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Numerical Analysis

$P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$

↳ Algebraic eqn

eg: $x^2 + 5x - 7 = 0$

$\tan x - 3x^2 + 5 = 0$

↳ Transcendental function

* An eqn of the form,
  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$
  is called algebraic equation.

Bisection - Method

Let $f: I \to \mathbb{R}$ be continuous function

Step 1: Find $(a, b) \in I$ such that $f(a) \cdot f(b) < 0$

Step 2: Choose c as $\frac{a+b}{2}$

Step 3: If:

* $f(c) = 0$ then c is a root
* $f(c) f(b) < 0$ root lies in $[c, b]$
  put $a = c$ ~~repeat~~
* $f(c) f(a) < 0$ root lies in $[a, c]$
  put $b = c$

Step 4: Repeat step 2 & 3 till $f(c) = 0$

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example :

Consider,

$f(x) = x^3 - 3x + 1 = 0$

find root of eqn.

$a = 0 \qquad f(0) = 1$

$b = 1 \qquad f(1) = 1 - 3 + 1 = -1$

$c_1 = 1/2$

$f(c_1) = 1/8 - 3/2 + 1$

$= 9/8 - 12/8 = -3/8$

$f(c_1) \cdot f(a) < 0$

$\therefore \; b = c_1$

$b = 1/2$

$c_2 = 1/4$

$f(c_2) = \frac{1}{64} - \frac{3}{4} + 1 > 0$

$f(c_2) \cdot f(b) < 0$

root $\in [b, c_2]$ $\qquad a = c_2$

$c_3 = \frac{1/2 + 1/4}{2} = 3/8$

$a = 1 \qquad f(1) = -1$

$b = 2 \qquad f(2) = 3$

$c = \frac{1+2}{2} = 3/2$

$\frac{27}{8} - 9/2 + 1$

$\frac{27}{8} + \frac{8}{8} - \frac{36}{8}$

$c_1 = 3/2$

$f(3/2) > 0$

$f(1) \cdot f(3/2) < 0$

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root exists b/w [1, 3/2]

$c_2 = \frac{1+3/2}{2}$

$= 5/4$

$f(5/4) = \cancel{5/4}$

$x^3 - 3x + 1 = 0$

$f(1) = -1$

$f(2) = 3$

$x_1 = \frac{1+2}{2} = 3/2$

$f(x_1) = (3/2)^3 - 3 \cdot 3/2 + 1$

$= 27/8 - 9/2 + 1$

$= 27/8 - 7/2 > 0$

$f(x_1) \cdot f(1) < 0$

$\therefore x_2 \in (1, x_1)$

$x_2 = \frac{1+x_1}{2} = \frac{1+3/2}{2} = 5/4$

$f(x_2) = (5/4)^3 - 5/4 \cdot 3 + 1$

$= \frac{125}{64} - \frac{15}{4} + 1 < 0$

$f(x_2) \cdot f(x_1) < 0$ $(x_2, x_1) \ni$ root

$\therefore$ ~~(1, x₂) → root~~

$\cancel{x_3 = \frac{1+x_2}{2} = \frac{1+5/4}{2} = 9/8}$

$\cancel{f(x_3) = (9/8)^3 - 3 \cdot 9/8 + 1}$

$x_3 = \frac{x_1+x_2}{2}$

$= \frac{3/2+5/4}{2} = 11/8$

$$8\overline{)11}\ 1.037$$
8
30
24
60

root at 3rd approx = 11/8
= 1.37

Prob. 28.1 27-6-5

1) i) $x^3-2x-5=0$

$\cancel{x_1=0}$ $a=0$ $f(a)=-5$

$b=3$ $f(b)=16$

$f(a)\cdot f(b)<0$

$x_1=3/2$

$f(x_1)=\frac{27}{8}-3-5<0$

$f(x_1)\cdot f(3)<0$

root $\in(x_1,3)$

$x_2=\frac{x_1+3}{2}$

$=\frac{3/2+3}{2}=9/4$

$f(x_2)=(9/4)^3-2\cdot 9/4-5$

$=\frac{729}{64}-\frac{18}{4}-5$

$=\frac{729-608}{64}>0$

$f(x_1)\cdot f(x_2)<0$

root $\in(x_1,x_2)$

$x_3=\frac{x_1+x_2}{2}=\frac{9/4+3/2}{2}=\frac{15}{8}=1.875$

$f(x_3)=\left(\frac{15}{8}\right)^3-2\cdot\frac{15}{\cancel{8}4}-5$

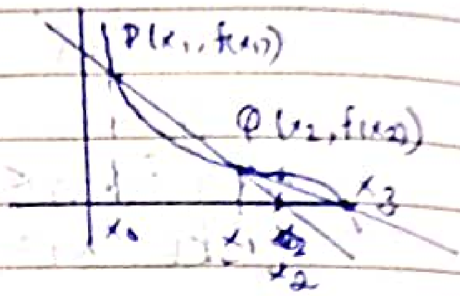
$=\frac{3375}{512}-\frac{15}{4}-5<0$

$(x_2,x_3)\to$ root

$x_4=\frac{x_2+x_3}{2}=2.0625$

Secant Method

let $f(x) = 0$



Algorithm

Step ① Consider x_0 & x_1 as initial approximation of a root $f(x) = 0$
Denote P for $(x_0, f(x_0))$ and Q for $(x_1, f(x_1))$

Step 2 Find eqn ~~for~~ for st. line PQ

$$\frac{y - f(x_1)}{x - x_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{— (1)}$$

Step 3 Put $y = 0$ in (1) & solve for x

$$\frac{-f(x_1)}{x - x_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$x = \frac{(x_1 - x_0)(-f(x_1))}{f(x_1) - f(x_0)} + x_1$$

$$\left[x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \right]$$

Step 4 : Repeat procedure for x_1 & x_2

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eg →

Find a root of $x^3 - 2x - 5 = 0$ using Secant method. $x_0 = 2$, $x_1 = 3$

$f(x_0) = 8 - 4 - 5 = -1$

$f(x_1) = 27 - 6 - 5 = 16$

$$\boxed{\frac{16}{3-2} = \cancel{x-}}$$

$$\frac{16+1}{1} = \frac{-16}{x-3}$$

$17x - 51 = -16$

1st Approx $x_2 = 2.0588$

$f(x_2) = -0.3907$ ... and so on.

$$\frac{16.3907}{0.9412} = \frac{0.3907}{x - 2.0588}$$

$17.414x - 35.853 = 0.3677$

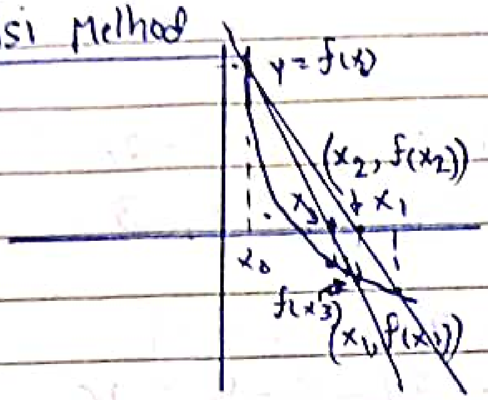
2nd Approx $x_3 = 2.079$

Criteria to stop → $\epsilon = 0.001$

$f(x^*) < \epsilon$

2nd stopping criteria
$|x - x_1|$
$|x_2 - x_1|$.
$|x_3 - x_2| < \epsilon_2 = 0.0005$

Regulor - Folsi Method



Step 1 : Consider x_0 & x_1 as 2 approximation of initial root such that $f(x_1) \cdot f(x_0) < 0$

Repeat for step 2 of the Secant method

Step 4 : Check whether $f(x_0)\, f(x_1) < 0$ or $f(x_1)\, f(x_2) < 0$

Step 5 : Repeat

Eg : Find a root of $x^3 - 2x - 5$ using Regular - Fals Method.

$x_0 = 2$ $8 - 4 - 5$

$x_1 = 3$

$f(x_1) \cdot f(x_0) < 0$

$$x_2 = 3 - \frac{16(1)}{16+1}$$

1st Approx $x_2 = 3 - \frac{16}{17} = 2.0588$

$f(x_2) = -0.3907 < 0$

$f(x_1) \cdot f(x_2) < 0$

$\therefore\; x_3 = 2.0588 + \frac{(0.3907)(\cancel{-}\,0.9412)}{\cancel{+}\,16.3907}$

2nd Approx $x_3 = 2.081$

$\cancel{f(x_3)}$

Secant method → root can be found outside the domain of (x_0, x_1) unlike the other 2 methods.

Problem 28.1

2) i) $\cos x = x e^x$

$\cancel{a=0 \quad f(0)=}$

$x e^x - \cos x = 0$

$a = 0 \quad f(0) = -1$

ii) $2\log_{10} x = 1.2$

$a = 2 \quad f(a) = 2\log 2 - 1.2 < 0$

$b = 3 \quad f(b) = 2\log 3 - 1.2 > 0$

$f(a) \cdot f(b) < 0$

$x_1 = \frac{2+3}{2} = 2.5$

$f(x_1) < 0$

$\therefore \quad f(x_1) \cdot f(b) < 0$

$x_2 = \frac{2.5+3}{2} = 2.75$

$f(x_2) > 0$

$\therefore \quad f(x_1) \cdot f(x_2) < 0$

$x_3 = \frac{x_1+x_2}{2} = 2.625$

$f(2.625) < 0$

$f(x_3) \cdot f(x_2) < 0$

$\therefore \quad x_4 = \frac{x_3+x_2}{2} = 2.6725$

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Newton - Ralphson Method

* function must be differentiable

$y = f(x)$

~~eqn of tangent~~

slope $= \dfrac{dy}{dx} = f'(x)\Big|_{(x_0,y_0)}$

$\dfrac{y-y_0}{x-x_0} = f'(x)$

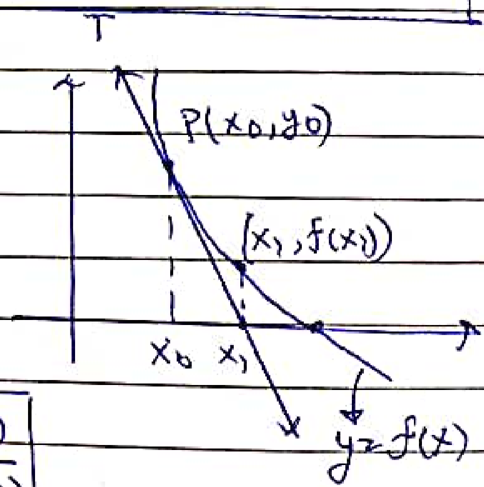
$y-y_0 = f'(x)(x-x_0)$

$\boxed{y = y_0 + f'(x)(x-x_0)}$

Eqn of tangent at (x_0,y_0)

$$\boxed{y = \underset{m}{\underbrace{f'(x)}}\,x - \underbrace{f'(x)(x_0)+y_0}_{c}}$$

put $y=0$



$\dfrac{\cancel{f'(x)}\,x}{\cancel{f'(x)}\cdot x_0}$

$f'(x)\cdot x = f'(x)(x_0) + y_0$

$$\boxed{x = x_0 - \frac{f(x)}{f'(x)}}$$

∴ General form : $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$

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Q :- $x^3-3x-5=0$

$f(x) = x^3-3x-5$

$f'(x) = 3x^2-3$

$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$

$f(0) = -5$
$f(1) = -7$
$f(2) = -3$
$f(3) = \cancel{13}\ 13$
$\cancel{f'(3)}$

27
-9
18-5
13

$x_{n+1} = \cancel{x_n^3-3x_n}\ x_n + \frac{x_n^3-3x_n-5}{3x_n^2-3}$

$x_0 = 2$

$x_{0+1} = 2 + \left(\frac{-3}{\cancel{21}\ 9}\right)$

$= 2 + 1/3 = 7/3 = 2.33$

$\cancel{x_2}\ f(7/3) = \cancel{0.70370}\ 0.6593$ $\quad f'(7/3) = \cancel{4.33}\ 13.2867$

$x_2 = \frac{7}{3} + \frac{0.7037}{4.33}$

$= 2.4925$

$f(x_2) = 3.0075$

$f'(x_2)$

$x_2 = 2.33 + \frac{\cancel{0.7037}\ 0.6593}{13.2867} = 2.3796$

Rate of convergence

An iterative method is said to be of order p has rate of convergence p if p is the largest +ve real number for which there exist a finite constant c is non-zero such that

$$|\epsilon_{k+1}| < c|\epsilon_k|^p$$

where $\epsilon_k = x_k - \xi$ error in k^{th} iteration and c is called as the asymptotic error constant.

a c usually depends upon ~~derivative~~ $f'(x)$ at $x =$

Problems 28.1

6) i) $x^3 - 3x + 1 = 0$

$f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

$f(0) = 1$

$f(1) = -1$

$x_0 = 0.5$

~~$f(0.5) = 0.631$~~ $\quad x_1 = x_0 + \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = 0.33$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 0.34718$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 0.34729$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7935$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7983$$

——×——×——×——×

Q8)

Rate of convergence

→ $|x_n - r| < \epsilon$

→ $|x_{n+1} - x_n| < \epsilon$

→ $|f(x_n)| < \epsilon$

let r be a root of $f(x) = 0$

Now, $x_k \to k^{th}$ iteration

$$\epsilon_k = x_k - r$$

error

let ϕ be the ~~ree~~ largest real number such that

$$|\epsilon_{k+1}| \leq c\,|\epsilon_k|^{\phi}$$

where ϕ is called rate of convergence

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i) Secant Method

$$\left[x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(k-1)} \right] \quad -(1)$$

Assume r k root of $f(x) = 0$

let

$$\epsilon_{k+1} = x_{k+1} - \gamma$$
$$\epsilon_k = x_k - \gamma$$
$$\epsilon_{k-1} = x_{k-1} - \gamma$$

Substitute value of (1) x_{k+1}, x_k, x_{k-1}

$$\epsilon_{k+1} + \gamma = \epsilon_k + \gamma - f(\epsilon_k+\gamma) \cdot \left[\frac{((\epsilon_k + \gamma) - (\epsilon_{k-1} + \gamma))}{f(\epsilon_k+\gamma) - f(\epsilon_{k-1}+\gamma)} \right] \quad -(2)$$

