \right) - \mathbf{e}_i, 1 \le i \le n-1$. Now $D = A_1 \cup A_2 \cup A_3 \cup A_4$ where

$$A_1 = \{\alpha \mathbf{e}_0, \alpha \neq 0\}$$
$$A_2 = \{\mathbf{e}_0 + \alpha \mathbf{u}_i : \alpha \neq 0\}$$
$$A_3 = \{\alpha \mathbf{e}_0 + \beta \mathbf{u}_i : \beta \neq 0; \ \alpha \neq -1; \ 1 \le i \le n - 1\}$$
$$A_4 = \{\alpha \mathbf{u}_i + \beta \mathbf{u}_j : \alpha, \beta \neq 0; \ 1 \le i, j \le n - 1; \ \alpha \neq \beta \text{ or } i \neq j\}$$

where $\alpha, \beta \in \mathbb{F}_q$. The elements of the forbidden set F_2 are given by

$$R_1 = \{a(\mathbf{e}_m + \mathbf{e}_{m+1}) + b\mathbf{e}_0 + (a - b)\underline{1} : (a, b) \neq (0, 0)\}$$
$$R_2 = \{a(\mathbf{e}_0 + \mathbf{e}_1) + b\mathbf{e}_{m+1} : (a, b) \neq (0, 0)\}$$
$$R_3 = \{a(\mathbf{e}_0 + \mathbf{e}_{n-1}) + b\mathbf{e}_m : (a, b) \neq (0, 0)\}$$
$$R_4 = \{a(\mathbf{e}_{j+m} + \mathbf{e}_{j+m+1}) + b\mathbf{e}_j + a\underline{1} : (a, b) \neq (0, 0); \ j \neq 0, j \neq m, j \neq m + 1\}$$

Now it can be verified that $A_i \cap R_j$ is empty for every $1 \le i, j \le 4$. It follows that $D \cap B = \emptyset$. ∎

6 Examples of 1-error correcting nonstabilizer codes

In this section we give explicit constructions of a $((33, 155, 3))$ code and a $((15, 8, 3))$ code. The codes we construct will be over the field $\mathbb{F}_2$.

Let $n = 2m + 1$ be any odd integer. Let $\mathbf{x}$ be the vector in $\mathbb{F}_2^n$ with zeros at all positions except $m + 1$ and $m + 2$. As in the previous section, let

$$S = \begin{pmatrix} \mathbf{x} \\ \sigma \mathbf{x} \\ \vdots \\ \sigma^i \mathbf{x} \\ \vdots \\ \sigma^{n-1} \mathbf{x} \end{pmatrix} \quad L = \left(\begin{array}{c|c} I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 1 \ldots 1 & 0 \end{array} \right),$$

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where σ is the cyclic shift. Let J be the $n \times n$ matrix all of whose entries are 1's. Recall that the generalized Laflamme code is the stabilizer code associated with the stabilizer group given by

$$\mathcal{S} = \{U_{L\mathbf{a}}V_{M\mathbf{a}} : \mathbf{a} \in \mathbb{F}_2^n\},$$

where $M = SL + J$. The corresponding 2-forbidden set is given by

$$F_2 = \{aM^Te_i + bL^Te_i : a, b \in \mathbb{F}_2\}.$$

Let $F_d + F_2 = \{\mathbf{u} \mid \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \text{such that } \mathbf{u}_1 \in F_d, \mathbf{u}_2, \in F_2\}$.

It can be easily verified that $F_{d+1} = \{\mathbf{u} \mid \mathbf{u} \in F_d + F_2, \mathbf{u} \neq 0\}$. Now, let $W_d = \{wt(\mathbf{a}) : \mathbf{a} \in F_d\}$. It can be easily checked that $W_2 = \{1, 2, 3, n-3, n-2, n-1\}$ and $W_3 = \{1, 2, 3, 4, 5, 6, n-6, n-5, n-4, n-3, n-2, n-1\}$.

If B is a subset of $\mathbb{F}_2^n$ such that for $u \in B - B$ we have $wt(u) \notin W_3$ then B is the Fourier description of a 1-error correcting quantum code. A natural approach to finding large Fourier descriptions B is to solve the following combinatorial problem.

Problem 6.1 *Construct a family of subsets $\mathfrak{F}$ of $\{1, 2, \ldots, n\}$ such that for all $S_1, S_2 \in \mathfrak{F}$, $S_1 \neq S_2$ we have*

$$\#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \notin W_3.$$

Given such a collection of subsets $\mathfrak{F}$, it is clear that the set B defined as

$$B = \left\{ \sum_{i \in S} e_i : S \in \mathfrak{F} \right\} \tag{4}$$

will yield the Fourier description of a 1-error correcting quantum code by Theorem 2.11. For, the condition on the family of subsets $\mathfrak{F}$ will ensure that the weight of any element in $B - B$ does not lie in the set $\{1, 2, 3, 4, 5, 6, n-6, n-5, n-4, n-3, n-2, n-1\}$ and hence $(B - B) \cap F_2 = \emptyset$.

