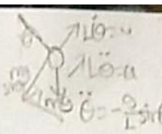
Oscillations

SHM

$m\ddot{x}+kx=0 \Rightarrow \ddot{x}+\omega^2x=0,\ \omega=\sqrt{k/m}$



period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}},\ f=\frac{1}{T},\ \omega=2\pi f=\sqrt{\frac{k}{m}}$

G.S.: $x(t)=C\cos(\omega t)+D\sin(\omega t)$

$x(t)=A\cos(\omega t-\delta)=A\cos\omega t\cos\delta+A\sin\omega t\sin\delta$

$A\cos\delta=C,\ A\sin\delta=D,\ C^2+D^2=A^2,\ \tan\delta=D/C$

Suppose $x(0)=x_0,\ x'(0)=v(0),\ A=\sqrt{x_0^2+\left(\frac{v_0}{\omega}\right)^2}$

$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \Rightarrow f(x)-f(a)\approx f'(a)(x-a)$

Damped, Unforced

① 2 real roots (both $-$ve): $b^2>4mk$, $\lambda=\frac{-b\pm\sqrt{b^2-4mk}}{2m}$ overdamped

② 1 root, $b^2=4mk$ (critical)

③ $b^2<4mk \Rightarrow$ complex $\Rightarrow x(t)=e^{-\frac{b}{2m}t}[c_1\cos\omega t+c_2\sin\omega t]$

$\omega=\frac{1}{2m}\sqrt{4mk-b^2},\ x(t)=Ae^{-\frac{b}{2m}t}\cos(\omega t-\delta)$

Forced w/o damping

$m\ddot{x}+kx=F_0\cos\alpha t \Rightarrow \ddot{x}+\omega^2x=0,\ \omega=\sqrt{k/m}$

$x(t)=x_h+x_p=A\cos(\omega t-\delta)+\frac{F_0/m}{\omega^2-\alpha^2}\cos(\alpha t),\ \omega\neq\alpha$

if $x(0)=0,\ \dot{x}(0)=0,\ x(t)=\frac{F_0/m}{\omega^2-\alpha^2}[\cos\alpha t-\cos\omega t]$

Beating: $\alpha\approx\omega$, $=\frac{2F_0/m}{\alpha^2-\omega^2}\sin\left[\frac{\alpha-\omega}{2}t\right]\sin\left[\frac{\alpha+\omega}{2}t\right]$

$\sin\left[\frac{\alpha+\omega}{2}t\right]$ is modulated by a slower one.

$A(t)=\frac{2F_0/m}{\alpha^2-\omega^2}\sin\left(\frac{\alpha-\omega}{2}t\right)$

Resonance: $\lim_{\alpha\to\omega}A(t)=\frac{F_0}{m\omega}\frac{t}{2}$ (st. line), c.f.: $x=\frac{B/t}{2m\omega}\sin(\omega t)$

Forced, damped:

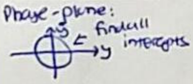
$m\ddot{x}+b\dot{x}+kx=F_0\cos(\alpha t)$, $x_p=B\sin\alpha t+C\cos\alpha t$, $x(t)=x_h+x_p$; $A(\alpha)=\frac{F_0/m}{\sqrt{(\omega^2-\alpha^2)^2+\frac{b^2}{m^2}\alpha^2}}$

Conservation $\frac{d}{dx}\left(\frac{1}{2}(\dot{x})^2\right)=\ddot{x}$

$m\ddot{x}=m\frac{d}{dx}\left(\frac{1}{2}(\dot{x})^2\right)$

$\frac{1}{2}m(\dot{x})^2+\frac{1}{2}kx^2=E$

Phase-plane: find all intercepts



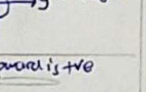
Cantilevers, $\frac{d^4y}{dx^4}=\frac{w(x)}{EI}$ ← wt per unit l, upward is +ve; Young's modulus, second moment

soln: $y=\frac{\alpha L^4}{2EI}\left[-\frac{1}{12}\left(\frac{x}{L}\right)^4+\frac{1}{3}\left(\frac{x}{L}\right)^3-\frac{1}{2}\left(\frac{x}{L}\right)^2\right],\ w=-\alpha$ (constant)

