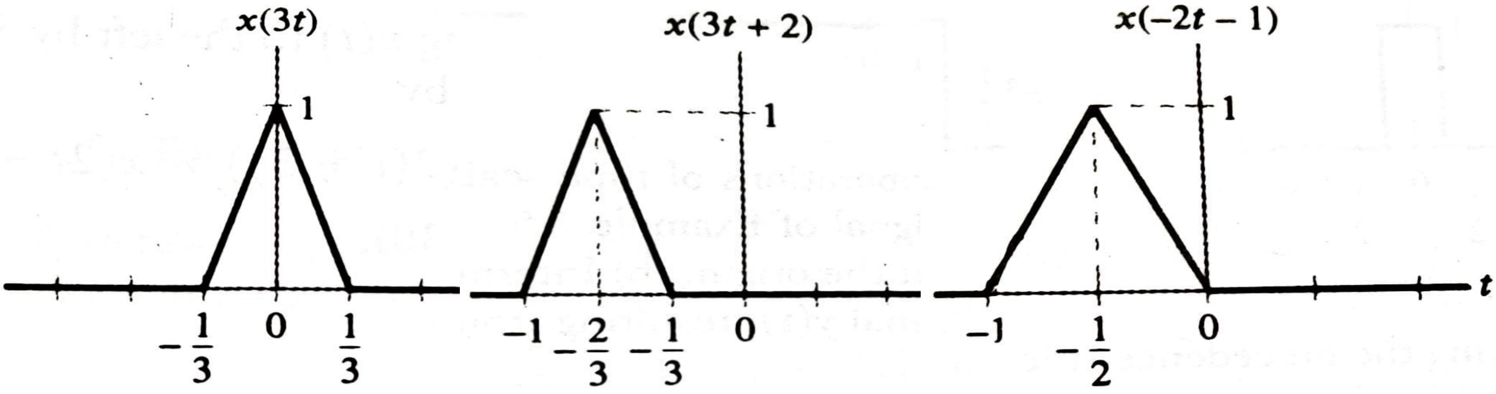
SIGNAL AND SYSTEMS (ECL 211)
Mid Sem Exam Solutions W24
B.Tech.(ECE), Semester III

October 18, 2024

1. **Solution:**



2. **Solution:**

   (a) Nonperiodic.
   Comparing with $x[n] = \text{A} \cos(\Omega n + \phi)$, here we get $\Omega = 2$. Thus Fundamental period $N = \frac{2\pi m}{\Omega} = \pi m$ is irrational. Hence given signal is non periodic.

   (b) Periodic, fundamental period = 10.
   Comparing with $x[n] = \text{A} \cos(\Omega n + \phi)$, here we get $\Omega = 0.2\pi$. Thus Fundamental period $N = \frac{2\pi m}{\Omega} = 10m$ is rational. Hence given signal is periodic with N=10.

   (c) Periodic, fundamental period = 35.
   Comparing with $x[n] = \text{A} \cos(\Omega n + \phi)$, here we get $\Omega = \frac{6\pi}{35}$. Thus Fundamental period $N = \frac{2\pi m}{\Omega} = \frac{35}{3}m$ is rational. Hence given signal is periodic with N=35.

3. **Solution:** Letting $x[n] = \delta[n]$, we find that impulse response is

$$h[n] = \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{2} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Write $x[n]$ as the weighted sum of time-shifted impulses:

$$x[n] = 2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$$

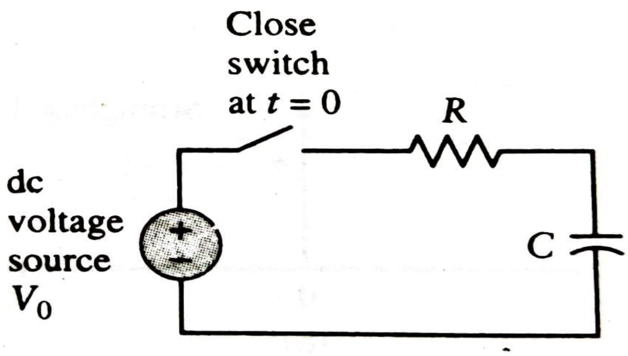
Here the input is decomposed as a weighted sum of three time-shifted impulses because the input is zero for $n < 0$ and $n > 2$. Since a weighted time-shifted impulse input, $\gamma\delta[n-k]$, results in a weighted, time-shifted impulse response output, $\gamma h[n-k]$, indicates that the system output may be written as

$$y[n] = x[n] * h[n] = 2h[n] + 4h[n-1] - 2h[n-2] (\text{since } x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0])$$