As our first example we describe a $((15, 8, 3))$ code. For $n = 15$ it suffices to construct a family of 8 subsets $\mathfrak{F}$ such that for any two distinct subsets $S_1, S_2 \in \mathfrak{F}$ we have $\#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \in \{7, 8\}$. Then, B defined by Equation 4 will be the Fourier description of a $((15, 8, 3))$ code. The eight subsets of $\{1, 2, \ldots, 15\}$ that we pick are as follows:

$$S_1 = \{1, 2, 3, 4, 13\}, S_2 = \{5, 6, 7, 8, 13\}, S_3 = \{9, 10, 11, 12, 13\},$$

$$S_4 = \{1, 2, 5, 6, 9, 10\}, S_5 = \{1, 2, 7, 8, 11, 12\}, S_6 = \{3, 4, 7, 8, 9, 10\},$$

$$S_7 = \{3, 4, 5, 6, 11, 12\}, S_8 = \{14, 15\}.$$

In order to construct such nonstabilizer codes for general n we need to construct explicit set families $\mathfrak{F}$ as a solution to Problem 6.1. To this end we describe a general method and use it to construct a $((33, 155, 3))$ code. More precisely, we will seek a special solution of Problem 6.1 in which all the sets in $\mathfrak{F}$ are of the same cardinality.

Let $n = 33$. In our construction we will consider only subsets of $\{1, 2, \ldots, 32\}$. Notice that, $\mathbb{F}_2^5$ and $\{1, 2, \ldots, 32\}$ are sets of the same size and can be identified using any 1-1 correspondence. Thus, our goal is essentially to find a family of subsets of $\mathbb{F}_2^5$ that yields a solution to Problem 6.1.

Consider the vector space $\mathbb{F}_2^5$. Let $\mathfrak{F}$ be the family of all 3 dimensional subspaces of $\mathbb{F}_2^5$. Since any $S \in \mathfrak{F}$ is a vector space over $\mathbb{F}_2$ of dimension three, we have $\#S = 2^3 = 8$. Moreover any two distinct subspaces can have at most 4 vectors in common. Hence for every pair of distinct sets $S_1, S_2 \in \mathfrak{F}$ we have $\#S_1 \cap S_2 \leq 4$. Consequently, for distinct sets $S_1, S_2 \in \mathfrak{F}$ we have

$$8 \leq \#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \leq 14.$$

Thus, for distinct sets $S_1, S_2 \in \mathfrak{F}$ $\#(S_1 \setminus S_2) + \#(S_2 \setminus S_1) \notin W_2$ and hence the corresponding Fourier description B gives rise to a 1-error correcting code.

Now, to find the size of the set B which is the dimension of the code we have to find the size of $\mathfrak{F}$. The following general theorem gives the exact size.

Theorem 6.2 *Consider the vector space $\mathbb{F}_q^m$. The number of subspaces of dimension r is*

$$\frac{(q^m - 1)(q^{m-1} - 1) \ldots (q - 1)}{(q^r - 1)(q^{r-1} - 1) \ldots (q - 1)(q^{m-r} - 1)(q^{m-r-1} - 1) \ldots (q - 1)}$$

Proof. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ be the standard basis for $\mathbb{F}_q^m$. Let $\mathfrak{T}$ be the family of all r dimensional subspaces of $\mathbb{F}_q^m$. We want to find $\#\mathfrak{T}$. Let R be the subspace of $\mathbb{F}_q^m$ spanned by the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r$. Consider the group $G = GL_m(\mathbb{F}_q)$. G acts on $\mathfrak{T}$ transitively and hence the orbit of R under the G action is the whole of $\mathfrak{T}$. Hence the number of elements in $\mathfrak{T}$ is given by

$$\#\mathfrak{T} = \frac{\#G}{\#G_R}$$

where G_R is the subgroup of G that leaves R invariant.

Any element of G_R is of the form

$$\left(\begin{array}{c|c} A & * \\ \hline 0 & B \end{array} \right)$$

where A and B are $r \times r$ and $(m-r) \times (m-r)$ nonsingular matrices respectively and $*$ is any $r \times (m-r)$ matrix. If $g(m) = \#GL_m(\mathbb{F}_q)$ then we have

$$\#G_R = g(r)g(m-r)q^{(m-r)r}.$$

So the problem reduces to finding $g(m)$. Let A be any matrix in $GL_m(\mathbb{F}_q)$. The first column of A can be any one of the nonzero vectors is $\mathbb{F}_q^m$. There are $q^m - 1$ nonzero vectors. Having fixed the first column $\mathbf{a}_1$, we have $q^m - q$ choices for the second column $\mathbf{a}_2$. Similarly there are $q^m - q^2$ choices for the third column and so on. Therefore the number of elements in $GL_m(\mathbb{F}_q)$ is given by

$$g(m) = \prod_{i=0}^{m-1} (q^m - q^i).$$

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Therefore the size of $\mathfrak{T}$ is given by

$$
\#\mathfrak{T} = \frac{g(m)}{g(r)g(m-r)q^{(m-r)r}}
$$

$$
= \frac{(q^m - 1)(q^{m-1} - 1)\dots(q-1)}{(q^r - 1)(q^{r-1} - 1)\dots(q-1)(q^{m-r} - 1)(q^{m-r-1} - 1)\dots(q-1)}
$$

$\blacksquare$

From Theorem 6.2 we have

$$
\#\mathfrak{F} = 155
$$

and hence the set B defined as

$$
B = \left\{\sum_{i \in S} e_i : S \in \mathfrak{F}\right\}
$$

gives a $((33, 155, 3))$ code.

