Differentiation

$(ax+b)^n \Rightarrow (n)(ax+b)^{n-1}(a)$

$\sin f(x) \Rightarrow f'(x)\cos f(x)$

$\cos f(x) \Rightarrow -f'(x)\sin f(x)$

$\tan f(x) \Rightarrow \sec^2 f(x)\, f'(x)$

$\mathrm{cosec}\, f(x) \Rightarrow -f'(x)\,\mathrm{cosec} f(x)\cdot\cot f(x)$

$\sec f(x) \Rightarrow f'(x)\sec f(x)\tan f(x)$

$\cot f(x) \Rightarrow -f'(x)\,\mathrm{cosec}^2 f(x)$

$\ln f(x) \Rightarrow \frac{f'(x)}{f(x)}$

$e^{f(x)} \Rightarrow f'(x)e^{f(x)}$

$\sin^n x \Rightarrow (n)\sin^{n-1}x\cos x$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \;\|\; \frac{d^2y}{dx^2} = \frac{d/dt\left(\frac{dy}{dx}\right)}{dx/dt}$

$f''(x)<0 \Rightarrow$ concave down $f''(x)>0 \Rightarrow$ concave up

$\sin^{-1}x \Rightarrow \frac{1}{\sqrt{1-x^2}},\ |x|<1$

$\sin^{-1}f(x) \Rightarrow \frac{f'(x)}{\sqrt{1-f(x)^2}},\ |f(x)|<1$

$\cos^{-1}f(x) \Rightarrow \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}\ |f(x)|<1$

$\tan^{-1}f(x) \Rightarrow \frac{f'(x)}{1+[f(x)]^2}$

$a^x \Rightarrow a^x.\ln a$

$a^{f(x)} \Rightarrow f'(x)a^{f(x)}.\ln a$

$\log_a f(x) \Rightarrow \frac{f'(x)}{f(x)}\log_a e$

Quotient Rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ or $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Trigonometric Identities

$\sin^2\theta+\cos^2\theta = 1,\ \sec^2\theta - \tan^2\theta = 1$

$\mathrm{cosec}^2\theta - \cot^2\theta = 1$

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$.

$\sin(A+B) = \sin A\cdot\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$

$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Integration

$\int(Ax+B)^n dx = \frac{(Ax+B)^{n+1}}{(n+1)(A)} + c,\ n\neq -1$

$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n\neq -1$

$\int e^{Ax+B}dx = \frac{e^{Ax+B}}{A}$

$\int f'(x)e^{f(x)}dx = e^{f(x)}$

$\int\frac{1}{Ax+B} = \frac{\ln|Ax+B|}{A}$ $\int\cot x = \ln(\sin x)$

$\int\frac{f'(x)}{f(x)} = \ln|f(x)| + c$

$\int\cos(Ax+B)dx = \frac{\sin(Ax+B)}{A}$

$\int f'(x)\cos f(x)dx = \sin f(x)$

$\int\sin(Ax+B)dx = -\frac{\cos(Ax+B)}{A}$

$\int f'(x)\sin f(x)dx = -\cos f(x)$

$\int\sec^2(Ax+B)dx = \frac{\tan(Ax+B)}{A}$

$\int f'(x)\sec^2 f(x)dx = \tan f(x)$

$\int\sec(Ax+B)\tan(Ax+B)dx = \frac{\sec(Ax+B)}{A}$

$\int f'(x)\sec f(x)\tan f(x)dx = \sec f(x)$

$\int\mathrm{cosec}(Ax+B)\cot(Ax+B)dx = -\frac{\mathrm{cosec}(Ax+B)}{A}$

$\int f'(x)\mathrm{cosec} f(x)\cot f(x)dx = -\mathrm{cosec} f(x)$

$\int a^{Ax+B}dx = \frac{a^{Ax+B}}{A\ln a} + c,\ a>0$ | $\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c,\ a>0$

Inverse Trigo:

$\int\frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c,\ |x|<|a|$, $\int\frac{f'(x)}{\sqrt{1-f(x)^2}} = \sin^{-1}f(x),\ |f(x)|<1$

$\int\frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$, $\int\frac{f'(x)}{a^2+f(x)^2} = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right)$

Partial Fractions:

$\int\frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c,\ |x|>a$

$\int\frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c,\ |x|<a$

Area under curve: $\int_p^q g(y)dy \,/\, \int_a^b y\,dx$

Bounded: $\int_a^b f(x) - g(x)dx$.

