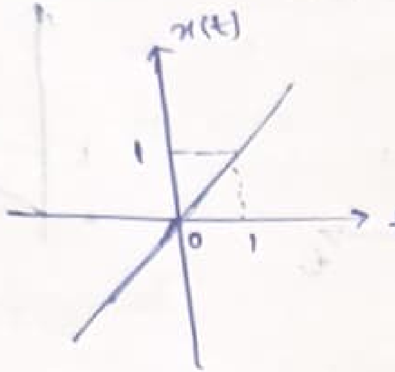
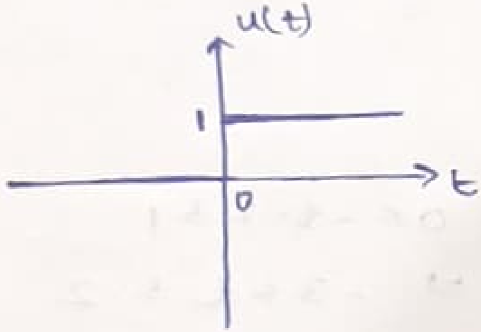
Signals and System Analysis

* $x(t) = t$

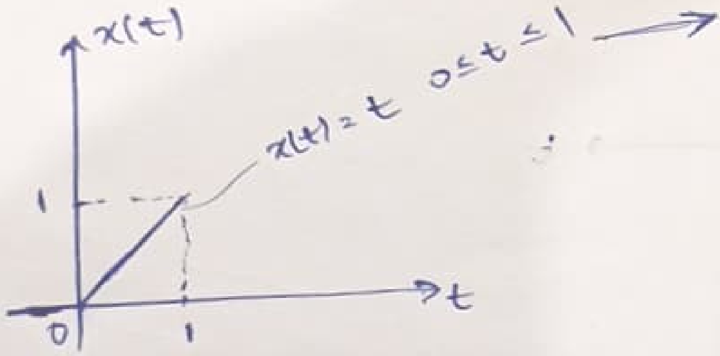


* $x(t) = u(t) = \begin{cases} 1 \; ; \; t > 0 \\ 0 \; ; \; t < 0 \end{cases}$ → unit STEP Signal

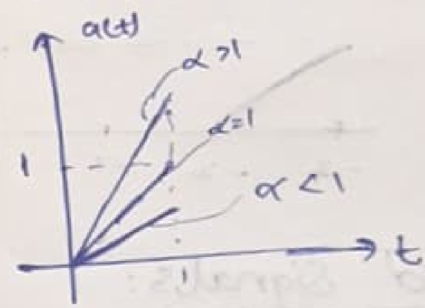


t at '0' is not defined

*

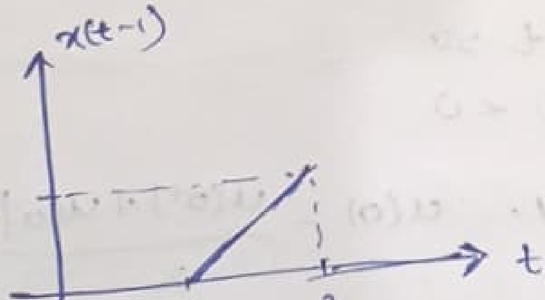


$a(t) = \alpha\, x(t)$

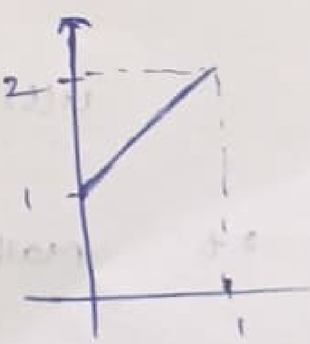


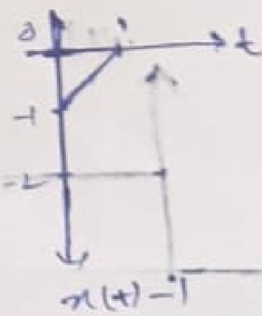
* $x(t-1) = t-1 \quad 0 \le t-1 \le 1$
$\rightarrow 1 \le t \le 2$

Delay

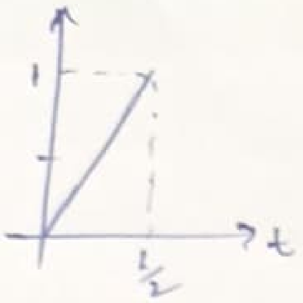
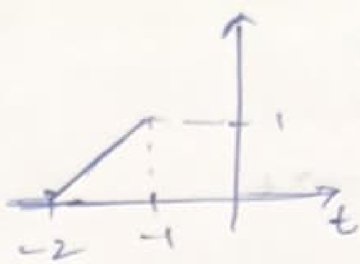
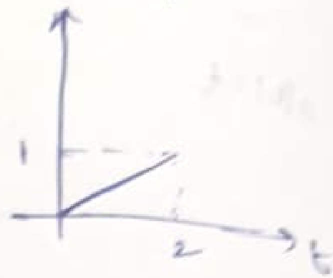


* $x(t) + 1$ * $x(t) - 1$

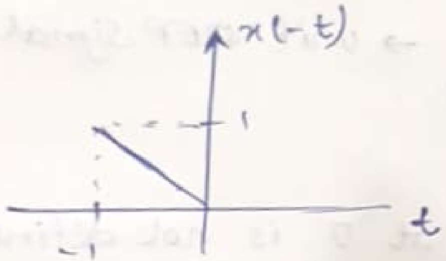




* $x(2t)$ * $x(t+2)$ ← Advanced   * $x(0.5t)$

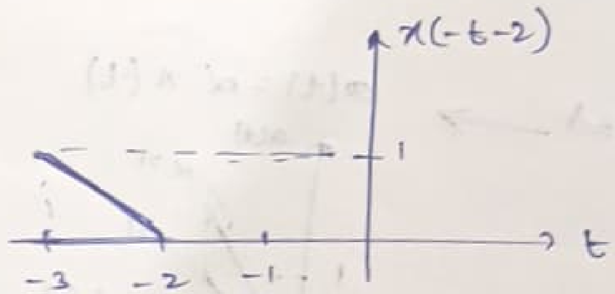




* $x(-t) = -t \quad 0 \le -t \le 1 \Rightarrow 0 \ge t \ge -1$



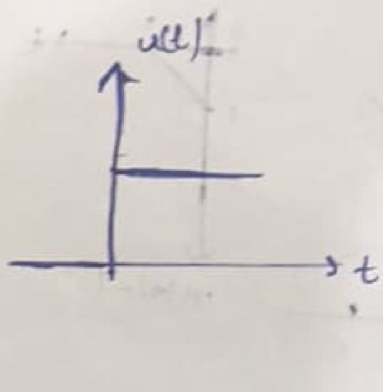
* $y(t) = x(-t-2) = -t-2 \quad 0 \le -t-2 \le 1$

$\Rightarrow -3 \le t \le -2$



Standard Signals:

1) Unit step signal :



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

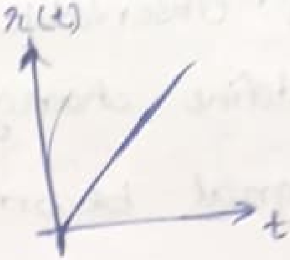
Mathematically, $u(0) = \dfrac{u(0^+) + u(0^-)}{2}$

$= \frac{1}{2}$

At Engineering → Value at '0' is appearing for zero time

2) Ramp signal:

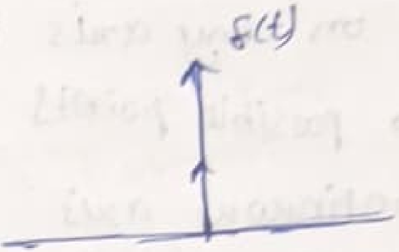
$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



3) Impulse Signal

$\delta(t) = 0 \quad t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 = \int_{0^-}^{0^+} \delta(t)\, dt$$



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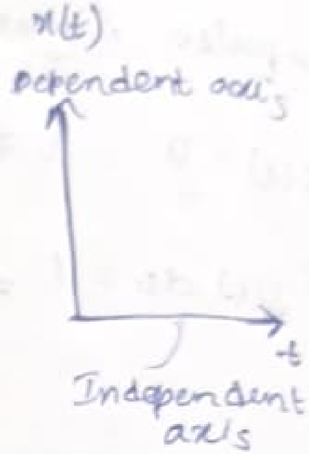
<u>Signals</u>: Observable change in any physical thing.

We define change w.r.t time. (t)

So signal becomes some function of time

<u>ex</u>: $x(t) \rightarrow f(t)$



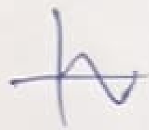
* If on any axis in ANY interval
→ ∞ possible points
continuous axis
finite
Discrete axis

Neighbour can be defined for all points.

there are no continuous signal or discrete signal.

A - Analog
D - digital

Time	Amp	
C	C	- Analog
C	D	- BoxCar
D	C	- sampled ~~dat~~ data
D	D	- ~~discrete~~ digital



* Sampling is a process of converting ~~CA~~ CT to DT

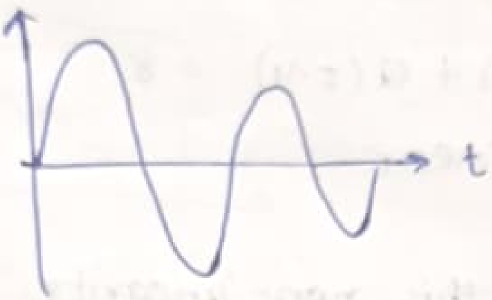
* Quantization → CA to DA

*. A to D conversion (ADC)
↳ sampling followed by Quantization.

( ) → continuous

[ ] → discrete

ex: $x(t) = 2e^{-0.3t} \sin t \quad t \geq 0$: find $x[nT]$



for $T = 0.5$ sec

$x(t = 0 \cdot T) = x[0T] = x[0] \rightarrow x(0)$

$x(t = 1 \cdot T) = x[T] = x[1] \rightarrow x(0.5)$ / $x(1)$

$\vdots$

$x(t = n \cdot T) = x[nT] = x[n]$

Assume $T = 0.5$ sec

$x[0] = x(0) = 0$

$x[1] = x(0.5) = 0.8253$

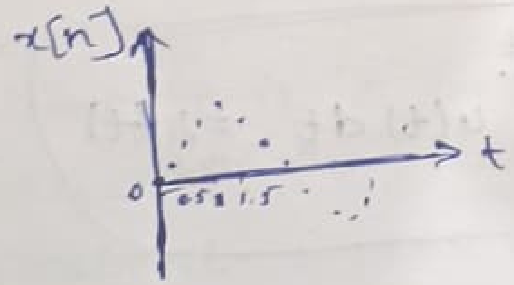
$x[2] = x(1) = 1.2467$

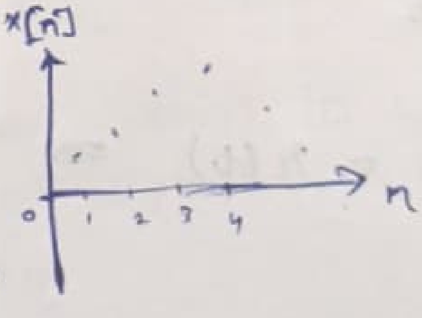
$x[3] = x(1.5) = 1.2721$

$x[4] = x(2) = 0.9981$

$x[5] = x(2.5) = 0.5654$

$x[n] = \{0, 0.8253, 1.2467, 1.2721, \ldots\}$





both are same if $T = 1$ sec

Quantization:

$Q(3.2) = 3$, $Q(6.7) = 7$ $Q(5.4) = 5$

$Q(6.7 + 5.4) = 12 = Q(6.7) + Q(5.4)$

$Q(3.2 + 5.4) = 9 \neq Q(3.2) + Q(5.4) = 8$

Quantization is Non-linear

$\Downarrow$

Digital signal incorporates this non-linearity

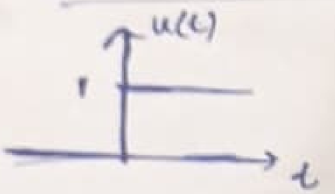
Elementary signals:

+ve sided signals → exists only for $t \geq 0$

-ve sided signal → exists only for $t \leq 0$

Standard signals:

Unit step signal



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

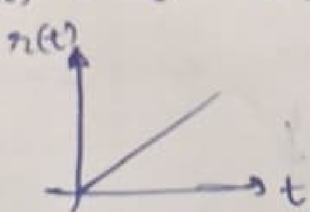
Mathematically

$$u(0) = \frac{u(0^+) + u(0^-)}{2} = \frac{1}{2}$$

At Engineering → Value at '0' is appearing for zero time.

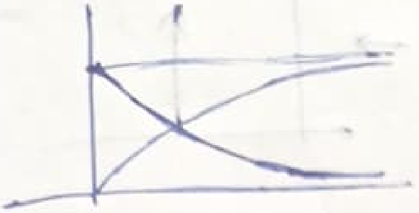
Ramp signal

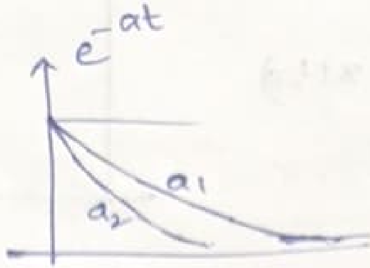
$r(t) = t \quad t \geq 0$



$$\int_{-\infty}^{t} u(t)\,dt = r(t) \Rightarrow \boxed{\int_{0}^{t} u(t)\,dt = r(t)}$$

Real exponential ce^{a}

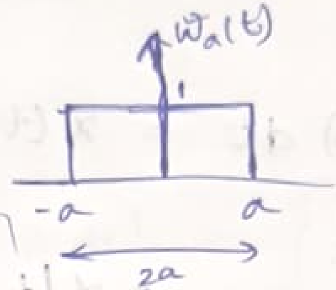


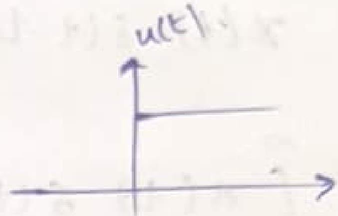


$a_2 > a_1$

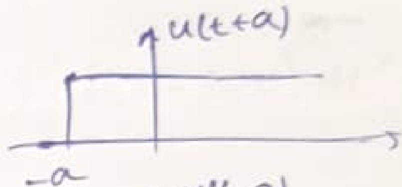
Windows and Pulses:

Two sided windows:

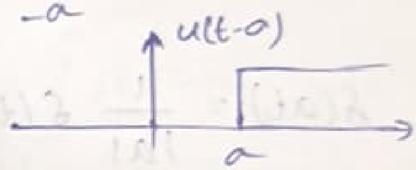




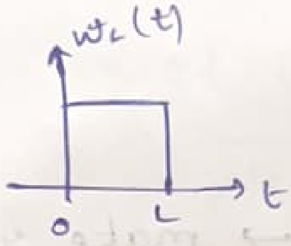
$$w_a(t) = u(t+a) - u(t-a)$$



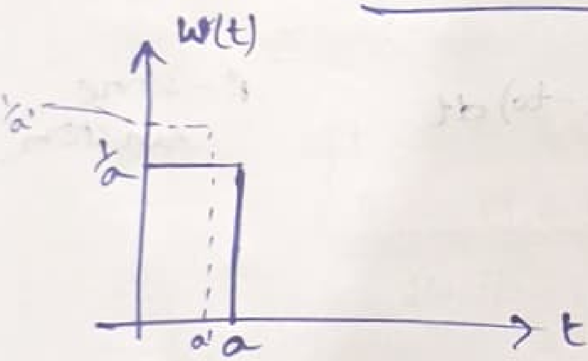
one sided window

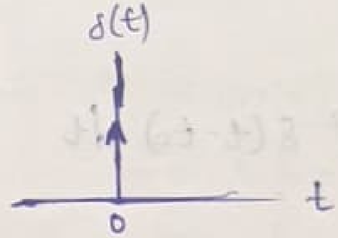


$$w_L(t) = u(t) - u\left(t - \tfrac{L}{a}\right)$$



* A pulse of unit area





$$\boxed{\delta(t) = 0 \quad t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1} \; *$$