Remark 6.3 *We can actually obtain a $((31, 155, 3))$ code as follows: construct the same family of 3-dimensional subspaces of $\mathbb{F}_2^5$ which are 155 in number. Now, drop the extra coordinate, which will still result in 155 distinct subsets such that the symmetric difference of any pair of these has weight in the range 7 to 14. Thus, we have a $((31, 155, 3))$ nonstabilizer code. Notice that we can easily extend this puncturing argument to other nonstabilizer codes.*

7 Encoding circuits for a class of nonstabilizer codes

In this section we discuss the encoding algorithm for the class of non-stabilizer codes defined in Section 5 and the asymptotically good codes of Corollary 4.5. Recall that given a Gottesman subgroup $\mathcal{S}$, the code $\mathcal{C}(\mathcal{S})$ can be specified by giving its Fourier description $B \subseteq \mathcal{S}$. We fix our encoding space to be $L^2(\mathbb{F}_q^n)$. Let $C = \{\mathbf{a} | \sum_i a_i = 0\}$ and $C^\perp$ the set $\{\mathbf{b} | \mathbf{b}^T \mathbf{a} = 0 \text{ for all } \mathbf{a} \in C\}$. We restrict attention to maximal Gottesman subgroups of the form

$$
\mathcal{S} = \{s_{a,b} = \omega(\mathbf{a}^T D \mathbf{a}) U_{\mathbf{a}} V_{L\mathbf{a}+\mathbf{b}} | \mathbf{a} \in C, \mathbf{b} \in C^\perp\},
$$

where D is an upper triangular matrix and $L = D + D^T$. Consider a code $\mathcal{C}_n$ with Fourier description $B \subseteq \mathcal{S}$. Recall that the dimension of the code is $\#B$. Due to the isomorphism $\mathcal{S} \cong C \times C^\perp \hookrightarrow \mathbb{F}_q^n \times \mathbb{F}_q^n$ we have the character group of $\mathcal{S}$ as

$$
\chi_{s_{\mathbf{c},\mathbf{d}}}(s_{\mathbf{a},\mathbf{b}}) = \chi_{\mathbf{c},\mathbf{d}}(\mathbf{a}, \mathbf{b}) = \omega(\mathbf{a}^T \mathbf{d} + \mathbf{b}^T \mathbf{c}).
$$

For $u \in \mathcal{S}$ define $\mathcal{S}_u$ to be the abelian group $\{\chi_u(s)s | s \in \mathcal{S}\}$. It is easy to see that the Gottesman subgroup $\mathcal{S}_u$ is also maximal. In this notation we have

$\mathcal{S}_1 = \mathcal{S}$. Let $\mathcal{C}_{\mathcal{S}_u}$ denote the stabilizer (one dimensional) code corresponding to the Gottesman subgroup $\mathcal{S}_u$. Let $\{|\varphi_u\rangle\}$ denote a (singleton) orthonormal basis for $\mathcal{C}_{\mathcal{S}_u}$ for each $u \in \mathcal{S}$.

Theorem 7.1 *The vectors $|\varphi_u\rangle$, $u \in B$ form an orthonormal basis for the code with Fourier description B.*

Proof. The projection operators for the code with Fourier description B is given by

$$P = \frac{1}{\#\mathcal{S}} \sum_{u \in B} \sum_{s \in \mathcal{S}} \chi_u(s)s = \sum_{u \in B} P_u$$

where $P_u = \frac{1}{\#\mathcal{S}} \sum_{s \in \mathcal{S}} \chi_u(s)s$. Note that P_u is the projection operator corresponding to the stabilizer code $\mathcal{C}_{\mathcal{S}_u}$. To prove that $\{|\varphi_u\rangle \,|\, u \in B\}$ forms an orthonormal basis for the code given by B it suffices to show that

$$P_u P_v = \begin{cases} 0 & \text{if } u \neq v, \\ P_u & \text{otherwise,} \end{cases}$$

which is an immediate consequence of the following:

$$P_u P_v = P_{\chi_u * \chi_v} = \begin{cases} 0 & \text{if } u \neq v, \\ P_u & \text{otherwise.} \end{cases}$$

It follows that $\{|\varphi_u\rangle \,|\, u \in B\}$ is an orthonormal basis for $\mathcal{C}_n$. ∎

For $s_{\mathbf{a},\mathbf{b}} \in \mathcal{S}$ instead of writing $\mathcal{S}_{s_{\mathbf{ab}}}$ we will write $\mathcal{S}_{\mathbf{a},\mathbf{b}}$. Similarly $|\varphi_{\mathbf{a},\mathbf{b}}\rangle$ will be used to denote $|\varphi_{s_{\mathbf{a},\mathbf{b}}}\rangle$. It is easy to see that

$$|\varphi_{\mathbf{c},\mathbf{d}}\rangle = \sqrt{\frac{1}{\#C}} \sum_{\mathbf{x} \in C} \omega((\mathbf{x}+\mathbf{d})^T D(\mathbf{x}+\mathbf{d}))\overline{\omega}(\mathbf{x}^T c) |\mathbf{x}+d\rangle .$$

Since the code has dimension $\#B$, we will assume that the encoding message space is a Hilbert space of dimension $\#B$ with basis indexed by elements of B i.e. $\{|\mathbf{c},\mathbf{d}\rangle\,|s_{\mathbf{c},\mathbf{d}} \in B\}$. To summarize, we have the following observation.

Proposition 7.2 *For a code with Fourier description B the encoding procedure is given by linear map with the following property $|\mathbf{c},\mathbf{d}\rangle \mapsto |\varphi_{\mathbf{c},\mathbf{d}}\rangle$ for all $s_{\mathbf{c},\mathbf{d}} \in B$.*

In the rest of this section we define the basic quantum gates using which we construct the encoding circuit for the code described above. We will also outline the encoding algorithm. It is to be noted that for a fixed q, these basic quantum gates over $L^2(\mathbb{F}_q)$ can be implemented easily over any universal set of quantum gates.