Using Taylor series

$$f(\epsilon_k+\gamma) = f(\gamma) + \epsilon_k f'(\gamma) + \frac{\epsilon_k^2}{2!} f''(\gamma) + \dots$$

$$f(\epsilon_{k-1}+\gamma) = f(\gamma) + \epsilon_{k-1} f'(\gamma) + \frac{\epsilon_{k-1}^2}{2!} f''(\gamma) \dots$$

$$f(\epsilon_k+\gamma) - f(\epsilon_{k-1}+\gamma) = (\epsilon_k - \epsilon_{k-1}) f'(\gamma) + (\epsilon_k^2 - \epsilon_{k-1}^2) \frac{f''(\gamma)}{2!} \quad -(3)$$

Use (3) in (2)

$$\epsilon_{k+1} + \cancel{\gamma} = \epsilon_k + \cancel{\gamma} - \frac{(\epsilon_k - \epsilon_{k-1})(\overset{0}{f(\gamma)} + \epsilon_k f'(\gamma) + \frac{\epsilon_k^2}{2!} f''(\gamma) \dots)}{(\epsilon_k - \epsilon_{k-1}) f'(\gamma) + (\epsilon_k^2 - \epsilon_{k-1}^2) \frac{f''(\gamma)}{2!}}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{\cancel{(\epsilon_k - \epsilon_{k-1})} (\epsilon_k f'(\gamma) + \frac{\epsilon_k^2}{2!} f''(\gamma))}{f'(\gamma) + \left(\frac{\epsilon_k + \epsilon_{k-1}}{2}\right) f''(\gamma)}$$

$$= \epsilon_k - f'(\gamma)\left(\epsilon_k + \frac{\epsilon_k^2 f''(\gamma)}{2! \, f'(\gamma)}\right) \Big/ f'(\gamma)\left(1 + \frac{(\epsilon_k + \epsilon_{k-1}) f''(\gamma)}{2 f'(\gamma)}\right)$$

use binomial expension of $(1+a)^{-1}$

$$= \varepsilon_k - \left[1 - \frac{1}{2}(\varepsilon_k + \varepsilon_{k-1})\frac{f''(r)}{f'(r)} + \cdots\right]\left[\varepsilon_k + \frac{1}{2}\frac{f''(r)}{f'(r)}\varepsilon_k^2\right]$$

$$= \varepsilon_k - \varepsilon_k + \frac{1}{2}\varepsilon_k \cdot \varepsilon_{k-1}\frac{f''(r)}{f'(r)}$$

$$\boxed{|\varepsilon_{k+1}| = \frac{1}{2}|\varepsilon_k||\varepsilon_{k-1}|\frac{f''(r)}{f'(r)}}$$

$$|\varepsilon_{k+1}| \leq c|\varepsilon_k|^{\phi}$$

$$|\varepsilon_{k}| \simeq c|\varepsilon_{k-1}|^{\phi}$$

$$|\varepsilon_{k-1}| = c^{-1/\phi}|\varepsilon_k|^{1/\phi}$$

$$c|\varepsilon_k|^{\phi} = |\varepsilon_{k+1}|$$

$$= \frac{1}{2}\left|\frac{f''(r)}{f'(r)}\right| c^{-1/\phi}|\varepsilon_k||\varepsilon_k|^{1/\phi}$$

$$= \frac{1}{2}\left|\frac{f''(r)}{f'(r)}\right|^{-1/\phi} c^{1+1/\phi}$$

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi \simeq 1.62$$

$$\cancel{f(\ ) = f(r) + \phi\,\varepsilon}$$

HW

Q) Obtai

Q)

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HW Calculate the rate of convergence for Regula-Falsi Method

Q) Obtain the value of $\sqrt[3]{25}$ correct for two decimal places using Newton raphson

$$r = \sqrt[3]{25}$$
$$r^3 = 25$$
$$r^3 - 25 = 0$$
$$f(x) = x^3 - 25$$
$$f'(x) = 3x^2$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2.925$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.924$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.924$$
$$\therefore \boxed{r = 2.92}$$

Q) How many steps would you expect to carry out solve:

i) $e^x - 3x = 0$ $a = 0$ & $b = 1$

ii) $x^3 - 3x + 1 = 0$ $a = 1$ & $b = 2$

correct to two decimal places

Hint $I \in (a_0, b_0)$ $I_1 \in (a_1, b_1)$, $I_2 \in (a_2, b_2)$

[for bisection method]

$x_1 = \frac{a_0+b_0}{2}$

$|x_1 - r| \leq 1/2\,(b_0 - a_0)$

$|x_2 - r| \leq 1/2\,(b_1 - a_1) = 1/2^2\,(b_0 - a_0)$

$\vdots$

$|x_n - r| \leq 1/2^n\,(b_0 - a_0) < 0.001$

ϵ ↓ 2 decimal places

$$2^n > \frac{b_0 - a_0}{\epsilon}$$

$$\left[\, n > \frac{\log(b_0 - a_0) - \log \epsilon}{\log 2} \,\right]$$

<u>Ex 28.1</u>

12) $r = \sqrt{N}$

$r^2 = N$

$r^2 - N = 0$

$\therefore f(x) = x^2 - N$

i) $f(x) = x^2 - 13$

$\{x_0 = 4\}$ $f(4) = 16 - 13 = 3$

$f'(x) = 2x$

$f'(4) = 8$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - 3/8$$

$= 3.625$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$= 3.6056$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.6055 = 3.606$$

13)

14)

15) $r = (28)^{-1/4}$

$\frac{1}{r} = (28)^{1/4}$

$\frac{1}{r^4} = 28$

$28r^4 - 1 = 0$

$f(x) = 28x^4 - 1$

$f'(x) = 112x^3$

$x_0 = 1/3$

$x_1 = x_0 - \frac{f(x_1)}{f'(x_1)}$

$= 0.4959$

$x_2 = 0.4451$

$x_3 = 0.4350$

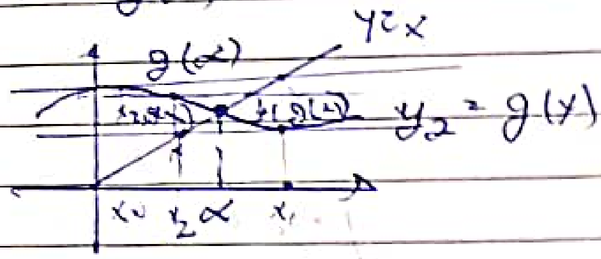
$x_4 =$

Fixed point iteration

Let g be a continuous function. A point $\alpha \in R$ is said to be fixed point of g if $\boxed{g(\alpha) = \alpha}$

$\boxed{f(x) = 0} \longrightarrow x = g(x)$

$g: R \rightarrow R$



Write $f(x) = 0$ as $x = g(x)$ for some $g(x)$ closer to initial approximation x_0

$$\boxed{x_{n+1} = g(x_n)}$$

eg: $f(x) = x^3 - 5x + 1 = 0$

$f(2) = -1 < 0$

$f(3) = 13 > 0$

$x_0 = 2$

$f(x) = x^3 - 5x + 1 = 0$

$x = g(x)$

$x^3 = 5x - 1$ or $x^3 + 1 = 5x$ or $x^3 - 4x + 1 = x$

$x = (5x - 1)^{1/3}$ $\qquad x = \frac{x^3 + 1}{5}$

$g_1(x)$ $\qquad g_2(x)$ $\qquad g_3(x)$

$x_1 = (5x - 1)^{1/3}$	$x_1 = \frac{x_0^3 + 1}{5}$	$x_1 = x_0^3 - 4x_0 + 1$
$x_0 = 2$		$= 1$
$x_1 = 9^{1/3}$	$= 1.8$	$x_2 = -2$
$= 2.08$	$x_2 = \frac{x_1^3 + 1}{5}$	Not b/w 2 & 3
$x_2 = (5x - 1)^{1/3}$	$= 1.384$	so discard
$x_2 = 2.11$	not b/w 2 & 3	
$x_3 = (5x - 1)^{1/3}$	so discard	
$= 2.121$		
$x_4 = (5x - 1)^{1/3}$		
$= 2.1257$		

M-I

How to pick g

Assume g is diffrentiable

$$\varepsilon_{k+1} = x_{n+1}^{k} - r \qquad \left\{ \frac{f(a)-f(b)}{a-b} = f'(c) \right\}$$

$$= g(x_k) - g(r)$$

$$= (x_k - r)\, g'(\alpha_k) \qquad \{ r < \alpha_k < x_k \}$$

$$\cancel{= \varepsilon_k\, g'(\alpha_k)}$$

$$\cancel{g(x_{k+1})}$$

$$= [g(x_{k-1}) - g(r)]\, g'(\alpha_k)$$

$$= (x_{k-2} - r)\, g'(\alpha_k)\, g'(\alpha_{k-2})$$

$$\vdots$$

$$= (x_0 - r)\, g'(\alpha_0)\, g'(\alpha_1) \cdots g'(\alpha_k)$$

$$|\varepsilon_{k+1}| \le \varepsilon_0 |c|^{k+1}$$

$$|g'(x)| \le c$$

if $c < 1$

then

$$|\varepsilon_{k+1}| \le c^{k+1} \cdot \varepsilon_0$$

choose function $g(x)$ where $f(x) = 0$ can be written as $x = g(x)$ such that ~~max~~ $\max |g'(x)| \le c < 1$

So using same in previous example

$$g_1(x) = (5x-1)^{1/3}$$

$$g_1'(x) = \frac{5}{3(5x-1)^{2/3}} \text{ at } x = 2 \quad < 1$$

M-II Findin

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M-II

Finding g

$f(x) = 0 \qquad x = g(x)$

$g(x) = x + \cancel{\alpha(x)}\ \alpha f(x)$

(1) $x = x + \alpha f(x)$

(2) $|g'(x)| < 1$

$\cancel{x+\alpha}\ |1 + \alpha f'(x)| < 1$

Choose $\alpha \in R$ in such a way that

eg// $f(x) = x^2 - 5$

$x = x + \alpha f(x)$

$= x + \alpha(x^2 - 5)$

$|1 + \alpha f'(x)| < 1$

$|1 + \alpha(2x)| < 1$

$x_0 \in (2, 3)$

$x_0 \sim 2.5$

$|1 + 2\alpha x_0| < 1$

$|1 + 5\alpha| < 1$

$-1 < 1 + 5\alpha < 1$

$\left\{ \frac{-2}{5} < \alpha < 0 \right\}$

?

let $\alpha = -0.1$

$g(x) = x \cancel{+} - 0.\cancel{8}(x^2 - 5)$ ~~0~~

$- \cancel{0.5x^2 + x\cdot 8}$

x^2

Newton Ralphson rate of convergence

$\epsilon_{n+1} = x_{n+1} - r$

$|\epsilon_{n+1}| \leq c|\epsilon_n|^p$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\epsilon_{n+1} + \cancel{r} = \epsilon_n + \cancel{r} - \frac{f(\epsilon_n + r)}{f'(\epsilon_n + r)}$

let f be twice diffrentiable

$f(\epsilon_n + r) = f(r) + \epsilon_n f'(r) + \frac{\epsilon_n^2}{2!} f''(r) \ldots$

$f'(\epsilon_n + r) = f'(r) + \epsilon_n f''(r) + \frac{\epsilon_n^2}{2!} f'''(r) \ldots$

$$\epsilon_{n+1} = \epsilon_n - \frac{\cancelto{0}{f(r)} + \epsilon_n f'(r) + \epsilon_n^2/2!\, f''(r) + \ldots}{f'(r) + \epsilon_n f''(r) + \frac{\epsilon_n^2}{2!} f'''(r) + \ldots}$$

$$= \epsilon_n - \frac{\cancel{f'(r)}\left[\epsilon_n + \epsilon_n^2/2!\, \frac{f''(r)}{f'(r)} + \ldots\right]}{\cancel{f'(r)}\left[1 + \epsilon_n \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2!} \frac{f'''(r)}{f'(r)}\right]}$$

$$= \epsilon_n - \left[\epsilon_n + \epsilon_n^2/2!\, \frac{f''(r)}{f'(r)}\right]\left[1 - \epsilon_n \frac{f''(r)}{f'(r)}\right.$$

$$= \cancel{\frac{\epsilon_n^3}{2!} f} \quad \cancel{\epsilon_n} - \left[\cancel{\epsilon_n} - \epsilon_n^2 \frac{f''(r)}{f'(r)} + \frac{\epsilon_n^2}{2!} \frac{f''}{f'}\right.$$

$$- \epsilon_n^3 / f'(r)^2 \}$$

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$$|\epsilon_{n+1}| \leq |\epsilon_n^2| \left|\frac{f''(r)}{2f'(r)}\right|$$

$$\boxed{c = \frac{f''(r)}{2f'(r)}} \qquad \boxed{p = 2}$$

Newton raphson method has ~~2~~ second order of convergence

$x_{n+1} = \phi(x)$

Using newton's formula

$$x_{n+1} = g(x_n)$$

$$e_{n+1} = x_{n+1} - r \qquad \text{where } \begin{Bmatrix} f(r) = 0 \\ g(r) = r \end{Bmatrix}$$

$$e_{n+1} + r = g(\epsilon_n + r)$$

$$= \cancel{r + \epsilon_n g(r)}$$

$$e_{n+1} + \cancel{r} = \cancel{g(r)} + \epsilon_n g'(r) + \frac{\epsilon_n^2}{2!} g''(r)$$

$$\{g(r) = r\}$$

$$e_{n+1} = \epsilon_n g'(r) + \frac{\epsilon_n^2}{2!} g''(r)$$

We know that if maximum of $|g'(x)| \leq c < 1$ then the sequence $\{x_n\}$ converges to r.