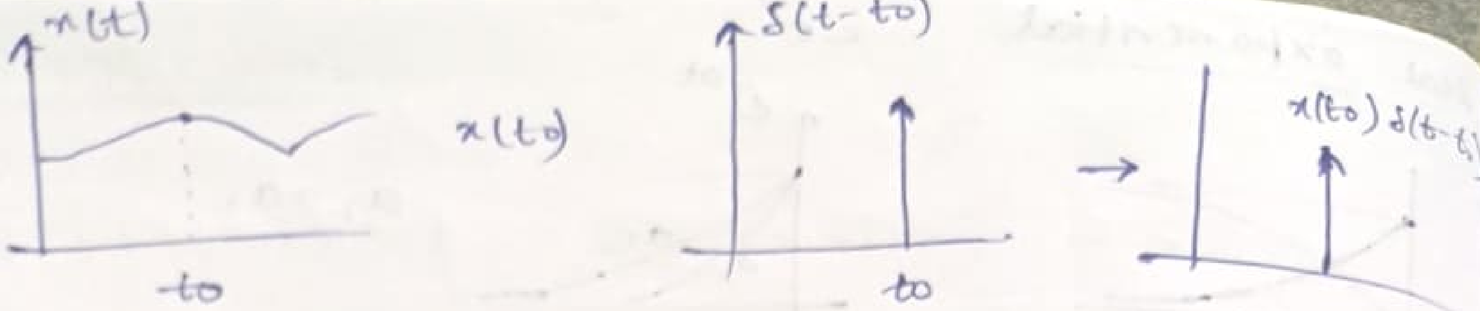
$$\delta(t) = \lim_{a \to 0} w(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

AND Area = 1

→ IMPULSE – not an ordinary function

Dirac Delta function

Impulse is a functional that is used to describe the value of signal at a given instant (ZERO WIDTH)



$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

$$\int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty}\delta(t-t_0)\,dt$$

$$= x(t_0)$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

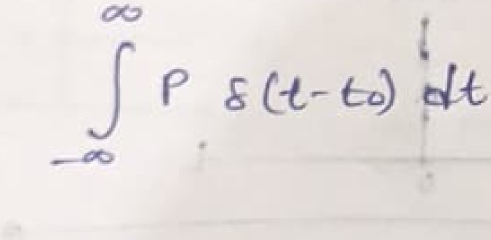
Area calculation → make sure integration touches the impulse

$$\int_{-\infty}^{\infty} P\,\delta(t-t_0)\,dt = \int_{t_0^-}^{t_0^+} P\,\delta(t-t_0)\,dt$$

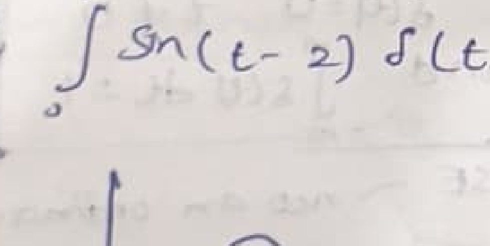
P – some function

eg: $\int_{0}^{\infty} \sin(t-2)\,\delta(t-3)\,dt$

$t_0 = 3$



$$\int_{0}^{\infty} \sin(t-2)\,\delta(t-3)\,dt$$



$$= \sin(3-2)$$

$$= \sin(1)$$

eg: $\int_{-\infty}^{\infty} x(t-\tau)\,\delta(\tau-3)\,d\tau$

$= \int_{-\infty}^{\infty} x(t-3)\,\delta(\tau-3)\,d\tau = x(t-3)$

eg. $\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau-3)\,d\tau$

$-\tau = t'$

$\int_{-\infty}^{\infty} x(-t')\,\delta(t+t'-3)\,dt' = \int_{-\infty}^{\infty} x(-t')\,\delta(t'-(3-t))\,dt'$

$= x(-(3-t)) \int_{-\infty}^{\infty} \delta(t'-(3-t))\,dt'$

$= x(t-3)$

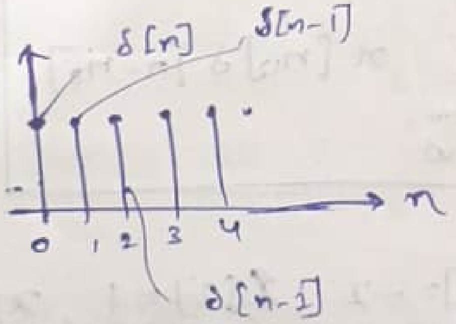
Elementry DT sequence :

i, step sequence $\quad x[n] \triangleq x(nT)$

$n = 0, 1, 2 \ldots$

$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n \leq 0 \end{cases}$

T is fixed for all 'n' and is determined by sample.



$\because u[n-1]$ = is possible but $u[n-0.7]$ ✗ not possible

$$\boxed{u[n] = \sum_{n_0=0}^{\infty} \delta[n-n_0]}$$

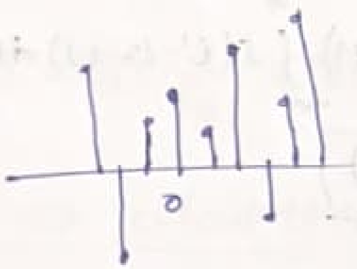
ii) DT impulse

$\delta[n] = 1 \quad n=0$
$\quad\quad\;\; 0 \quad \text{elsewhere}$

$$\boxed{\delta[n] = u[n] - u[n-1]}$$

differentiation in CT ⇒ difference in DT
integration in CT ⇒ Summation in DT

* An arbitary sequence



→ $x[n]u[n] + x[n]u[-n] - x[n]\delta[n]$

$x[n]u[n]$ ↓ for +ve side

$x[n]u[-n]$ ↓ for -ve side

$x[n]\delta[n]$ ↓ ∵ At '0' the value is there twice

Also :- $x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \dots$

$$\Rightarrow \sum_{n_0=-\infty}^{\infty} x[n_0]\,\delta[n-n_0]$$

$$\therefore x[n] = \sum_{n_0=-\infty}^{\infty} x[n_0] \quad = \boxed{\sum_{n_0=-\infty}^{\infty} x[n_0]\,\delta[n-n_0]}$$

eg : $x[0]=1$; $x[1]=0$; $x[2]=-2$; $x[3]=1$, $x[4]=-3$

sol) $x[n] = x[0]\delta[n-0] + x[1]\delta[n-1] + \dots x[4]\delta[n-4]$

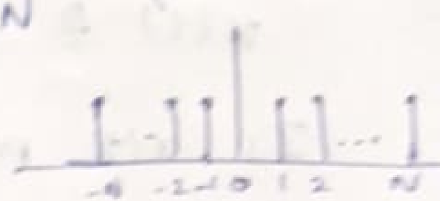
$$\underline{x[n] = \delta[n] - 2\delta[n-2] + \delta[n-3] - 3\delta[n-4]}$$

* For a +ve integer 'N' sequence

$$w_N[n] = \begin{cases} 1 & \forall \; -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

is an window sequence.

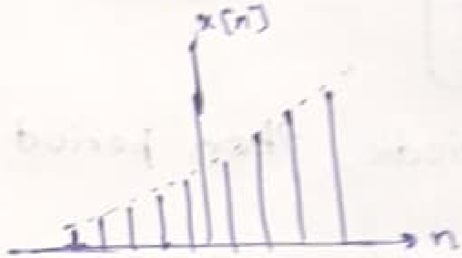
$u[n+N] - u[n-N-1]$?



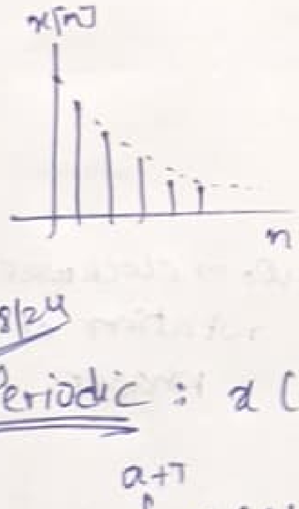
Real exponentials :

$$x[n] = c\alpha^n \;,\; \cancel{\alpha \in t}$$

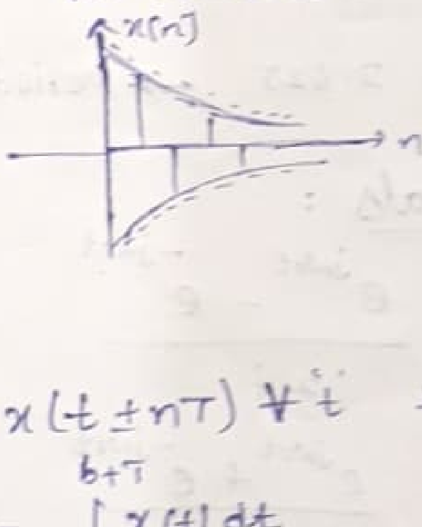
For $\alpha > 1$:



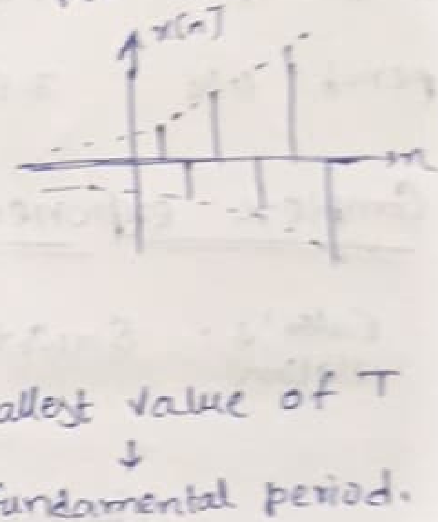
For $0 < \alpha < 1$:



For $-1 < \alpha < 0$:



For $\alpha < -1$



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Periodic : $x(t) = x(t \pm nT) \; \forall t$ smallest value of T $\downarrow$ fundamental period.

$$\int_a^{a+T} x(t)\,dt = \int_b^{b+T} x(t)\,dt$$

For DT signals : $x[n] = x[n+N] \quad \forall n$

* Sum of M CT signals (each periodic with period 'T_i' $i = 1, 2, \ldots M$) is periodic with period T iff

$$\frac{T}{T_i} \rightarrow \text{integer.}$$

Pb: $x_1(t) = x_1(t+T_1)$ $x_2(t) = x_2(t+T_2)$

$x(t) \triangleq x_1(t) + x_2(t)$

Is $x(t)$ periodic?

Soln: for $x(t)$ to be periodic:

$\frac{T}{T_1} = q$ $\frac{T}{T_2} = p$

$\rightarrow \boxed{\frac{T_1}{T_2} = \frac{p}{q}}$

If $x(t)$ is periodic, then period $= LCM(T_1, T_2)$

Pb: $x_1(t)$ $x_2(t)$ $x_3(t)$

period: 4 1.25 $\sqrt{2}$ $\rightarrow$ not periodic

period: 1.08 3.6 2.025 $\rightarrow$ periodic

Complex exponentials:

Euler's relation: $\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$-\omega_0 \Rightarrow$ clockwise rotation of phasor

$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

DT complex exponentials: represents DT frequency

$x[n] = e^{j\Omega_0 n}$

$e^{j\Omega_0 n} = \cos\Omega_0 n + j\sin\Omega_0 n$

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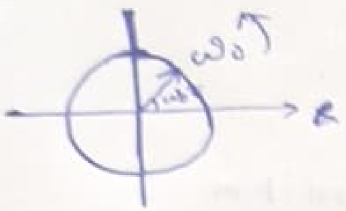
CT:

$-\infty < \omega_0 < \infty$ (cw, ccw)

frequency - no. of cycles per second



$\cos(\phi(t))$ - rad

$\phi(t) = \omega_0 t + \phi_0$

rad ω_0 - units - rad/sec

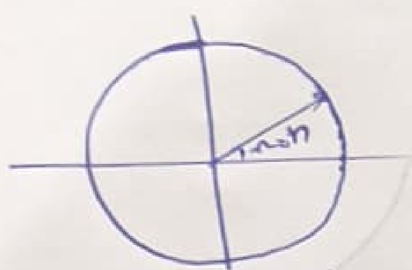
As the phasor comes back to original position after 2π rotation

$\Rightarrow$ Always Periodic.

DT:

$x[n] = e^{j\Omega_0 n}$

$\cos \phi[n]$



$\phi[n] = \Omega_0 n + \phi_0$

rad rad

Not necessarily periodic.

$0 < \Omega_0 < 2\pi$

or

$-\pi < \Omega_0 < \pi$

Difference between DT and CT complex exponentials:

1) $e^{j\omega_0 t}$ is always periodic

* $e^{j\Omega_0 n}$ is not necessarily periodic. for it to be periodic; $e^{j\Omega_0 n} = e^{j\Omega_0 (n+N)}$

$\Rightarrow N = \frac{2\pi}{\Omega_0}(m)$ - integer

$\Rightarrow \boxed{\frac{\Omega_0}{2\pi} = \frac{p}{q}}$ - for it to be periodic

* $e^{j\omega_0 t}$ will be a "distinct" exponential for all values of 'ω_0'

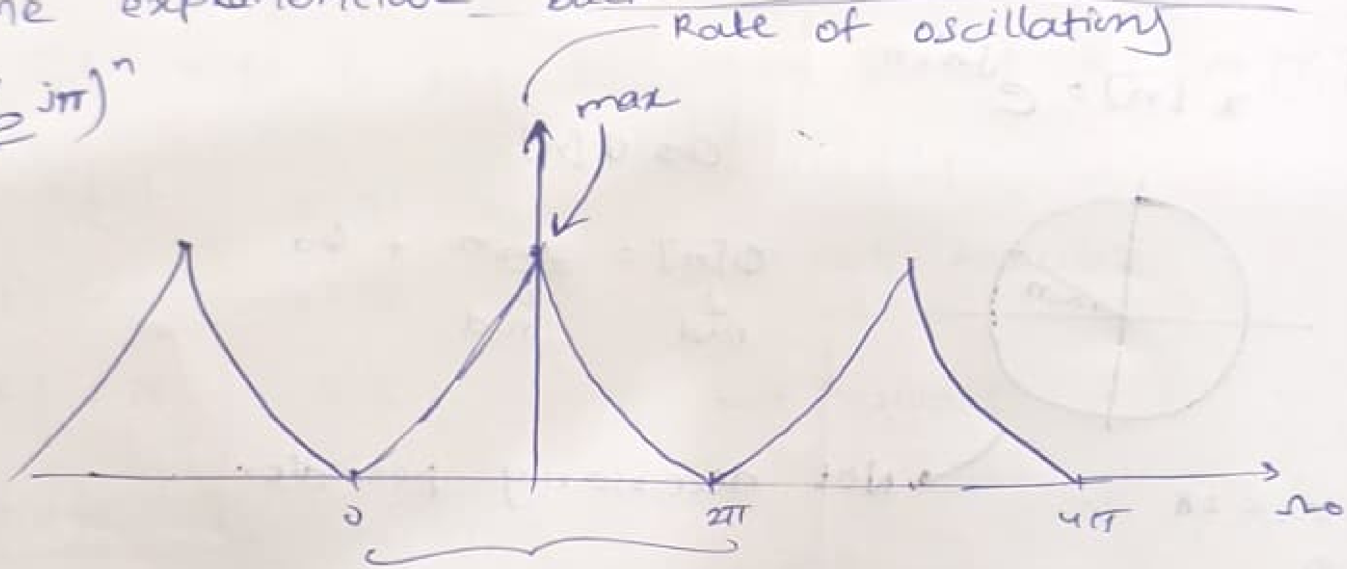
For ~~$e^{j\omega_0 n}$~~ $e^{j\Omega_0 n}$:

$$e^{j(\Omega_0 + k.2\pi)n} = \underbrace{e^{j\Omega_0 n} \cdot e^{j2\pi.kn}}_{\downarrow}$$

If kn - is integer $\rightarrow e^{j\Omega_0 n}$

Shifting Ω_0 by an integer multiple of 2π you get same exponential back.