Summing the weighted and shifted impulse responses over n gives

$$y[n] = \begin{cases} 0, & n < 0 \\ 2, & n = 0 \\ 5, & n = 1 \\ 0, & n = 2 \\ -1, & n = 3 \\ 0, & n \geq 4 \end{cases}$$

4. **Solution:**
PART A :
Assuming a switch is placed between voltage source and RC circuit. The switching oper-



ation is represented by a step function. Hence the step response of the RC circuit $y(t)$is calculated when $x(t)$ is unit step signal. After the switch is closed, the capacitor cannot charge suddenly so being initially uncharged, we have $y(0) = 0$. For $t = \infty$, $y(\infty) = 1$. As time constant for the circuit τ=RC, step response of the circuit

$$y(t) = [y(\infty) + (y(0) - y(\infty))e^{(-t/\tau)}]u(t)$$

After substituting the corresponding values, we get

$$y_{step}(t) = (1 - e^{-\frac{t}{RC}})u(t)$$

Impulse response can be calculated as the derivative of step response.

$$y_{impulse}(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

PART B :
Impulse response is given as, $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$. We first graph $x(\tau)$ and $h(t-\tau)$ as a function of τ such that:

$$x(\tau) = \begin{cases} 1, & 0 < \tau < 2 \\ 0, & otherwise \end{cases}$$

and

$$h(t-\tau) = \frac{1}{RC}e^{-(t-\tau)}u(t-\tau) = \begin{cases} \frac{1}{RC}e^{-(t-\tau)}, & \tau < t \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$w_t(\tau) = x(\tau)h(t-\tau) = \begin{cases} \frac{1}{RC}e^{-(t-\tau)}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

Identifying intervals of time shifts t, for $t < 0$ we have $w_t(\tau)$=0. At t = 0,$w_t(\tau)$=1. Hence the second and third intervas are $0 \le t < 2$ and $t \ge 2$.

$$y(t) = \int_{-\infty}^{\infty} w_t(\tau) d\tau$$

for interval $0 \le t < 2$

$$y(t) = \int_0^t \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau = 1 - e^{-\frac{t}{RC}}$$

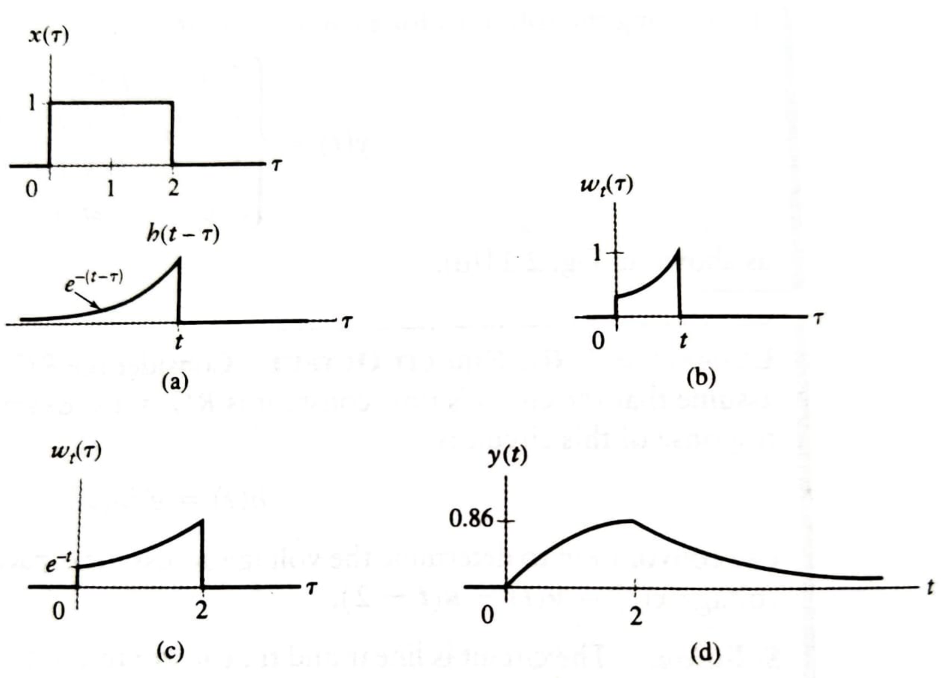
for interval $t \ge 2$

$$y(t) = \int_0^2 \frac{1}{RC} e^{-\frac{(t-\tau)}{RC}} d\tau = (e^{\frac{2}{RC}} - 1)e^{-\frac{t}{RC}}$$