$|g(r)| \leq 1$

Now, <u>case 1</u> : $g'(r) \neq 0$ {~~Neglect~~ higher order terms}

<u>case 2</u> : $g'(r) = 0$ but $g''(r) \neq 0$

<u>case 3</u> : $g'(r) = 0, g''(r) = 0$ & $g'''(r) \neq 0$

case 4 : $g'(r) = 0, = g''(r) = \cdots g^{(p-1)}(r) = 0$ but $g^{(p)}$

case 1: $|\epsilon_{n+1}| \leq |g(x)||\epsilon_n|$

case 2: $g'(r) = 0$ but $g''(r) \neq 0$

$$|\epsilon_{n+1}| \leq |\epsilon_n|^2 \left|\frac{g''(r)}{2!}\right|$$

case 3 : $$|\epsilon_{n+1}| \leq |\epsilon_n|^3 \left|\frac{g'''(r)}{3!}\right|$$

$$|\epsilon_{n+1}| \leq \left|\frac{g^{(p)}}{p!}\right| |\epsilon_n|^p$$

System of Linear Equations

$$2x + 3y = 7$$
$$4x + 5y = 8$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

~~Gauss~~ Method

Cramer's rule

$$\left\{ \begin{aligned} x &= \frac{\Delta_1}{\Delta_0} \\ y &= \frac{\Delta_2}{\Delta_0} \end{aligned} \right\}$$

$$|\Delta_1| = \left|\begin{bmatrix} 7 & 3 \\ 8 & 5 \end{bmatrix}\right|$$

$$|\Delta_2| = \left|\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}\right|$$

$$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 \cdots + a_{nn}x_n = b_n$$

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$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\boxed{AX = b}$ → $r_1 =$ no of LI column vectors

$m \times n$

$r_2 =$ no of LI row vectors

$\dim(\mathbb{R}^2) = 2$

basis $(\mathbb{R}^2) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ → any element can be written as linear combination of these two elements

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \end{pmatrix} \right]_{2 \times 5}$$

$$\left\{ \begin{array}{l} r_2 = 2 \\ r_1 = 2 \end{array} \right\}$$

No of LI column vector = No of LI row vector

Solutions of Linear System

1) A linear system of m equation in n-unknowns is of the form -

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1} + a_{m2} + a_{m3} \cdots + a_{mn}x_n = b_n$$

$a_{jk} \in R$ $1 \leq j \leq m$, $1 \leq k \leq n$ → coefficients, $b_j \in R$ → constants A $x_1, x_2 \cdots x_n$ are unknown

An element $\underline{x} \in (x_{12}, x_n)$ is said to be a solution of ① if $A\underline{x} = b$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{m1} & & & & a_{mn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

<u>case 1</u> (Homogeneous System) The system ① is called homogeneous system if $b_1 = b_2 = b_3 \cdots = b_n$

In this case, solution has atleast 1 solution (zero), $x_1 = x_2 = \cdots x_n = 0$.

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Case 2: (non-homogeneous system).

The system is said non-homogeneous system if at-least one of b_i is non-zero:

In this case

- → may not have solution $\begin{Bmatrix} 2x+4y=5 \\ 4x+6y=8 \end{Bmatrix}$
- → may have unique solution $\begin{Bmatrix} y-x=0 \\ y-2x=0 \end{Bmatrix}$
- → may have infinite many solution

$3x+y=5$

$~~3x~~\ 10x+\frac{2}{5}y=25$

Special Case:

Assume $a_{11}\neq 0$

$a_{22}\neq 0 \quad \cdots \quad a_{mn}\neq 0$

$$a_{11}x_1+a_{12}x_2 \cdots a_{1n}x_n=b_1$$
$$a_{22}x_2+a_{23}x_3 \cdots a_{2n}x_n=b_2$$
$$a_{mn-1}x_{n-1}+a_{mn}x_n=b_{n-1}$$
$$a_{mn}x_n=b_n$$

By backsubstitution,

$$x_n=\frac{b_n}{a_{mn}}$$

$$x_{n-1}=\frac{1}{a_{mn-1}}\left(b_{n-1}-\frac{b}{a_{mn}}a_{mn-1}\right)$$

eg// Consider the linear system,

$$x_1-x_2+x_3=0$$
$$-x_1+x_2-x_3=0 \quad (1)$$
$$10x_2+25x_3=90 \quad (2)$$
$$20x_1+10x_2=80 \quad (3)$$

Step (1): Eliminate x_1 from (2) & (1) eqn

$$-20x_1+20x_2-20x_3=0$$
$$30x_2-20x_3=80 \quad (4)$$

$$\left(3x_2 - 2x_3 = 80\right)^{25/2}$$
$$10x_2 + 25x_3 = 90$$

$$\frac{75x_2}{2} + 10\,[illegible]\,\frac{x_2}{2} - 100 + 90$$

$$\frac{95x_2}{2} = 190$$

$$\boxed{x_2 = 4}$$

$$\boxed{x_3 = \frac{50}{25} = 2}$$

$$\boxed{x_1 = 2} \quad \boxed{x_1 = 2}$$

$\boxed{\text{Gauss Elimination}}$

$$x_1 - x_2 + x_3 = 0 \quad \text{—(1)}$$

$$10x_2 + 25x_3 = 90 \quad \text{—(2)}$$

$$20x_1 + 10x_2 = 80 \quad \text{—(3)}$$

Eliminate x_1 from (3)

$$x_1 - x_2 + x_3 = 0 \quad \text{—(4)}$$
$$10x_2 + 25x_3 = 90 \quad \text{—(5)}$$
$$30x_2 - 20x_3 = 80 \quad \text{—(6)}$$

Eliminate x_2 from (2) & (3)

$$x_1 - x_2 + x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$95x_3 = 190$$

Using back-substitution

$x_3 = 2$

$x_2 = \dfrac{190 - \cancel{50}\ 90}{\cancel{100} \times 25} = 4$

$x_1 = 2$

Consider the agumented matrix

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$R_4 \to R_4 - 20 \times R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$25 + 20/3$

$90 - 80/3$

$193/$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2/3$

$x_1 - 4 + 2 = 0$

$\boxed{x_1 = 2}$

$\frac{95}{3} x_3 = 190/3$

$\boxed{x_3 = 2}$

$30x_2 - 20 \cdot 2 = 80$

$\boxed{x_2 = 4}$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95/3 & 190/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$$

Such a matrix is called,
Row - Echelon form

$A_1 x = b_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 90 \\ -190 \\ 0 \end{bmatrix}$$

A matrix A is said to be in row-Echelon form if the following condition are satisfied.

i) If i th row contains all zeros it is true for all subsequent rows.

ii) If i^{th} and $(i+1)^{th}$ row are both non-zero rows initial non-zero entry of $(i+1)^{th}$ row appears in a later column that of the i^{th} row.

iii) Rows containing all zeros occur only after all non-zero rows.

Def : If a row is non-zero in a matrix. Then the first non-zero element in that row is called pivot.

Elementary Row operations

(i) Interchange of any two rows ($R_i \leftrightarrow R_j$)

(ii) Addition of a scalar multiple of a row to another row. $R_i \to R_i + \alpha R_j$

(iii) Multiplication / Division of any row by non-zero scalar $R_i = \alpha R_i \ (\alpha \neq 0)$

eg// Find rank of matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$ by reducing it to the row-echelon form.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Rank} = 2$$

eg// $\mathbb{R}^3 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \ni a, b, c \in \mathbb{R} \right\}$

$x_1 = (1 \ 3 \ 5)^T$

$x_2 = (2 \ -1 \ 4)^T$ Check whether they are linearly dependent / independent?

$x_3 = (-2, 8, 2)^T$

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Steps:

* reduce to row-Echelon
* look at row-Echelon form and look at no. of non-zero rows

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_2 + 2R_1$

$$\sim \begin{pmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{pmatrix}$$

$R_3 \to R_3 + 2R_2$

$$= \begin{pmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

If no. of non zero row ≠ ~~no. of~~ order of matrix then the eqns are linear dependent.

else, they are independent.

Gauss Elimination Method

This is an algorithm to transform a system linear equation into its row-echelon form and then back-substitution method is applied to solve

Steps: ① Step up agumented matrix
② Transform to its row-echelon form
③ Use back-substitution to solve

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eg: Use Gauss Elimination method to solve:

$$6x_2 + 13x_3 = 61$$
$$6x_1 - 8x_3 = -38$$
$$13x_1 - 8x_2 = 79$$

$$\begin{bmatrix} 0 & 6 & 13 \\ 6 & 0 & -8 \\ 13 & -8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 61 \\ -38 \\ 79 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 6 & 13 & 61 \\ 6 & 0 & -8 & -38 \\ 13 & -8 & 0 & 79 \end{array}\right]$$

$R_3 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 6 & 0 & -8 & -38 \\ 0 & 6 & 13 & 61 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - \frac{6}{13} R_1$

$$\frac{474}{13}\ -38 - \frac{474}{13} = -\frac{968}{13}$$

$$\left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 0 & 0 & 48/13 & \cancel{228/13}\ -968/13 \\ 0 & 6 & 13 & 61 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 0 & 6 & 13 & 61 \\ 0 & 0 & 48/13 & \cancel{228/13}\ -968/13 \end{array}\right]$$

$R_2 \to R_2 + \cancel{6}/8\, R_1$ ($3/4$)

$$\left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 0 & 0 & 13 & 481/13 \\ 0 & 0 & 48/13 & \cancel{228}/13 \end{array}\right]$$

-968

$$\frac{48}{13} x_3 = -\frac{968}{13}$$

$$6x_2 + 13 \times \frac{121}{6} = 61$$

$$\boxed{x_3 = \frac{121}{6}}$$

Rough work:

$494 + 474 = 968$

$38 \times 13 = 114 + 38\cdot = 494$

$\frac{79 \times 6}{474}$

$\cancel{38} \times 6 = 228$, $\cancel{228/13}$

$79 \times 3 = 237$

$\frac{237 + 61}{4}$

$244 + 237 = 481$

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$$\left[\begin{array}{ccc|c} 0 & 6 & 13 & 61 \\ 6 & 0 & -8 & -38 \\ 13 & -8 & 0 & 79 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 6 & 0 & -8 & -38 \\ 0 & 6 & 13 & 61 \end{array}\right]$$

$0 + \frac{6}{13} \times 8$

$R_2 \rightarrow R_2 - \frac{6}{13} R_1$

$-38 - \frac{79 \times 6}{13}$

$-38 - \frac{474}{13}$

$$\left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 0 & \frac{48}{13} & -8 & -968/13 \\ 0 & 6 & 13 & 61 \end{array}\right]$$

494
474
968

$R_3 \rightarrow R_3 - \left(\frac{13 \times 6}{48}\right) R_2$

$13 + \left(\frac{13 \times 6}{48}\right) 8$

$61 + \left(\frac{13 \times 6}{48}\right) 8$

121
61
182

$$\left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 0 & 48/13 & -8 & -968/13 \\ 0 & 0 & 26 & 182 \end{array}\right]$$

$26 x_3 = 182$

$\boxed{x_3 = 7}$

56
13
168
56
728
968
728
240

$x_2 = \left(\frac{-968}{13} + 8 \times 7\right) \frac{13}{48}$

$= \frac{-240}{48} = -5$

$13 x_1 + 40 = 79 \quad x_1 = 39/13 = 3$

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$Ax = b$

$A\bar{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$A\bar{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A\bar{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

{Gauss Jordan Method}

$\bar{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}$

$\begin{bmatrix} A\bar{x}_1 & A\bar{x}_2 & A\bar{x}_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow I$

$A\bar{x} = I$

$\downarrow$

A^{-1}

eg

$A = \begin{bmatrix} -2 & 4 & -1 \\ -2 & 3 & 0 \\ 7 & -12 & 2 \end{bmatrix}$

$\det(A) \neq 0$

convert this to identity matrix

$(A|I) = \left[\begin{array}{ccc|ccc} -2 & 4 & -1 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 1 & 0 \\ 7 & -12 & 2 & 0 & 0 & 1 \end{array}\right]$

$\rightarrow (I|X)$

this becomes inverse

$$\left[\begin{array}{ccc|ccc} -2 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 2 & -3/2 & 7/2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} -2 & 4 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1/2 & 3/2 & 2 & 1 \end{array}\right]$$

$R_1 \to -1/2\, R_1$
$R_2 \to -R_2$
$R_3 \to 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{array}\right]$$

$1/2 - 2$

$R_1 \to R_1 + 2R_2$

$-1/2 + 2 \cdot 1$
$0 + (-2)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3/2 & 3/2 & -2 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{array}\right]$$

$R_1 \to R_1 + 3/2\, R_3$
$R_2 \to R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 4 & 3 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{array}\right]$$

$3/2 + 3/2 \cdot 3$
$3/2 + 9/2$

$-2 + 3/2 \cdot 4$ 2
$3/2$ $0 + 3/2 \cdot 2$

1 1+
$-1 + 4$

$$A^{-1} \begin{bmatrix} 6 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

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LU-Decomposition Method

→ factorization method
→ triangulation method

In this method, we decompose the coefficient matrix A is $A = LU$ & L is lower triangular matrix U is upper triangular matrix. of the form.