* $(e^{j\pi})^n$



For DT complex exponentials signals only in range $0 < \Omega_0 < 2\pi$ are unique.

We can <u>not</u> say that this is periodic. Since it is w.r.t Ω_0 (not time).

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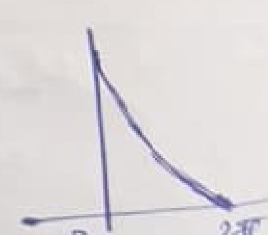
* π interval is sufficient for total

$\boxed{\text{BASEBAND}}$



Pb :- 1) $x[n] = \cos \frac{n}{6}$ $\longrightarrow$ $\Omega_0 = \frac{1}{6}$

$$\frac{\Omega_0}{2\pi} = \frac{1}{12\pi} \quad \text{- irrational}$$

$\Rightarrow$ Not periodic.

2) $y[n] = \cos \frac{2\pi n}{12}$ $\longrightarrow$ $\Omega_0 = \frac{\pi}{6}$ $\Rightarrow$ $\frac{\Omega_0}{2\pi} = \frac{1}{12}$ - rational

$\Rightarrow$ Periodic

3) $z[n] = \cos\left(\frac{2n}{13\pi}\right)$ $\longrightarrow$ $\Omega_0 = \frac{2}{13\pi}$ $\Rightarrow$ $\frac{\Omega_0}{2\pi} = \frac{1}{13\pi^2}$

- irrational.

$\Rightarrow$ Not periodic

Pb: $x[n] = \cos\left(\frac{8\pi n}{31}\right)$

$\Omega_0 = \frac{8\pi}{31}$ $\Rightarrow$ $\frac{\Omega_0}{2\pi} = \frac{4}{31}$ - periodic

$$\text{period} = \frac{2\pi}{\Omega_0} m = \frac{31}{4} m$$

$\therefore$ periodic with fundamental period $= 31$

Pb: $x(t) = \cos\left(\frac{8\pi t}{31}\right)$

$$\text{period} = \frac{31}{4}$$

27/8/24

Energy and Power :

$$E = \lim_{L\to\infty} \int_{-L}^{L} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E = \lim_{L\to\infty} \sum_{n=-L}^{L} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

} Energy

$$\text{Power, } P = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} |x(t)|^2 \, dt = \frac{E}{2L}$$

$$\cancel{P = Lt} \quad P = \lim_{L \to \infty} \frac{1}{2(L+1)} \sum$$

$$P = \lim_{L \to \infty} \frac{1}{2L+1} \sum_{n=-L}^{L} |x[n]|^2$$

zeroth sample

$$P = \lim_{L \to \infty} \frac{E}{2L+1}$$

* Energy signals are those which have finite energy and zero power.

energy signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \text{FINITE} \quad \text{signal}$$

for such signals $\quad P = \lim_{L \to \infty} \frac{1}{2L} E$ finite

$$P \to 0$$

* Power signal $\to$ P - finite
  $(E \to \infty)$ $\quad$ $E \to \infty$

eg: $x(t) = \begin{cases} 1 & 2 \le t \le 3 \\ 0 & \text{otherwise} \end{cases} \Rightarrow$ energy signal.

Even and odd:

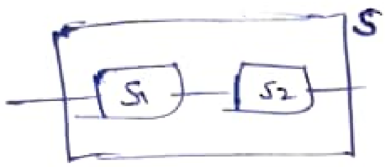
$x(t) = x(-t)$ - even

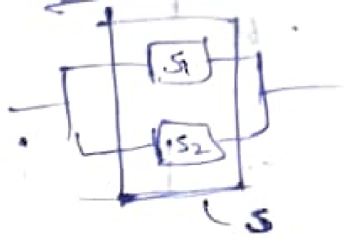
$x(-t) = -x(t)$ - odd

SYSTEMS :

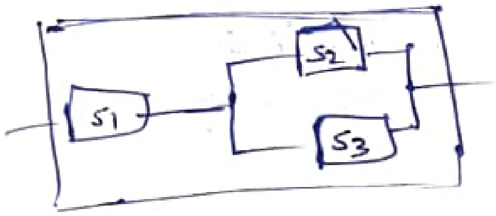
Entity that processes a signal.

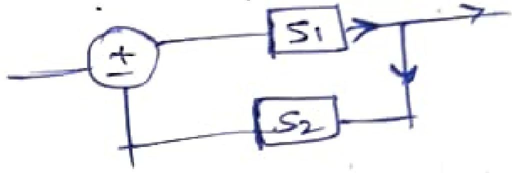
Cascaded (series) | Parallel





hybrid | Feedback





Memory / Memoryless :

The system which depends only on present time is memoryless.

Memory - depends on other inputs also.

eg. $y(t) = 3x(t) + 2x(t-1) + 4.5$ → depends on past memory

$y(t) = x(t+3)$ — anticipatory
(⇒ future memory required)

—||— , mmm → with memory.
C   L

* Accumulator : $y[n] = \sum_{i=k}^{n} x[k]$

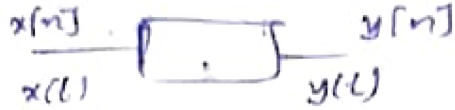
↳ It has memory.

memoryless

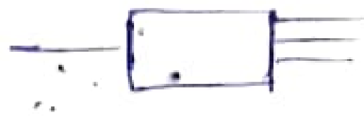
* $y[n] = \frac{1}{2}\left[x[n] - x^2[n]\right]^{1/2}$

No memory

* SISO



* SIMO



* MISO

* MIMO

→ ongoing wireless communication technology.

<u>Invertible / Non-invertible</u> :

If two subsystems cascades to a UNITY.

$y(t) = R\,x(t)$

$x(t) = \frac{y(t)}{R}$

$y = x^2$ $y = 0$

non invertible.

If system produces distinct outputs for distinct inputs is invertible.

$y[n] = \sum_{k=-\infty}^{n} x[k]$ → invertible

$x[n] = y[n] - y[n-1]$

∵ x[n] can be written in terms of y[n]

Casual / Non-casual :

predicts future.

All systems except anticipatory system are causal.

If output is functions of present / past input $\Rightarrow$ Causal.

$y[n] = x[2-n]$ → Non casual ($\because y[-1] = x[3]$)

$y(t) = \int_{-\infty}^{t} x(t)\,dt$ → Causal

$y[n] = \sum_{k=1}^{M} x(k)$ → $\begin{cases} M > n \Rightarrow \text{Non causal} \\ M < n \Rightarrow \text{Causal} \end{cases}$

* Causality is MUST for realizability.

implemented.

Concept of States :

$\rightarrow \boxed{S} \rightarrow$

Natural response → $x = 0$ − $y_{natural}$

Forced response → when excitation / input is provided.

$y(t) = x(t) + 3$

Response = Natural response + Forced response.

30/3/24

Stability (BIBO) - bounded input bounded output

$x \rightarrow \boxed{S} \rightarrow y$

Bounded input (Not ∞ in Amp / time) $\Rightarrow$ Output should also be bounded.

$y[n] = \sum_{k=-\infty}^{N} x[k]$ − Not stable / Unstable.

$y(t) = e^{t}$ – can not be determined

$y(t) = e^{x(t)}$ – Unstable.

Useful unstable system : Oscillator.

generates periodic signals.

<u>Time Invariance</u> :

A system can be time varying or time invariant if

some time shift at input $\Rightarrow$ same time shift at output

$x(t) \rightarrow y(t)$

$x(t-t_0) \rightarrow y(t-t_0)$

$x[n] \rightarrow y[n]$

$x[n-n_0] \rightarrow y[n-n_0]$

eg: $y[n] = \sin[n]$ $\rightarrow$ Time invariant

$y[n-n_0] = \sin[n-n_0]$

eg: $y[n] = n\, x[n]$

If we change input $x[n]$ to $x[n-n_0]$ then

$y[n-n_0] \neq n\, x[n-n_0]$

$= (n-n_0)\, x[n-n_0]$

$\therefore$ Not time invariant

Linear / Non-linear :

linear $\Rightarrow$ Superposition and Homogeneity.

$$x_1 \longrightarrow \boxed{S} \longrightarrow y_1$$

$$x_2 \qquad\qquad y_2$$

$(x_1 + x_2) \longrightarrow (y_1 + y_2) \Rightarrow$ Superposition

$\alpha x_1 \longrightarrow \alpha y_1 \Rightarrow$ Homogenity

System is linear if $(\alpha x_1 + \beta x_2) \longrightarrow (\alpha y_1 + \beta y_2)$

eg: $y = mx + c$ → Non linear system

for linear system : Zero input $\Rightarrow$ Zero output.

eg: $x(t) \longrightarrow \boxed{S} \longrightarrow y(t) = \int_{-\infty}^{3t} x(\tau) d\tau$

sol:

$$k\, y(t) = k \int_{-\infty}^{3t} x(\tau) d\tau$$

$$\int \alpha x + \beta y = \alpha \int x + \beta \int y \Rightarrow \boxed{\text{Linear}}$$

Memory
Invertible
Non ~~causa~~ casual
Unstable
Time variant.

Linear Time Invariant Systems: (LTI system)

[Most of the practical systems can be approximated as LTI systems]

Impulse Response

Impulse response

$$\delta(t),\ \delta[n] \longrightarrow \boxed{\begin{matrix}LTI\\ S\end{matrix}} \longrightarrow h(t),\ h[n]$$

$$\delta[n-k] \longrightarrow h[n-k] \quad (TI)$$

time invariance

sample value

$$x[k]\,\delta[n-k] \longrightarrow x[k]\,h[n-k] \quad (\text{Homogeneity})$$

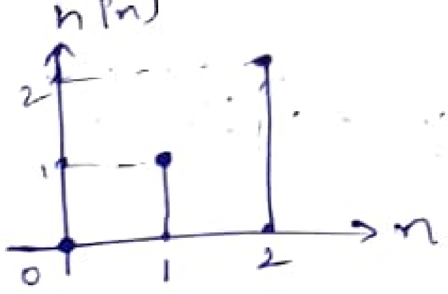
$$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}_{x[n]} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \quad (\text{superposition})$$

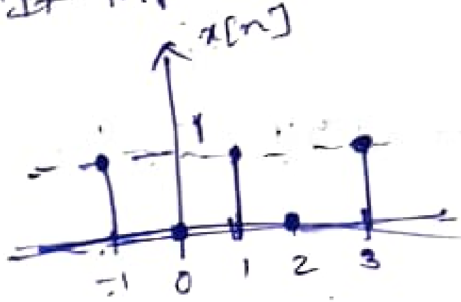
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

31/8/24

Ex: An ~~LTI~~ LTI system

If input is:





$h[0]=0,\ h[1]=1,\ h[2]=2$

$h[n] = [\underset{\uparrow}{0}, 1, 2]$

$x[n] = [1, \underset{\uparrow}{0}, 1, 0, 1]$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}$$ - Convolution - ($*$)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$$\boxed{x[n] * h[n] = h[n] * x[n]}$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k]$$

$$= \sum_{k=-4}^{4} x[k]\, h[-k]$$

$$= \overset{\text{does not exist}}{x[-4]\, h[4]} + \overset{\text{does not exist}}{x[-3]\, h[3]} + \overset{\text{does not exist}}{x[-2]\, h[2]} +$$

$$x[-1]\, h[1] + x[0]\, h[0] + x[1]\, h[-1] +$$

$$x[2]\, h[-2] + x[3]\, h[-3] + x[4]\, h[-4]$$

$$= x[-1]\, h[1] + x[0]\, h[0]$$

$$= (1)(1) + (0)(0)$$

$$\boxed{y[0] = 1}$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k]$$

$$= \sum_{k=-1}^{1} x[k]\, h[1-k]$$

$$= x[-1]\, h[2] + x[0]\, h[1] + x[1]\, h[0]$$

$$= (1)(2) + (0)(1) + (1)(0)$$

$$\boxed{y[1] = 2}$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\, h[2-k]$$

$$= \sum_{k=-2}^{2} x[k]\, h[2-k]$$

$y[2] = x[-2]\,h[4] + x[-1]\,h[3] + x[0]\,h[2] + x[1]\,h[1]$

$+ x[2]\,h[0]$

$= (0)(2) + (1)(1) + (0)(0)$

$\boxed{y[2] = 1}$

$y[3] = \sum_{k=-\infty}^{\infty} x[k]\,h[3-k] = \sum_{k=-3}^{3} x[k]\,h[3-k]$

$= x[-3]\,h[6] + x[-2]\,h[5] + x[-1]\,h[4] + x[0]\,h[3] + x[1]\,h[2]$

$+ x[2]\,h[1]\;\; x[3]\,h[0]$

$= (1)(2) + 0 + (1)(0) = \boxed{2}$

$y[4] = \sum_{k=-\infty}^{\infty} x[k]\,h[4-k] = \sum_{k=-4}^{4} x[k]\,h[4-k]$

$= x[-1]\,h[5] + x[0]\,h[4] + x[1]\,h[3] + x[2]\,h[2] + x[3]\,h[1]$

$= (0)(2) + (1)(1) = \boxed{1}$

$y[5] = \sum_{k=-\infty}^{\infty} x[k]\,h[5-k] = \sum_{k=-5}^{5} x[k]\,h[5-k]$

$= x[3]\,h[2] + x[4]\,h[1] \qquad = (1)(2) = \boxed{2}$

$y[6] = \sum_{k=-\infty}^{\infty} x[k]\,h[6-k]$

$0 \le 6-k \le 2 \Rightarrow \underbrace{4 \le k \le 6}_{\text{does not exist for } x[k]}$

$\Rightarrow$ does not exist

$y[-1] = \sum_{k=-\infty}^{\infty} x[k]\,h[-1-k] = \sum_{k=-3}^{3} x[k]\,h[-1-k]$

$= x[-1]\,h[0] + x[0]\,h[-1] = 1(0) = \boxed{0}$

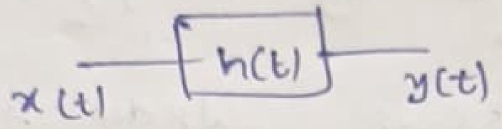
$y[-2] = \sum_{k=-\infty}^{\infty} x[k]\,h[-2-k]$ ✗