Gate	Definition
Invertor	$\mathcal{I}\,|a\rangle = |-a\rangle\,,\ a \in \mathbb{F}_q$
Control-U	$C\text{-}U\,|a\rangle\,|b\rangle = |a\rangle\,|a+b\rangle\,,\quad a,b \in \mathbb{F}_q$
Control-V	$C\text{-}V\,|a\rangle\,|b\rangle = \omega(ab)\,|a\rangle\,|b\rangle\,,\quad a,b \in \mathbb{F}_q$
Control-Control-U	$CC\text{-}U\,|a,b,c\rangle = |a,b,c+ab\rangle\,,\quad a,b,c \in \mathbb{F}_q$
Control-Control-V	$CC\text{-}V\,|a,b,c\rangle = \omega(c+ab)\,|a,b,c\rangle\,,\quad a,b,c \in \mathbb{F}_q$
Fourier Transform	$F\,|a\rangle = \frac{1}{\sqrt{q}}\sum_{x\in\mathbb{F}_q}\omega(ax)\,|x\rangle\,,\quad a \in \mathbb{F}_q$

Next, for any gate $C\text{-}f$ of two arguments $a,b \in \mathbb{F}_q$ defined as

$$C\text{-}f\,|a\rangle\,|b\rangle = |a\rangle\,|f(a,b)\rangle\,,$$

we extend it to an operator $C\text{-}f_n$ with two arguments $\mathbf{a},\mathbf{b} \in \mathbb{F}_q^n$ as follows (see Figure 2)

$$C\text{-}f_n\,|\mathbf{a}\rangle\,|\mathbf{b}\rangle = C\text{-}f_n\,|a_1 a_2 \ldots a_n\rangle\,|b_1 b_2 \ldots b_n\rangle = |\mathbf{a}\rangle\,|f(a_1,b_1), f(a_2,b_2), \ldots, f(a_n,b_n)\rangle\,.$$

In a similar fashion we extend the gate $CC\text{-}f$ with three arguments $a,b,c \in \mathbb{F}_q$ to $CC\text{-}f_n$ defined on $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{F}_q^n$. In the circuits we draw a $C\text{-}f_n$ gate with thick wires to indicate that it takes a tuple from $\mathbb{F}_q^n$.

Given the message $\mathbf{c},\mathbf{d}$, we can describe the main steps of the encoding algorithm as follows.

1. Initialize $|R\rangle := |0^n\rangle$

2. Apply C on R so that

$$|R\rangle = \sqrt{\frac{1}{q^{n-1}}}\sum_{\mathbf{x}\in C}|\mathbf{x}\rangle\,.$$

3. Apply $C\text{-}V_n$ on $|\mathbf{c}\rangle\,|R\rangle$ so that

$$|\mathbf{c}\rangle\,|R\rangle \mapsto |\mathbf{c}\rangle \otimes \Big(\frac{1}{\sqrt{q^{n-1}}}\sum_{\mathbf{x}\in C}\omega(\mathbf{c}^T\mathbf{x})\,|\mathbf{x}\rangle\Big).$$

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4. Apply $C\text{-}U_n$ on $|\mathbf{d}\rangle |R\rangle$

$$|\mathbf{d}\rangle |R\rangle \mapsto |\mathbf{d}\rangle \otimes (\frac{1}{\sqrt{q^{n-1}}} \sum_{\mathbf{x}\in C} \omega \mathbf{c}^T \mathbf{x} |\mathbf{x}+\mathbf{d}\rangle .$$

5. Apply the circuit in Figure 4 on $|R\rangle |D\rangle |0^n\rangle$

$$|R\rangle |D\rangle |0^n\rangle \mapsto \frac{1}{\sqrt{q^{n-1}}} \sum_{\mathbf{x}\in C} \omega(\mathbf{c}^T \mathbf{x}) |\mathbf{x}+\mathbf{d}, D(\mathbf{x}+\mathbf{d})\rangle |D\rangle .$$

6. Apply $C\text{-}V_n$ on $|R_1\rangle = \frac{1}{\sqrt{q^{n-1}}} \sum_{\mathbf{x}\in C} \omega(\mathbf{c}^T \mathbf{x}) |\mathbf{x}+\mathbf{d}, D(\mathbf{x}+\mathbf{d})\rangle$ to get

$$|R_1\rangle \mapsto \frac{1}{\sqrt{q^{n-1}}} \sum_{\mathbf{x}\in C} \omega((\mathbf{x}+\mathbf{d})^T D(\mathbf{x}+\mathbf{d}) + \mathbf{c}^T \mathbf{x}) |\mathbf{x}+\mathbf{d}, D(\mathbf{x}+\mathbf{d})\rangle .$$

The complete encoding circuit along with the building blocks is given in Figures 1, 2, 3, 4, 5, and 6.

8 Decoding for a class of nonstabilizer codes

Let $\mathcal{C} \subseteq \mathcal{H}^{\otimes^n}$ be a t-error correcting quantum code satisfying the conditions of Theorem 2.11, with Fourier description B, and such that its projection P has support in the Gottesman subgroup $\mathcal{S}$ of the error group $\mathcal{E}$. The two conditions that $\mathcal{C}$ satisfies are:

1. For each $g \in \mathcal{S}$ such that $wt(g) \leq 2t$

$$\chi_g(u_1^{-1} u_2) = 1 \text{ for all } u_1, u_2 \in B.$$

2. For each $g \in \mathcal{E} \setminus \mathcal{S}$ such that $wt(g) \leq 2t$, and for $s \in S$ we have

$$\sum_{u_1, u_2 \in B} \chi_{u_2}(s) \sum_{s_1} \gamma(g, s_1) \chi_{s_1}(u_1^{-1} u_2) = 0,$$

where $\gamma(g, s)$ is the scalar such that $\gamma(g, s)sg = gs$.