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & 0 \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & \cdots & & l_{nn} \end{bmatrix}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & & u_{1n} \\ 0 & u_{22} & \cdots & 0 & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots 0 & u_{nn} \end{pmatrix}$$

Dolittle's Method

In this method we choose $l_{11} = l_{22} = l_{nn} = 1$
$l_{ii} = 1 \quad \forall 1 \le i \le n$

Croudx Method

In this method we choose $u_{1} = u_{22} \cdots = u_{nn}$
$u_{ii} = 1 \quad \forall 1 \le i \le n$

Choleski's Method

On this method we choose $U = L^T$

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Q) Solve system of linear equations by using Dolittle Method:

$$\left.\begin{aligned} 1.8x_1 + 2.6x_2 &= 13.2 \\ 0.36x_1 + 3.782x_2 &= 12.24 \end{aligned}\right\} \text{(1)}$$

Steps

(1) write eqn as $Ax = b$

$$A = \begin{pmatrix} 1.8 & 2.6 \\ 0.36 & 3.782 \end{pmatrix}$$

$$b = \begin{pmatrix} 13.2 \\ 12.24 \end{pmatrix}$$

$$A = LU$$

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

$$LU = \begin{pmatrix} u_{11}+0 & u_{12}+0 \\ l_{21} \cdot u_{11} & l_{21}u_{12}+u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 1.8 & 2.6 \\ 0.36 & 3.782 \end{pmatrix}$$

$u_{11} = 1.8$

$u_{12} = 2.6$

$$l_{21} = \frac{0.36}{1.8} = \frac{\overset{2}{\cancel{36}} \times 10^{-2}}{\cancel{18} \times 10^{-1}} = 0.2$$

$l_{21} \cdot u_{22} = 3.782$

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Step 2 :

$Ux = Z$

$Ax = b$

$LUx = b$

$Lz = b$

$$Z = \left[\begin{array}{cc|c} 1 & 0 & 13.2 \\ 0.2 & 1 & 12.24 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$= \left[\begin{array}{cc|c} 0.2 & 1 & 12.24 \\ 1 & 0 & 13.2 \end{array}\right]$$

$R_2 \to R_2 \times 0.2$

$$\left[\begin{array}{cc|c} 0.2 & 1 & 12.24 \\ 0.2 & 0 & 2.64 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{cc|c} 0.2 & 1 & 12.24 \\ 0 & 0 & 2.64 \end{array}\right]$$

$$\begin{pmatrix} 1 & 0 \\ 0.2 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 13.2 \\ 12.24 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ 0.2z_1 + z_2 \end{pmatrix} = \begin{pmatrix} 13.2 \\ 12.24 \end{pmatrix}$$

$$\begin{cases} z_1 = 13.2 \\ z_2 = 9.6 \end{cases}$$

$Ux = Z$

$$\begin{pmatrix} 1.8 & 2.6 \\ 0 & 3.202 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 13.2 \\ 9.6 \end{pmatrix}$$

~~1.8x1~~

$$\begin{pmatrix} 1.8x_1 + 2.6x_2 \\ 3.202x_2 \end{pmatrix} = \begin{pmatrix} 13.2 \\ 9.6 \end{pmatrix}$$

$$\begin{cases} x_2 = 2.998 \\ x_1 = 3.003 \end{cases}$$

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Summary for Dolittle / Crout Method

$$\left\{\begin{array}{l}\text{(i) Write } A = LU \\ \text{(ii) Solve } Lz = b \text{ for } z \\ \text{(iii) } Ux = Z \text{ for } x\end{array}\right\}$$

Solve system of linear eqns using Crout's Method

$$3x_1 + 2x_2 = 18$$
$$18x_1 + 17x_2 = 123$$

$$Ax = b$$

$$A = \begin{pmatrix} 3 & 2 \\ 18 & 17 \end{pmatrix}$$

$$b = \begin{pmatrix} 18 \\ 123 \end{pmatrix}$$

$$A = LU$$

$$L = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \qquad U = \begin{pmatrix} 1 & u_{12} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 18 & 17 \end{pmatrix} = LU = \begin{pmatrix} l_{11} + 0 & l_{11} \cdot u_{12} \\ l_{21} & l_{21} \cdot u_{12} + l_{22} \end{pmatrix}$$

$l_{11} = 3$ $\qquad l_{21} = 18$

$u_{12} = 2/3$ $\qquad \overset{6}{\cancel{18}} \cdot 2/\cancel{3} + l_{22} = 17 \Rightarrow l_{22} = 5$

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$$L = \begin{pmatrix} 3 & 0 \\ 18 & 5 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 2/3 \\ 0 & 1 \end{pmatrix}$$

$Ax = b$

$L\underbrace{Ux}_{z} = b$

$Lz = b$

$$\begin{pmatrix} 3 & 0 \\ 18 & 5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 18 \\ 123 \end{pmatrix}$$

$\cancel{3z_1} = \cancel{18}\ 6$

$18z_1 + 5z_2 = 123$

$$\left[\begin{aligned} z_1 &= 6 \\ z_2 &= 3 \end{aligned}\right.$$

$Ux = z$

$$\begin{pmatrix} 1 & 2/3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$x_1 + \frac{2}{3}x_2 = 6$

$\boxed{x_2 = 3}$

$x_1 + 2 = 6$

$\boxed{x_1 = 4}$

$\cancel{x_1 - \frac{34}{3} = 6}$

$\cancel{x_1 = 52/3 = 17.33}$

$$\cancel{\left\{\begin{aligned} x_1 &= 17.33 \\ x_2 &= -17 \end{aligned}\right.}$$

Q) Solve the system of equation by using Cholesky's method.

$$4x_1 + 6x_2 + 8x_3 = 0$$
$$6x_1 + 34x_2 + 52x_3 = -160$$
$$8x_1 + 52x_2 + 129x_3 = -452$$

$$A = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

$$= \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 & 8 \\ 6 & 34 & 52 \\ 8 & 52 & 129 \end{pmatrix}$$

$$\cancel{L = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & \end{pmatrix}}$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 8 & 7 \end{pmatrix}$$

$$\left\{ \begin{array}{l} Lz = b \\ L^T x = z \end{array} \right\}$$

Note : Choleski's Method is not possible if
1. U_{nn} is ~~non~~ negative
2. Matrix is not symmetric
3. Principle matrix Δ is negative

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots + a_{1n}x_n = b_1 \quad -(i)$$
$$a_{21}x_1 + a_{22}x_2 \quad + a_{2n}x_n = b_2 \quad -(ii)$$
$$a_{n1}x_1 + a_{n2}x_2 \cdots + a_{nn}x_n = b_n \quad -(iii)$$

$$x_1 = \frac{1}{a_{11}}\left(b_1 - a_{1n}x_n - \cdots a_{12}x_2\right)$$

$$x_2 = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1 - a_{23}x_3 \cdot a_{2n}x_n\right)$$

$$x_n = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1 - a_{n2}x_2 \cdots a_{nn-1}x_{n-1}\right)$$

iterative formula

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^k - a_{13}x_3^k \cdots a_{1n}x_n^k\right)$$

$$x_n^{k+1} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^k - a_{n2}x_2^k \cdots a_{nn}x_n^k\right)$$

$$\bar{x}_0 = (x_1^0, x_2^0, x_3^0 \cdots x_n)$$

and so on

$$\bar{x}_1 = (x_1^1, x_2^1, x_3^1 \cdots x_n^1)$$
$$\bar{x}_2 = (x_1^2, x_2^2, x_3^2 \cdots x_n^2)$$

This Method is called Gauss Jacobi

Iterative Method

$$x_{k+1} = \phi(x_k)$$

Consider the linear system,
$Ax = b$ —① where A is $n \times n$ matrix wh diagonal elements are non-zero. Aim is to write ① in the form

$$x^{k+1} = Hx^k + C \qquad k = 0, 1, 2 \ldots \quad \text{—②}$$

for some $n \times n$ matrix H and some vector

→ Gauss Jordan

→ Gauss Siedel

Here H is the called the iterative matrix

Remark

The iterative formula ② converges to solution of ① if & only if $\rho(H) < 1$

$\rho(H) \rightarrow$ spectral radius of H

$\rho(H) = \max\{|\lambda| : \lambda \text{ is eigenvalue of}$

Note A suitable initial approximation for ② can be taken as $x_i = \dfrac{b_i}{a_{ii}}$ $\quad 1 \le i \le n$

Stopping criterion: A stopping criterion can be chosen as

$$|x_i^{k+1} - x_i^{k}| < \epsilon \quad \forall \; i \in n$$

Gauss Jacobi

(For both methods Diagonal Dominance is Mandatory)

Consider the linear system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \quad (1)$$

with the assumption that $a_{ii} \neq 0 \ \forall \ 1 \leq i \leq n$

Now consider iterative formula,

$$x_1^{k+1} = \frac{1}{a_{11}}\left(b_1 - \left(a_{12}x_2^k + a_{13}x_3^k + \cdots\right)\right)$$

$$x_2^{k+1} = \frac{1}{a_{22}}\left(b_2 - \left(a_{21}x_1^k + a_{23}x_3^k \cdots a_{2n}x_n^k\right)\right)$$

$$\vdots$$

$$x_n^{k+1} = \frac{1}{a_{nn}}\left(b_n - \left(a_{n1}x_1^k + \cdots + a_{nn}x_n^k\right)\right)$$

The iterative formula can be written as

$$x^{k+1} = Hx^k + C$$

where $H = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots \\ -\frac{a_{21}}{a_{12}} & \cdots & & \end{pmatrix}$

$$c = \begin{pmatrix} b_1/a_{11} \\ b_2/a_{22} \\ b_3/a_{33} \\ \vdots \end{pmatrix}$$

let (1) can be written as

$$Ax = b$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$L+D+U = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ a_{21} & 0 & 0 & \cdots & \\ a_{31} & a_{32} & 0 & 0 & \cdots \\ a_{41} & a_{42} & a_{43} & 0 & \\ \vdots & & & & 0 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 & \cdots & \\ & a_{22} & 0 & 0 & 0 \\ & & a_{33} & 0 & \cdot \\ & & & a_{44} & \\ 0 & 0 & \cdots & & a_{nn} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} & \\ 0 & 0 & a_{23} & a_{24} \cdots & a_{2n} \\ \vdots & & & & \\ 0 & 0 & 0 & 0 \cdots & 0 \\ & & & & 0 \end{pmatrix}$$

$$(L+D+U)x = b$$

$$Dx = -(L+U)x + b = H$$

(D is invertiable since $a_{ii} \neq 0$)

$$x = \underbrace{-D^{-1}\left[(L+U)x\right]}_{H} + D^{-1}b$$

$$[x_{i+1}] = [x_i] + D^{-1}([b] - [A]$$

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Thus the iterative formula (Gauss - Siedel)

$$x^{k+1} = -D^{-1}(L+U)x^k + D^{-1}b \quad k=0,1,2\ldots$$

$$a_1x_1 + b_1x_2 + c_1x_3 = d_1$$
$$a_2x_1 + b_2x_2 + c_2x_3 = d_2$$
$$a_3x_1 + b_3x_2 + c_3x_3 = d_3$$

$x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \rightarrow$ initial guess

$$x_1^{(1)} = \frac{1}{a_1}\left(d_1 - b_1x_2^{(0)} - c_1x_3^{(0)}\right)$$

put this value

$$x_2^{(1)} = \frac{1}{b_2}\left(d_2 - a_2x_1^{(1)} - c_2x_3^{(0)}\right)$$

$$x_3^{(1)} = \frac{1}{c_3}\left(d_3 - a_3x_1^{(1)} - b_3x_2^{(1)}\right)$$

$$(L+D+U)x = b$$
$$(L+D)x = -Ux + b \qquad [x_{i+1}] = [L']([b]-[U]x_i)$$

$$x^{k+1} = Hx^k + c$$

where $H = -(L+D)^{-1}U$

$L+D$ is invertible

$$c = (L+D)^{-1}b$$

Definition — A $n \times n$ matrix is said to be diagonally dominant if

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}| \quad \forall \quad 1 \leq i \leq n$$

eg $\begin{pmatrix} 3 & 2 & 0 \\ 4 & 15 & 3 \\ 7 & 9 & 20 \end{pmatrix}$ $\begin{matrix} 3 > 2+0 \\ 15 > 4+3 \\ 20 > 9+7 \end{matrix}$

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Note: $\rho(H) < 1$ is automatically satisfied if the matrix A is diagonally dominant.

Rate of convergence

If the iterative formula converges ($\rho(A) < 1$) then

$r = -\log_{10}(\rho(H))$ is called rate/order of convergence.

eg// Write the iterative formula for Gauss Jacobi method for the system

$$7x_1 - 15x_2 - 21x_3 = -2$$

$$7x_1 - x_2 - 5x_3 = -3$$

$$7x_1 + 5x_2 + x_3 = 1$$

Without performing iterations show that sequence does not convergence to exact soln of system.

Youtube

Jacobi iteration → should be diagonally dominates

$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$

Q) System of non-linear equations

Let us consider $f(x,y) = 0$ and $g(x,y) = 0$ be two non-linear equations with unknown x and y. Let (x_0, y_0) be an initial approximation of ①. ~~write expansion~~ Approximate f and g using taylor's formula.

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \quad -①$$

$$f(x,y) = f(x_0,y_0) + \left[(x-x_0) f'_x(x_0) + (y-y_0) f'_y(y_0)\right] + \left(\frac{(x-x_0)^2}{2} f''_x(x_0) + \frac{(y-y_0)^2}{2!} f''_{yy} + (x-x_0)(y-y_0) f''_{xy}\right) + \dots$$

linear approximation

$$\left.\begin{aligned} f(x,y) &= f(x_0 y_0) + \left[(x-x_0) f'_x(x_0) + (y-y_0) f'_y(y_0)\right] \\ g(x,y) &= g(x_0 y_0) + \left[(x-x_0) g'_x(x_0) + (y-y_0) g'_y(y_0)\right] \end{aligned}\right\} ②$$

Equate RHS quantities in ② to zero.

$$\left.\begin{aligned} f(x_0,y_0) + (x-x_0) f'_x(x_0) + (y-y_0) f'_y(y_0) &= 0 \\ g(x_0,y_0) + (x-x_0) g'_x(x_0) + (y-y_0) g'_y(y_0) &= 0 \end{aligned}\right\} ③$$

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Solve ③ for x & y and denote them as x_1 & y_1 respectively

$$\frac{f(x_0,y_0)}{f'_x(x_0)} \cdot g'_x(x_0) + (x-x_0)\, g'_x(x_0) + \frac{(y-y_0) f'_y(y_0)\, g'_x(x_0)}{f'_x(x_0)} = 0$$

$$g(x_0,y_0) + (x-x_0)\, g'(x_0) + (y-y_0)\, g'_y(y_0) = 0$$

$$\frac{f(x_0,y_0)}{f'(x_0)} g'_x(x_0) + \frac{(y-y_0) f'_y(y_0)\, g'_x(x_0)}{f'_x(x_0)} - g(x_0,y_0) - (y-y_0)\, g'_y(y_0) = 0$$

$$y - y_0 = \left[\frac{f(x_0,y_0)\, g'_x(x_0)}{f'_x(x_0)} - g(x_0,y_0)\right] \frac{f'_x(x_0)}{-f'_y(y_0) g'_x(x_0) + g'_y(y_0) f'_x(x_0)}$$

Δy

$$\boxed{y = y_0 - \frac{f(x_0,y_0)\, g'_x(x_0) - g(x_0,y_0) f'_x(x_0)}{g'_y(y_0) f'_x(x_0) - f'_y(y_0) g'_x(x_0)}}$$

④

$$\boxed{x = x_0 - \frac{f(x_0,y_0)\, g_y(x,y_0) - g(x_0,y_0) f_y(x_0,y_0)}{f_x(x_0,y_0)\, g_y(x,y_0) - g_x(x_0,y_0) f_y(x,y_0)}}$$

Δx

$$\left.\begin{aligned} x_1 &= x_0 + \Delta x \\ y_1 &= y_0 + \Delta y \end{aligned}\right\} \rightarrow (5)$$

$$\underbrace{\begin{bmatrix} f_x(x_0 y_0) & f_y(x_0 y_0) \\ g_x(x_0 y_0) & g_y(x_0 y_0) \end{bmatrix}}_{J_0} \underbrace{\begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix}}_{\underline{\Delta x}^{(0)}} = - \underbrace{\begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}}_{F(x_0, y_0)}$$

$$J_0 \underline{\Delta x}^{(0)} = -F(x_0 y_0)$$

Assume $J_0 \rightarrow$ invertible

$$\boxed{\Delta x^{(0)} = -J_0^{-1} F(x_0, y_0)} \quad (6)$$

From (5) & (6) we obtain

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \Delta x_0 \\ \Delta y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - J_0^{-1} \{ F(x_0 y_0)$$

<u>General iterative formula</u>

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J_k^{-1} \{ F(x_k y_k)$$

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$$J_k = \begin{bmatrix} f_x(x_k\ y_k) & f_y(x_k\ y_k) \\ g_x(x_k\ y_k) & g_y(x_k\ y_k) \end{bmatrix}$$

$$F(x_k\ y_k) = \begin{bmatrix} f(x_k\ y_k) \\ g(x_k\ y_k) \end{bmatrix}$$

Example – The system of equation

$$f(x,y) = y\cos(xy) + 1 = 0$$
$$g(x,y) = \sin(xy) + x - y = 0$$

has one solution close to (1,2). Calculate approximate solution using Newton-Raphson method (2 iterations).