$0 \le -2-k \le 2 \Rightarrow -4 \le k \le -2$ - does not exist

* Convolution with an impulse: (same signal)

$$y[n] = \sum_k x_1[k]\,\delta[n-k] = x_1[n]$$

* Convolution for Continuous time domain



$$y(t) = h(t) * x(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

3/9/24

Pb: Find $x(t) * y(t)$ if 1) $x(t) = e^{-t^2}$ Ans: $5.3t^2 + 2.659$

$y(t) = 3t^2$

2) $x(t) = 3\cos 2t$ Ans: $\frac{6}{5}\cos 2t$

$y(t) = e^{-|t|}$

$$x(t) * y(t) = \int_{-\infty}^{\infty} (e^{-\tau^2})\,3(t-\tau)^2\,d\tau$$

$\frac{\tau}{-2}(-2\tau e^{-\tau^2})$

$$= 3\left[t^2\int_{-\infty}^{\infty} e^{-\tau^2}\,d\tau + \int_{-\infty}^{\infty} \tau^2 e^{-\tau^2}\,d\tau - t\int_{-\infty}^{\infty} 2\tau e^{-\tau^2}\,d\tau\right]$$

$$= 3\left[t^2\left(\int_{-\infty}^{\infty} e^{-\tau^2}\,d\tau\right) + \left[\frac{\tau}{-2}e^{-\tau^2}\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-\tau^2}\,d\tau\right.$$

$\rightarrow 0$ $\quad - t\left[e^{-\tau^2}\right]_{-\infty}^{\infty} \rightarrow 0$

$$= \left(3t^2 + \frac{1}{2}\right)\int_{-\infty}^{\infty} e^{-\tau^2}\,d\tau$$

$$\int_{-\infty}^{\infty} e^{-\tau^2} d\tau = I$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

$x = r\cos\theta,\ y = r\sin\theta$, $dx\,dy = r\,d\theta\,dr$

$$\therefore I^2 = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2} dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-r^2} dr = (2\pi)\left(-\frac{1}{2}\right)\left[e^{-r^2}\right]_0^{\infty}$$

$$I^2 = \pi$$

$$\Rightarrow \boxed{I = \sqrt{\pi}}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \sqrt{\pi}$$

$$x(t) * y(t) = \left(3t^2 + \frac{1}{2}\right)(\sqrt{\pi}) = 3\sqrt{\pi}\, t^2 + \frac{\sqrt{\pi}}{2}$$

$$\boxed{= 5.3t^2 + 2.659}$$

2) $x(t) = 3\cos 2t$, $y(t) = e^{-|t|}$

$$x(t) * y(t) = \int_{-\infty}^{\infty} 3(\cos(2)(t-\tau))\, e^{-|\tau|} d\tau$$

$$= 3\int_{-\infty}^{\infty} \cos 2t \cos 2\tau\, e^{-|\tau|} + \sin 2t \sin 2\tau\, e^{-|\tau|} d\tau$$

$$= 3\cos 2t \int_{-\infty}^{\infty} \cos 2\tau \, e^{-|\tau|} d\tau + 3\sin 2t \cancel{\int_{-\infty}^{\infty} \sin 2\tau \, e^{-|\tau|} d\tau}^{\,0}$$

$$= 3\cos 2t \left(2\int_0^{\infty} \cos 2\tau \, e^{-|\tau|} d\tau \right)$$

$$x(t) * y(t) = 3\cos 2t \left(2 \overbrace{\int_0^{\infty} \underset{v}{\cos 2\tau} \, \underset{u}{e^{-\tau}} d\tau}^{I} \right)$$

$$I = \left[\left[e^{-\tau} \frac{\sin 2\tau}{2} \right]_0^{\infty} + \int_0^{\infty} \underset{u}{e^{-\tau}} \underset{v}{\frac{\sin 2\tau}{2}} d\tau \right]$$

$$I = \frac{1}{2} \left[\left[-e^{-\tau} \frac{\cos 2\tau}{2} \right]_0^{\infty} - \int_0^{\infty} e^{-\tau} \frac{\cos 2\tau}{2} d\tau \right]$$

$$= \frac{1}{2} \left(\frac{1}{2} - \frac{I}{2} \right) \Rightarrow 4I = 1 - I$$

$$\Rightarrow I = 1/5$$

$$x(t) * y(t) = (3\cos 2t)(2)(1/5)$$

$$\boxed{x(t) * y(t) = \frac{6}{5} \cos 2t}$$

$\Rightarrow$

Properties of Convolution :

1) Communitative for both continuous and discrete

$$h * x = x * h$$

2) Associative for both continuous and discrete.

$$h_1 * (h_2 * h_3) = (h_1 * h_2) * h_3$$

3) Distributive for both continuous and discrete

$$h_1 * (h_2 + h_3) = h_1 * h_2 + h_1 * h_3$$

*Shifting :

CT : If $x(t) * h(t) = y(t)$
(continuous)

$$x(t-t_0) * h(t) = x(t) * h(t-t_0) = y(t-t_0)$$

DT : $x[n-n_0] * h[n] = x[n] * h[n-n_0] = y[n-n_0]$
(discrete)

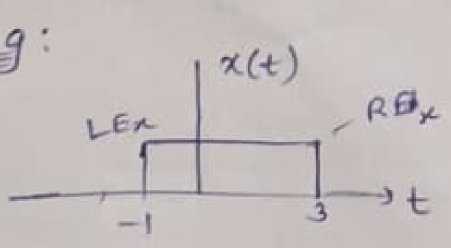
* $x * \delta = x$
impulse

* Width : $x(t) \longrightarrow W_x$

$h(t) \longrightarrow W_h$

$x(t) * h(t) \longrightarrow W_x + W_h$

eg:

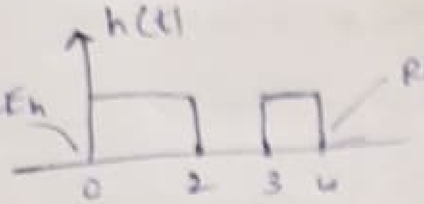


$W_x = RE_x - LE_x$

$= 3 - (-1)$

$= 4$

RE - right edge LE - left edge



$W_h = 4 - 0$
$= 4$

$y = x * h$

$$\left.\begin{aligned} W_y &= W_x + W_h = 8 \\ RE_y &= RE_x + RE_h = 7 \\ \cancel{RE}\, LE_y &= LE_x + LE_h = -1 \end{aligned}\right\} \quad W_y = RE_y - LE_y$$

In DT Convolution : $N_y = N_x + N_h - 1$

* <u>Area</u> :

For any signal $A = \int_{-\infty}^{\infty} x(t)\, dt$

$$x(t) \longrightarrow A_x$$
$$h(t) \longrightarrow A_h$$
$$y(t) = x(t) * h(t) \longrightarrow A_x \cdot A_h$$

$$x[n] \longrightarrow S_x = \sum_{-\infty}^{\infty} x[n]$$
$$h[n] \longrightarrow S_h$$
$$y[n] = x[n] * h[n] \longrightarrow S_x \cdot S_h.$$

* <u>Differentiation</u> :

$$\frac{d}{dt}\left(x(t)\right) * h(t) = x(t) * \frac{d}{dt} h(t) = \frac{d}{dt} y(t)$$

* <u>Time scaling</u> :

$$x(at) * h(at) = \frac{1}{|a|}\, y(at)$$

Characterization of LTI systems:

Memory:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= \ldots\ldots x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \ldots\ldots$$

$$\left.\begin{aligned} * \; h[n] &= k\,\delta[n] \\ h(t) &= k\,\delta(t) \end{aligned}\right\} \text{memoryless}$$

Causality:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$y[n] = \ldots\ldots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \ldots\ldots$$

eg: $y[1] = \underbrace{\ldots x[-1]h[2] + x[0]h[1]}_{\text{Past}} + \underbrace{x[1]h[0]}_{\text{Present}} + \underbrace{x[2]h[-1] \ldots}_{\text{future input}}$

future input are not allowed

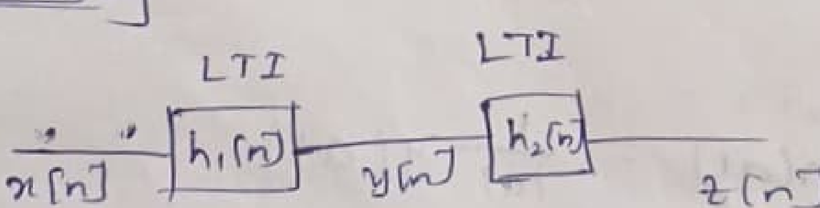
$\Rightarrow$ condition for causality:

$$h[n] = 0 \quad n < 0 \qquad h[-1] = \ldots \& \; h[-\infty] = 0$$

(IR) Impulse response should be positive sided

* Output appearing up before applying the input, $\Rightarrow$ Non-realisable.

Invertibility:



$y[n] = x[n] * h_1[n]$

$z[n] = y[n] * h_2[n]$

$= x[n] * h_1[n] * h_2[n]$

System is invertible if $z[n] = x[n]$ it is possible

iff $h_1[n] * h_2[n] = \delta[n]$

Stability: (BIBO)

$\Sigma x[n] < \infty$, $\Sigma y[n] < \infty$

BIBO stable : $\boxed{\sum_{k=-\infty}^{\infty} h[k] < \infty \rightarrow \text{Finite}}$

* For continuous time ; discrete time

$\left.\begin{aligned} h[n] &= k\,\delta[n] \\ h(t) &= k\,\delta(t) \end{aligned}\right\}$ Memoryless

$\left.\begin{aligned} h[n] &= 0 \quad n<0 \\ h(t) &= 0 \quad t<0 \end{aligned}\right\}$ Causality

$\left.\begin{aligned} h_1[n] * h_2[n] &= \delta[n] \\ h_1(t) * h_2(t) &= \delta(t) \end{aligned}\right\}$ Invertibility.

$\Sigma h[n] < \infty$ if IR is summable

$\int_{-\infty}^{\infty} h(t)\,dt < \infty$ if IR is integrable

} BIBO stability

$\Rightarrow$

6/9/24

↗ It is the output of any LTI system.

Convolution:

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)$$

shifting.
flipping
multiplication

CORRELATION: Cross correlation

$$\text{Cor}(x(t), y(t)) = R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t+\tau)\, dt.$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, y(t)\, dt$$

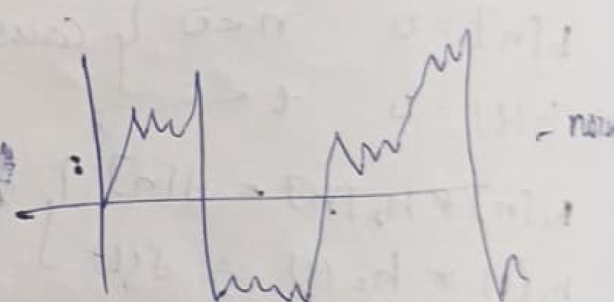
$$R_{yx}(\tau) = R_{xy}(-\tau)$$

* More correlation ⟹ better relation between signals.

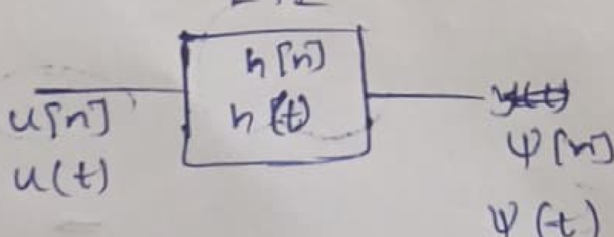
Auto Correlation:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

$$R_n(\tau) = \begin{cases} 0 & \tau \neq 0 \\ \delta(t) \end{cases}$$



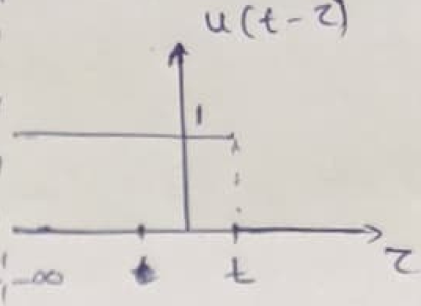
Step response of LTI ~~respon~~ system:



$$u[n] = \sum_{0}^{\infty} \delta[n]$$ step response

CT case: $\psi(t) = u(t) * h(t)$

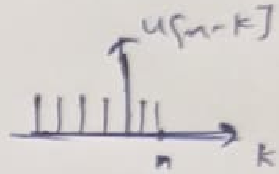
$$\psi(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$



$$\psi(t) = \int_{-\infty}^{t} h(\tau) \cdot 1 \cdot d\tau + \int_{t}^{\infty} \overbrace{h(\tau) \cdot 0\, d\tau}^{0}$$

$$u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$$

$$\boxed{\psi(t) = \int_{-\infty}^{t} h(\tau)\, d\tau}$$



<u>DT case</u>: $\psi[n] = h[n] * u[n]$

$$= \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]$$

$$u[n-k] = \begin{cases} 1 & k < n \\ 0 & k > n \end{cases}$$

$$= \sum_{k=-\infty}^{n} h[k] \cdot 1 + \sum_{k=n}^{\infty} h[k] \cdot 0$$

$$\boxed{\psi[n] = \sum_{k=-\infty}^{n} h[k]}$$

<u>exponential</u>:

$s = \sigma + j\omega$

$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = e^{st} * h(t)$

$$y(t) = \int_{-\infty}^{\infty} e^{s\tau}\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{s(t-\tau)}\, h(\tau)\, d\tau$$

$$y(t) = e^{st} \underbrace{\int_{-\infty}^{\infty} e^{-s\tau}\, h(\tau)\, d\tau}_{\text{independent of 't'}} = H(s)$$

$$y(t) = e^{st}\, H(s)$$

1/9/24 · $\mathcal{F}$ (FS)

Fourier Series :

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = x(t) * h(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \underbrace{\delta(t-\tau)}_{\text{elementary}}\, d\tau$$

input = superposition of weighted elementary signals such as impulses (δ)

(sum) (homogenity)

* Decomposing any signal interms of weighted sum of orthogonal basis signal is GRAM - SCHMIDT orthogonalization.

* e^{st} ; $s = j\omega$

$$e^{j\omega t} \longrightarrow \boxed{} \longrightarrow e^{j\omega t} H(j\omega)$$

$m = 0, 1, 2, 3, \ldots$ $e^{jm\omega t} \longrightarrow$

$m=0$ 1 . dc signal

$m=1$ $e^{j\omega t} \longrightarrow P$

$m=2$ $e^{j2\omega t} \longrightarrow P/2$

$m=3$ $e^{j3\omega t} \longrightarrow P/3$

$\vdots$

Harmonically related.

$$\int_{a}^{a+P} e^{jm\omega_0 t}\, dt = \left[\frac{e^{jm\omega_0 t}}{jm\omega_0} \right]_{a}^{a+P}$$

$$= \frac{1}{jm\omega_0}\left(e^{ja\omega_0 t + jp\omega_0 t} - e^{ja\omega_0 t}\right)$$

$$\int_a^{a+p} e^{jm\omega_0 t}\,dt = \begin{cases} p & m=0 \\ 0 & m\neq 0 \end{cases}$$

$$\int_p e^{jm\omega t}\left(e^{jk\omega_0 t}\right)^* dt = \int_p e^{j(m-k)\omega t}\,dt$$

$$= \begin{cases} p & m=k \\ 0 & m\neq k \end{cases}$$

Complex exponentials satisfy the property of orthogonality.