(ucf) $\Delta=-\frac{\alpha L^4}{8EI},\ w=-\alpha$ $w(x)$ is $-$ve as it is downwards

Modelling:

① Malthus: $\frac{dN}{dt}=(B-D)N=kN,\ k=B-D$, $N=\tilde{N}e^{kt}$

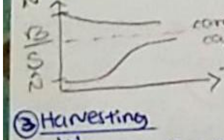


② Logistic: $D=sN$

$\frac{dN}{dt}=BN-sN^2$ $\frac{1}{N}=\frac{1}{N_\infty}+Ce^{-Bt}$

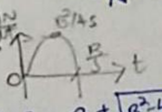
$C=\frac{1}{\tilde{N}}-\frac{1}{N_\infty},\ N_\infty=\frac{B}{s}$

$\therefore N=\frac{N_\infty}{1+\left(\frac{N_\infty}{\tilde{N}}-1\right)e^{-Bt}}$

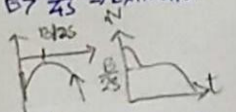


③ Harvesting

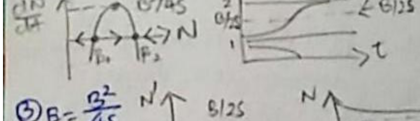
$\frac{dN}{dt}=(B-sN)N-E$



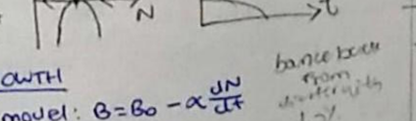
① $E>\frac{B^2}{4s} \Rightarrow$ Extinction $\beta_1,\beta_2=\frac{B\pm\sqrt{B^2-4Es}}{2s}$



② $E<\frac{B^2}{4s} \Rightarrow$ double roots $\frac{\beta_1+\beta_2}{2}=\frac{B}{2s}$ $\therefore \beta_1+\beta_2=\frac{B}{s}$



③ $E=\frac{B^2}{4s}$



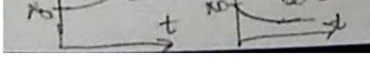
STEADY GROWTH

modified model: $B=B_0-\alpha\frac{dN}{dt}$ bounce back from disturbance 10%

$\left(\frac{1}{N}+\alpha\right)\frac{dN}{dt}=B_0-D$ $\beta_1=90\%\ \text{of}\ \beta_2$

$\frac{dN}{dt}=\frac{(B_0-D)N}{1+\alpha N}$

$\frac{dN}{dt}\approx\frac{(B_0-D)}{\alpha}$, assuming $B_0>D$, else graph just decreases rapidly



Integration:

$\int(Ax+B)^n dx=\int\frac{(Ax+B)^{n+1}}{(n+1)(A)}+C,\ n\neq-1$

$\int f'(x)f(x)^n=\frac{[f(x)]^{n+1}}{n+1}+C$

$\int e^{Ax+B}=\frac{e^{Ax+B}}{A}$

$\int f'(x)e^{f(x)}=e^{f(x)}$

$\int\frac{1}{Ax+B}=\frac{\ln|Ax+B|}{A}$ $\int\frac{f'(x)}{f(x)}=\ln|f(x)|+C$

$\int\cos(Ax+B)=\frac{\sin(Ax+B)}{A},\ \int f'(x)\cos f(x)=\sin f(x)$

$\int\sin(Ax+B)=-\frac{\cos(Ax+B)}{A},\ \int f'(x)\sin f(x)=-\cos f(x)$

$\int\sec^2(Ax+B)=\frac{\tan(Ax+B)}{A},\ \int f'(x)\sec^2 f(x)=\tan f(x)$

$\int\sec(Ax+B)\tan(Ax+B)=\frac{\sec(Ax+B)}{A},\ f'(x)\sec\cdot\tan=\sec f(x)$

$\int\mathrm{cosec}(Ax+B)\cot(Ax+B)=-\frac{\mathrm{cosec}(Ax+B)}{A}$

$\int a^{Ax+B}=\frac{a^{Ax+B}}{A\ln a}\ a>0,\ \int f'(x)a^{f(x)}=\frac{a^{f(x)}}{\ln a}$