Combined solution is

$$y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\frac{t}{RC}}, & 0 \le t < 2 \\ (e^{\frac{2}{RC}} - 1)e^{-\frac{t}{RC}}, & t \ge 2 \end{cases}$$

Figure for reference :



METHOD 2:

For given RC circuit, unit step response is

$$y_{step}(t) = (1 - e^{-\frac{t}{RC}})u(t)$$

For finding unit impulse response,

$$input = x(t) = \delta(t) = \lim_{\Delta \to 0} \frac{u(t + \Delta/2) - u(t - \Delta/2)}{\Delta}$$

$$y_{impulse}(t) = h(t) = \lim_{\Delta \to 0} \frac{y_{step}(t + \Delta/2) - y_{step}(t - \Delta/2)}{\Delta}$$

Hence,

$$h(t) = \lim_{\Delta \to 0} \frac{(1 - e^{-\frac{t+\Delta/2}{RC}})u(t + \Delta/2) - (1 - e^{-\frac{t-\Delta/2}{RC}})u(t - \Delta/2)}{\Delta}$$

After putting limits we get $\frac{0}{0}$ form, hence using L'Hospital's rule we get,

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

OR
Solving further

$$\begin{aligned}
h(t) &= \lim_{\Delta \to 0} \frac{u(t + \Delta/2) - u(t - \Delta/2) + e^{-\frac{t-\Delta/2}{RC}} u(t - \Delta/2) - e^{-\frac{t+\Delta/2}{RC}} u(t + \Delta/2)}{\Delta} \\
&= \delta(t) + \lim_{\Delta \to 0} \frac{e^{-\frac{t-\Delta/2}{RC}} u(t - \Delta/2) - e^{-\frac{t+\Delta/2}{RC}} u(t + \Delta/2)}{\Delta} \\
& \text{when} \Delta \to 0, e^{\frac{\Delta}{2RC}} = 1 + \frac{\Delta}{2RC} \\
&= \delta(t) + \lim_{\Delta \to 0} \frac{e^{-\frac{t}{RC}} u(t - \Delta/2)(1 + \frac{\Delta}{2RC}) - e^{-\frac{t}{RC}} u(t + \Delta/2)(1 - \frac{\Delta}{2RC})}{\Delta} \\
&= \delta(t) - e^{-\frac{t}{RC}} \delta(t) + \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \\
& as \delta(t) = 0 for t \neq 0 \\
h(t) &= \frac{1}{RC} e^{-\frac{t}{RC}} u(t)
\end{aligned}$$

METHOD 3:
Applying KVL in loop and applying Laplace Transform, we get

$$X(s) - I(s)R - \frac{1}{Cs} I(s) = 0$$

but$Y(s) = \frac{I(s)}{Cs}$, hence putting value of $I(s)$ in KVL equation, we get

$$\frac{Y(s)}{X(s)} = \frac{1}{1 + RCs}$$

Using inverse Laplace Transformation,

$$h(t) = \mathcal{L}^{-1} H(s) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

5. **Solution:** Here the fundamental period $T = 4$. As given signal has even symmetry, integrating over a period that is symmetric about the origin, $-2 < t \leq 2$ where, $X[k] = c_k$ = Fourier series coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$
\begin{aligned}
X[k] &= \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{4}\int_{-2}^{2} \delta(t)e^{-jk(\pi/2)t}dt \\
&\text{(since the impulse only exist at origin, putting t=0)} \\
&= \frac{1}{4}\int_{-2}^{2} \delta(t)dt \\
&\text{(for any real number } a > 0, \int_{-a}^{a} \delta(t)dt = 1) \\
&= \frac{1}{4}
\end{aligned}
$$

The magnitude spectrum is constant since it is independent of k and the phase spectrum is zero.

6. **Solution:** Here, fundamental period T=2, $X[k] = c_k =$ Fourier series coefficients

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} (-j\delta[k-2] + j\delta[k+2] + 2\delta[k-3] + 2\delta[k+3])e^{jk\pi t} \\
&= j(e^{-j2\pi t} - e^{j2\pi t}) + 2(e^{-j3\pi t} + e^{j3\pi t}) \\
x(t) &= 2sin(2\pi t) + 4cos(3\pi t)
\end{aligned}
$$