11/9/24

* Decomposing Periodic signals as weighted sum of complex exponentials.

Let $x(t)$ be periodic signal with period 'p' where $\omega_0 = \frac{2\pi}{p}$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$ FS SYNTHESIS EQN

k is frequency index (integer)

$c_k \rightarrow$ CTFS coefficients

↳ continuous time fourier series

(or)

frequency component of signal

Finding CTFS coefficient, c_k's:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

multiply with $e^{-jm\omega_0 t}$

$$\Rightarrow x(t)\, e^{-jm\omega_0 t} = \sum_{k=-\infty}^{\infty} e^{j(k-m)\omega_0 t} \cdot c_k$$

$$\int_P x(t)\, e^{-jm\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} \int_P e^{j(k-m)\omega_0 t}\, c_k\, dt$$

$$= c_m \cdot P$$

$$\Rightarrow c_m = \frac{1}{P} \int_P x(t)\, e^{-jm\omega_0 t}\, dt$$

$$\Rightarrow \boxed{c_k = \frac{1}{P} \int_P x(t)\, e^{-jk\omega_0 t}\, dt}$$ Fourier series Analysis equation.

Ex:

$x(t) = \cos 2\omega t$

$$= \frac{e^{j2\omega t} + e^{-j2\omega t}}{2} = \frac{1}{2} e^{j2\omega t} + \frac{1}{2} e^{j(-2\omega)t}$$

$\Rightarrow c_2 = \frac{1}{2}$

$c_{-2} = \frac{1}{2}$

* $x(t) = 1 + \cos(2\omega_0 t) + \sin(\omega_0 t)$

$$= 1 + \frac{1}{2} e^{j2\omega_0 t} + \frac{1}{2} e^{-j2\omega_0 t} + \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

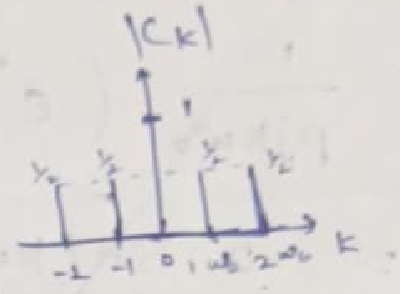
$C_0 = 1$

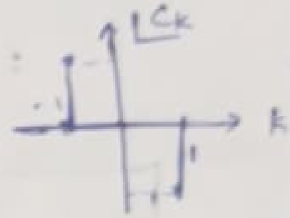
$C_1 = \frac{1}{2j}$

$C_{-1} = \frac{-1}{2j}$

$C_2 = \frac{1}{2}$

$C_{-2} = \frac{1}{2}$





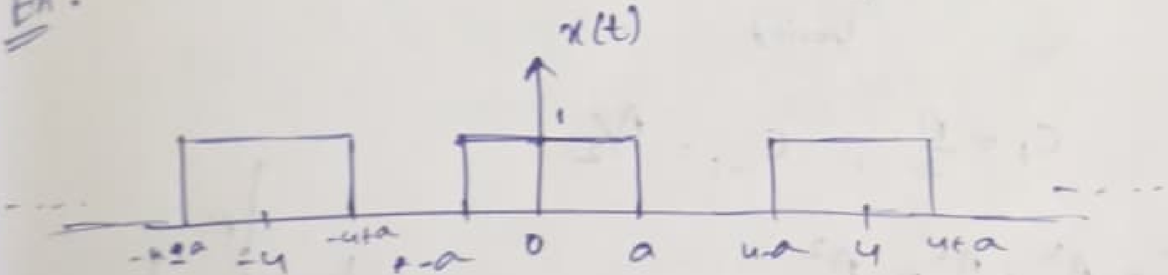
* C_k is complex number:

$$|C_k| = |C_{-k}|$$

$$\angle C_k = -\angle C_{-k}$$

In CTFS the spectra is called line spectra

Ex:



find C_k's

Here $P = 4$

$$C_0 = \frac{1}{P} \int_{-a}^{a} 1 \cdot 1 \, dt = \frac{2a}{P} = \frac{a}{2}$$

$$C_k = \frac{1}{P} \int_{-a}^{a} 1 \cdot e^{-jk\omega_0 t} \, dt$$

$$C_k = \frac{1}{P} \left[\int_{-a}^{a} \cos(k\omega_0 t) - j \sin(k\omega_0 t) \, dt \right.$$

$$\frac{2\sin(k\omega_0 a)}{k\omega_0 p}$$

$$C_k = \frac{1}{P}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-a}^{a}$$

$$= \frac{1}{Pjk\omega_0}\left(e^{jak\omega_0} - e^{-jak\omega_0}\right)$$

$$\boxed{C_k = \frac{2\sin(k\omega_0 a)}{k\omega_0 p}}$$

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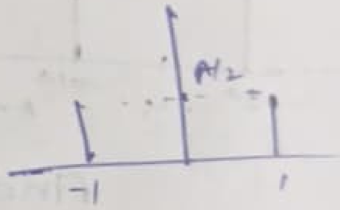
$$x(t) = A\cos\omega_0 t = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t}$$

$$\text{Power} = \frac{1}{P}\int_{P} (A\cos\omega_0 t)^2\, dt = \frac{A^2}{2}$$

(period)

$$C_1 = \frac{A}{2}, \quad C_{-1} = A/2$$



$$C_1^2 + C_{-1}^2 = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

If there is periodic signal $x(t)$ with period P.

Its Power $= \frac{1}{P}\int_{P} |x(t)|^2\, dt$

$$\boxed{\frac{1}{P}\int_{P}|x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2}$$: PARSEVAL'S THEOREM

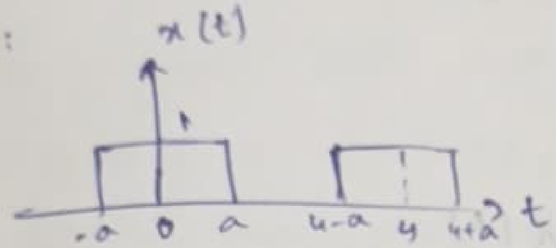
* $x(t) \longrightarrow P$

$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

So the basis $e^{jk\omega_0 t}$ are also periodic with period 'P'.

$\omega_0 = \frac{2\pi}{P}$

Pb:



$a = 1$
$P = 4$

Find % of power lying inside frequency range of $[-2, 2]$ rad/sec.

Sol:

$\omega_0 = \frac{2\pi}{P} = \frac{2\pi}{4} = \pi/2$

Here $2\omega_0 > 2$
So for power we should find C_1, C_0, C_{-1}

$C_0 = \frac{2a}{P} = \frac{1}{2}$

$C_k = \frac{2 \sin(k\omega_0 a)}{k\omega_0 P} \Rightarrow C_1 = \frac{2(1)}{\frac{\pi}{2}(4)} = \frac{1}{\pi}$

$C_{-1} = \frac{-1}{\pi}$

$P = \frac{1}{\pi^2} + \frac{1}{4} + \frac{1}{\pi^2} = \frac{2}{\pi^2} + \frac{1}{4} = 0.458 \rightarrow$ in $[-2, 2]$

Total power $= \frac{1}{4} \int_{-a}^{4-a} x(t)^2 \cdot dt = \frac{1}{4}\left[\int_{-1}^{1} 1^2 dt + \int_{1}^{3} 0\, dt\right]$

$= \frac{1}{4}(2) = \frac{1}{2}$

% power lying in $[-2, 2]$ $= \frac{0.458}{0.5} \times 100 = \boxed{91.6\ \%}$

Some Uncomfortable Questions :

1) What if $C_k \to \infty$ for some k ?

→ Fourier series does not exist.

2) How do we know that $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ converges to $x(t)$?

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \xrightarrow[\text{to}]{\text{converges}} x(t)$$

Let $$\sum_{k=-M}^{M} c_k e^{jk\omega_0 k} \xrightarrow[\text{to}]{\text{converges}} x'(t)$$

error : $e(t) = x(t) - x'(t)$ $\qquad \int |e(t)|^2 dt$

$$C_k = \frac{1}{P} \int_P x(t) e^{-jk\omega_0 t} dt$$

It is <u>not</u> gauaranted that at any 't' $x(t)$ is equal to its Fourier series expansion. Only gaurantee is the average squared error between FS expansion and $x(t)$ is almost zero.

A 19/24

Drichlet's Conditions : [sufficient (not necessary) condition

1) BOUNDED VARIATIONS :

Finite numbers of maxima and minima

Counter ex : $x(t) = \sin\left(\frac{2\pi}{t}\right)$

$x(t) = \sin(1/t)$

2) Finite number of finite discontinuities.

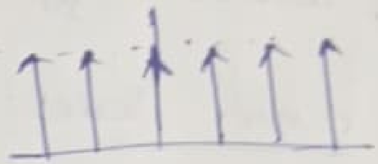
3) Absolute Integrability : $\int_P |x(t)|\, dt < \infty$.

counter ex: $x(t) = \tan x$

* A signal that satisfies DCs must have FS expansion. However, if it does not satisfy
→ it may/may not have FS expansion.

So Dirichlet's Conditions are sufficient but not necessary.

eg: Find FS expansion



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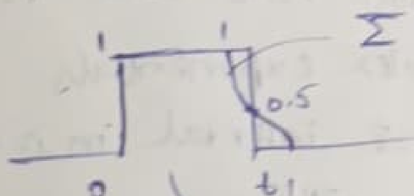


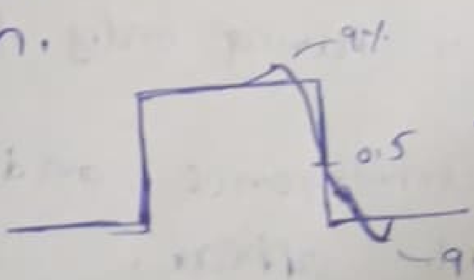
$\sum C_k e^{jk\omega_0 t_1}$

$\sum C_k e^{jk\omega_0 t} \longrightarrow \frac{x(t_1^-) + x(t_1^+)}{2}$

$x(t_1) \neq \sum C_k e^{jk\omega_0 t_1}$

Point of time at which discontinuity is appearing

* When we reconstruct a square wave from FS expansions near the discontinuity, there is a oversoot to left and undersoot to the right.
This % does not depend on no. of terms in summation.



→ Gibb's Phenomenon

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Discrete Time Fourier Series:

$x[n] = x[n+N]$

$\Omega_0 = \frac{2\pi}{N}$

So, $e^{j\left(\frac{2\pi}{N}\right)n}$ will have total 'N' distinct complex exponentials exist over 2π interval in Ω.

Complex exponential, $e^{j\Omega_0 n} = e^{j\left(\frac{2\pi}{N}\right)n}$

n, N - integer.

FS

$$x[n] = \sum_{k=k_0}^{k_0+N-1} C_k e^{jk\Omega_0 n} = \sum_{\langle N \rangle} C_k e^{jk\Omega_0 n}$$

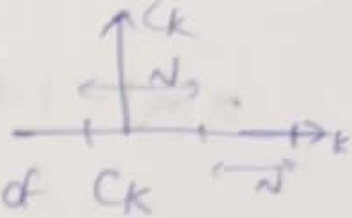
where,

$$C_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$$

As DTFS expansion is having only finite terms in sum

→ No question of convergence and Gibb's phenomenon does not appear.

So N distinct C_k's are possible which will repeat after N values



* Periodicity of ~~DC~~ DTF's coefficient of C_k

$$C_k = C_{k+N}$$

$$\Rightarrow C_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$$

$$C_{k+N} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j(k+N)\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n} \cdot e^{-j\Omega_0 n N}$$

$$\Omega_0 = \frac{2\pi}{N}$$

$$\Rightarrow e^{-j\Omega_0 n N} = 1$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$$

$$\boxed{C_{k+N} = C_k}$$

* $C_k(\Omega_0) = C_k(\Omega_0 + 2\pi)$

C_k repeat after every 2π in Ω.

Pb: ~~x[n]~~ Find DTFS expansion for

$$x[n] = 1 + \sin\left[\frac{2\pi}{N} n\right] + 3\cos\left[\frac{2\pi}{N} n\right] + \cos\left[\frac{4\pi}{N} n + \frac{\pi}{2}\right]$$

Sol: $$x[n] = 1 + \frac{1}{2j}\left[e^{j\Omega_0 n} - e^{-j\Omega_0 n}\right] + \frac{3}{2}\left[e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right] + \frac{1}{2}\left[e^{j(2\Omega_0 n + \pi/2)} + e^{-j(2\Omega_0 n + \pi/2)}\right]$$

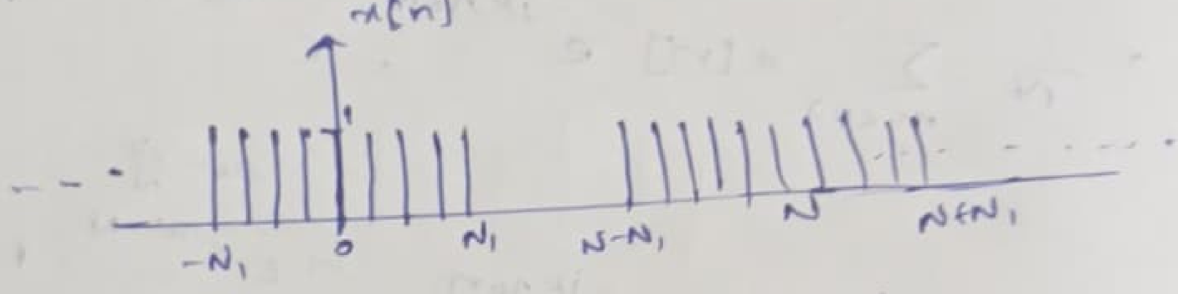
$$x[n] = 1 + e^{j\Omega_0 n}\left[-\frac{j}{2}+\frac{3}{2}\right] + e^{-j\Omega_0 n}\left[\frac{j}{2}+\frac{3}{2}\right]$$

$$+ e^{j2\Omega_0 n}\left[\frac{j}{2}\right] + e^{-j2\Omega_0 n}\left[-\frac{j}{2}\right]$$

$C_0 = 1 \qquad C_1 = -\frac{j}{2} + 3/2 \qquad C_{-1} = \frac{j}{2} + 3/2$

$C_2 = j/2 \qquad C_{-2} = -j/2$

Pb:



$$x[n] = 1 \qquad -N_1 \le n \le N_1$$

Sol:

$$C_k = \frac{1}{N}\sum_{n=\langle N\rangle} e^{-jk\Omega_0 n}$$

$$C_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\Omega_0 n} \qquad \text{put } m = n+N_1$$

$$\Rightarrow C_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\Omega_0(m-N_1)}$$

$$= \frac{1}{N} e^{jk\Omega_0 N_1} \sum_{m=0}^{2N_1} e^{-jk\Omega_0 m}$$

20/9/24

Properties of DTFS :

$x[n] \rightarrow C_k$

$y[n] \rightarrow d_k$

1) $a x[n] + b y[n] \longrightarrow a C_k + b d_k$

2) $x[n-n_0] \longrightarrow e^{-jk\Omega_0 n_0} C_k$ (time shift)

3) $e^{jM\Omega_0 n} x[n] \longrightarrow C_{k-M}$ (frequency shift)

