Visvesvaraya National Institute of Technology, Nagpur
Department of Mathematics
Assignment: Probability Theory(PT-1) **Topics: D.F., P.M.F., P.D.F.**

Notations: DF-distribution function, PMF-probability mass function, PDF-probability density function

1. Do the following functions define DFs?
   (i) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$ (ii) $F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 1 - \dfrac{1}{x} & \text{if } x > 1 \end{cases}$ (iii) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \ge 0 \end{cases}$

2. Does the function $f_\theta(x) = \begin{cases} \theta^2 x e^{-\theta x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$ where $\theta > 0$, define a PDF? Find the DF associated with $f_\theta(x)$; if X is RV with PDF $f_\theta(x)$, find $P\{X \ge 1\}$.

3. Are the following functions distribution functions? IF so, find the corresponding density functions.
   (i) $F(x) = 0$ if $x < -\theta$, $= \frac{1}{2}\left(\frac{x}{\theta} + 1\right)$ if $|x| \le \theta$, and 1 for $x > \theta$ where $\theta > 0$.

   (ii) $F(x) = 0$ if $x < 0$, and $= 1 - e^{-x^2}$ if $x \ge 0$.

   (iii) $F(x) = 0$ if $x < 0$, and $= 1 - (1 + x)e^{-x}$ if $x \ge 0$.

   (iv) $F(x) = 0$ if $x < 1$, and $= \frac{(x-1)^2}{8}$ if $1 \le x < 3$, and 1 for $x \ge 3$.

4. Determine the constant A in the following function, so that the function is PMF and find the DF $F(x)$ in each case. Plot PMF and DF.
   (i) $f(x) = A\left(\dfrac{1}{4}\right)^x$, $x = 1, 2, 3, 4, \ldots$ (ii) $f(x) = \begin{cases} A\begin{pmatrix} 6 \\ x \end{pmatrix}\left(\dfrac{1}{4}\right)^x\left(\dfrac{3}{4}\right)^{6-x}, & x = 0, 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$

   (iii) $f(x) = \begin{cases} A\dfrac{3^x}{x!}, & x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$ (iv) $f(x) = \begin{cases} A\left(\dfrac{2}{3}\right)^{x-1}, & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$

5. Determine the constant A in the following function, so that the function is PDF and find the DF $F(x)$ in each case. Plot PDF and DF.
   (i) $f(x) = \begin{cases} A\left[2 - |2 - x|\right], & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$ (ii) $f(x) = \begin{cases} Ax^{-\frac{1}{2}}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

   (iii) $f(x) = \begin{cases} A\dfrac{1}{\sqrt{4 - x^2}}, & |x| < 2 \\ 0 & \text{otherwise} \end{cases}$ (iv) $f(x) = \begin{cases} \dfrac{A}{x}e^{-\dfrac{(\log x)^2}{2}}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$

   (v) $f(x) = Ae^{-|x|}$, $-\infty < x < \infty$ (vi) $f(x) = \begin{cases} Ax^2, & 1 \le x \le 2 \\ Ax & 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

   (vii) $f(x) = Ae^{-\frac{\left(\frac{x-2}{2}\right)^2}{2}}$, $-\infty < x < \infty$.

6. A box contains good and defective items. If an item drawn is good , we assign the number 1 to the drawing, otherwise the number 0. Let p be the probability of the drawing at random a good item. Then find the DF $F(x)$.

7. Suppose that $P(X \geq x)$ is given for a RV X (of the contineous type) for all x. Find the PDF in each of the following cases:
(i) $P(X \geq x) = 1$ if $x < 0$, and $= \left(1 + \frac{x}{\lambda}\right)^{-\lambda}$if $x \geq 0, \lambda > 0$ us a constant.

(ii) $P(X \geq x) = 1$ if $x < x_0$, and $= \left(\frac{x_0}{x}\right)^{\alpha}$ if $x \geq x_0, x_0 > 0$ and $\alpha > 0$ are constants.

8. Suppose one is told that the time, one has to wait (in mins.) for a bus on a certain bus stop is a random phenomenon, with the probability function specified by the PDF $f(x)$, given by
(i) $f(x) = \begin{cases} 4x - 2x^2 - 1 & \text{for } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$ (ii) $f(x) = \begin{cases} 4x - 2x^2 & \text{for } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$
(iii) $f(x) = \begin{cases} \frac{3}{8}\left(4x - 2x^2\right) & \text{for } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$.
Is it possible for $f(x)$ to be a PDF, if so find the probability that one has to wait for a bus is (a) more than 1 min. (b) less than 1 min.

9. The length of time (in minutes) that a certain young lady speaks on the telephone is found to be a random phenomenon with the probability function specified by the PDF $f(.)$, given by

$$f(x) = \begin{cases} Ae^{-\frac{x}{5}}, & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

(i) Find the value of A that makes f a PDF. (ii) Show that the telephone conversation will last more than $a + b$ minutes given that it has lasted at least a minutes is equal to the unconditional probability that it will last more than b minutes.

10. The number of times a certain equipment operates before having to be discarded is found to be a random phenomenon with the probability function specified by the PMF $p(.)$, given by

$$p(x) = \begin{cases} A\left(\frac{1}{3}\right)^x, & x = 0, 1, 2, 3, ... \\ 0 & \text{otherwise} \end{cases}.$$

(i) Find the value of A that makes $p(.)$ the PMF. (ii) What is the probability that equipment will operate more than 15 times given that it has operated more than 5 times.

11. For a certain random variable X the PDF $f(.)$ is given by $f(x) = \dfrac{k}{1 + x^2}, \ -\infty < x < \infty$.
(i) Find the value of k that makes $f(.)$ the PDF and the DF $F(x)$ of X. (ii) Find $P\left(\frac{1}{3} < X^2 < 1\right)$.

12. For a certain random variable(RV) X the DF F is given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ k(x-1)^4 & \text{if } 1 < x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}.$$

(i) Find the value of k which makes F a DF. (ii) Find the PDF $f(x)$ and $P\{1.5 \leq X \leq 3.5\}$.