Parametric: $\int_{x_1}^{x_2} y\,dx = \int_{t_1}^{t_2} y\left(\frac{dx}{dt}\right)dt$

By parts:

$\int uv\,dx = u\left(\int v\,dx\right) - \int\left(\int v\,dx\right)\left(\frac{du}{dx}\right)dx$.

$\int\cos^n dx, \int\sin^n dx \to$ even pwr to cosines of double

$\to$ odd pwr: $\sin^2 x + \cos^2 x = 1$

$\int\tan^n dx \to 1 + \tan^2 x = \sec^2 x$.

Fourier Series.

$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,\ n=1,2\ldots$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx,\ n=1,2\ldots$

$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx = 0$ if $n\neq m$; $=1$ if $n=m$

$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)\cos(mx)dx = 0$ if $n\neq m$; $=1$ if $n=m$

$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\cos(mx)dx = 0$ for all n, m.

$\int_{-L}^{L}(\text{even})dx = 2\int_0^L \text{even}\,dx$.

$\left(\sum_{n=1}^{\infty} b_n\sin nx\right)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n b_m\sin(nx)\sin(mx)$

Half Range (func on interval $0\le x\le L$)

Sine: $f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$

cosine: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$

a_0: $\frac{1}{L}\int_0^L f(x)dx$.

a_n: $\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx,\ n=1,2\ldots$

b_n: $\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,\ n=1,2\ldots$

$\int x\sin(\lambda x)dx = \frac{\sin(\lambda x)}{\lambda^2} - \frac{x\cos(\lambda x)}{\lambda}$

$\int x^2\sin(\lambda x)dx = \frac{2x\sin(\lambda x)}{\lambda^2} + \frac{(2-\lambda^2x^2)\cos(\lambda x)}{\lambda^3}$

$\int x\cos(\lambda x)dx = \frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}$

$\int x^2\cos(\lambda x)dx = \frac{2x\cos(\lambda x)}{\lambda^2} + \frac{(\lambda^2x^2-2)\sin(\lambda x)}{\lambda^3}$

E×E=E E+E=E
O×O=E O+O=O
E×O=O E+O=?

Odd ⇒ symm. abt origin (x & y)
even ⇒ symm. abt y-axis.

Func. of Sev. Var.

partial der.: $\frac{d}{dx}f(x,b)\Big|_{x=a} = \lim_{h\to 0}\frac{(f(a+h,b) - f(a,b))}{h}$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

Directional Derivative (Rate of Δ in a spec. direct.)

$D_u f(a,b) = \lim_{h\to 0}\frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$

$D_u f = f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$ $\boxed{\text{* u is a unit vector}}$

$D_u f = \nabla f\cdot \underset{\sim}{u}$, $D_u f = \|\nabla f(a,b)\|\cos\theta$.

∇f (gradient vector) $= f_x\vec{i} + f_y\vec{j}$

function increases most rapidly in direction of $\nabla f(a,b)$ & decreases most in $-\nabla f(a,b)$

Δ in $df = D_u f(a,b)\cdot dt$ (how much Δ in direction of $\underset{\sim}{u}$).

Critical points:

$\Rightarrow f_x(a,b) = 0$ AND $f_y(a,b) = 0$

$\Rightarrow f_x(a,b)$ OR $f_y(a,b)$ doesn't exist.

2nd Der. TEST:

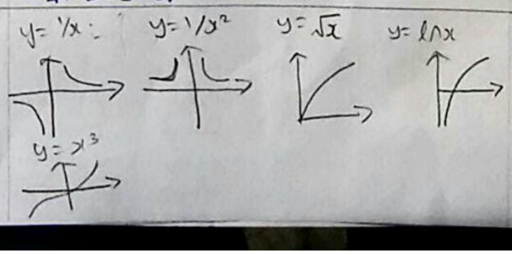
- $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$