For $u \in S$ let $\mathcal{S}_u$ be the Gottesman group $\{\chi_u(s)s | s \in \mathcal{S}\}$.
The projection operator for the code $\mathcal{C}_n$

$$P = \frac{1}{\#\mathcal{S}} \sum_{u \in B} \sum_{s \in \mathcal{S}} \chi_u(s)s = \sum_{u \in B} P_u,$$

where $P_u = \frac{1}{\#\mathcal{S}} \sum_{s \in \mathcal{S}} \chi_u(s)s$. Note that P_u is the projection operator corresponding to the stabilizer code $\mathcal{C}_u$ with $\mathcal{S}_u$ as stabilizer group for each $u \in B$. As argued in Theorem 7.1, $P_u P_v = 0$ for distinct elements $u, v \in B$.

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Let D_u denote an orthonormal basis for $\mathcal{C}_u$, $u \in B$. Then $D = \bigcup_{u \in B} D_u$ is an orthonormal basis for $\mathcal{C}$.

It suffices to describe the decoding procedure for the encoded message as a basis element from D and error $g \in \mathcal{E}$ such that $wt(g) \leq t$. With out loss of generality let $|\psi\rangle \in D_u$ for some $u \in B$ be the encoded message, and let $g \in \mathcal{E}$ of weight at most t be the error operator. The decoding procedure takes $g |\psi\rangle = |\psi'\rangle$ as input and outputs $|\psi\rangle$. Let $\{s_1, s_2, \ldots, s_k\}$ be an independent generator set for $\mathcal{S}$. Notice that for $1 \leq i \leq k$

$$s_i g |\psi\rangle = \overline{\gamma(s_i, g)} \chi_u(s_i) g |\psi\rangle.$$

Thus, $g |\psi\rangle$ is an eigen vector for operator s_i with eigen value $\overline{\gamma(s_i, g)} \chi_u(s_i)$.

The decoding procedure will carry out the following steps. It uses as subroutine the phase estimation algorithm of Kitaev[12].

1. Let $|\psi'\rangle$ be the received state.

2. Apply s_i successively, for each $1 \leq i \leq k$, and when s_i is applied run Kitaev's phase estimation algorithm to compute the eigen value $\alpha_i = \overline{\gamma(s_i, g)} \chi_u(s_i)$. Let $|\rho\rangle$ be the resulting state.

3. If $g = I$ and some $u \in B$ constitute a solution to the system of k group equations (using a classical algorithm that searches through B):

$$\alpha_i = \overline{\gamma(s_i, g)} \chi_u(s_i), 1 \leq i \leq k,$$

then apply s_i^{-1} for each $1 \leq i \leq k$ to the state $|\rho\rangle$ and output that as the decoded state.

4. If $g = I$ does not give a solution to the k equations, find (using a classical algorithm that searches through B) a $g \neq I \in \mathcal{E}$ and the corresponding unique $u \in B$ which are a solution to the k equations. Apply g^{-1} to the current state. Then apply s_i^{-1} for each $1 \leq i \leq k$, and output that as the decoded state.

We now argue the correctness of the procedure. Firstly, notice that if the error operator is $g \in \mathcal{S}$ such that $wt(g) \leq t$, then by assumption $\chi_g(u_2) = \chi_g(u_1)$ for all $u_1, u_2 \in B$. Denote this scalar by λ. Notice that for any state $|\varphi\rangle \in \mathcal{C}$, $g |\varphi\rangle = \lambda |\varphi\rangle$. Which means that g introduces only an overall phase. We establish the following claim from which the correctness of the procedure follows.

Claim 8.1 *1. If $g = I$ and $u \in B$ is a solution to the k group equations given above, then $u \in B$ is the unique solution, and there is no $g \notin \mathcal{S}$ which is a solution to the equations.*

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2. *If $g = I$ is not part of a solution to the k equations, then there is a unique $u \in B$ and some $g \notin \mathcal{S}$ that form a solution such that $wt(g) \leq t$.*

Proof of Claim. For the first part, assume that $g = I$ and $u \in B$ is a solution to the k group equations, and some $g \notin \mathcal{S}$ and $u' \in B$ is another solution. Then we have $\gamma(g, s_i)\chi_u(s_i)\overline{\chi_{u'}(s_i)} = 1, 1 \leq i \leq k$. Since $s_1, s_2, \ldots, s_k$ generate $\mathcal{S}$, it implies $\gamma(g, s)\chi_u(s)\overline{\chi_{u'}(s)} = 1$ for all $s \in \mathcal{S}$. It is easy to see that this contradicts the second condition of Theorem 2.11 for the element $g \notin \mathcal{S}$ of $wt(g) \leq t$.