HW

$$x_0 = 1$$
$$y_0 = 2$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - J_0^{-1} F(x_0 y_0)$$

$$\begin{bmatrix} f(x_0 y_0) \\ g(x_0 y_0) \end{bmatrix} = \begin{bmatrix} 2\cos 2 + 1 \\ \sin 2 - 1 \end{bmatrix}$$

$$J_0 = \begin{bmatrix} f_x(x_0\ y_0) & f_y(x_0\ y_0) \\ g_x(x_0,\ y_0) & g_y(x_0\ y_0) \end{bmatrix}$$

$f_x = -y^2\sin(xy)$
$f_y = \cos(xy) - xy\sin(xy)$
$g_x = y\cos(xy) + 1$
$g_y = x\cos(xy) - 1$

$$= \begin{bmatrix} -4\sin(2) & \cos(2) - 2\sin(2) \\ 2\cos(2) + 1 & \cos(2) - 1 \end{bmatrix}$$

$R_1 \to R_2 - 2R_1$
$R_1 \to R_1 - 3R_2$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$J_0^{-1} = \frac{1}{|J_0|} \begin{bmatrix} \cos(2) - 1 & 2\sin(2) - \cos(2) \\ -2\cos(2) - 1 & -4\sin(2) \end{bmatrix}$$

$$\begin{bmatrix} 1.0756 \\ 1.4231 \end{bmatrix} - \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix}$$

General form

Consider,

$$\left.\begin{aligned} f_1(x_1, x_2, x_3 \cdots x_n) &= 0 \\ f_2(x_1, x_2, x_3 \cdots x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \cdots x_n) &= 0 \end{aligned}\right\} \quad (7)$$

as n-system of non-linear equation

Then Newton - Ralphson iterative formula for solving (7) is:-

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} - J_k^{-1} F(x_1^{(k)}, x_2^{(k)} \cdots x_n^{(k)})$$

$(k = 0, 1, 2 \cdots$ iterations$)$ $\rightarrow$ (8)

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$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_{21}}{\partial x_2} & \cdots & \frac{\partial f_{n1}}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_{n2}}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$F(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)}) = \begin{bmatrix} f_1(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)}) \\ f_2(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)}) \\ \vdots \\ f_n(x_1^{(k)}, x_2^{(k)} \ldots x_n^{(k)}) \end{bmatrix}$$

<u>Note</u> : A necessary and sufficient condition for convergence of 8 is

$$\rho(J_k^{-1}) < 1$$

↳ spectral radius

Stopping criterion (for given tolerance)

$$|x_i^{k+1} - x_i^{k}| < 1$$

—— x —— x —— x ——

Eigen Value Problems

Def^n – let A be a general $n \times n$ matrix
A number λ is said to be an eigen value of A if there exists a non zero vector $v \in \mathbb{R}^n$ such that

$$\det(A - \lambda I) = 0 \quad \text{or} \quad Av = \lambda v$$

$\downarrow$ polynomial in λ of order 'n'

Dominant Eigen Value – An eigen value λ of $n \times n$ matrix is said to be dominant eigen value if $|\lambda| = \max\{|z| : z \text{ is an eigen value of } A\}$

Power Method – This method is used to obtain a specific eigen value called dominant eigen value and the corresponding eigen vector for a given $n \times n$ matrix.

Steps –

1) let X_0 be the initial vector which is generally chosen as vector with all components as 1, that is $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

2) Calculate AX_0 and write it in the form $AX_0 = \lambda_1 X_1$, where X_1 is normalized by taking out the largest component of AX_0.

3) Calculate AX_1 and express AX_1 as $AX_1 = \lambda_2 X_2$ where x_2 is normalised as X_1 in step 2 by taking out largest (in magnitude) component of AX_1. Continue the process till the desired result is obtained.

4) Then we have a sequence of eqns

$$AX_0 = \lambda_1 X_1, \; AX_1 = \lambda_2 X_2, \; AX_2 = \lambda_3 X_3$$

The λ_n is an approximation of dominant eigen value & X_n is approximate of eqn.

Q) Find the largest (in magnitude) eigen value of $A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ using power method and compare your result with explicit solution.

Soln : $X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as initial vector.

$$AX_0 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \underbrace{5}_{\lambda_1} \underbrace{\begin{bmatrix} 1 \\ 4/5 \end{bmatrix}}_{X_1}$$

$$AX_1 = \lambda_2 X_2$$

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 4 + 4/5 \\ 1 + 12/5 \end{bmatrix} = \begin{bmatrix} 4.8 \\ 3.4 \end{bmatrix} = \underbrace{4.8}_{\lambda_2} \underbrace{\begin{bmatrix} 1 \\ 34/48 \end{bmatrix}}_{X_2}$$

$$AX_2 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.7083 \end{bmatrix} = \begin{bmatrix} 4 + 0.7083 \\ 1 + 2.1249 \end{bmatrix} = \begin{bmatrix} 4.7083 \\ 3.1249 \end{bmatrix}$$

$$= \underbrace{4.7083}_{\lambda_3} \underbrace{\begin{bmatrix} 1 \\ 3.1249 / 4.7083 \end{bmatrix}}_{X_3}$$

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$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6637 \end{bmatrix} = \begin{bmatrix} 4.6637 \\ 2.9242 \end{bmatrix}$$

$$= 4.6637 \begin{bmatrix} 1 \\ 0.63 \end{bmatrix}$$

:

12th iteration

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \lambda_4 X_4 = 4.6182 \begin{bmatrix} 1 \\ 0.6182 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$12 + \lambda^2 - 7\lambda - 1 = 0$$

$$\lambda^2 - 7\lambda + 11 = 0$$

$$\lambda = \frac{7 \pm \sqrt{49-44}}{2}$$

$$= \frac{7 \pm \sqrt{5}}{2} = 4.618 \rightarrow \text{Dominent}$$

$$= 7 \pm 2.23$$

$$\& \ \lambda_2 = 2.38$$

Note:

For a given tolerance one can stop when $|X_{i+1} - X_i| < \epsilon$ or $|\lambda_{i+1} - \lambda_i| < \epsilon$

$V = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow$ find using eg. value

$A\lambda = \lambda X$

$\{AV = \lambda V\}$

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Find the largest eigen value of $A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$ using power method.

$\lambda_1 = 1 \quad \lambda_2 = 0 \quad \lambda_3 = 3$

$$X_0 = \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix}$$

$$AX_0 = \begin{bmatrix} 3 \\ -4+3+0.5 \\ 16-0.5\times15-0.25\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -0.5 \\ 7.25 \end{bmatrix} = \underbrace{7.25}_{\lambda_1} \underbrace{\begin{bmatrix} 0.413 \\ -0.0689 \\ 1 \end{bmatrix}}_{X_1}$$

$AX_1 = \lambda_2 X_2$

$$\begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \begin{bmatrix} 0.0413 \\ -0.0689 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.246 \\ -0.0689 \\ 2.655 \end{bmatrix}$$

$$= \underbrace{2.655}_{\lambda_2} \underbrace{\begin{bmatrix} 0.4637 \\ +0.0359 \\ 1 \end{bmatrix}}_{X_2}$$

$AX_2 = \lambda X_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \begin{bmatrix} 0.4637 \\ -0.0359 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.0489 \\ -0.00905 \\ 1 \end{bmatrix}}_{X_3} \underbrace{2.87}_{\lambda_3}$$

$$AX_9 = \begin{bmatrix} 1.99952 \\ -0.0883 \\ 2.99936 \end{bmatrix} = 2.9996 \begin{bmatrix} 0.49985 \\ -0.02946 \\ 1 \end{bmatrix}$$

$\lambda_{(1)}$ — closer to actual $\lambda_2 = 3$

$X_{(10)}$ — closer to
eigen vector

eigen vector

$$V = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix}$$

$\cancel{\text{closer}}$

A

If A is a real square matrix

↳ eigen values are real? A

$\boxed{\text{No}}$

eg — $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\{\lambda = \pm i\}$

If A is a real symmetric matrix

↳ Are ~~all~~ eigen values real?

$\boxed{\text{Yes}}$

$\exists$ orthogonal matrix is s.t

$$S^{-1}AS = D$$

D — diagonal matrix

$S = [v_1, v_2, v_3, v_4]$ $\{v_1, v_2, v_3, v_4 \rightarrow$ eigen vectors / values$\}$

Jacobi Method for real symmetric matrix

Let A be a real symmetric matrix. Then the eigen values of A are real and there exists a real orthogonal matrix S such that

$S^{-1}AS = D$, where D is diagonal matrix.

These processes can be done by applying a series of orthogonal matrices $S_1, S_2 \ldots$

Simple case –

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ be a real symmetric matrix.

$S_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where θ is an angle to be determined.

Calculate $S^{-1}AS$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta & (a_{22}-a_{11})\sin\theta\cos\theta + a_{12}\cos 2\theta \\ (a_{22}-a_{11})\sin\theta\cos\theta + a_{12}\cos 2\theta & a_{11}\sin^2\theta + a_{22}\cos^2\theta - a_{12}\sin 2\theta \end{bmatrix}$$

To make off-diagonal entries zero, we choose θ in such a way that

$$\frac{1}{2}(a_{22}-a_{11})2\sin\theta\cos\theta + a_{12}\cos 2\theta = 0.$$

$$\frac{1}{2}(a_{22}-a_{11})\sin 2\theta + a_{12}\cos 2\theta = 0$$

$$\tan 2\theta = \frac{2a_{12}}{a_{11}-a_{22}}$$

$$\boxed{\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right)}$$

let A be $m \times n$ ~~matrix~~ real symmetric matrix.

let $A = (a_{ij})_{1 \le i \le n}$ $a_{ij} \in R$ $a_{ij} = a_{ji}$

Step 1: Find out the largest (in magnitude) of the diagonal element in A, a_{ij}

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 0 & 7 \\ 4 & 7 & -4 \end{pmatrix} \begin{array}{c} ij \\ (2,3) \end{array}$$

Step 2 Calculate $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{ij}}{a_{ii}-a_{jj}}\right)$

Step 3 Set $S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \underset{a_{ii}}{\cos\theta} & \underset{a_{ij}}{-\sin\theta} & 0 & 0 \\ 0 & \underset{\sin\theta}{a_{ji}} & \underset{\cos\theta}{a_{jj}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

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$\left\{ \begin{array}{l} s_{ii} = s_{jj} = \cos\theta, \text{ other diagonal entries} = 1 \\ s_{ij} = -\sin\theta, \; s_{ji} = \sin\theta \quad \text{all other elements} = 0 \end{array} \right\}$

Observe ~~that~~ that S_1 is inverted

Step 4 :-

Compute $D_1 = S_1^{-1} A S_1$

Steps : let ϵ be the given tolerance. If sum of squared of off-diagonal ~~matrix~~ ~~matrix~~ entries is less than ϵ.

$$\sum_{\substack{i=1 \\ j \neq i}} (a_{ij})^2$$

Or else repeat steps 1-4 for matrix D_1.

$$\begin{array}{c} \qquad \quad a_{12} \\ A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix} \end{array}$$

~~Step 1~~ $a_{ij} = 2$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}} \right)$$

$$= \pi/4$$

$$S_1 = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 & 0 \\ \sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_1^{-1} A S_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 3 & 0 & 1/\sqrt{2} \\ 0 & -1 & 3/\sqrt{2} \\ 1/\sqrt{2} & 3/\sqrt{2} & 1 \end{pmatrix}$$

(Atleast 2 more iteration)

Q) Using Jacobi Method to find all eigen values of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2(2)}{1-1}\right) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \pi/4$$

$$S_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$D = S^{-1}AS = \begin{bmatrix} 3 & 0 & 1/\sqrt{2} \\ 0 & -1 & 2.1213 \\ 1/\sqrt{2} & 2.1213 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 1/\sqrt{2} \\ 0 & -1 & 3/\sqrt{2} \\ 1/\sqrt{2} & 3/\sqrt{2} & 1 \end{bmatrix} \rightarrow d_{23}$$

$$\theta = 1/2 \tan^{-1}\left(\frac{2 \cdot 3/\sqrt{2}}{-1-1}\right)$$

$$= -0.565$$

$\cos\theta = 0.84$

$\sin\theta = -0.535$

$$S = \begin{bmatrix} 0.84 & 0.535 & 0 \\ -0.535 & 0.84 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.84 & 0.535 \\ 0 & -0.535 & 0.84 \end{bmatrix}$$

$$D_2 = S^{-1} A_2 S = \begin{bmatrix} 3 & -0.378 & 0.5939 \\ -0.381 & -2.345 & -0.009 \\ 0.5988 & -0.009 & 2.345 \end{bmatrix}$$

Eigen values

$$D_6 = \begin{bmatrix} 3 & 0.00048 & 0 \\ 0.0048 & -2.37 & 0 \\ 0 & 0 & 1.99 \end{bmatrix}$$