⇒ If $D>0$ & $f_{xx}(a,b)>0$ ⇒ Local min

⇒ If $D>0$ & $f_{xx}(a,b)<0$ ⇒ Local max

⇒ If $D<0$, ⇒ Saddle point

⇒ If $D=0$ ⇒ No conclusion



Lagrange Multipliers:

$z = f(x,y) = $ Eg: $12x - 16y + 50$

① Identify constraint

② Write it as $g(x,y) = x^2+y^2 = 25$

③ construct function:

$F(x,y) = f(x,y) - \lambda(g(x,y))$

④ Set $F_x = 0, F_y = 0, F_z = 0$

⑤ Simultaneous Eqns to get $\lambda/x/y/z$

⑥ Subst. back into equation to get other values (Eg: x/y)

⑦ Subst. these values into the equation to see if ans is local max or local min.

Multiple Integrals

$\iint_R 1\,dA = \iint 1\,dA =$ Area of R

$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$, $R = R_1 \cup R_2$

If $m \le f(x,y) \le M$, $mA(R) \le \iint_R f(x,y)\,dA \le MA(R)$.

$\iint_R g(x)h(y)\,dx\,dy = \left(\int_a^b g(x)\,dx\right)\left(\int_a^b h(y)\,dy\right)$

Type A

$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

Type B

$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

* Reverse order / iterate integral
  $\Rightarrow$ draw region & change boundaries.

Polar Coordinates

R: $0 \le r \le 1$, $0 \le \theta \le 2\pi$

conv: $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$

volume: $= \iint_R f(x,y)\,dA$.
region under func. $z = f(x,y)$

Triple Integral: $\iiint 1\,dV =$ vol. of D

Avg. value of function: $\frac{1}{\text{Area of R}} \iint_R f(x,y)\,dA$

Surface Area: $S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$
use original eqn.

Surface Integrals

① Parametric: planes: Let the missing varbe u or v
eg: $2y + x = 7$, $\Rightarrow x = 7 - 2y$, $y = v$, $z = u$

② Surfaces of form $z = f(x,y)$
$r(u,v) = u\vec{i} + v\vec{j} + f(u,v)\vec{k}$.

③ Sphere: $x^2 + y^2 + z^2 = a^2$, radius $= a$.
$x = a\sin u\cos v$, $0 \le u \le \pi$ (measures from N pole)
$y = a\sin u\sin v$, $0 \le v \le 2\pi$ (from x-axis)
$z = a\cos u$
$r(u,v) = a\sin u\cos v\,\vec{i} + a\sin u\sin v\,\vec{j} + a\cos u\,\vec{k}$.
Hemisphere: $0 \le u \le \pi/2$, $0 \le v \le 2\pi$.

④ Circular Cylinders
$x^2 + y^2 = a^2$ (abt z-axis) : $0 \le u \le 2\pi$
$r(u,v) = (a\cos u)\vec{i} + (a\sin u)\vec{j} + v\vec{k}$: v is the height of cylinder
$x^2 + z^2 = a^2$ (abt y-axis)
$r(u,v) = (a\cos u)\vec{i} + v\vec{j} + (a\sin u)\vec{k}$.
$y^2 + z^2 + a^2$ (abt. x-axis)
$r(u,v) = v\vec{i} + (a\cos u)\vec{j} + (a\sin u)\vec{k}$.

Tangent planes & normal vectors (given surface $r(u,v)$).
① r_u, ② r_v ③ $r_u \times r_v$ (normal to tangent plane).
④ Find u, v using (pt.) and original eqn. pt.: $(1,4,-1)$
⑤ Subst. u and v into normal eqn. ($r_u \times r_v$) eg: $-8\vec{i} + \vec{j} + 4\vec{k}$.
⑥ $r \cdot n = (x-1)(-8) + (y-4)(1) + (z+1)(4) = 0$

S.I. of scalar functions
$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,\|r_u \times r_v\|\,dA$

S.I. of vector fields: Let F be a cont. vec. field on a surface S with a unit normal vec. n.
$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\,dA$

orientation: $r_v \times r_u = -r_u \times r_v$, $\iint_{-S} \vec{F} \cdot d\vec{S} = -\iint_S \vec{F} \cdot d\vec{S}$.
+ve k component of normal vector $\Rightarrow$ upward normal vector

Arc. Length vector: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$
$L = \int_a^b \|r'(t)\|\,dt$.