For the second part, notice that any solution g of $wt(g) \leq t$ to the equations is not in $\mathcal{S}$. For, if $g' \in \mathcal{S}$ and $u \in B$ are a solution then so is $g = I$ and $u \in B$, because $\gamma(g', s) = 1$ for all $s \in \mathcal{S}$. Assume to the contrary that there are two distinct solutions $g_1 \notin \mathcal{S}$ and $u \in B$ and $g_2 \notin \mathcal{S}$ and $u' \in B$, where $wt(g_1) \leq t$ and $wt(g_2) \leq t$. Then, as before, the k equations will yield

$$\gamma(g_1, s)\chi_u(s) = \gamma(g_2, s)\chi_{u'}(s), \quad \forall s \in \mathcal{S}.$$

By rearranging terms we get $\gamma(g_1^{-1}g_2, s)\chi_s(u'u^{-1}) = 1$ for all $s \in \mathcal{S}$.

Now, if $g_1^{-1}g_2 \notin \mathcal{S}$, then this again contradicts the second condition of Theorem 2.11 for the element $g_1^{-1}g_2 \notin \mathcal{S}$ of $wt(g_1^{-1}g_2) \leq 2t$.

Next, suppose that $g_1^{-1}g_2 \in \mathcal{S}$. Then we get $\chi_s(u'u^{-1}) = 1$ for all $s \in \mathcal{S}$ implying that $u = u'$. Thus u is unique. Notice that by the first condition of Theorem 2.11, since $wt(g_1^{-1}g_2) \leq 2t$, the effect of applying the error $g_1^{-1}g_2$ to a state in $\mathcal{C}$ only introduces an overall phase. Thus, decoding with either g_1^{-1} or g_2^{-1} will coincide upto an overall phase. This completes the proof of the claim and correctness of the decoding procedure.

To analyze the efficiency of the decoding procedure, we recall from Nielsen and Chuang[12] that the phase estimation quantum circuit is efficient (polynomial size in n). However, solving the k group equations involves exhaustive enumeration. This takes time $O(n^{O(d)}.\#B)$, which is also the dominant term in the entire time bound.

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See Figure 1 for the basic gates. From these gates we can construct a circuit that computes for any $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n$ the dot product $\sum_i a_i b_i$ (see Figure 3). The second circuit in Figure 3 is a symbolic representation of the circuit (note the thick lines). Using this inner product circuit we can also define the circuit that takes a $n \times n$ matrix D and a vector $\mathbf{a} \in \mathbb{F}_q^n$ and computes the vector $D\mathbf{a}$ (see Figure 4 note the thick lines and the cut).

We also need a circuit which will take the vector $|0^n\rangle$ and generate the uniform superposition $\frac{1}{\sqrt{q^{n-1}}} \sum_{\mathbf{x} \in C} |\mathbf{x}\rangle$. This circuit is given in Figure 5.

See Figure 6 for a complete circuit. The other figures are the building blocks. Note that the extra $|D(\mathbf{x} + \mathbf{d})\rangle$ can be removed by inverting it and then applying the circuit in fig 4 on $|\mathbf{x} + \mathbf{d}, D(\mathbf{x} + \mathbf{d})\rangle |D\rangle$.

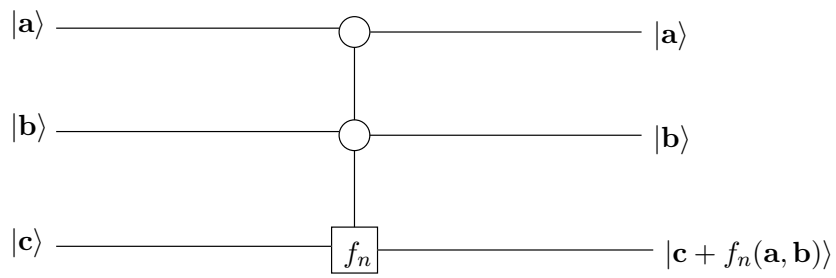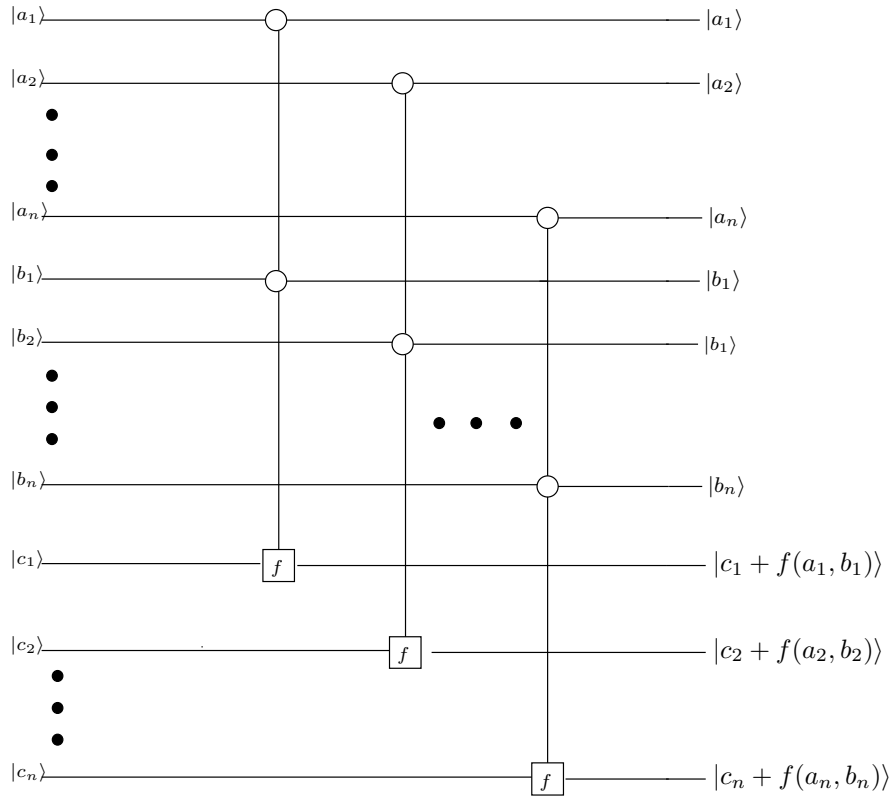$$|a\rangle \longrightarrow \boxed{\mathcal{I}} \longrightarrow |-a\rangle$$