$\int\frac{1}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right),\ \int\frac{f'(x)}{\sqrt{1-f(x)^2}}=\sin^{-1}f(x)$

$\int\frac{1}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right),\ \int\frac{f'(x)}{a^2+f(x)^2}=\frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right)$

$\int\frac{1}{x^2-a^2}=\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right),\ \int\frac{1}{a^2-x^2}=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$

$\int x\sin(\lambda x)=\frac{\sin(\lambda x)}{\lambda^2}-\frac{x\cos(\lambda x)}{\lambda}$

$\int x^2\sin(\lambda x)=\frac{2x\sin(\lambda x)}{\lambda^2}+\frac{(2-\lambda^2x^2)\cos(\lambda x)}{\lambda^3}$

$\int x\cos(\lambda x)=\frac{\cos(\lambda x)}{\lambda^2}+\frac{x\sin(\lambda x)}{\lambda}$

$\int x^2\cos(\lambda x)=\frac{2x\cos(\lambda x)}{\lambda^2}+\frac{x\sin(\lambda x)}{\lambda}$

$\int\tan x=-\ln|\cos x|+C,\ \int\tanh=\ln(\cosh x)+C$

$\int\sinh=\cosh,\ \int\cosh=\sinh,\ \int\mathrm{sech}^2=\tanh$

$\int\mathrm{cosech}^2=-\coth,\ \int\mathrm{sech}\tanh=-\mathrm{sech},\ \int\mathrm{csch}\coth=-\mathrm{csch}$

Int. by parts: $\int uv\,dx=u\int v\,dx-\int\left(\int v\right)\left(\frac{du}{dx}\right)dx$

$\int\cos^n,\int\sin^n$ → even pwr: to cosines of $x2\not{x}$ → odd pwr: use $\cos^2+\sin^2=1$

$\int\tan^n dx$: $1+\tan^2x=\sec^2x$

LAPLACE TRANSFORM:

$F(s)=L(f)=\int_0^\infty e^{-st}f(t)\,dt$? convergent

Properties: $L(\alpha f+\beta g)=\alpha Lf+\beta Lg,\ L^{-1}(\alpha f+\beta h)=\alpha L^{-1}f+\beta L^{-1}h$

methods: ① Decompose into partial fract., ② complete sq.

$L(f^n)=s^nL(f)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots f^{n-1}(0)$

$=s^nL(f)-\sum_{k=0}^{n-1}s^{n-1-k}f^k(0)$, for $n=1,2,3\ldots$

$*L(f')=sL(f)-f(0),\ L(f'')=s^2L(f)-sf(0)-s^0f'(0)$

Trick: Eg: $\frac{\alpha}{s^2+\alpha^2}=L(\sin\alpha t)=\int_0^\infty e^{-st}\sin\alpha t\,dt$

method: Eg: $y''+y=e^{2t},\ y(0),y'(0)\Rightarrow$ take LT on both sides

$*L\left(\int_0^t f(u)\,du\right)=\frac{1}{s}L(f(t))$

*s-shifting: If $L(f(t))=F(s)$, $L(e^{ct}f(t))=F(s-c)$

Heaviside/Unit-Step Function: to cut to reg. from a to b only

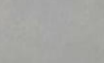
$*\ u(t-a)=\begin{cases}0 & t<a\\ 1 & t>a\end{cases}$ $*g(t)\{u(t-a)-u(t-b)\}=\begin{cases}0 & t<a\\ g(t) & a<t<b\\ 0 & b<t\end{cases}$

