Assignment - I

1) $\tan x = 0$ in $[1.5, 1.6]$

$a = 1.5$ and $b = 1.6$

$f(a) \cdot f(b) = f(1.5) \cdot f(1.6) < 0$ but,

$\tan x$ is ~~cos~~ not continuous at $\frac{\pi}{2} \approx 1.57$

So, we can't solve the equ^n $\tan x = 0$

2) $f(\cancel{x}\,h) = 3\cos^{-1}(1-h) - 3(1-h)\sqrt{2h-h^2} - 1 = 0$

Lets, its root lies between $[0, 0.5]$

$a = 0$, $b = 0.5$

$$x_1 = \frac{a+b}{2} = \frac{0+0.5}{2} = 0.25$$

$f(a) \cdot f(x_1) > 0$ and $f(x_1) \cdot f(b) < 0$

So, Roots lies between $[x_1, b] \equiv [0.25, 0.5]$

$$x_2 = \frac{a+x_1}{2} = 0.375$$

$f(b) \cdot f(x_2) > 0$ and $f(x_2) \cdot f(x_1) < 0$

So, Roots lies between $[x_1, x_2] \equiv [0.25, 0.375$

$$x_3 = \frac{x_1+x_2}{2} = 0.3125$$

$f(x_1) \cdot f(x_3) > 0$ and $f(x_3) \cdot f(x_2) < 0$

So, Roots lies between $[x_3, x_2] \equiv [0.3125, 0.375]$

$$x_4 = \frac{x_2+x_3}{2} = 0.344$$

$f(x_3)\cdot f(x_4) < 0$ and $f(x_1)\cdot f(x_2) > 0$
So, Roots lies between $[x_3, x_4] \equiv [0.3125, 0.344]$

$$x_5 = \frac{x_3 + x_4}{2} = 0.3285$$

$f(x_3)\cdot f(x_5) < 0$ and $f(x_4)\cdot f(x_5) > 0$
So, Roots lies between $[x_3, x_5] \equiv [0.313, 0.329]$

$$x_6 = \frac{x_3 + x_5}{2} = 0.321$$

$f(x_3)\cdot f(x_6) > 0$ and $f(x_6)\cdot f(x_5) < 0$
So, Root lies between $[x_6, x_5] \equiv [0.321, 0.329]$

$$x_7 = \frac{x_6 + x_5}{2} = 0.325$$

$\therefore$ Root of this eq^n is 0.325

3) $x^3 + 4x^2 - 10 = 0$ in $[1, 2]$
$a = 1,\ b = 2$

$$x_1 = \frac{a+b}{2} = 1.5$$

$f(a)\cdot f(x_1) < 0$ and $f(x_1)\cdot f(b) > 0$
So, Roots lies between $[a, x_1] \equiv [1, 1.5]$

$$x_2 = \frac{a + x_1}{2} = 1.25$$

$f(a)\cdot f(x_2) > 0$ and $f(x_2)\cdot f(x_1) < 0$
So, Roots lies between $[x_2, x_1] \equiv [1.25, 1.5]$