4) $x[-n] \longrightarrow C_{-k}$

5) $x[n] * y[n] \longrightarrow N C_k d_k$

6) First difference

$x[n] - x[n-1] \longrightarrow (1 - e^{-jk\Omega_0}) C_k$

7) If $x[n]$ is real

$C_k = C_{-k}^*$

$|C_k| = |C_{-k}|$

$\angle C_k = -\angle C_{-k}$

If $x[n]$ is real and even then

$C_k \rightarrow$ real and even

If $x[n]$ is real and odd then

$C_k \longrightarrow$ pure imaginary and odd

Parseval's relation :

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |C_k|^2$$

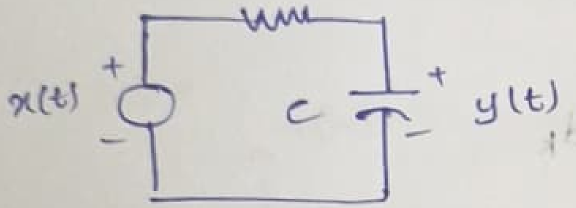
Pb: Check memory, causality and stability

1) $h(t) = u(t+1) - u(t-1)$ - Memory, Non-causal, stable

2) $h[n] = e^{2n} u[n-1]$ - Memory, causal, unstable

3) $h[n] = \cos(\frac{\pi}{2} n)\, u[n+3]$ → Memory, non-causal, unstable

Pr: Consider an RC circuit



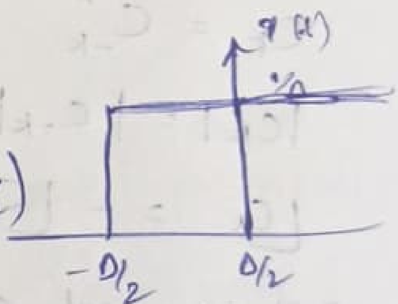
Find impulse response if $RC = 1$ (time constant)

Sol:

$$y(t) = (1 - e^{-t/RC})\, x(t)$$

Response to $x(t) = u(t)$

$$y(t) = (1 - e^{-t/RC})\, u(t)$$



$$x_\Delta(t) = \frac{1}{\Delta} u(t + \Delta/2) - \frac{1}{\Delta} u(t - \Delta/2)$$

$$x_\Delta(t) = \frac{1}{\Delta}\left[u(t+\Delta/2) - u(t - \Delta/2)\right]$$

$$y_1(t) = \frac{1}{\Delta}\left(1 - e^{-(t+\Delta/2)}\right) u(t + \Delta/2)$$

$$y_2(t) = \frac{1}{\Delta}\left(1 - e^{-(t-\Delta/2)}\right) u(t - \Delta/2)$$

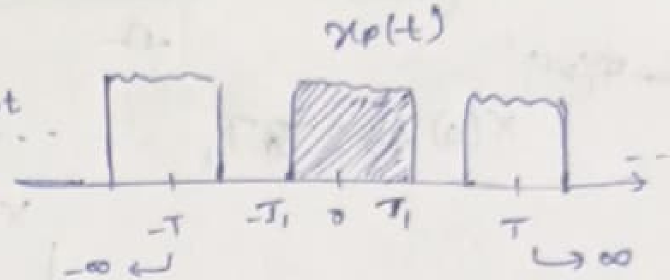
response to $x_\Delta(t)$ (owing to linearity)

$$y_\Delta(t) = y_1(t) - y_2(t)$$

8/10/24

$$\Omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T} X(jk\Omega_0)$$

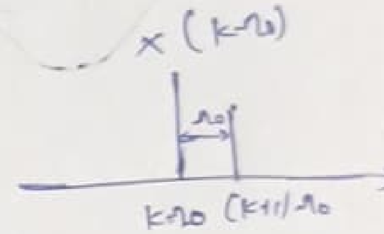


$$x_p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\Omega_0) e^{jk\Omega_0 t}$$

$$x_p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(jk\Omega_0) e^{jk\Omega_0 t}$$

As $T \to \infty$

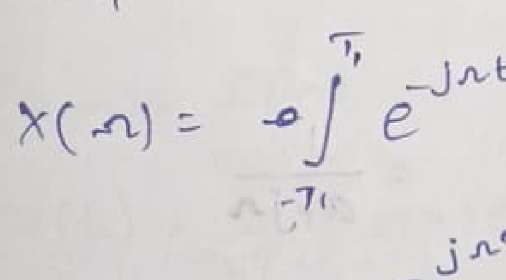
$$x_p(t) = \frac{\Omega_0}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\Omega_0) e^{jk\Omega_0 t}$$



$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} \, d\Omega}$$

$$\boxed{X(k\Omega_0) = X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} \, dt}$$

* 1)



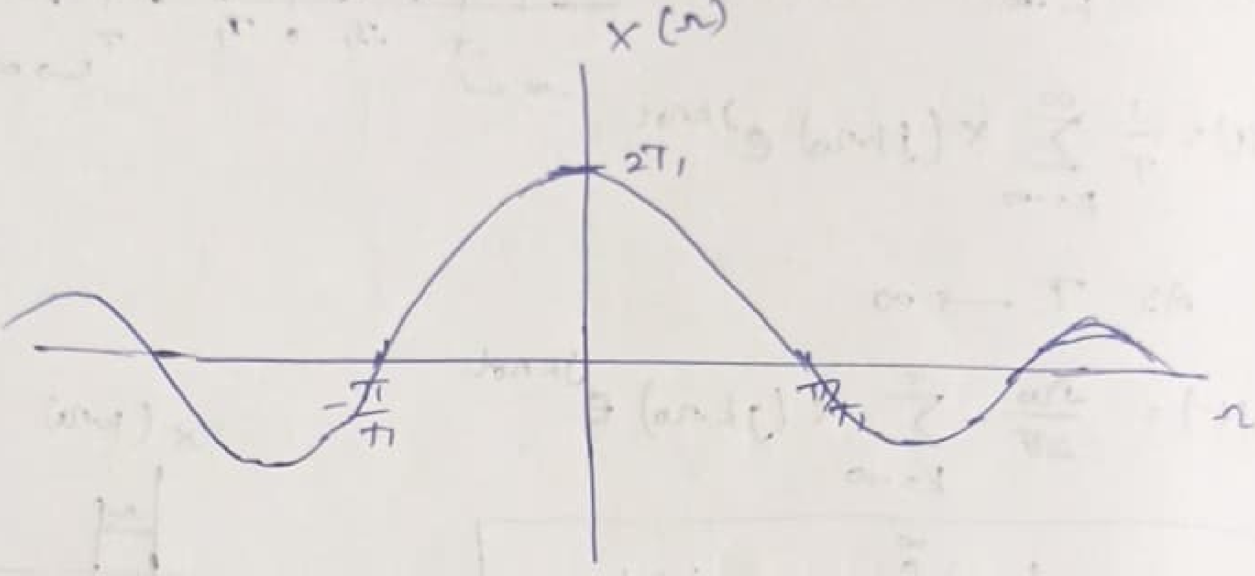
$$X(\Omega) = \int_{-T_1}^{T_1} e^{-j\Omega t} \, dt = \left[\frac{e^{-j\Omega t}}{-j\Omega} \right]_{-T_1}^{T_1}$$

$$X(\Omega) = \frac{e^{j\Omega T_1} - e^{-j\Omega T_1}}{j\Omega} = \frac{2\sin(\Omega T_1)}{\Omega}$$

$$X(\Omega) = \frac{2\sin(\Omega T_1)}{\Omega}$$

for

$X(0) \to 2T_1$



2) $x(t) = e^{-at}\,u(t)$; $a > 0$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, e^{-j\Omega t} \cdot u(t)\, dt$$

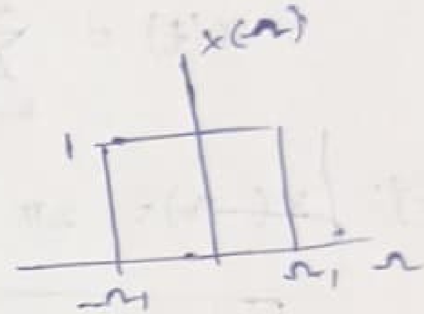
$$= \int_{0}^{\infty} e^{t(-a-j\Omega)}\, dt$$

$$= \left[\frac{e^{-t(a+j\Omega)}}{-(a+j\Omega)} \right]_0^{\infty} = \frac{1}{a+j\Omega}$$

$$X(\Omega) = \frac{a - j\Omega}{a^2+\Omega^2} = \left(\frac{a}{a^2+\Omega^2}\right) - j\left(\frac{\Omega}{a^2+\Omega^2}\right)$$

magnitude $= \dfrac{1}{\sqrt{a^2+\Omega^2}}$ & phase $= -\tan^{-1}\left(\dfrac{\Omega}{a}\right)$

3)



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\Omega_1}^{\Omega_1} e^{j\Omega t} \cdot 1 \, d\Omega$$

$$= \frac{1}{2\pi j t}\left(e^{j\Omega_1 t} - e^{-j\Omega_1 t}\right) = \boxed{\frac{\sin(\Omega_1 t)}{\pi t}}$$

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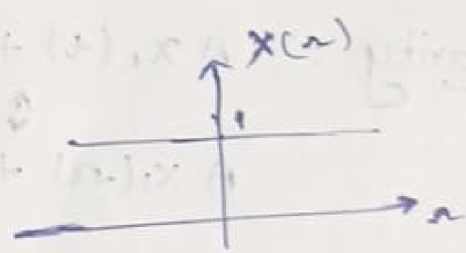
eg. $x(t) = \delta(t)$



$$X(\Omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\Omega t}\, dt$$

$$= 1$$

eg. $X(\Omega) = \delta(\Omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}$$

eg. $X(\Omega) = \delta(\Omega - \Omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega - \Omega_0)\, e^{j\Omega t}\, d\Omega$$

$$x(t) = \frac{1}{2\pi} e^{j\Omega_0 t}$$

eg: $X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0) \, e^{j\Omega t} \, d\Omega$$

$$x(t) = \sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t}$$

eg: $X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0)$ } Fourier Transform

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}}$$ Fourier series representation

Fourier transform is applicable for periodic as well as aperiodic.

1) Linearity : $A x_1(t) + B x_2(t)$

$$\updownarrow$$

$$A X_1(\Omega) + B X_2(\Omega)$$

2) Time-shift : $x(t) \longleftrightarrow X(\Omega)$

$$x(t - t_0) \longleftrightarrow e^{-j\Omega t_0} X(\Omega)$$

3) Derivative in time : $x(t) \longleftrightarrow X(\Omega)$

$$\frac{d}{dt} x(t) \longleftrightarrow j\Omega X(\Omega)$$

for $\frac{d}{dt}(x(t))\Omega$ $X(\Omega) = \int_{-\infty}^{\infty} \underbrace{\frac{d}{dt}x(t)}_{v} \underbrace{e^{-j\Omega t}}_{u} dt$

$$= \left[e^{-j\Omega t}\, x(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-j\Omega)\, e^{-j\Omega t}\, x(t)$$

$$= j\Omega\, X(\Omega)$$

$$\boxed{\frac{d}{dt}x(t) \longleftrightarrow j\Omega\, X(\Omega)}$$

4) $x(t) * h(t) \longleftrightarrow X(\Omega)\, H(\Omega)$

5) Time - multiplication :

$$x(t)\, h(t) \longleftrightarrow \frac{1}{2\pi}\left[X(\Omega) * H(\Omega)\right]$$

eg: $h(t) = \cos(\Omega_0 t)$

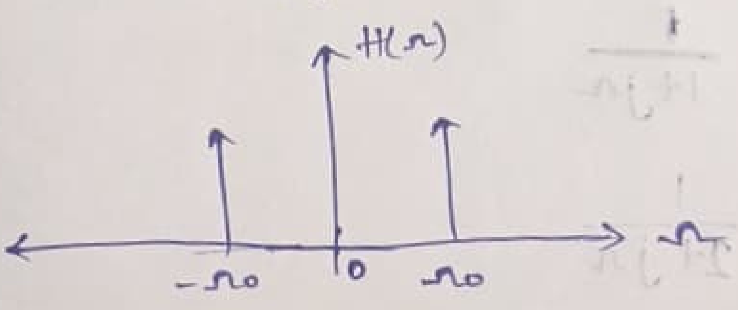
$H(\Omega) = ?$

$$h(t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} = \frac{1}{2}e^{j\Omega_0 t} + \frac{1}{2}e^{-j\Omega_0 t}$$

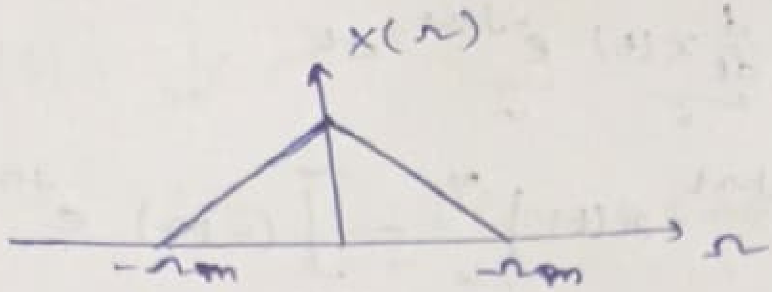
$k = \pm 1$ $\quad a_1 = \frac{1}{2} \quad a_{-1} = \frac{1}{2}$

$$H(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{2}\, \delta(\Omega - k\Omega_0)$$

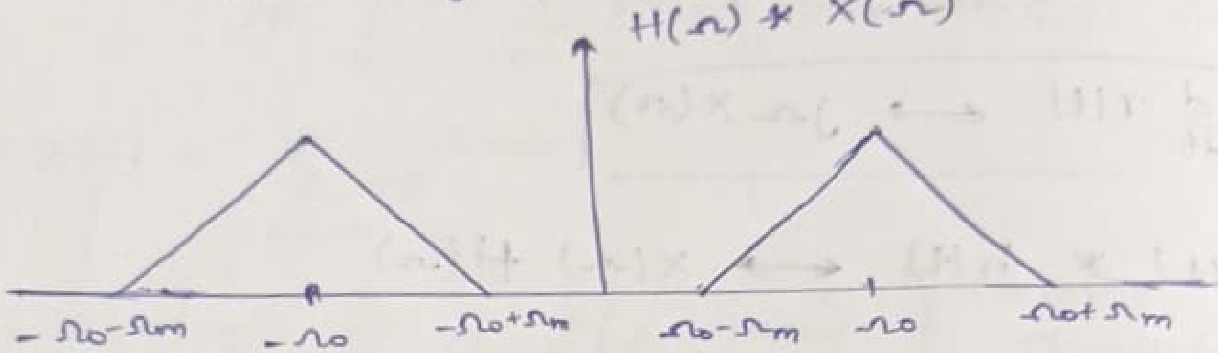
$$H(\Omega) = \pi\left(\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)\right)$$