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Exact eigenvalues

$$\begin{Bmatrix} \lambda_1 = 3.37 \\ \lambda_2 = -2.37 \\ \lambda_3 = 2 \end{Bmatrix}$$

Q)

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

(1,1) (1,3)
(3,1) (3,3)

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{ij}}{a_{ii}-a_{jj}}\right) = \pi/4$$

$$S = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

(1,2)

$$A_2 = D_1 = \begin{bmatrix} 5 & 2.8284 & 0 \\ 2.8284 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta = 0.5\tan^{-1}\left(\frac{1}{0}\right) = \pi/4$$

$$S_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D_2 = \begin{bmatrix} 7.8284 & 0 \\ 0 & 2.1716 \\ 0 & 0 \end{bmatrix}$$

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eigen values of A are diagonal elements

$$\begin{cases} \lambda_1 = 7.8284 \\ \lambda_2 = 2.1716 \\ \lambda_3 = 1 \end{cases}$$

Q) Use Power Method to find smallest eigen value for same matrix A

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\cancel{AX_0 = \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Concept → $A = (|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|)$ $A^{-1} = (1/\lambda_1 \; 1/\lambda_2 \cdots 1/\lambda_n)$

$$|1/\lambda_n| < |1/\lambda_{n-1}|$$

$$A^{-1} = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}$$

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$A^{-1} =$

Steps //

1. Check $\det(A) \neq 0$
2. Find largest value of A^{-1} using power method
3. $\frac{1}{\lambda}$ will be smallest

$$A^{-1}x = \begin{bmatrix} 1.5 \\ 2 \\ 1.5 \end{bmatrix} = 2\begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix} \rightarrow X_2$$

$$A^{-1}x_2 = \begin{bmatrix} 1.25 \\ 1.75 \\ 1.25 \end{bmatrix} = \begin{bmatrix} 0.714 \\ 1 \\ 0.714 \end{bmatrix} 1.75$$

$$A^{-1}x_3 = \begin{bmatrix} 1.214 \\ 1.714 \\ 1.214 \end{bmatrix} = \begin{bmatrix} 0.708 \\ 1 \\ 0.708 \end{bmatrix} \times 1.714$$

$$A^{-1}x_4 = \begin{bmatrix} 1.208 \\ 1.708 \\ 1.208 \end{bmatrix} = 1.708 \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

λ

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Numerical Solution of ordinary differential equation (ODE)

Let us consider the following ODE

$$y'(x) = f(x, y) \quad \text{--- (1)}$$

dependent independent ~~indep~~

where $y(x)$ is unknown, $f: R \times R \to R$ is a given function $y'(x) = dy/dx$

The aim is ~~f~~ to find a solution of (1) with the initial condition

$$y(x_0) = y_0 \quad \text{--- (2)}$$

Finding the solution of (1) with initial condition (2) is called initial value problem (IVP).

Lemma: A continious function y defined in on interval I containing the point x_0 is a solution of IVP (1) if and only if y satisfies the integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds \quad \forall\ x \in I$$

We partition the interval $I = [a, b]$ $x_0 = a$ into the points $x_0 = a < x_1 < x_2 \dots < x_N = b$ with $x_i = x_0 + ih$, $i = 1, 2, 3 \dots N$.

Single Step Method

- explicit $y(x_{n+1}) = \phi(x_0, y_0, f(x_0, y_0), h)$
- implicit $y(x_{n+1}) = \phi(x_n, x_{n+1}, y_n, y_{n+1}, h, f(x_{n+1}, y_{n+1}))$

Taylor Series Method

Taylor Series expansion of $y(x)$ around,

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!}\, y''(x_0) + \cdots$$

$$\frac{(x-x_0)^n}{n!}\, y^n(x_0)$$

Q)
$$y' = -2xy^2$$
$$y(1) = 1$$

there $x_0 = 1$ and $y(x_0) = 1$

$h = 0.1$ $x_1 = 1.1$ $x_2 = 1.2$ $x_3 = 1.3$

$y''(x_0) = -\{4xy\,y' + 2y^2\} = 6$

$$y(x_1) = y(x_0) + h\, y'(x_0) + \frac{h^2}{2!}\, y''(x_0)$$

$$= 1 + (0.1)(-2 \times 1 \times 1) + \frac{(0.1)^2}{2!} \times 6$$

$$= 1 - 0.2 + 0.03$$

$$= 0.8 + 0.03$$

$$= 0.83$$

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$$y(x_2) = y(x_1) + h y'(x_1) + \frac{h^2}{2!} y''(x_1)$$

$$= 0.83 + (0.1)(-2 \times (0.83)^2 \times 1.1) + \frac{(0.1)^2}{2!}\left[-4(0.1\cdot1)(0.83)(y') - 2(0.83)^2\right]$$

$$= \cancel{0.901} \quad 0.692$$

$$y(x_3) = 0.5967$$

Put $x_n = x_{n+1}$

and $h = x_{n+1} - x_n$

$$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(x_n) \cdots$$

If we retain terms upto the pth power of h in the above equation, we get an approximate

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2!} y''_n \cdots + \frac{h^p}{p!} y_n^{(p)}$$

and error as

$$E = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(x_n) + \frac{h^{p+2}}{(p+2)!} y^{(p+2)}(x_n)$$

$$E = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi) \quad x_n < \xi < x_{n+1}$$

The pth order Taylor series method for solving an ODE of the form (1) is $y'(x) = f(x, y)$, $y(x_0) = y_0$

is given as

Consider the ODE, {Runge kutta Method}

$y'(x) = f(x,y)$

$y(x_0) = y_0$

Assume that $y(x_n)$ is known. Use Taylor Series expension of $y(x_{n+1})$ around x_n upto second order

$$y(x_{n+1}) = y(x_0) + y'(x_0) + \frac{h}{2!} y''(x_0) + 0 \quad \text{—} (1)$$

$$y'(x) = \frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y) \cdot \frac{\partial y}{\partial x}$$

$$y''(x) = \frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0) + f(x_0,y_0) \quad \text{—} (2)$$

Substitute (2) in (1)

$\cancel{y'(x) = y(x_{n+1})}$

$$y(x_{n+1}) = y(x_n) + hf(x_0 y_0) + \frac{h^2}{2!}\left[\frac{\partial f}{\partial x}(x_0 y_0) + \frac{\partial f}{\partial y}(x_0 y_0) f(x_0 y_0)\right] + (0)h^3$$

$$= y(x_n) + \frac{h}{2}(x_0 y_0) + \frac{h}{2}\underbrace{\left[f(x_0 y_0) + h\frac{\partial f}{\partial x}(x_0 y_0) + h f(x_0 y_0)\frac{\partial f}{\partial y}(x_0 y_0)\right]}_{f[x_n+h,\ y_n + hf(x_n y_n)]} + 0(h^3) \quad \text{—} (3)$$

$$= y(x_n) + \frac{h}{2} f(x_n, y_n) + \frac{h}{2} f(x_n + h, y_n + h f(x_n, y_n))$$

$$= y(x_n) + \frac{h}{2} f(x_n, y_n) + \frac{h}{2} f(x_n + h, y_n + h f(x_n, y_n)) + O(h^3)$$

lets denote

$$\left.\begin{aligned} k_1 &= h f(x_n, y_n) \\ k_2 &= h f(x_{n+1}, y_n + k_1) \end{aligned}\right\} \quad (4)$$

Thus we can approximate $y(x_{n+1})$ by

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \quad (5)$$

where k_1, k_2 are in (4)

{ This equation is called as the Runge - kutta Method of order two }

$$\text{Error} = \frac{h^3}{3!} y'''(\xi) \qquad x_n \leq \xi < x_{n+1}$$

{ Rate of convergence = 3 }

The truncation error of Runge - kutta method is of order $O(h^3)$.

Example: Solve the ODE

$$y' = y$$
$$y(0) = 1$$

using R-K method of order 2 with step size $h = 0.01$

$\{f(x,y) = y\}$

n	x	y	k_1	k_2
0	0	1	0.01	0.0101
1	0.01	1.01005	0.0101005	0.010201505
2	0.02	1.0202013	0.0102021	0.015000
3	0.03	1.30454	0.01202	0.010304
4	0.04	1.040510	0.010305	0.010408

$n = 0$

$k_1 = hf(x_0, y_0)$

$= 0.01 \times 1$

$= 0.01$

$k_2 = hf(x_1, y_0 + k_1)$

$= (0.01)$

$f(0.01, y + 0.01)$

$= (0.01)(1.01)$

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

$= 1 + 1/2(0.01 + 0.0101)$

$k_1 = hf(x_1, y_1) = hy_1$

$k_2 = hf(x_2, y_1 + k_1)$

$= 0.01 \times [1.01005 + 0.0101005]$

$= 0.010201505$

$y_2 = y_1 + 1/2(k_1 + k_2)$

Runge Kutta Method of order four

We state without deviation — the formule for the Runge - kutta method of order four.

$$~~y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + k_4)~~$$

where

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_n + k_1/2, y_n + k_1/2)$

$k_3 = hf(x_n + k_2/2, y_n + k_2/2)$

$k_4 = hf(x_n + h, y_n + k_3)$

HW

eg// Solve the ODE,

$y' = y \qquad y(0) = 1$

using Runge-kutta method of order four with step-size $h = 0.01$. Compare with exact solution and find the error.

Ex 32.1

6) $y' = 1 - 2xy \qquad x_0 = 0 \qquad x = x_0 + h$

$y(0) = 0 \qquad h = 0.2 \qquad = 0.2$

$y(x) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0)$

$y'(0) = 1$

$y''(x) = 1 - 2(y + xy')$

$y''(0) = 1 - 2(0 + 0) = 1$

$y'''(x) = 1 - 2(y' + y' + xy'')$

$y'''(0) = 1 - 2(1 + 1 + 0)$

$= 1 - 4 = -3$//

$y(0.2) = 0 + (0.2)(1) + \frac{(0.2)^2}{2}(1) + \frac{(0.2)^3}{3!}(-3)$

$= 0.216$ //

Runge kutta Method of order four

The local transition error of the R-k method of order 4 is $O(h^5)$.

$$y' = y$$
$$y(0) = 1$$
$$h = 0.01$$
$$y(0.04) = ?$$

Solve with assignment

Euler's Method

Recall the Taylor series expansion $y(x_{n+1})$ is around x_n as

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) \cdots + \frac{h^p}{p!}y^{p}(x_n) + O(h^{p+1})$$

Take $p < 1$ and approximate $y(x_{n+1})$ as

$$y_{n+1} = y_n + hy'_n$$
$$= y_n + hf(x_n, y_n) \qquad \{x = 0, 1, 2 \cdots\}$$

Truncation error

$$T.E = \frac{h^2}{2!}y''(\xi_p) \qquad x_n \leq \xi < x_{n+1}$$

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Eg// Consider the IVP,

$$y'(x) = y$$
$$y(0) = 1$$

and solve using Euler's method to find the approximate value of $y(0.04)$ with step size $h = 0.01$. Compare with exact solution and find the error.

Sol^n Consider $f(x,y) = y$ here $x_0 = 0$, $y_0 = 1$

The Euler formula is given by

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$1 + (0.01)(1)$$

$$= y_n + hy_n$$

$y'(x) = y \quad \therefore y = e^x$

n	x_n	y_n	Exact values	Error
0	0	1	1	0
1	0.01	1.01	1.01005	0.00505
2	0.02	1.0201	1.02020	0.00020
3	0.03	1.0303	1.03045	0.00154
4	0.04	1.0406	1.04081	0.00207

Modified Euler Method

Consider the IVP,

$$\left.\begin{aligned} y'(x) &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}\right\} \text{—} (1)$$

Integral representation of (1) over $[x_n, x_{n+1}]$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

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$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x,y)\,dx \quad — ①$$

approximate

$$y(x_n) + (x_{n+1} - x_n) f(x,y) — ②$$

If we approximate ① by ② then

$$h = x_{n+1} - x_n$$

$$y(x_{n+1}) = y(x_n) + h f(x,y)$$

Using the integral formula over the integral $[x_{n}, x_{n+1}]$ and using midpoint formula

$$\int_{x_{n-1}}^{x_{n+1}} f(x,y) \simeq f(x_n, y_n) \cdot (x_{n+1} - x_{n-1})$$

$$= 2h(x_n, y_n)$$

We get Euler's ~~method~~ midpoint

$$y_{n+1} = y_{n-1} + 2hf(x_n, y_n) \qquad n = 0, 1, 2, 3 \ldots$$

$$y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(x,y)\,dx$$

we approximate

$$\int_{x_{n-1}}^{x_{n+1}} f(x,y)\,dx = (x_{n+1} - x_n) f(x_n, y_n)$$

$$= 2h\, f(x_0, y_0)$$

is given by

$y_{n+1} = y_{n-1} + 2h(f(x_n, y_n))$ — ②

put $n=1$ in ②

$y_2 = y_0 + 2h(x_1, y_1)$

$= y_0 + 2hy_1 = 1 + 2 \times 0.01 \times 0.01$

put $n=2$ in ②

$\{ y_3 = \underline{y_1} + 2hy_2 \}$

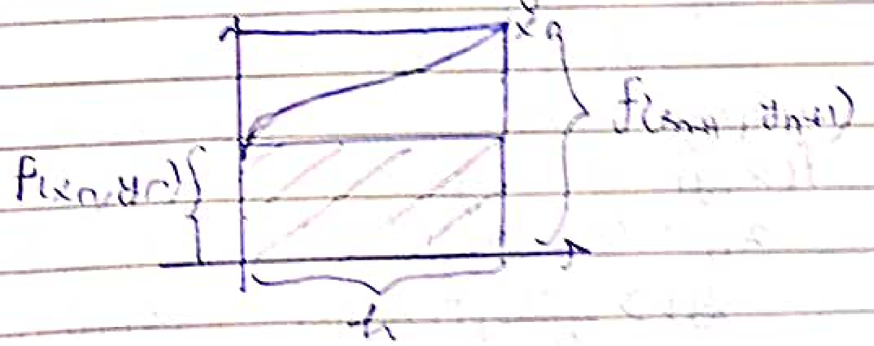
n	x_n	y_n		
0	0	1		
1	0.01	1.01		
2	0.02	~~1.0201~~	~~1.0201~~	1.202
3	0.03	~~1.0406~~	~~1.0202~~	1.0304

Consider the IVP

$y' = f(x, y)$

$y(x) = y$

The integral of ① of over the interval $[x_0, x_{n+1}]$ is $y(x_{n+1}) = y(x_0) + \int^{x_{n+1}} f(x_n, y_n)\,dx$



If we approximate the integral of

$$\int_{x_n}^{x_{n+1}} f(x_n y_n) \simeq h(x_n, y_n)$$

we approximate y_{n+1} as

$y_{n+1} = y_n + f(x_n, y_n) \cdot h$ {Euler's method}

Now, if we approximate as $\int_{x_n}^{x_{n+1}} f(x_n y_n)\,dx = h f(x_{n+1}, y_{n+1})$

we can approximate y_n as

$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$ [Backward Euler's Method]

$n = 0, 1, 2$

If we take $\int_{x_n}^{x_{n+1}} f(x, y) = \frac{h}{2} [f(x_n y_n) + f(x_{n+1}, y_{n+1})]$

we approximate value of y_n as

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n y_n) + f(x_{n+1}, y_{n+1})\right]$$

$n = 0, 1, 2$

[Euler's trapezoid Method]

eg

Find the approximate value of $y(0.4)$ of the IVP.