* t-shifting: $L\{f(t-a)u(t-a)\}=e^{-as}F(s)$, $L(f(t))=F(s)$

$\therefore L(u(t-a))=\frac{e^{-as}}{s}$

* Dirac Delta: For any cont' fnc. g, $\int_0^\infty\delta(t-a)g(t)\,dt=g(a)$

$L(\delta(t-a))=e^{-as}$ (impulsive force at $t=a$)



$L(\delta(t))=1$ Eg $100\,\delta(t-2\pi)$: 100 units at $t=2\pi$

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$\frac{1}{s}$	$\sin\omega t$	$\omega/(s^2+\omega^2)$
t^n	$n!/s^{n+1}$	$\cos\omega t$	$s/(s^2+\omega^2)$
$\sqrt{t}$	$\sqrt{\pi}/2s^{3/2}$	$e^{ct}\sin\omega t$	$\omega/(s-c)^2+\omega^2$
e^{at}	$1/(s-a),\ s>a$	$e^{ct}\cos\omega t$	$(s-c)/(s-c)^2+\omega^2$
$e^{ct}t^n$	$n!/(s-c)^{n+1}$	$\sinh\omega t$	$\omega/s^2-\omega^2$
$u(t-a)$	e^{-as}/s	$\cosh\omega t$	$s/s^2-\omega^2$
$u(t)$	$1/s$	$\cos(at+b)$	$\frac{s\cos b-a\sin b}{s^2+a^2}$
$\delta(t-t_0)$	e^{-st_0}	$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\delta(t)$	1	$\sinh(at)$	$\frac{a}{s^2-a^2}$

Differentiation

$((ax+b)^n)\Rightarrow n(ax+b)^{n-1}(a)$

$\sin f(x)=f'(x)\cos f(x),\ \cos f(x)=-f'(x)\sin f(x)$

$\tan f(x)=\sec^2 f(x)f'(x),\ \mathrm{cosec}f(x)=-f'(x)\mathrm{cosec}\cot f(x)$

$\sec f(x)=f'(x)\tan f\sec f,\ \cot f=-f'\mathrm{cosec}^2f$

$\ln f=\frac{f'}{f},\ e^{f(x)}=f'(x)e^{f(x)}$ $\frac{d^2y}{dx^2}=\frac{d/dt\left(\frac{dy}{dx}\right)}{dx/dt}$

$\sin^nx=n\sin^{n-1}x\cos x$ $\frac{d^2x}{dy^2}=\left[\frac{d}{dx}\left(\frac{dx}{dt}\right)\right]\frac{dx}{dt}$

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}\ \Big|\Big|\ \frac{d^2y}{dx^2}=\frac{d(d/dt)\left(\frac{dy}{dx}\right)}{dx/dt}$

$f''(x)<0\Rightarrow$ concave down, $>0\Rightarrow$ concave up

$\sin^{-1}x\Rightarrow\frac{1}{\sqrt{1-x^2}},\ \sin^{-1}f(x)=\frac{f'(x)}{\sqrt{1-f(x)^2}}$

$\cos^{-1}f(x)\Rightarrow\frac{-f'(x)}{\sqrt{1-f(x)^2}},\ \tan^{-1}f(x)=\frac{f'(x)}{1+f(x)^2}$

$a^x\Rightarrow a^x\cdot\ln a,\ a^{f(x)}\Rightarrow f'(x)a^{f(x)}\cdot\ln a$

$\log_a f(x)\Rightarrow\frac{f'(x)}{f(x)}\log_a e$

Quotient Rule: $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ or $=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$