Invertor gate

$$|a\rangle \longrightarrow \circ \longrightarrow |a\rangle$$
$$|b\rangle \longrightarrow \boxed{U} \longrightarrow |a + b\rangle$$

$C - U$ gate

$$|a\rangle \longrightarrow \circ \longrightarrow |a\rangle$$
$$|b\rangle \longrightarrow \boxed{V} \longrightarrow \omega(ab) |b\rangle$$

$C - V$ gate

$$|a\rangle \longrightarrow \boxed{F} \longrightarrow F |a\rangle$$

Fourier Transform gate
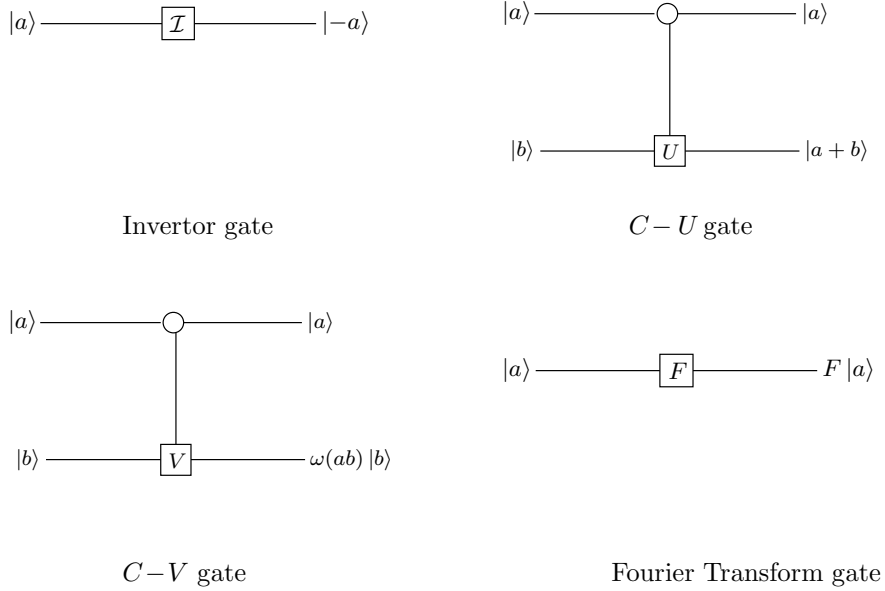
Figure 1. Basic Gates
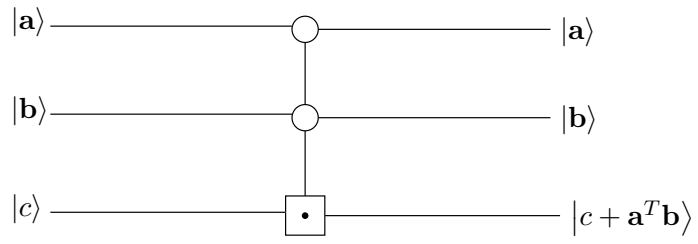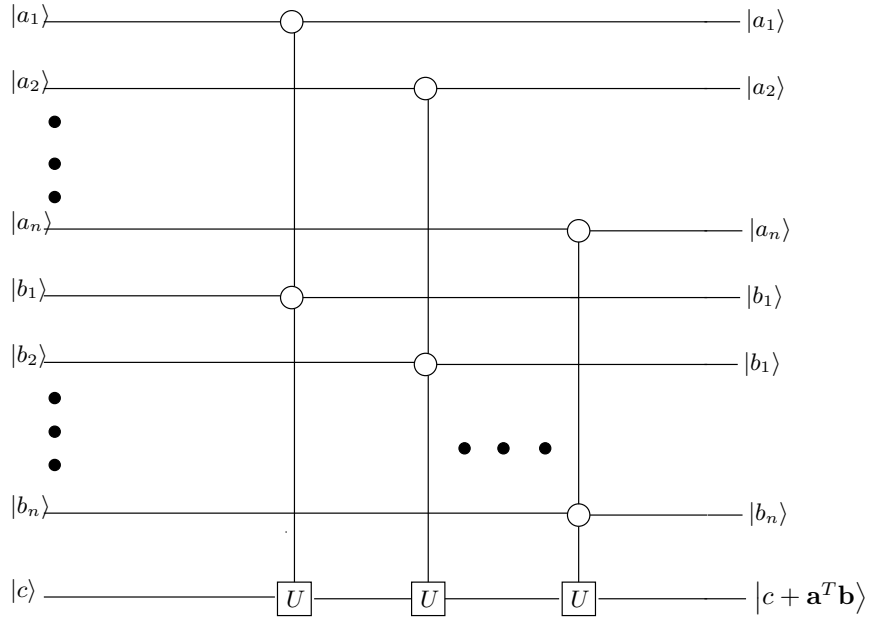
Figure 2. Circuit computing C-f_n from C-f

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Figure 3. Circuit computing the dot product $\mathbf{a}^T\mathbf{b}$