$y' = xy$

$y(0) = 1$

using Euler's trapezoid method with $h = 0.2$

$f(x, y) = xy$

$x_0 = 0, \quad y_0 = 1, \quad x_1 = 0.2, \quad x_2 = 0.4$

$$y_1 = \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] + y_0$$

$$y_1 = \frac{0.2}{2}\left[0 + (0.2) y_1\right] + 1$$

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$y_1 = 0.1\,(0.2)y_1 + 1$

$y_1 = 0.02y_1 + 1$

$0.98\,y_1 = 1$

$y_1 = \frac{100}{98} = 1.0204$

$y_2 = y_1 + \frac{h}{2}\left(f(x_n\, y_n) + f(x_{n+1}\, y_{n+1})\right)$

$= 1.0204 + \cancel{0.0}\,(0.1)\left((0.2)(1.0204) + 0.4y\right)$

$y_2 = 1.0204 + 0.0204 + 0.04y_2$

$0.96\,y_2 = 1.0408$

$\boxed{y_2 = 1.0841}$

$\left\{ y(0.4) = 1.0842 \right\}$ Hence approximate value of $Y(0.4) = 1.0842$

$y_{n+1}^{(0)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$

$y_1^{(1)} = y_0 + h\left[f(x_0\, y_0) + f(x_{0+1}, y_{0+1})\right]$

more accuracy $\Downarrow$ $y_1^{(2)} = y_0 + h\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

{correction step}

Predictor - Corrector Method for Euler's trapezoid

$P: \quad y_{n+1}^{(0)} = y_n + h\left[f(x_n, y_n) \cancel{+ f(x_{n+1}, y_{n+1})}\right]$

$C: \quad y_{n+1}^{(1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(0)})\right]$

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Eg Obtain an approximate value of $y(0.2)$ for IVP

$$y' = x^2 + y^2$$

$$y(0) = 1$$

using Euler trapezoidal Predictor corrector method with $h = 0.1$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$y_1 = 1 + \frac{0.1}{2}\left[1 + (0.1)^2 + y_1^2\right]$$

$$y_{1\,pred} = 1 + 0.05 + 0.0005 + 0.05y^2$$

$$0.05y^2 - y + 1.05005 = 0$$

$$y^{(0)} = 1.11$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[1 + (0.1)^2 + (1.1)^2\right]$$

$$y_1^{(1)} = 1.111$$

$$y_1^{(2)} = 1 + \frac{0.1}{2}\left[1 + 0 + (0.1)^2 + (1.11)^2\right]$$

$$y_1^{(2)} = 1.112$$

$$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2)\right)$$

$$= 1.112 + 0.1\left((0.1)^2 + 1.1\ldots\right)$$

$$y_2 = 1.236$$

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$$y_2^{(1)} = \cancel{y_0}\,y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^{(0)})\right)$$

$$= 1.112 + \frac{0.1}{2}\left((0.1)^2 + (1.112)^2 + (0.2)^2 + (1.235)^2\right)$$

$$y_2^{(1)} = 1.2527$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^{(1)})\right)$$

$$= 1.254$$

Predictor Correction method for Backward Euler

$$\cancel{P\ \ y_{n+1}^{(0)} =}\ y_n + \frac{h}{\cancel{2}}\left[f(x_n, \cancel{y_n) + f(x_{n+1}}, y_{n+1})\right]$$

$\cancel{C:\ \ y_{n+1}}$

$$\underline{\underline{P}}:\ y_{n+1} = y_n + h\left[f(x_n, y_n)\right]$$

$$\underline{\underline{C}}:\ y_{n+1} = y_n + h\left[x_{n+1}, y_{n+1}\right]$$

Consider the IVP,

$$\left.\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned}\right\} (1)$$

Solve (1) on $[x_0, b]$ into n parts

Divide $[x_0, b]$ into n-sub intervals

$$x_0 < x_1 < x_2 \ldots \leq x_n = b$$

Milne's P-C method

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3}\left[2f(x_n, y_n) - f(x_{n-1}, y_{n-1}) + 2f(x_{n-2}, y_{n-2})\right]$$

$$n = 3, 4, 5 \ldots$$

$$y_{n+1}^{(C)} = y_{n-1} + \frac{h}{3}\left[f(x_{n-1}, y_{n-1}) + 4f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

We have,

$$y_4^{(0)} = y_0 + \frac{4h}{3}\left(2f(x_3, y_3) - f(x_2, y_2) + 2f(x_1, y_1)\right)$$

eg// Find an approximate value of $y(0.8)$ using Milne's P-C method for IVP.

$$y' = (x^2 + y^2)$$
$$y(0) = 1$$
with $h = 0.2$.

→ $h = 0.2$

$x_0 = 0 \quad x_1 = 0.2 \quad x_2 = 0.4 \quad x_3 = 0.6 \quad x_4 = 0$

$y_0 = 1$

Using Euler's method,

$$y_{n+1} = y_n + f(x_n, y_n)$$

to calculate y_1, y_2, y_3

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.2)(0^2 + 1^2) = 1.2$$

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.2 + (0.2)(0.2^2 + 1.2^2) = 1.496$$

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$Y_3 = Y_2 + hf(x_2, y_2)$

$= 1.496 + (0.2)(0.4^2 + 1.496^2)$

$= 1.9756$

put $n=3$

$$Y_4^{(0)} = Y_0 + \frac{4h}{3}\left(2f(x_3, y_3) - f(x_2, y_2) + 2f(x_1, y_1)\right)$$

$$= 1 + \frac{4 \times 0.2}{3}\left(2 \times 1.9756 - 1.496 + 2 \times 1.2\right)$$

$= 3.424$

$$y_4^{(1)} = y_2 + \frac{h}{3}\left[f(x_2, y_2) + 4f(x_3, y_3) + f(x_4, y_4^{(0)})\right]$$

$= 3.6169$ $\{k = 0, 1, 2 \dots\}$

$$y_4^{(2)} = y_2 + \frac{h}{3}\left[f(x_2, y_2) + 4f(x_3, y_3) + f(x_4, y_4^{(1)})\right]$$

$y_4^{(2)} = 3.7074$

$y_4^{(3)} = 3.7078$

$y_4^{(8)} = 3.7958$

$y_1 \Rightarrow$ approximates $Y(x_1)$

$Y_2 \Rightarrow Y(x_2)$

$\vdots$

$Y_n \Rightarrow Y(x_n)$

To compute for y_5 first

$y_1\ y_2\ y_3 \to$ Euler

$y_4 \to$ P-C milline method

then use y_5 from ↑ method

Adam Bash. & Adam Milt Moulton Method

method

$$P \quad y_{n+1} = y_n + \frac{h}{24}\left[55 f(x_n, y_n) - 59 f(x_{n-1}, y_{n-1}) + 37 f(x_{n-2}, y_{n-2}) - 9 f(x_{n-3}, y_{n-3})\right] \quad \text{--- (1)}$$

$n = 3, 4, 5 \ldots$

$$C \quad y_{n+1} = y_n + \frac{h}{24}\left[f(x_{n-2}, y_{n-2}) - 5 f(x_{n-1}, y_{n-1}) + 19 f(x_n, y_n) + 9 f(x_{n+1}, y_{n+1}^{(k)})\right] \quad \text{--- (2)}$$

$n = 3, 4, 5 \ldots$ $\quad k = 0, 1, 2$

Use eulers method to find y_1, y_2 & y_3

Use (1) to find $y_4^{(0)}$

$y_4[(k+1)]$ can be computed by (2)

$y_{k+1}^{(0)}$ can be computed by (1) using $y_k^{(0)}$

y_{k+1}^{n} can be compted by (2) using $y_k^{(n-1)}$

Q) Find approximate value of $y(1.4)$ using Adam's P-C method for IVP $y' = x^2 + y^2$

$y(1) = 1$

with $h = 0.1$

[Assignment 4]

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Tut

* To find eigen value of A closest to 5 (using power method)
so $B = A - 5I$
smallest eigen value of B is closest eigen value of A
$\therefore B^{-1}$ (power method)

Q. if $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n \Rightarrow$ Eigen values of A
$\lambda_1 - 5, \lambda_2 - 5, \lambda_3 - 5 \cdots \lambda_n - 5 \Rightarrow$ Eigen values of B

Boundary Value Problem (BVP)

Consider the differential equation

$$y'' = f(x, y, y') \quad x \in (a, b)$$

with boundary

Type 1 :
$$y(a) = A$$
$$y(b) = B$$

Type 2 :
$$y'(a) = A$$
$$y'(b) = B$$

Type 3 :
$$\left.\begin{aligned} a_1 y(a) + a_2 y'(a) &= A \\ b_1 y(b) + b_2 y'(b) &= B \end{aligned}\right\} \text{Mixed}$$

$a_1, a_2, b_1, b_2, A, B \rightarrow$ constants

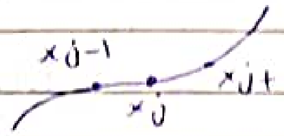
$$\left\{ \begin{array}{l} y''_{(x_n)} = \dfrac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \\ y'(x_n) = \dfrac{y_{n+1} - y_{n-1}}{2h} \end{array} \right\}$$

Finite Difference Method (FDM)

Consider, the BVP,

$$\left.\begin{array}{l} a(x)y'' + b(x)y' + c(x) = d(x) \quad x \in (a,b) \\ y(a) = A \\ y(b) = B \end{array}\right\} \text{(1)}$$

let $a = x_0 < x \ldots < x_N < x_{N+1} = b$ be a uniform partition of $[a,b]$ such that $x_{j+1} - x_j = h$



MVT, Ro...

$\therefore\ y(x_{j+1}) = y(x_j) + hy'(x_j) + \dfrac{h^2}{2!} y''(x_j)$ $\quad y'(x_j) = \dfrac{y(x_{j+1}) - y(\ldots)}{x_{j+1} - x_{j}}$

Euler

Substitute : $Y'(x_j) = \dfrac{y(x_{j+1}) - y(x_j)}{2h}$ $\quad Y'(x_j) = \dfrac{y(x_{j+1}) - y(x_{j-1})}{2h}$

Add : $y(x_{j+1}) + y(x_{j-1}) = 2hy(x_j) + h^2 y'(x_j)$

Backward Euler

$$\left\{ y'(x_j) = \frac{y(x_{j+1}) - 2y(x_j) + y(x_{j+1})}{h^2} \right\}$$

Approx : $y_{j-1} - (2 + h^2) y_j + y_{j+1} = h^2 x_j$

$(j = 1, 2, \ldots$

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let $y''(x)$

We can approximate $y''(x_j)$ by $\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}$

condi $y(x_j)$ by

$\rightarrow \frac{y_{j+1} - y_{j-1}}{2h}$ (Central fr)

$\rightarrow \frac{y_{j+1} - y_j}{h}$ (Euler)

$\rightarrow \frac{y_j - y_{j-1}}{h}$ (Backward Euler)

Thus we can approximate ① by

$$a(x_j)\left[\frac{y_{j+1} - 2y_j + y_{j-1}}{2h^2}\right] + b(x_j)\left(\frac{y_{j+1} - y_{j-1}}{2h}\right)$$

$$+ c(x_j)y_j = d(x_j) \qquad j = 1, 2, 3 \ldots N$$

$$\left[a(x_j) + \frac{h}{2} b(x_j)\right] y_{j+1} + \left[- \cancel{2h}\, 2ax_j + h^2 c(x_j)\right] y_j$$

$$+ \left[a(x_j) - \frac{h}{2} b(x_j)\right] y_{j-1} = d(x_j) \;\forall\, j = 1, 2 \ldots$$

There are N eqns and N unknown variables $(y_1, y_2 \ldots y_N)$ which can be solved by using any known technique.

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<u>example</u> — Solve the BVP,

$$y'' = y + x_j \qquad x \in (0,1)$$

$$y(0) = 0$$

$$y(1) = 0$$

with $h = 1/4$ using second order FDM.