Trigonometry

$\sin^2+\cos^2=1,\ \sec^2-\tan^2=1,\ \mathrm{cosec}^2-\cot^2=1$

$\cosh^2-\sinh^2=1,\ \tanh^2+\mathrm{sech}^2=1,\ \coth^2-\mathrm{csch}^2=1$

$\sin2x=2\sin x\cos x,\ \cos2x=\cos^2-\sin^2=2\cos^2-1=1-2\sin^2$

$\sinh2x=2\sinh\cosh,\ \cosh2x=\cosh^2+\sinh^2x$

$\cosh^2x=\frac{\cosh2x+1}{2},\ \sinh^2x=\frac{\cosh2x-1}{2}$

$\tan2x=\frac{2\tan x}{1-\tan^2x}$

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\sin(A-B)=\sin A\cos B-\cos A\sin B$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

Hyperbolic Approx.:

$\sinh x=\frac{e^x-e^{-x}}{2}$

$\cosh x=\frac{e^x+e^{-x}}{2}$

$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$2\cos A\sin B=\sin(A+B)-\sin(A-B)$

$2\cos A\cos B=\cos(A+B)+\cos(A-B)$

$2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Fourier Series:

$f(x)=a_0+\sum_{n\geq1}^{\infty}a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}$

$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx$

$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\ n=1,2\ldots$

$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx,\ n=1,2\ldots$

$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)=\begin{cases}0 & \text{if } n\neq m\\ 1 & \text{if } n=m\end{cases}$

$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)=\begin{cases}0 & \text{if } n\neq m\\ 1 & \text{if } n=m\end{cases}$

$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)=0$ for any n/m

$\int_{-L}^{L}\text{odd}=0,\ \int_{-L}^{L}\text{even}=2\int_0^L\text{even}$

$\cos n\pi=(-1)^n,\ n=1,2,3\cdots$

$\cos(2n-1)\frac{\pi}{2}=0,\ n=1,2,3\cdots$

$\cos\left(\frac{n\pi}{2}\right)=\begin{cases}0 & n=\text{odd}\\ 1 & n=\text{even}=2m\end{cases}$

$\sin(2n-1)\frac{\pi}{2}=(-1)^{n+1},\ n=1,2,3\cdots$

$\sin n\pi=0$

$\sin\left(n\frac{\pi}{2}\right)=\begin{cases}0 & n=\text{even}=2m\\ (-1)^{m+1} & n=\text{odd},=2m+1\end{cases}$

Let $L(f(t))=F(s)$

Then, $L(tf(t))=-F'(s)$

$L(t^nf(t))=(-1)^nF^n(s)$

Let $\vec{v}(t)=\begin{bmatrix}x(t)\\ y(t)\end{bmatrix},\ \frac{d\vec{v}}{dt}=\begin{bmatrix}dx/dt\\ dy/dt\end{bmatrix}$

Then $L\left(\frac{d\vec{v}}{dt}\right)=sL(\vec{v})-\vec{v}(0)$

MATRICES

Transpose → $\begin{bmatrix}1&2\\4&8\end{bmatrix}^T = \begin{bmatrix}1&4\\2&8\end{bmatrix}$ row→col, col→row

$(AB)^T = A^TB^T$

Symmetric if $A^T=A$, $A+A^T$ is also symm.

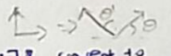
Anti-symmetric if $A^T=-A$, Any mat $B-B^T$ is AS, $A-A^T$ is AS

if A is AS, BAB^T is also AS

Length of vector $|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{\vec{u}^T\vec{u}}$

Orthogonal matrix: if it satisfies $B^TB=I \Rightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$ Note: $IA=A=AI$

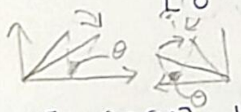
Rotational matrix: $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, AC through θ



$*[R(\theta)]^{1506} = \begin{bmatrix}\cos(1506\theta) & -\sin(1506\theta)\\ \sin(1506\theta) & \cos(1506\theta)\end{bmatrix}$ $*\begin{bmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}^8 \Rightarrow$ convert to rotational 45° & compute

Involutory matrix: $AA=I$, $A^9=A^8A=IA=A$, $A^{100}=I$

Shear matrix: $\begin{bmatrix}1 & \tan\theta\\ 0 & 1\end{bmatrix}$, maps $\hat{i}$ → $\hat{i}$, $\hat{j}$ to $(\tan\theta\,\hat{i}+\hat{j})$. clockwise θ.