Let



X convolution with δ gives X



b) <u>Parsevals Theorem</u> : Energy remain same in time and frequency domain.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

eg: $x(t) = e^{-t} u(t)$

$h(t) = e^{-2t} u(t)$

$y(t) = x(t) * h(t) \Rightarrow Y(\Omega) = X(\Omega) . H(\Omega)$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{-t(1+j\Omega)} dt$$

$$X(\Omega) = \int_{-\infty}^{0} 0 \, + \int_{0}^{\infty} e^{-t(1+j\Omega)} dt = \left[\frac{e^{-t(1+j\Omega)}}{-(1+j\Omega)} \right]_0^{\infty}$$

$$X(\Omega) = \frac{1}{1+j\Omega}$$

$$H(\Omega) = \frac{1}{2+j\Omega}$$

$$Y(\Omega) = X(\Omega).H(\Omega) = \frac{1}{(1+j\Omega)(2+j\Omega)} \quad ①$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau)\, e^{-2(t-\tau)} u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} e^{-2t+\tau}\, d\tau = e^{-2t}(e^{t}-1)$$

$$\underline{y(t) = (e^{-t} - e^{-2t})\, u(t)}$$

$$Y(\Omega) = \int_{-\infty}^{\infty} (e^{-t} - e^{-2t})(e^{-j\Omega t})\, dt$$

$$= \int_{-\infty}^{\infty} \left(e^{-t(1+j\Omega)} - e^{-t(2+j\Omega)}\right) dt \quad u(t)$$

$$= \int_{0}^{\infty} e^{-t(1+j\Omega)} u(t) - e^{-t(2+j\Omega)} u(t)\, dt$$

$$= \frac{1}{1+j\Omega} - \frac{1}{2+j\Omega} = \underline{\underline{\frac{1}{(1+j\Omega)(2+j\Omega)}}} \quad ②$$

① = ② ✓

Pb: $$\frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} + 3 = y(t)$$

$h(t) = ?$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$(j\omega)^2 X(\Omega) + 2(j\omega) X(\Omega) + 3\,\delta(\Omega) = Y(\Omega)$$

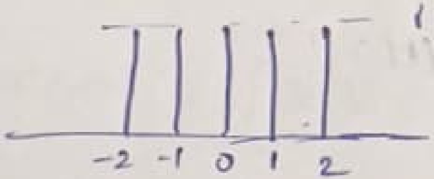
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\Omega) e^{j\Omega t}\, d\Omega$$

DTFT : Discrete Time Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$ – continuous, periodic

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega)\, e^{j\omega n}\, d\omega$$ – Discrete, Aperiodic

*



$x[n] = 1 \ \forall \ n \in [-2, 2]$

$$X(\omega) = e^{-j\omega} + e^{-j2\omega} + e^{j\omega} + e^{j2\omega} + 1$$

$$\boxed{X(\omega) = 1 + 2\cos\omega + 2\cos(2\omega)}$$

* $\Omega = 2\pi F$, $\omega = 2\pi f$

$$F = \frac{f}{F_s}$$

$$X(\omega) = X(\omega \pm k\,2\pi)$$

eg: 1) $x[n] = e^{-an} u[n]$; $a > 0$

$$X(\omega) = \sum x[n] e^{-j\omega n}$$

$$= \sum e^{-(a+j\omega)n} u[n]$$

$$= 1 + e^{-(a+j\omega)} + e^{-2(a+j\omega)} + \dots$$

$$\boxed{X(\omega) = \frac{1}{1 - e^{-(a+j\omega)}}}$$

2) $X(\omega) = \delta(\omega)$

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} \delta(\omega) e^{j\omega n} d\omega = \boxed{\frac{1}{2\pi}}$$

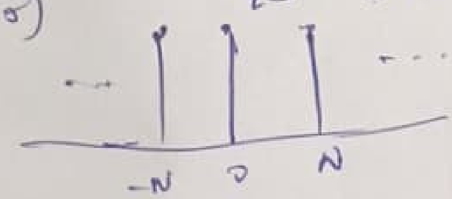
3) $X(\omega) = \delta(\omega - k\omega_0)$

$$x[n] = \frac{1}{2\pi} \int e^{jk\omega_0 n} = \frac{e^{jk\omega_0 n}}{2\pi}$$

4) $x[n] = \delta[n]$

$$X(\omega) = 1$$

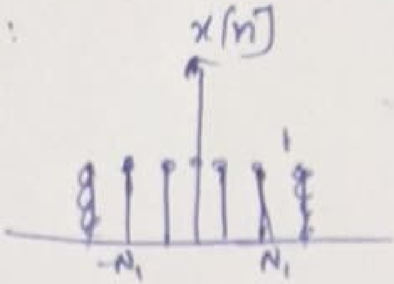
5) $\tilde{x}[n]$ periodic



$$a_k = \frac{1}{N} \sum x[n] e^{-jk\omega_0 n} = \frac{1}{N}$$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

eg:



$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= e^{j\omega N_1} + e^{j\omega(N_1-1)} + \ldots + e^{-j\omega N_1}$$

$$= \frac{e^{j\omega N_1}\left(e^{-j\omega(2N_1+1)} - 1\right)}{e^{-j\omega} - 1}$$

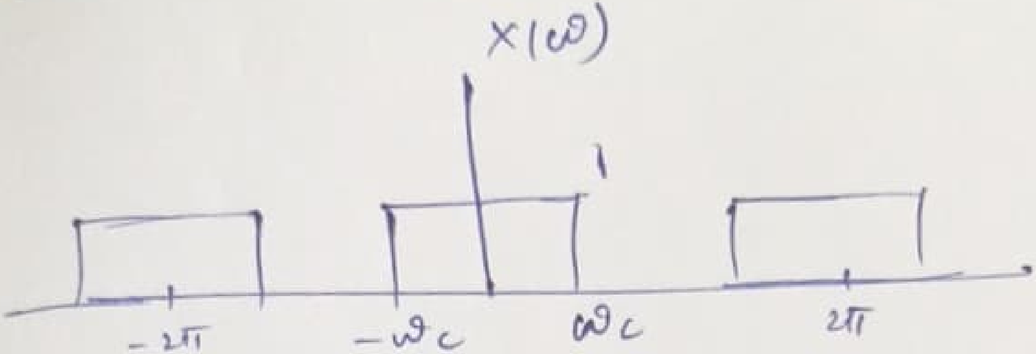
$$= \frac{e^{j\omega N_1} e^{-j\omega(N_1+\frac{1}{2})}\left(e^{-j\omega(N_1+\frac{1}{2})} - e^{j\omega(N_1+\frac{1}{2})}\right)}{e^{-j\omega/2}\left(-e^{j\omega/2} + e^{-j\omega/2}\right)}$$

$$= \frac{-2j\sin(\omega(N_1+\frac{1}{2}))}{-2j\sin\left(\frac{\omega}{2}\right)}$$

$$\boxed{X(\omega) = \frac{\sin(\omega(N_1+\frac{1}{2}))}{\sin(\omega/2)}}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

eg:



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi(jn)}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$x[n] = \frac{1}{2\pi jn} \cdot \frac{2j\sin(\omega_c n)}{1}$$

$$\boxed{x[n] = \frac{\sin(\omega_c n)}{\pi n}}$$

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$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega t}$$

$$a_k = \frac{1}{N} \int_{\langle N \rangle} x_p(t) e^{-jk\omega_0 t} dt$$

CTFS

$$x_p[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$ } Discrete and periodic

DTFS

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x_p[n] e^{-jk\omega_0 n}$$ } Discrete and periodic

---

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$ } continuous and aperiodic

CTFT

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$ } continuous and aperiodic

---

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{j\omega n} d\omega$$ } discrete aperiodic

DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ } continuous periodic

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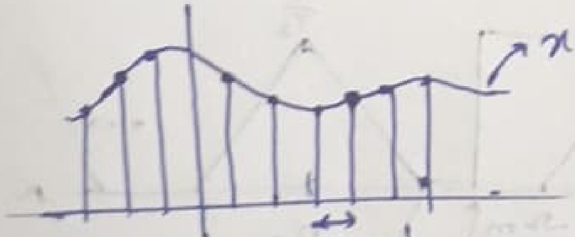
*



$x(t) \to x[nT_s] \to x[n]$ ...

F_s

$T_s = \frac{1}{F_s}$

$$\boxed{F_s \geq 2F_m}$$

F_s: sampling frequency

F_m: maximum frequency

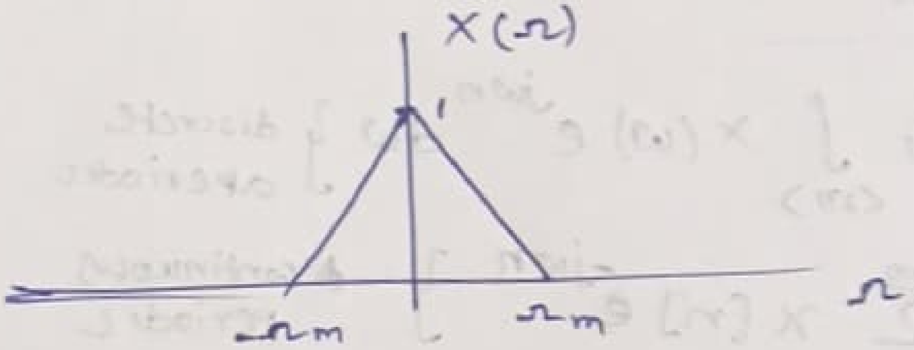
$$* \; x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

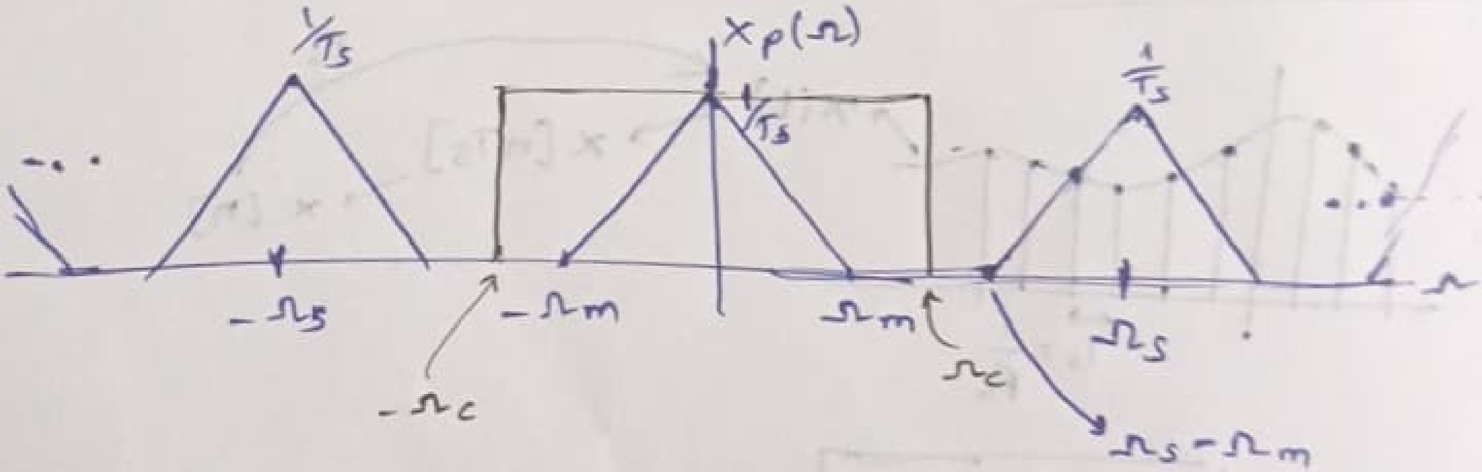
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t-k\,t_0)$$

$$x(t)\cdot p(t) = x_p(t)$$

$$= x(t)\left(\sum_{k=-\infty}^{\infty} \delta(t-kT_s)\right)$$

$$P(\Omega)\ \cancel{p(t)} = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\Omega-k\Omega_s)$$





$$\Omega_s - \Omega_m \geq \Omega_m$$

$$\Rightarrow \boxed{\Omega_s \geq 2\Omega_m}$$

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$x(T) = A\cos(2\pi F T)$

$x[nT_s] = A\cos(2\pi F nT_s)$

$= A\cos\left(2\pi \frac{F}{F_s} n\right)$

$f = \frac{F}{F_s} \Rightarrow -\frac{1}{2} < f < \frac{1}{2}$

* $x[n] = \cos(2\pi\ 0.3n)$ $\qquad x[n] = \cos(2\pi\ 0.7n)$

$= \cos(2\pi\ (1-0.3)n)$

$= \cos(2\pi\ 0.3\ n)$

$f = 0.7$ will appear as $f = 0.3$

0.8 $\qquad$ 0.2

0.9 $\qquad$ 0.1

$\vdots$ $\qquad$ $\vdots$

* $x(t) = e^{2t} u(t)$

$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt = \int_{-\infty}^{\infty} e^{2t} u(t) e^{-j\Omega t}\, dt$

$X(\Omega) = \int_{0}^{\infty} e^{t(2-j\Omega)}\, dt = \cancel{\left[\frac{e^{t(2-j\Omega)}}{2-j\Omega}\right]_0^{\infty}}$

$e^{2t} \rightarrow$ does not satisfy Dirichlet's condition $\qquad \cancel{\frac{1}{j\Omega - 2}}$

* $X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt$

$X(\sigma + j\Omega) = \int_{-\infty}^{\infty} \{x(t) e^{-\sigma t}\} e^{-j\Omega t}\, dt$ $\qquad$ attenuation

$= \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$ $\qquad \boxed{s = \sigma + j\Omega}$

eg: $x(t) = e^{-at} u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-t(a+s)}\, dt$$

$$= \int_{0}^{\infty} e^{-at} \cdot e^{-(\sigma+j\Omega)t}\, dt \qquad = \frac{1}{s+a}$$

$$= \int_{0}^{\infty} e^{-(a+\sigma+j\Omega)t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+\sigma)t} \cdot e^{-j\Omega t}$$

Its the angle

& is the cause of diverging

$$X(s) = \frac{1}{s+a} \quad \text{if} \quad \underbrace{\underset{(\sigma)}{Re\{s\}} > -a}_{\text{region of convergence}}$$

Range of values of σ in which laplace transform exists.

eg: $x(t) = -e^{-at} u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = -\int_{-\infty}^{0} e^{-t(a+s)}\, dt$$

$$= \frac{1}{s+a}$$

$$= -\int_{-\infty}^{0} e^{-t(a+\sigma+j\Omega)}\, dt$$

$$= -\int_{-\infty}^{0} e^{-t(a+\sigma)} \cdot e^{-tj\Omega}\, dt$$

$a + \sigma < 0$

$a + Re\{s\} < 0 \Rightarrow \underbrace{Re\{s\} < -a}_{\text{region of convergence}}$

$$\boxed{X(s) = \frac{1}{s+a} \quad \text{if} \quad Re\{s\} < -a}$$

eg: $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \int_0^\infty 3e^{-t(2+\sigma+j\Omega)} - 2e^{-t(1+\sigma+j\Omega)}\,dt$$

$$= \underbrace{3\int_0^\infty e^{-t(2+\sigma)} \cdot e^{-jt\Omega}\,dt}_{} - 2\underbrace{\int_0^\infty e^{-t(1+\sigma)} \cdot e^{-tj\Omega}\,dt}_{}$$

$2 + Re\{s\} > 0$ $\quad\quad$ $1 + Re\{s\} > 0$

$\Rightarrow Re\{s\} > -2$ and $Re\{s\} > -1$

region of convergence $\rightarrow Re\{s\} > -1$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

$$\boxed{X(s) = \frac{s-1}{(s+2)(s+1)} \; ; \; Re\{s\} > -1}$$

eg: $x(t) = e^{-2t}u(t) - e^{-3t}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} \left(e^{-2t}u(t) - e^{-3t}u(-t)\right) e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-t(2+s)}\, dt - \int_{-\infty}^{0} e^{-t(3+s)}\, dt$$