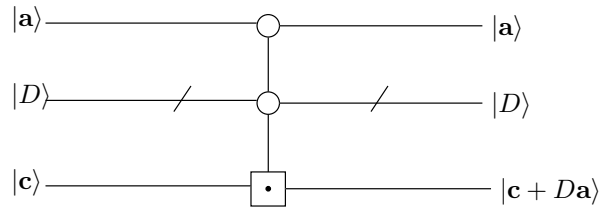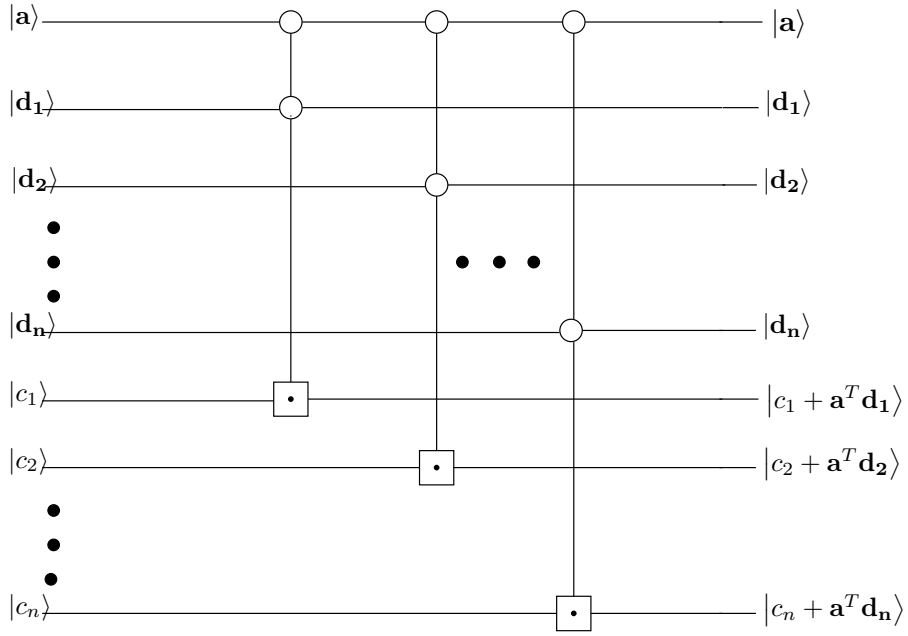Figure 4. Circuit to compute $D\mathbf{x}$
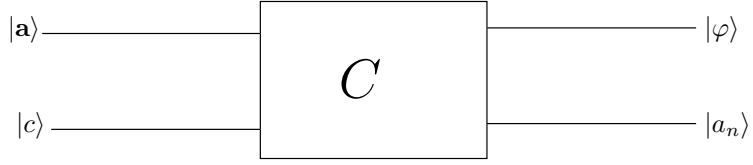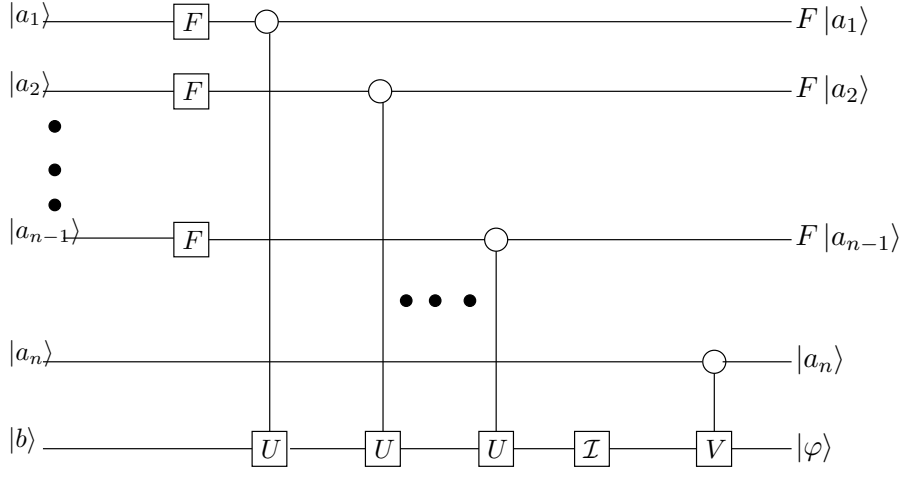
$$|a_1\rangle \boxed{F} \quad F|a_1\rangle$$
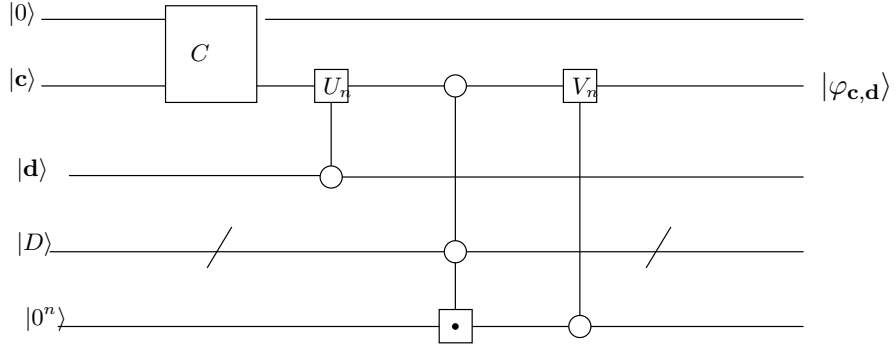

Figure 5. Circuit generating $\sqrt{\frac{1}{q^{n-1}}}\sum_{x\in C}|x\rangle$

Figure 6. Complete encoding circuit