Rotate, shear, rotate
3rd 2nd 1st

$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ $\begin{vmatrix}+&-&+\\-&+&-\\+&-&+\end{vmatrix}$

$\det A = a_{11}\begin{bmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{bmatrix} - a_{12}\begin{bmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{bmatrix} + a_{13}\begin{bmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}$

Steps for inverse: ① work out cofactor for each slot with correct sign.
② Take det & slot back
③ Transpose
④ Divide by determinant of A (initial mat.)

*if det=0, no inv.
*if A=upp/low. Δ, det A = prdt of diagonal entries

Inverse of 2x2 mat: $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$\det(ST)=\det(S)\times\det(T)$, $\det M^T=\det M$, $\det(cM)=c^n\det M$, n=size of M.

Leontief I/O model: Form Alg. matrices.

E.val: $\det(T-\lambda I)=0$, E.vec. $\Rightarrow (T-\lambda I)u=0$

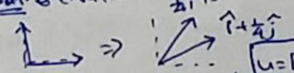
Diagonalisation: $A=PDP^{-1}$ (A must be n×n with non-// vec.)

Eg: $e_1=\begin{bmatrix}1\\1/2\end{bmatrix}$, $e_2=\begin{bmatrix}1\\-2\end{bmatrix}$, $\Rightarrow D=\begin{bmatrix}2&0\\0&-3\end{bmatrix}$, $P=\begin{bmatrix}1&1\\1/2&-2\end{bmatrix}$, P^{-1}=inv.
$\lambda=2$, $\lambda=-3$

$M^n=PD^nP^{-1}=P\begin{bmatrix}\lambda_1^n&0\\0&\lambda_2^n\end{bmatrix}P^{-1}$ $\boxed{\mathrm{Tr}(A)=\Sigma\text{ e.val}}$

Trace: sum of diagonal values., $\mathrm{Tr}(MN)=\mathrm{Tr}(NM)=\mathrm{Tr}(PDP^{-1})=\mathrm{Tr}(D)$

*Note: $\det(PDP^{-1})=\det(PP^{-1}D)=\det(D)$ For $\mathrm{Tr}(M)=\mathrm{Tr}\,D \Rightarrow$ sum of e.val
det * = prdt. of e.val.

Linear T: eg: $T=\begin{bmatrix}1&1/4\\1/4&1\end{bmatrix}\Rightarrow$  $\hat{i}+\frac{1}{4}\hat{j}$

Area of //logram $=|\vec{u}||\vec{v}|\sin\theta=|\vec{u}\times\vec{v}|=\det A=|ad-bc|$ $\vec{u}=\begin{bmatrix}a\\c\end{bmatrix}$, $\vec{v}=\begin{bmatrix}b\\d\end{bmatrix}$, $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$A=[\vec{u},\vec{v},\vec{w}]$

T = n×n mat A if det A=0, T is not 1-1 (singular)

vol. of //lepiped = base A × ht. $=|\vec{u}\times\vec{v}||\vec{w}|\cos\theta=|(\vec{u}\times\vec{v})\cdot\vec{w}|=\det A$

Rank: ① det T≠0 3d onto 3d ⇒ 3 ② det T=0 a) 3d onto plane R2 b) 3d onto line R1 $T\hat{i}=T\hat{j}=T\hat{k}$ | No. of non-zero e.val. of mat. = rank

FIRST/SECOND ORDER ODE:

methods: sep. var., subs., bernoulli ($z=y^{1-n}$), I.F. $I(x)=e^{\int P(x)dx}$ $\Rightarrow y=\frac{1}{I}\int r I(x)$