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} <1 \Rightarrow C \\ >1 \Rightarrow D \\ =0 \Rightarrow ?? \end{cases}$

Line Integrals

Gradient Field: $\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$
$\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y\vec{j} + f_z\vec{k}$. cont' diff^{ble} func of x

Conservative field: A vector field is called a cons. vec. field if it is the grad. field of some scalar function. $\vec{F} = \nabla f$. (f: potential func.)

Criteria: $F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$
$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$.

Scalar func

$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,\|r'(t)\|\,dt$

$\|r'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$.

	AC	C
Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$	$x = a\cos t$, $y = b\sin t(-1)$, $0 \le t \le 2\pi$.
Circle $x^2 + y^2 = 1$	$x = r\cos t$, $y = r\sin t$, $0 \le t \le 2\pi$	$x = r\cos t$, $y = -r\sin t$, $0 \le t \le 2\pi$.
$y = f(x)$	$x = t$, $y = f(t)$	
$x = g(y)$	$x = g(t)$, $y = t$	

Length of a curve: $\int_a^b ds$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Line segment (x_0, y_0, z_0) to (x_1, y_1, z_1):
$x = (1-t)x_0 + tx_1$
$y = (1-t)y_0 + ty_1$
$z = (1-t)z_0 + tz_1$
$0 \le t \le 1$

Vector fields:

$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$ ⋆ dot prdt

Orientation of C: $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$.

Component Forms: $F = P\vec{i} + Q\vec{j} + R\vec{k}$.
$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_C P\frac{dx}{dt}\,dt + \int_C Q\frac{dy}{dt}\,dt + \int_C R\frac{dz}{dt}\,dt$

Eg: C is a line seg from $(-5,-3)$ to $(0,2)$ $\Rightarrow r(t) = (5t-5)\vec{i} + (5t-3)\vec{j}$, $0 \le t \le 1$

$\int_C y^2\,dx + x\,dy = \int_0^1 (5t-3)^2 \frac{dx}{dt}\,dt + \int_0^1 (5t-5)\left(\frac{dy}{dt}\right)dt$.
$= \int_0^1 (5t-3)^2(5)\,dt + \int_0^1 (5t-5)(5)\,dt$.

Fundamental Thm. for L.I.
Let C be a smooth curve w/ vec. function $r(t)$, $t \in [a,b]$, if f is a scalar function whose gradient ∇f is continuous, then,
$\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$ [end - start]

If F is a cons. vec. field, i) $\int_C \vec{F} \cdot d\vec{r}$ is indp. of path: $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$
ii) $\oint_C \vec{F} \cdot d\vec{r} = 0$ (for a closed curve, start = end)

Let D be a bounded region in the xy plane and ∂D the boundary of D in positive orientation. Suppose $P(x,y)$ & $Q(x,y)$ have continuous partial derivatives on D:
$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$ GREEN'S THM.

$\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$
$\operatorname{curl}\vec{F} = \nabla \times \vec{F}$
$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$
$\operatorname{curl}\vec{F} = 0 \Rightarrow \vec{F}$ is a conservative field

STOKES' THM: S is an oriented piecewise smooth surface which is bounded by a closed, piecewise-smooth bounding curve C. Let $\vec{F}$ be a vec. field whose components have continuous partial der. on S.
$\oint_C \vec{F} \cdot d\vec{r} = \iint (\operatorname{curl}\vec{F}) \cdot d\vec{S}$

Orientation of C: walk in the direct^n around C with head pointing in direct^n of normal vector of S, S should be on your left.

Tech: Multiple Int / Polar Coord

Divergence Thm: Let E be a solid region and let S be the boundary of E, given with the outward orientation. Let $\vec{F}$ be a vector field whose component func. have cont' partial derivat
$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$

orientation: outward vector pointing out from E

Eg: $\operatorname{div} F = 3x$.
$\int_0^1\int_0^2\int_{-3}^0 3x\,dx\,dy\,dz = 3\int_0^1 x\,dx \int_0^2 dy \int_{-3}^0 dz$
$= 9$

Intersection find z of int.
pt: $\_\,\vec{i} + \_\,\vec{j} + z\vec{k}$