$$x_3 = \frac{x_1 + x_2}{2} = 1.375$$

$f(x_2) \cdot f(x_3) < 0$ and $f(x_3) \cdot f(x_1) > 0$

Roots lies between $[x_2, x_3] \equiv [1.25, 1.375]$

$$x_4 = \frac{x_2 + x_3}{2} = 1.313$$

~~Roots~~ $f(x_2) \cdot f(x_4) > 0$ and $f(x_4) \cdot f(x_3) < 0$

Root lies between $[x_4, x_3] \equiv [1.313, 1.375]$

$$x_5 = \frac{x_4 + x_3}{2} = 1.344$$

$f(x_4) \cdot f(x_5) > 0$ and $f(x_5) \cdot f(x_3) < 0$

Root lies between $[x_5, x_3] \equiv [1.344, 1.375]$

$$x_6 = \frac{x_5 + x_3}{2} = 1.360$$

$f(x_5) \cdot f(x_6) > 0$ and $f(x_6) \cdot f(x_3) < 0$

Root lies between $[x_6, x_3] \equiv [1.36, 1.375]$

$$x_7 = \frac{x_6 + x_3}{2} = 1.368 \approx 1.37$$

$\therefore$ Root of equ^n $(x^3 + 4x^2 - 10 = 0)$ is 1.37

4) (i) $e^x - 3x = 0 \Rightarrow x = \underbrace{\frac{e^x}{3}}_{g(x)}$

$g(x) = \frac{e^x}{3} \Rightarrow g'(x) = \frac{e^x}{3} < 1 \quad \forall x \in [0, 1]$

$g(x) = \frac{e^x}{3}$

Lets, $x_0 = 0.5$

$x_1 = g(x_0) = g(0.5) = \frac{e^{0.5}}{3} = 0.55$

$x_2 = g(x_1) = g(0.55) = 0.578$

$x_3 = g(x_2) = g(0.578) = 0.594$

$x_4 = g(x_3) = g(0.594) = 0.604$

$x_5 = g(x_4) = g(0.604) = 0.610$

$x_6 = g(x_5) = g(0.61) = 0.613$

$\therefore$ Root of equ^n $(e^x - 3x = 0)$ is 0.61

(ii) $x^3 - 3x + 1 = 0$, $a = 1, b = 2$

$x = \underbrace{\frac{x^3+1}{3}}_{g(x)}$

$g(x) = \frac{x^3+1}{3} \Rightarrow g'(x) = x^2 \nless 1$ for $[1, 2]$

$x = \underbrace{(3x-1)^{1/3}}_{g(x)} \Rightarrow g'(x) = \frac{1}{\cancel{3}}(3x-1)^{-2/3} \cdot \cancel{3}$

$\Rightarrow g'(x) = \frac{1}{(3x-1)^{2/3}} < 1 \ \forall [1, 2]$

So, $g(x) = (3x-1)^{1/3}$

Lets, $x_0 = 1.5$

$x_1 = g(x_0) = g(1.5) = 1.518$
$x_2 = g(x_1) = g(1.518) = 1.526$
$x_3 = g(x_2) = g(1.526) = 1.529$
$x_4 = g(x_3) = g(1.529) = 1.531$
$x_5 = g(x_4) = g(1.531) = 1.532$
$x_6 = g(x_5) = g(1.532) = 1.532 \approx 1.53$

$\therefore$ Root of equ^n $(x^3 - 3x + 1 = 0)$ is 1.53

5) $0.5 \sin x - x + 3 = 0 \Rightarrow f(x) = 0.5 \sin x - x + 3$
$\Rightarrow \quad x = \underbrace{0.5 \sin x + 3}_{g(x)}$ (It is continuous)

$f(0) \cdot f(2\pi) < 0$. So, it has atleast one soln.
$g(x) = 0.5 \sin x + 3$
$\Rightarrow g'(x) = 0.5 \cos x < 1 \ \forall \ x \in [0, 2\pi]$

$g(x) = 0.5 \sin x + 3$
Lets, $x_0 = \pi$
$x_1 = g(x_0) = g(\pi) = 3$
$x_2 = g(x_1) = g(3) = 3.071$
$x_3 = g(x_2) = g(3.071) = 3.035$
$x_4 = g(x_3) = g(3.035) = 3.053$
$x_5 = g(x_4) = g(3.053) = 3.044$
$x_6 = g(x_5) = g(3.044) = 3.049$
$x_7 = g(x_6) = g(3.049) = 3.046$
$x_8 = g(x_7) = g(3.046) = 3.048 \approx 3.05$

$\therefore$ Root of equ^n $(0.5 \sin x - x + 3 = 0)$ is 3.05

$$x_{n+1} = \underbrace{6 - \frac{5}{x_n}}_{g(x)}$$

$$f(x) = x - g(x) = x - 6 + \frac{5}{x} = \frac{x^2 - 6x + 5}{x}$$

~~x_{n+1} will converges to root of f(x)~~

$$|g'(x)| = \frac{5}{x^2} < 1 \quad \forall\ x \in (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$$

Roots of $f(x)$ is 5 $> \sqrt{5}$.