2nd order: ① Homog → $y''+Py'+qy=0 \Rightarrow \lambda^2+P\lambda+q=0$
① distinct roots ⇒ $y=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}$
② repeat ⇒ $y=c_1e^{\lambda x}+c_2xe^{\lambda x}$
③ complex ⇒ $y=e^{ax}[c_1\cos bx+c_2\sin bx]$, $\lambda=a+bi$

② Non-homog:
① char. eqn. & solve
$k\neq\lambda_1, k\neq\lambda_2 \Rightarrow$ deg n
$k=\lambda_1/\lambda_2 \Rightarrow$ deg n+1
$k=\lambda_1=\lambda_2 \Rightarrow$ deg n+2

cases: ① poly ⇒ (try $Ax^2+Bx+C)e^{0x}$, ③ Try $z=ue^{ikx}$, $u=Ax+B$..
② Exp ⇒ $y=ue^{kx}$

③ Variation of Parameters:
① Find homog soln ② $y_p=uy_1+vy_2$ ③ Find y_1', y_2', ④ $W=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}=y_1y_2'-y_1'y_2$
⑤ $u=\int-\frac{y_2r}{W}$, $v=\int\frac{y_1r}{W}$, solve with IVP, Note: $e^{i\theta}=\cos\theta+i\sin\theta$

Newton's Cooling: $\frac{dT}{dt}=-K(T-T_{env})$, Heating: $\frac{dT}{dt}=K(T_{env}-T)$

Extra: Maclaurin series: $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^n(0)}{n!}x^n+\dots$

Taylor series: $f(x)=f(a)+f'(a)(x-a)+\dots+\frac{f^n(a)}{n!}(x-a)^n+\dots$

Partial fraction: $\frac{}{(x-1)^3}\Rightarrow\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$, $\frac{1}{x^2+3}=\frac{Ax+B}{x^2+3}$, $\frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$

Systems of First Order ODE.

$\frac{dx}{dt}=-2x+y$, $\frac{dy}{dt}=2x-3y \Rightarrow B=\begin{bmatrix}-2&1\\2&-3\end{bmatrix}$

① e.val: $\alpha=\frac{1}{2}[\mathrm{Tr}B\pm\sqrt{\mathrm{Tr}B^2-4\det B}]$

ⓐ distinct: $c_1u_1e^{\alpha_1t}+c_2u_2e^{\alpha_2t}$ where u_1 & u_2 are e.vectors

ⓑ complex: $c_1e^{\alpha t}[u\cos\beta t-v\sin\beta t]+c_2e^{\alpha t}[u\sin\beta t+v\cos\beta t]$.

$w=u+iv$ (only consider one complex pair e.vec.) → eg: $\binom{\alpha}{\beta}=\binom{1}{-i}$
$w=\binom{1}{0}+i\binom{0}{-1}$, u, v

② Laplace method

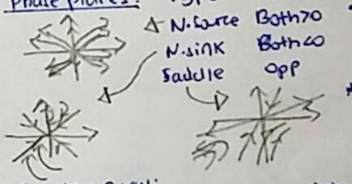
$\vec{J}=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$, $\frac{d\vec{J}}{dt}=\vec{J}'t=\begin{bmatrix}x'\\y'\end{bmatrix}=B\vec{J}$.

⇒ Apply L.T: $L(f')=sL(f)-f(0)$
$sL(\vec{J})-\vec{J}(0)=BL(\vec{J})$
$sL(\vec{J})-BL(\vec{J})=\vec{J}(0)$
$(sI-B)L(\vec{J})=\vec{J}(0)$
$L(\vec{J})=[sI-B]^{-1}\vec{J}(0)\Rightarrow$ $\begin{bmatrix}-\\-\end{bmatrix}$ → L(x), L(y) } take inverse

Stability: ① e.val ≤0, det B>0, Tr B<0 } real no. ② Tr(B)≤0, det B>0 } complex.