$\sigma > -2$ and $\sigma + 3 < 0$

$\Rightarrow \sigma < -3$

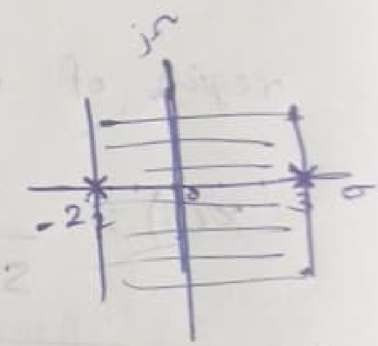
No common region of convergence

eg: $x(t) = e^{-2t}u(t) - e^{3t}u(-t)$

$$X(s) = \int_{0}^{\infty} e^{-t(2+s)}\, dt - \int_{-\infty}^{0} e^{-t(s-3)}\, dt$$

$\sigma > -2$ and $\sigma < 3$

$\rightarrow \boxed{-2 < \sigma < 3}$



$$X(s) = \frac{1}{s+2} + \frac{1}{s-3}$$

$$\boxed{X(s) = \frac{2s-1}{(s+2)(s-3)} \; ; \; -2 < Re\{s\} < 3}$$

* $X(s) = \dfrac{s-1}{(s+1)(s+2)} = \dfrac{N(s)}{D(s)}$ $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$Re\{s\} > -1$

$s = 1$ $(N(s) = 0) \to$ roots - zeros

$s = -1, -2$ $(D(s) = 0) \to$ poles

* $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$

$Y(s) = X(s) \cdot H(s)$

Transfer function: $H(s) = \dfrac{Y(s)}{X(s)}$

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Z - Transform

$s = \sigma + j\Omega$ $\underbrace{e^{-\sigma t}}_{\text{attenuation factor}} \cdot e^{-j\Omega t}$

$\int_{-\infty}^{\infty} (x(t)\, e^{-\sigma t})\, e^{-j\Omega t}\, dt$

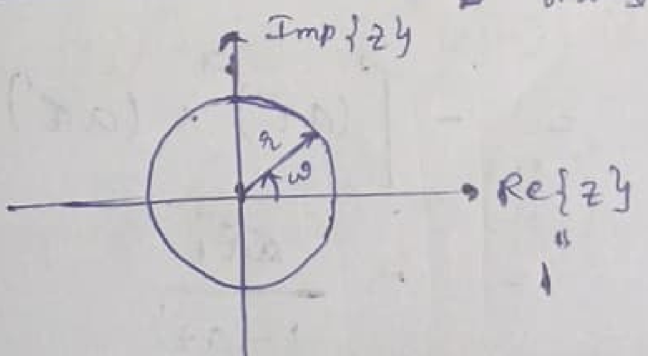
* $e^{-st} \longleftrightarrow z^{-n}$ $n = t$

$z = e^{s} = e^{j\omega}$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$ ← Z - transform

$z = re^{j\omega}$

$z = re^{j\omega}$



eg: ~~x[n]~~ $x[n] = a^n u[n]$; $a < 1$

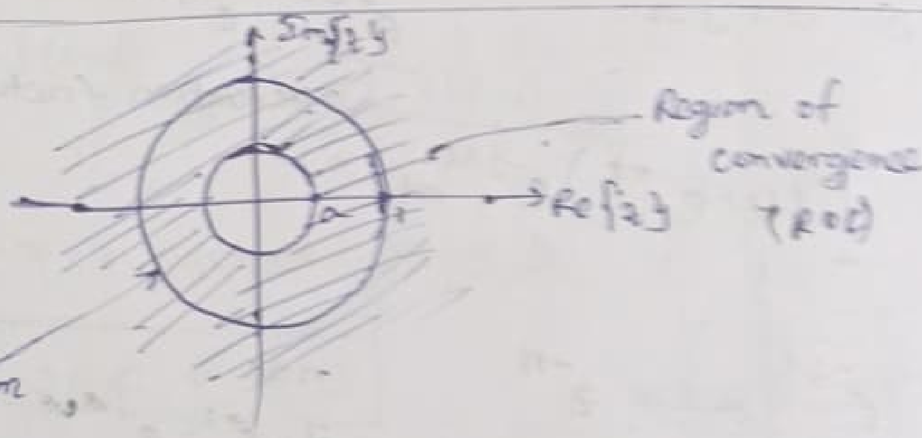
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z = re^{j\omega}$$

$$= \sum_{n=-\infty}^{\infty} (a^n u[n]) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \cdots$$

$$\boxed{X(z) = \frac{1}{1 - \frac{a}{z}} \quad ; \left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|}$$



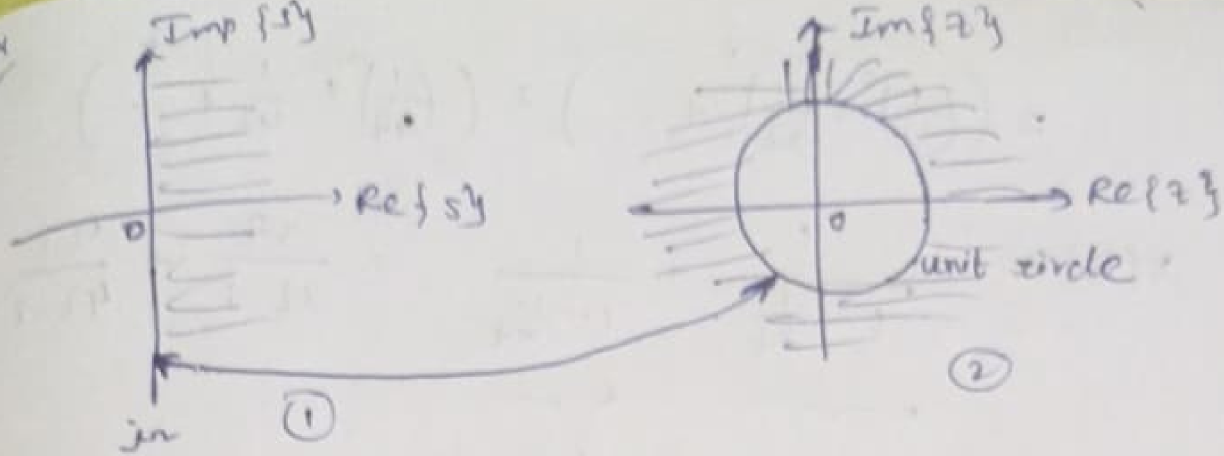
eg: $x[n] = -a^n u[-n]$

$$X(z) = \sum_{n=-\infty}^{0} -a^n z^{-n} = -\sum_{n=-\infty}^{0} (a z^{-1})^n$$

$$= -\left[(a z^{-1})^0 + (a z^{-1})^{-1} + \cdots\right]$$

$$= -\frac{a z^{-1}}{1 - a z^{-1}}$$

6/11/24



for stable it should include $j\Omega$ axis in ①
(or)
unit circle in ②

eg. $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

$X(z) = 1$; ROC : Entire z-plane

eg. $x[n] = \delta[n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = \frac{1}{z}$$

$\boxed{X(z) = \frac{1}{z} \text{ ; ROC : Entire z-plane except } z=0}$

eg: $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]\right) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n + \left(\frac{1}{4z}\right)^n$$

$$= \left(\left(\frac{1}{3z}\right)^0 + \left(\frac{1}{3z}\right)^1 + \cdots\right) + \left(\left(\frac{1}{4z}\right)^0 + \frac{1}{4z} + \cdots\right)$$

$$X(z) = \underbrace{\frac{1}{1-\frac{1}{3z}}} + \underbrace{\frac{1}{1-\frac{1}{4z}}} = \frac{3z}{3z-1} + \frac{4z}{4z-1}$$

ROC $|z| > 1/3$ and $|z| > 1/4$

$\Rightarrow$ ROC : $|z| > 1/3$

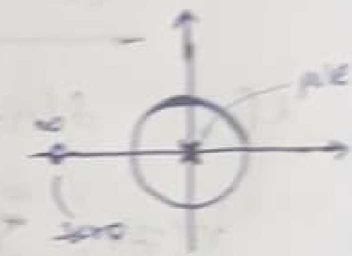
Property:

$$a\,x_1[n] + b\,x_2[n] \overset{z}{\longleftrightarrow} a\,\underbrace{X_1(z)}_{ROC - R_1} + b\,\underbrace{X_2(z)}_{R_2}$$

ROC is atleast $R_1 \cap R_2$

no. of zeros = no. of poles

eg: $X(z) = 1/z$ zero at $z = \infty$, pole at $z = 0$



Pb: $x[n] = \{1, \underset{\uparrow}{2}, 3, 4\}$

$X(z) = 1 \cdot z + 2\,z^0 + 3\,z^{-1} + 4\,z^{-2}$ $\boxed{X(z) = \sum x[n] z^{-n}}$

$X(z) = z + 2 + 3z^{-1} + 4z^{-2}$

ROC Entire z-plane except $|z| = 0, \infty$

Convolution Property:

$$x[n] * h[n]$$
$$\updownarrow$$
$$\underbrace{X(z)}_{R_1} \cdot \underbrace{H(z)}_{R_2}$$

$\therefore$ ROC : Atleast $R_1 \cap R_2$

eg: $x[n] = a^n$; $n = 0$ to $M-1$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n}$$

$$= \sum_{n=0}^{M-1} (a z^{-1})^n$$

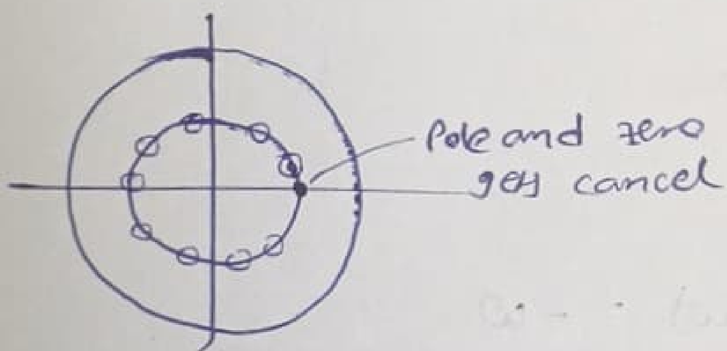
$$= 1 + (a z^{-1})^1 + \cdots + (a z^{-1})^{M-1}$$

$$= \frac{1(1 - (a z^{-1})^M)}{1 - a z^{-1}} = \frac{1 - (a z^{-1})^M}{1 - a z^{-1}}$$

$$X(z) = \frac{z^M - a^M}{(z^{M-1})(z - a)}$$

Zeros : $z^M = a^M$

$$z_k = (a e^{j 2\pi k})$$



eg: $x[n] = a^n u[n]$, $a = 1 \rightarrow$

$$X(z) = \frac{1}{1 - z^{-1}}$$

$a = 1$; $X(\omega) = \dfrac{1}{1 - e^{-j\omega}}$

$-\pi$ to π

8/4/24

eg: $y[n] = x[n-1]$

$\updownarrow z$

$Y(z) = X(z)\, z^{-1}$ — single unit delay element

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1] = y[n]$$

$$H(z) = \frac{Y(z)}{X(z)}\bigg|_{z = e^{j\omega}}$$

$H(\omega) = e^{-j\omega}$

$|H(\omega)| = 1$. $\angle H(j\omega) = -\omega$

$\angle H(j\omega) = -m\omega$ for $x[n-m]$

$\angle H(j\omega)$ is linear function of ω

Phase delay $= \dfrac{\angle H(j\omega)}{\omega}$

Group delay, $\frac{d\,\angle H(j\omega)}{d\omega}$

eg. $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

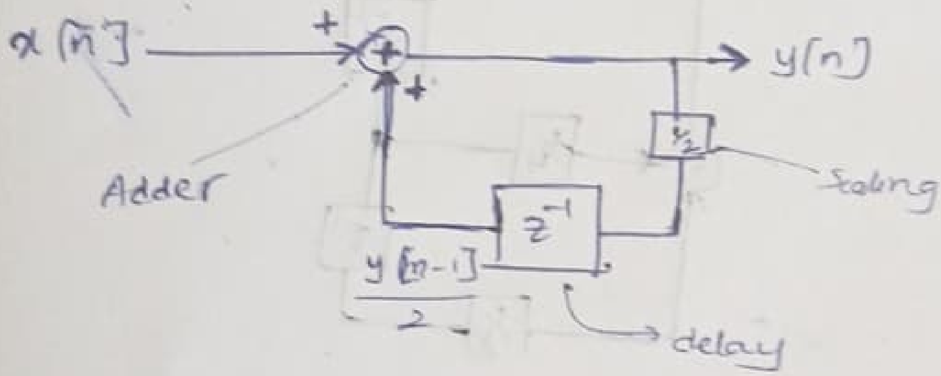
I

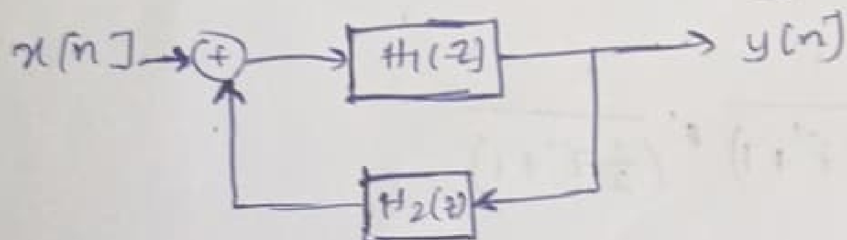
Inverse z transform

$\frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow Y(z)\left(1-\frac{1}{2}z^{-1}\right) = X(z).$

$\Rightarrow y[n] - \frac{1}{2}y[n-1] = x[n]$

$\Rightarrow y[n] = \frac{1}{2}y[n-1] + x[n]$





$$\boxed{H(z) = \frac{H_1(z)}{1+H_1(z)\,H_2(z)}}$$

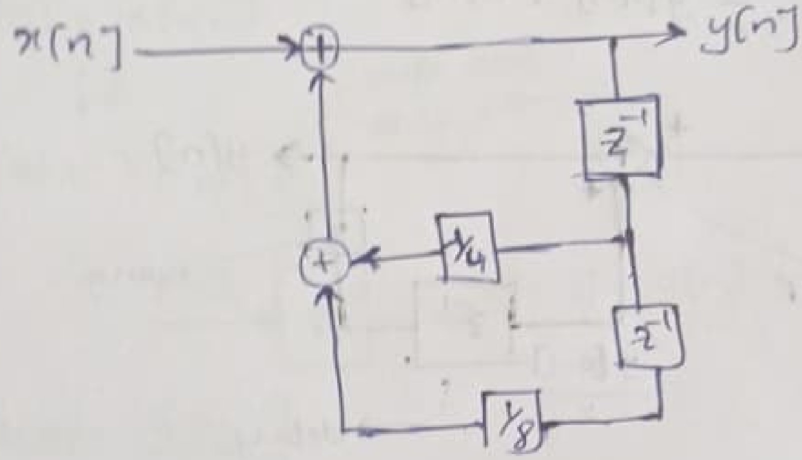
eg. $H(z) = \dfrac{1}{1 + \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}}$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 + \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}}$$

$$X(z) = Y(z)\left(1 + \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}\right)$$

$$x(n) = y[n] + \frac{1}{4} y[n-1] - \frac{1}{8} y[n-2]$$

$$y(n) = x[n] - \frac{1}{4} y(n-1] + \frac{1}{8} y[n-2]$$



$$H(z) = \frac{1}{\left(-\frac{1}{4} z^{-1} + 1\right)} \cdot \frac{1}{\left(\frac{1}{2} z^{-1} + 1\right)}$$