So, x_{n+1} converges to root of $x^2 - 6x + 5$

(ii) $$x_{n+1} = \underbrace{\frac{x_n^2 + 5}{6}}_{g(x)}$$

$$f(x) = g(x) - x = \frac{x^2 + 5}{6} - x = \frac{x^2 - 6x + 5}{6}$$

$$|g'(x)| = \left|\frac{x}{3}\right| < 1 \quad \forall\ -3 < x < 3$$

$\Rightarrow$ $|g'(x)| < 1 \quad \forall\ -3 < x < 3$

Roots of $f(x)$ is 1 < 3

So, x_{n+1} converges to root of $x^2 - 6x + 5$

(iii) $$x_{n+1} = \underbrace{\sqrt{6x_n - 5}}_{g(x)}$$

$$f(x) = x - \sqrt{6x - 5}$$

$$|g'(x)| = \frac{3}{\sqrt{6x-5}} = \frac{3}{\sqrt{6x-5}} < 1$$

$$6x - 5 > 9 \Rightarrow x > \frac{7}{3}$$

$f(x)$ converges for ~~x~~ $x > \frac{7}{3}$

$f(x) = 0 \Rightarrow \quad x = \sqrt{6x-5} \Rightarrow x^2 - 6x + 5 = 0$

So, Roots of $x^2 - 6x + 5$ is $5 > 7/3$

Hence, x_{n+1} converges to root of $x^2 - 6x + 5 = 0$

7) $$x_{n+1} = \underbrace{x_n^2 - \frac{1}{x_n}}_{g(x)}$$

$$f(x) = x - g(x)$$
$$= x - x^2 - \frac{1}{x}$$
$$= \frac{x^2 - x^3 - 1}{x}$$

$$|g'(x)| = \left|2x + \frac{1}{x^2}\right| > 1 \ \forall \ x \in (1, 2)$$

So, It will not converge.

8) $\sqrt[3]{25}$

$x^3 = 25$

Bisection method:

In $x \in [2, 3]$

$a = 2, \quad b = 3$

$x_1 = \frac{2+3}{2} = 2.5$

$f(a) \cdot f(x_1) > 0 \qquad f(x_1) \cdot f(b) < 0$

Root lies between $[x_1, b] \equiv [2.5, 3]$

$$x_2 = \frac{x_1 + b}{2} = \frac{5.5}{2} = 2.75$$

$f(x_1)\cdot f(x_2) > 0$ $\quad$ $f(x_2)\cdot f(b) < 0$

Root lies between $[x_2, b] \equiv [2.75, 3]$

$$x_3 = \frac{x_2+b}{2} = \frac{2.75+3}{2} = 2.875$$

$f(x_2)\cdot f(x_3) > 0$ $\quad$ $f(x_3)\cdot f(b) < 0$

Root lies between $[x_3, b] \equiv [2.875, 3]$

$$x_4 = \frac{x_3+b}{2} = 2.938$$

$f(x_3)\cdot f(x_4) < 0$ $\quad$ $f(x_4)\cdot f(b) > 0$

Root lies between $[x_3, x_4] \equiv [2.875, 2.938]$

$$x_5 = \frac{x_3+x_4}{2} = 2.907$$

$f(x_3)\cdot f(x_5) > 0$ $\quad$ $f(x_5)\cdot f(x_4) < 0$

Root lies between $[x_5, x_4] \equiv [2.907, 2.938]$

$$x_6 = \frac{x_5+x_4}{2} = 2.923$$

Newton-Raphson method:

$f(x) = x^3 - 25$ in $[2, 3]$

$f(x)$ is differentiable function $f'(x) = 3x^2$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Lets, $x_0 = 2.5$

$$x_1 = \cancel{2.5}\; x_0 - \left(\frac{x_0^3 - 25}{3x_0^2}\right)$$

$\Rightarrow$ $x_1 = 3$
$x_2 = 2.926$
$x_3 = 2.924$

Bisection – 6 iteration
Newton – 3 iteration

9) $f(x) = (x-1)^3(x-2) = x^4 - 5x^3 + 9x^2 - 7x + 2$
$f'(x) = 4x^3 - 15x^2 + 18x - 7$
Using Newton – Raphson method
$x_0 = 0.5$,
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\Rightarrow$ $x_1 = 0.65$
$x_2 = 0.757$
$x_3 = 0.833$
$x_4 = 0.886$
$x_5 = 0.923$
$x_6 = 0.948$
$x_7 = 0.965$
$x_8 = 0.977$
$x_9 = 0.984$
$x_{10} = 0.989$
$x_{11} = 0.993$
$x_{12} = 0.995$
$x_{13} = 0.997$
$x_{14} = 0.998$
$x_{15} = 0.999$
$x_{16} = 0.999 \approx 1$

So, This converges to root of $f(x)$ i.e. 1

$x_0 = 1.5$,
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\Rightarrow$ $x_1 = 1.25$
$x_2 = 1.156$
$x_3 = 1.1$
$x_4 = 1.066$
$x_5 = 1.043$
$x_6 = 1.029$
$x_7 = 1.019$
$x_8 = 1.013$
$x_9 = 1.008$
$x_{10} = 1.006$

$x_{11} = 1.004$
$x_{12} = 1.002$
$x_{13} = 1.002$
$x_{14} = 1.001$
$x_{15} = 1.001$
$x_{16} = 1.000$

So, This converges to root of $f(x)$. i.e. 1

$x_0 = 2.5,$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\Rightarrow$ $x_1 = 2.25$ $\quad x_2 = 2.094$
$x_3 = 2.019$ $\quad x_4 = 2.001$
$x_5 = 2.000$

So, This converges to root of $f(x)$ i.e. 2

Newton Raphson method for multiple root:

$$g(x) = \frac{f(x)}{f'(x)} = \frac{(x-1)^3(x-2)}{(x-1)^2\{(x-1)+3(x-2)\}}$$

$$\Rightarrow \quad g(x) = (x-1)\left(\frac{x-2}{4x-7}\right)$$

$$\Rightarrow \quad g(x) = \frac{x^2-3x+2}{4x-7}$$

$$g'(x) = \frac{(4x-7)(2x-3)-(x^2-3x+2)\cdot 4}{(4x-7)^2}$$

$$\Rightarrow \quad g'(x) = \frac{8x^2-26x+21-4x^2+12x-8}{(4x-7)^2}$$

$$\Rightarrow \quad g'(x) = \frac{4x^2 - 14x + 13}{(4x-7)^2}$$

So,
$$\frac{g(x)}{g'(x)} = \frac{(x^2-3x+2)(4x-7)}{(4x^2-14x+13)}$$

Lets, $x_0 = 0.5$

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

$\Rightarrow x_1 = 1.036$

$x_2 = 1.000$

Lets, $x_0 = 1.5$

$x_1 = 1.25$

$x_2 = 1.036$

$x_3 = 1.000$

Lets, $x_0 = 2.5$

$x_1 = 1.75$

$x_2 = 1.75$ (Fails)

10) $$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \dots$$

$f(0) = 1 > 0$

$f(1) > 0$

$f(2) < 0$

$\therefore f(1) \cdot f(2) < 0$

So, roots lies between $[1, 2]$

$f(x) = g(x) - x$

So, $g(x) = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} + \cancel{\text{term}(x^5 \dots)} \to 0$

$$|g'(x)| = \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{144} < 1 \quad \forall \; x \in [1, 2]$$

Lets, $x_0 = 1.5$

$x_1 = g(x_0)$
$\Rightarrow x_1 = 1.478$
$x_2 = 1.464$
$x_3 = 1.457$
$x_4 = 1.453$
$x_5 = 1.450$
$x_6 = 1.449$
$x_7 = 1.448$
$x_8 = 1.447$
$x_9 = 1.447 \approx 1.45$

This converges to root of $f(x)$
$\therefore$ Root $= 1.45$

11) $f(x) = \sin x - 5x + 2 = 0$

$f_1(x) = \cancel{x - g(x)}\ x - \sin^{-1}(5x-2)$
$g(x) = \cancel{\ldots}\ \sin^{-1}(5x-2)$

$$|g'(x)| = \frac{5}{\sqrt{1-(5x-2)^2}}$$

$\Rightarrow |g'(0.5)| \not< 1$

So, it do not converge to roots of $f(x)$.