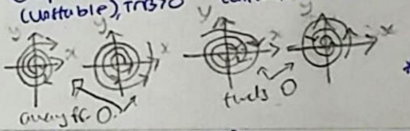
Phase planes:

Type	E.val.	Tr(B)	det(B)	Stability
N.source	Both>0	>0	>0	unstable
N.sink	Both<0	<0	>0	stable
Saddle	opp		<0	unstable

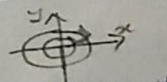


*Note: e.val <0: traj goes to 0
>0: goes to ∞
all traj. tend. to e.val. of bigger val.

Complex e.val:
① Spiral source (unstable), TrB>0 ② Spiral sink (unstable, TrB<0) ③ Centre, TrB=0



away fr. 0, tends 0



*To find direction, use $\frac{dy}{dt}/\frac{dx}{dt}$.
eg: $\frac{dy}{dt}=2x-2y$ if $y=0$, $y'=2x$
Positive x axis, y↑

M2: $\mathrm{Tr}B^2-4\det(B)$
+ve → TrB>0, detB>0 ↓ n.source; TrB<0, detB>0 ↓ n.sink; detB<0 ↓ saddle
-ve → TrB>0 → s.source; → TrB<0 → s.sink; → TrB=0 → centre

Partial Differential Eqns:

SEP VAR: Assume $u(x,y)=X(x)Y(y)$, ⇒ eg: $\frac{1}{x}\frac{X'(x)}{X(x)}=\frac{Y'(y)}{Y(y)}$ = constant
simplify & solve sep. by equating

Sturm-Liouville Eqn: $X''(x)+\lambda X(x)=0$, $X(0)=0$, $X'(0)=0$

cases: ① $\lambda<0$ or $\lambda=0 \Rightarrow$ 0 is the only soln
② $\lambda>0\Rightarrow$ ⓐ when $\lambda\neq(\frac{n\pi}{L})^2\Rightarrow$ 0 is the only soln.
ⓑ when $\lambda=(\frac{n\pi}{L})^2\Rightarrow\pi^2/L^2\dots$, there exists non-zero soln
$X(x)=C\sin[(n\pi/L)x]$ ← eigenfunction.

Wave Eqn: $c^2y_{xx}(x,t)=y_{tt}(x,t)$, $0<x<L$, $t>0$

G.S.: $y(x,t)=\sum_{n=1}^{\infty}A_n\sin[(\frac{n\pi}{L})x]\cos[(\frac{n\pi}{L}c)t]$

∴ $A_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$

General steps: ① Let $y(x,t)=u(x)v(t)$,
② $u''(x)+\lambda u(x)=0$, $v''(t)+\lambda c^2v(t)=0$
③ For u, use $\lambda=(\frac{n\pi}{L})^2$, $\Rightarrow u(x)=B_n\sin(\frac{n\pi}{L}x)$
④ For v, use initial cond., $v(t)=C_n\cos[\sqrt{\lambda c^2}\,t]$ = nct
⑤ Compare coeff. of f(x) with A_n

d'Alembert's soln for wave eqn: $y(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]$
B.Conds: $y(0,t)=0$, $y(L,t)=0$ for all $t>0$
$y(x,0)=f(x)$, $y_t(x,0)=0$, $0<x<L$

Heat Eqn: $u_t(x,t)=c^2u_{xx}(x,t)$, $0<x<L$, $t>0$, $u(x,0)=f(x)$, $u(0,t)=0$, $u(L,t)=0$

$u(x,t)=\sum_{n=1}^{\infty}A_n\sin[(\frac{n\pi}{L})x]e^{-c^2(\frac{n\pi}{L})^2t}$

$A_n=\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$

LAPLACE EQN: $u_{yy}(x,y)=-u_{xx}(x,y)$, $0<x<L$, $0<y<K$.

G.S.: $\sum_{n=1}^{\infty}A_n\sin[n\pi x]\sinh[n\pi(y-2)]$

$A_n=\frac{2}{\sinh(-2n\pi)}\int_0^1 f(x)\sin(n\pi x)dx$.