$f_2(x) = x - \frac{1}{5}(\sin x + 2)$
$g(x) = \frac{1}{5}(\sin x + 2)$
$|g'(x)| = |\frac{1}{5}\cos x| < 1 \quad \forall x$

So, it will converge to roots of $f(x)$.

Lets, $x_0 = 0.5$

$x_1 = g(x_0) = g(0.5) = 0.496$

$x_2 = g(x_1) = 0.495$

$x_3 = g(x_2) = 0.495$

12) $x^2 + ax + b = 0$, $\alpha + \beta = -a$, $\alpha\beta = b$

$$f_1(x) = x + \underbrace{\left(\frac{ax+b}{x}\right)}_{-g(x)}$$

$$g'(x) = -\left(\frac{a\cancel{x} - (a\cancel{x}+b)}{x^2}\right)$$

$$\Rightarrow |g'(x)| = \frac{|b|}{x^2} < 1 \quad \forall x \in R - (-\sqrt{b}, \sqrt{b})$$

If $x = \alpha$

$$\frac{|b|}{x^2} < 1$$

$$\Rightarrow \frac{|\alpha\beta|}{|\alpha^2|} < 1$$

$\therefore$ $|\beta| < |\alpha|$

$$f_2(x) = x + \underbrace{\frac{b}{x+a}}_{-g(x)}$$

$$g(x) = \left(\frac{-b}{x+a}\right)$$

$$|g'(x)| = \frac{|b|}{(x+a)^2} < 1$$

If $x = \alpha$

$$|b| < (\alpha + a)^2$$

$$\Rightarrow |b| < (\cancel{\alpha} - \cancel{\alpha} - \beta)^2 \Rightarrow |\alpha\cancel{\beta}| < \beta^2$$

$\therefore$ $|\alpha| < |\beta|$

13) Let $f(x)=0$ be the given eqn. Lets,
$f(x) = g(x) - x$, where g is differentiable
on $[a, b]$ such that $|g'(x)| \leq K < 1$.
Then g possesses a unique fixed point
α where α is a root of $g(x)$ on $[a, b]$

14) $x \log_{10} x = 1.2$

$f(x) = x \log_{10} x - 1.2$. This is a continous function

$f(1) < 0$
$f(3) > 0$
$f(1) \cdot f(3) < 0$
So, $f(x)$ has root in $[1, 3]$
Lets, $x_0 = 1$, $x_1 = 3$

$$x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) f(x_1)$$

$\Rightarrow \quad x_2 = 2.67767$

$x_0 \qquad x_2 \qquad x_1$

$f(x_0) \cdot f(x_2) > 0 \qquad f(x_2) \cdot f(x_1) < 0$

Root lies between $[x_2, x_1] \equiv [2.6767, 3]$

$$x_3 = x_1 - \left(\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right) f(x_1)$$

$\Rightarrow \quad x_3 = 2.7392$

~~Root lies between [x~~

$x_2 \qquad x_2 \qquad x_1$

$f(x_2) \cdot f(x_3) > 0 \qquad f(x_3) \cdot f(x_1) < 0$

Root lies between $[x_3, x_1] \equiv [2.7392, 3]$

$x_4 = 2.7406$

$x_3 \qquad x_4 \qquad x_1$

$f(x_3) \cdot f(x_4) > 0 \qquad f(x_4) \cdot f(x_1) < 0$

Root lies between $[x_4, x_1] \equiv [2.7406, 3]$

$x_5 = 2.7406$

$\therefore$ Root of $f(x) \equiv x \log_{10} x - 1.2$ is 2.7406

15) $x^2 e^{-x/2} = 1$ in $[0, 2]$

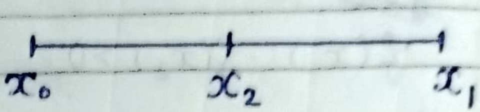
$f(x) = x^2 e^{-x/2} - 1$

$f(0) \cdot f(2) < 0$

Lets, $x_0 = 0$, $x_1 = 2$

$$x_2 = x_0 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) \cdot f(x_0)$$

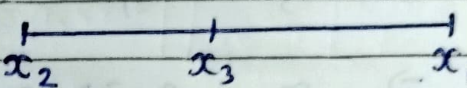
$\Rightarrow$ $x_2 = 1.359$



$f(0) \cdot f(x_2) > 0$ $\qquad$ $f(x_2) \cdot f(x_1) < 0$

Root lies between $[x_2, x_1] \equiv [1.359, 2]$

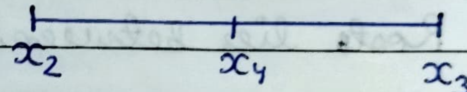
$x_3 = 1.435$



$f(x_2) \cdot f(x_3) < 0$ $\qquad$ $f(x_3) \cdot f(x_1) > 0$

Root lies between $[x_2, x_3] \equiv [1.359, 1.435]$

$x_4 = 1.4296$



$f(x_2) \cdot f(x_4) > 0$ $\qquad$ $f(x_4) \cdot f(x_3) < 0$

Root lies between $[x_4, x_3] \equiv [1.4296, 1.435]$

$x_5 = 1.42961 \approx 1.4296$

Root of $f(x) = x^2 e^{-x/2}$ is 1.4296

16) $F(x, y) = 0.2x^2 + 0.8$

Lets, (α, β) be the soln of $F(x, y)$

$\alpha - x_0 = h$, $\qquad$ $\beta - y_0 = K$

$\Rightarrow$ $\alpha = h + x_0$, $\qquad$ $\beta = K + y_0$

$$F(x_0 + h,\ y_0 + k) = 0$$

$$\Rightarrow F(x_0, y_0) + h \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} + k \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \approx 0$$

$$\Rightarrow \cancel{0.2x_0^2 + 0.8 + h \cdot 0.4x_0 + k \cdot 0 \approx 0}$$

$$h \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} + k \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} = -F(x_0, y_0) \quad \text{— (I)}$$

$$G(x,y) = 0.3xy^2 + 0.7$$

Lets, (α, β) be the soln of $G(x,y)$

$$\alpha - x_0 = h, \qquad \beta - y_0 = k$$

$$\Rightarrow \alpha = h + x_0, \qquad \beta = k + y_0$$

$$G(x_0 + h,\ y_0 + k) = 0$$

$$\Rightarrow G(x_0, y_0) + h \left.\frac{\partial G}{\partial x}\right|_{(x_0,y_0)} + k \left.\frac{\partial G}{\partial y}\right|_{(x_0,y_0)} \approx 0$$

$$\Rightarrow h \left.\frac{\partial G}{\partial x}\right|_{(x_0,y_0)} + k \left.\frac{\partial G}{\partial y}\right|_{(x_0,y_0)} = -G(x_0, y_0) \quad \text{— (II)}$$

Write equ^n (I), (II) in matrix form

$$\left.\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}\right|_{(x_0,y_0)} \begin{bmatrix} h \\ k \end{bmatrix} = -\begin{bmatrix} F(x_0,y_0) \\ G(x_0,y_0) \end{bmatrix}$$

$$\Rightarrow \left.\begin{bmatrix} 0.4x & 0 \\ 0.3y^2 & 0.6xy \end{bmatrix}\right|_{(x_0,y_0)} \begin{bmatrix} h \\ k \end{bmatrix} = -\left.\begin{bmatrix} 0.2x^2 + 0.8 \\ 0.3xy^2 + 0.7 \end{bmatrix}\right|_{(x_0,y_0)}$$

$$\Rightarrow \begin{bmatrix} h \\ k \end{bmatrix} = -\begin{bmatrix} 0.4x & 0 \\ 0.3y^2 & 0.6xy \end{bmatrix}^{-1}\Bigg|_{(x_0,y_0)} \begin{bmatrix} 0.2x^2+0.8 \\ 0.3xy^2+0.7 \end{bmatrix}\Bigg|_{(x_0,y_0)}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} h+x_0 \\ k+y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} 0.4x & 0 \\ 0.3y^2 & 0.6xy \end{bmatrix}^{-1}\Bigg|_{(x_0,y_0)} \cdot \begin{bmatrix} 0.2x^2+0.8 \\ 0.3xy^2+0.7 \end{bmatrix}\Bigg|_{(x_0,y_0)}$$

Lets,

17) $15^{1/3}$

$x^3 - 15 = 0$ $\quad f(x) = x^3 - 15$, $f'(x) = 3x^2$

$f(x)$ is continuous and differentiable function

Using Newton-Raphson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

~~$x_1 =$~~ Lets, $x_0 = 2.5$ in $[2, 3]$

$x_1 = 2.4$[redacted]667

$x_2 = 2.4$[redacted]66

$\therefore \quad 15^{1/3} = 2.466$

18)

* 19) $x_{n+1} = p x_n + q \frac{N}{x_n^2}$

$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(p x_n + q \frac{N}{x_n^2} \right)$

$\Rightarrow \quad l = pl + q \frac{N}{l^2}$

$\Rightarrow \quad (1-p) l = \frac{qN}{l^2}$

$\therefore \quad l = \left(\frac{qN}{1-p} \right)^{1/3}$

Given, $l = N^{1/3} = \alpha \Rightarrow \alpha^3 = N$

$\Rightarrow \cancel{N^{1/3}} = \left(\frac{q}{1-p} \right)^{1/3} \cdot \cancel{N^{1/3}}$

$\Rightarrow \quad 1 - p = q$

$\Rightarrow \quad p + q = 1 \qquad - (1)$

$x_n - \alpha = e_n \Rightarrow x_n = \alpha + e_n$

$\alpha + e_{n+1} = p(\alpha + e_n) + qN(\alpha + e_n)^{-2}$

$\Rightarrow \alpha + e_{n+1} = \alpha p + p e_n + \frac{qN}{\alpha^2} \left(1 + \frac{e_n}{\alpha} \right)^{-2}$

$\Rightarrow \alpha + e_{n+1} = \alpha p + p e_n + \frac{qN}{\alpha^2} \left\{ 1 - \frac{2e_n}{\alpha} + \frac{3e_n^2}{\alpha^2} - \dots \right\}$

$\Rightarrow \alpha + e_{n+1} = \alpha p + p e_n + \frac{qN}{\alpha^2} - \frac{2qN}{\alpha^3} e_n + \frac{qN}{\alpha^4} e_n^2$

$\Rightarrow \alpha + e_{n+1} = \left(\alpha p + \frac{qN}{\alpha^2} \right) + \left(p - \frac{2qN}{\alpha^3} \right) e_n + \frac{qN}{\alpha^4} e_n^2$

Let $N = \alpha^3$

$$\alpha + e_{n+1} = (\alpha p + q\alpha) + (p - 2q)e_n + \frac{q}{\alpha} e_n^2$$

For order = 2

$$\alpha + e_{n+1} = (p+q)\alpha + \underbrace{(p-2q)}_{=0} e_n + \frac{q}{\alpha} e_n^2$$

$$p + q = 1 \quad \text{and} \quad p - 2q = 0$$

So, $\quad 1 - q - 2q = 0$

$$\Rightarrow \quad q = 1/3 \quad , \quad p = 2/3$$

$$\lim_{n \to \infty} \frac{e_{n+1}}{e_n^2} = \frac{q}{\alpha} = \frac{1}{3\alpha}$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^2} = \frac{1}{3\alpha}$$