→ let $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$$y_0 = 0 \qquad \underbrace{y_1 \; y_2 \; y_3}_{\text{unknown}} \quad \cdots \quad y_4 = 0$$

Here y_1 y_2 y_3 are unknowns that approximate y at $1/4$, $1/2$, $3/4$ respectively

The approximate system will be,

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = y_j + x_j$$

$$j = 1, 2, 3$$

$$\frac{y_2 - 2y_1 + y_0}{h^2} = y_1 + x_1$$

$$y_2 - 2y_1 + 0 = \frac{1}{16}\left(y_1 + 1/4\right)$$

$$y_2 - 2y_1 - \frac{y_1}{16} = \frac{1}{64}$$

$$y_2 - \frac{33y_1}{16} = 1/64 \quad \text{—— } (1)$$

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$$\frac{y_3 - 2y_2 + y_1}{h^2} = y_2 + x_2$$

$$y_3 - 2y_2 + y_1 = \frac{1}{16}\left(y_2 + \frac{1}{2}\right)$$

$$y_3 - 2y_2 + y_1 - \frac{1}{16} \times y_2 = \frac{1}{32}$$

$$y_3 - \frac{33y_2}{16} + y_1 = \frac{1}{32} \quad \text{---} \ (2)$$

$$\frac{y_4 - 2y_3 + y_2}{h^2} = y_3 + x_3$$

$$0 - 2y_3 + y_2 = \frac{1}{16}\left(y_3 + 3/4\right)$$

$$y_2 - 2y_3 - \frac{1}{16} y_3 = \frac{3}{64}$$

$$y_2 - \frac{33y_3}{16} = 3/64 \quad \text{---} \ (3)$$

$$\left\{ \begin{array}{l} y_1 = -0.035 \\ y_2 = -0.036 \\ y_3 = -0.050 \end{array} \right\}$$

<u>Example</u>: Solve BVP

$$y'' = xy + 1 \qquad x \in [0, 1]$$

$$y'(0) = y(0) = 1 \quad (\text{or } y'(0) = 7)$$

$$y(1) = 1 \quad (\text{or } y(1) + 3y'(1) = 5)$$

use FDM, to find approximate value of $y(0)$, $y(0.25)$, $y(0.5)$, $y(0.75)$

Sol^n :

$h = 0.25$

$x_0 = 0 \quad x_1 = 0.25 \quad x_2 = 0.5 \quad x_3 = 0.75$

$y_0 \simeq y(0)$

$y_1 \simeq y(0.25)$

$y_2 \simeq y(0.5)$

$y_3 \simeq y(0.75)$

$\cancel{y_{j+1} = 2 + h}$

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{2h} = x_j y_j + 1$$

$$\frac{y_2 - 2y_1 + y_0}{2 \cdot \frac{1}{4}} = x_1 y_1 + 1$$

$$\frac{y_2 - 2y_1 + y_0}{1/2} = \frac{y_1}{4} + 1$$

$$y_2 - 2y_1 + y_0 = \frac{y_1}{8} + \frac{1}{2}$$

$$y_2 - \frac{17y_1}{8} + y_0 = \frac{1}{2} \quad \text{--- (1)}$$

$$\frac{y_3 - 2y_2 + y_1}{2h} = x_2 y_2 + 1$$

$$\frac{y_3 - 2y_2 + y_1}{1/2} = \frac{1}{2} y_2 + 1$$

$$y_3 - 2y_2 - \frac{1}{4} y_2 + y_1 = 1/2$$

$$y_3 - \frac{9y_2}{4} + y_1 = \frac{1}{2} \quad \text{--- (2)}$$

$$\frac{y_4 - 2y_3 + y_2}{2h = 1/2} = x_3 y_3 + 1$$

$$y_4 - 2y_3 + y_2 = \frac{3}{8} y_3 + \frac{1}{2}$$

$$y_4 - \frac{19 y_3}{8} + y_2 = 1/2 \quad \text{— (3)}$$

MVT

Use $y'(0) + y(0) = 1$

$$\frac{y_1 - y_{-1}}{2(1/4)} + y_0 = 1$$

$$2y_1 - 2y_{-1} + 2y_0 = 2 \quad \text{— (3)}$$

$$\frac{y_1 - 2y_0 + y_{-1}}{2h} = x_0 y_0 + 1$$

$$y_1 - 2y_0 - y_{-1} = \frac{1}{2} \cdot (0) + 1$$

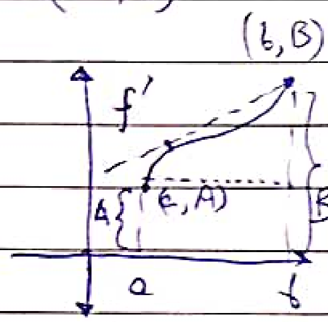
$$y_1 - 2y_0 - y_{-1} = 1 \quad \text{— (4)}$$

Shooting Method

Consider a second order BVP

$$y'' + p(x) y' + q(x) y = r(x) \qquad x \in (a, b)$$

$$y(a) = A \quad , \quad y(b) = B$$



eg: $y'' = 6y^2 - x$

$y(0) = 1$, $y(1) = 5$ $\quad h = 1/3$

Sol^n: Choose $y'(0) = \alpha_0 = 1.2$

let $y' = z$

$z' = 6y^2 - x$

$z(0) = 1.2$

We can use R.K method, Euler method, Taylor series to solve IVPS.

~~Here we can use R.K method~~

Here $x_0 = 0$ & $x_1 = 1/3$, $x_2 = 2/3$ ...

$$z' = 6y^2 - x$$

$$y_{n+1} = y_n + hz_n$$

$$\therefore z_{n+1} = z_n + h(6y^2 - x)$$

$$\cancel{y_1 = 1 + \frac{1}{3}(1.2)}$$

$$\cancel{= 1.4}$$

$$y_1 = 1 + \frac{1}{3}(1.2)$$

$$= 1.4$$

$$y_2 = y_1 + hz_1$$

$$= 1.4 + \frac{1}{3}(3.2)$$

$$= 3.466$$

$$z_1 = z_0 + h(6y_0^2 - x_0)$$

$$= 1.2 + \frac{1}{3}(6) = 3.2$$

$$z_2 = z_1 + h(6y_1^2 - x_1)$$

$$= 3.2 + \frac{1}{3}\left(6 \times (1.4)^2 - \frac{1}{3}\right)$$

$$= 7.01$$

$$y_3 = y_2 + hz_2$$

$$= 3.466 + \frac{1}{3} \times 7.01$$

$$= 4.7966$$

$y(x_3) = 4.7966$ {approx}

$y(x_3) = 5$

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$$\phi(\alpha_0) = y(1) - y(1;\alpha_0)$$
$$= 5 - 4.7966$$
Error $= 0.2034$

Now consider the initial guess
$$y'(0) = 1.5$$
Again solve ① with $\alpha_1 = 1.5$ by replacing α_0 by α_1

$y' = z \quad y(0) = 1$	$z_{n+1} = z_n + h(6y_n - x_n)$
$y_1 = y_0 + \cancel{z}(\cancel{\alpha_1})$	$z_1 = z_0 + h(6y_0 - x_0)$
$= 1 + \frac{1}{3} \times 1.5$	$= 1.5 + \frac{1}{\cancel{3}} \times \cancel{6}\, 2$
$= 1.5$	$= 3.5$
$y_2 = y_{01} + hz_1 \quad 1.166$	$z_2 = z_1 + h(6y_1 - x_1)$
$= 1.5 + \frac{1}{\cancel{3}} \times \cancel{3.5}$	$= 3.5 + \frac{1}{\cancel{3}}(\cancel{6}_2 \times 1.5)$
$= 1.5 + 1.167$	$= 6.5$
$= 1.667$	
$y_3 = \cancel{6.5}\ y_2 + hz_2$	
$= 1.67 + \frac{1}{\cancel{3}} \times \cancel{6.5}\ 2.167$	

$= 5.29$

$$\phi(\alpha_1) = 5 - 5.29$$
$$= -0.29$$

Note – If it is asked to use N.R method
$$\alpha_1 = \alpha_0 - \frac{\phi(\alpha_0)}{\phi'(\alpha_0)}$$

using secant method,

$$x_2 = x_1 - \frac{(x_1 - x_0)\cdot \phi(x_1)}{\phi(x_1) - \phi(x_0)}$$

$\Rightarrow \quad x_2 = 1.323$

$z(0) = x_2 = 1.32$

$y' = z \quad y(0) = 1$

$z' = 6y^2 - x$

$y_1 = y_0 + hz_0$	$z_1 = z_0 + h(6y_0^2 - x_0)$
$= 1 + \frac{1}{3} \times 1.32$ 0.44	$= 1.32 + \frac{1}{3}(6)$
$= 1.44$	$= 3.32$
$y_2 = y_1 + hz_1$ 1.106	$z_2 = z_1 + h(6y_1^2 - x_1)$
$= 1.44 + \frac{1}{3} \times 3.32$	$= 3.32 + \frac{1}{3}(6 \cdot 1$
$= 2.546$	$= 7.316$

$y_3 = y_2 + hz_2$

$= 2.546 + \frac{1}{3} \times 7.316$

$= 4.98$

Q) $$\frac{d^2y}{dx^2} - y = 3x \qquad x \in (0, 6)$$

$y(0) = 0$

$y(6) = 0$

using shooting method, $h = 2$

Use Euler's method to solve IVP.

Use linear-interpolation to obtain subsequent initial condition $y'(0) = 3$

$y''(0) = 3$

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let $z = y'$

$\cancel{z'}$ $\underline{\text{IVP 1}}$: $y' = z$

$y(0) = 0$

$\underline{\text{IVP 2}}$:

$z' - y = 3x$

$z(0) = 3$

$x_0 = 0 \quad x_1 = 2 \quad x_2 = 4 \quad x_3 = 6$

$h = 2$

IVP 1	IVP 2
$y' = z$	$z' - y = 3x$
$y(0) = 0$	$z(0) = 3$
$y_{n+1} = y_n + hz_n$	$z_{n+1} = z_n + (y_n + 3x_n)h$
$y_1 = y_0 + hz_0$	$z_1 = z_0 + (y_0 + 3x_0)(2)$
$= 0 + 2 \times 3$	$= 3 + (0 + 0)(2)$
$= 6$	$= 3$
$y_2 = y_1 + hz_1$	$z_2 = z_1 + h(y_1 + 3x_1)$
$y_2 = 6 + 2 \times 3$	$= 3 + 2(6 + 6)$
$= 12$	$= 27$

$y_3 = y_2 + hz_2$

$= 12 + 2 \times 27$

$= 54$

$ 12$

$ 66$

For $y'(0) = -3$

$y' = z$

$y_1 z$

$y(0) = 0$

$y_1 = y_0 + hz_0 = -6$

$y_2 = y_1 + hz_1$

$= -6 + 2 \times (-3)$

$= -12$

$y_3 = y_2 + hz_2$

$= -12 + 2 \times (-3)$

$= \cancel{-20}\ -18$

$z' = y' + 3x$

$z(0) = -3$

$z_1 = z_0 + h(y_0 + 3x_0)$

$= -3 + 2(0 + 0)$

$= -3$

$z_2 = z_1 + h(y_1 + 3x_1)$

$= -3 + 2(-6 + 3(2))$

$= -3$

$\alpha_0 = 3, \ \alpha_1 = -3$

$\varepsilon_0 = 66 - 0 \qquad \varepsilon_1 = -18 - 0$

$= 66 \qquad = -18$

Linear Interpolation Formula

$$\alpha_2 = \alpha_0 + \left(\frac{\alpha_1 - \alpha_0}{\varepsilon_1 - \varepsilon_0}\right)(y(6) - \varepsilon_0)$$

$= -1.7914$

$y' = z'$

$y'(0) = 0$

$y_1 = y_0 + hz_0$

$= 0 + 2 \times (-1.7914)$

$= -3.428$

$y_2 = y_1 + hz_1$

$= -3.428 + 2(-1.714)$

$= -6.856$

$z' = y + 3x$

$z(0) = -1.714$

$z_1 = z_0 + h(y_0 + 3x_0)$

$= -1.714 + 2(0 + 3(0))$

$= -1.714$

$z_2 = z_1 + h(y_0 + 3x_0)$

$= \cancel{-3.428 + 2(-1.714)}$

$= 3.429$

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$y_3 = y_2 + h z_2$
$= 0.104$

Shooting method using Newton-Ralphson method

let $y'(a) = \alpha$ be the additional initial form

let $y(b, \alpha) \rightarrow$ approximate value of $y(b)$ using shooting method with $y'(a) = \alpha$

let $\phi(\alpha) = y(b, \alpha) - \gamma$

γ → desired value

$\left\{ \begin{array}{l} b \Rightarrow \text{approximate value} \\ \alpha \Rightarrow \text{x value guess} \end{array} \right\}$

Compute $\phi'(\alpha)$

Denote $\frac{d\phi}{d\alpha} = \frac{dy}{dx}(b, \alpha) - \cancelto{0}{\frac{d\gamma}{dx}}$

$= \frac{dy}{dx}(b, \alpha) = \eta_\alpha(b)$

$\eta_\alpha'' = \frac{\partial f}{\partial y}\eta_\alpha + \frac{\partial f}{\partial y'}\eta_\alpha'$

$\eta_\alpha(a) = 0$

$\eta_\alpha'(a) = 1$

Example :- $y'' = y$

$y(0) = 1, \; y(1) = 0$

$f(x, y, y') = y$

$\left\{ \begin{array}{l} \eta_\alpha'' = \eta_\alpha \\ \eta_\alpha(0) = 0 \quad \eta_\alpha(1) = b \end{array} \right\}$

let $h = 1/3$

$\mu = x'_\alpha$

$x'_\alpha = \mu$

$x_\alpha(0) = 0$

$\mu(0) = x'_\alpha(0) =$

$x_{\alpha_1} = x_{\alpha_0} + h\mu_0$

$= 0 + \frac{1}{3}(1)$

$= 1/3$

$x_{\alpha_2} = x_{\alpha_1} + h\mu_1$

$= 1/3 + 1/3$

$= 2/3$

$x_{\alpha_3} = x_{\alpha_2} + h\mu_2$

$= 2/3 + \frac{1}{3} \times \frac{10}{9}$

$= \frac{18}{27} + 10/27$

$= 28/27 \rightarrow \phi'(\alpha_n)$

$\mu' = x_\alpha$

$\mu(0) = x'_\alpha(0) = 1$

$\mu_1 = \mu_0 + h(x_{\alpha_0})$

$= 1$

$\mu_2 = \mu_1 + h\, x_{\alpha_1}$

$= 1 + \frac{1}{3}(1/3)$

$= 10/9$

$$\alpha_{n+1} = \alpha_n - \frac{\phi(\alpha_n)}{\phi'(\alpha_n)}$$

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Probability

Random experiments

A random experiment is an experiment such that
1. All outcomes of experiment is known in advance.
2. Any performance of results is not known in advance.
3. The experiment can be repeated under identical condition.

Sample Space

A set S that consists of all possible outcome of a random experiment is called sample space and each outcome is called sample point.

If we toss a dice,
Sample space $S_1 = \{1, 2, 3, 4, 5, 6\}$
$S_2 = \{odd, even\}$

Event -

An event is a subset of the sample space, it is a subset of all possible outcomes.

example: $A = \{1, 6\}$ → is an event.

Class: Set of all subsets of a sample space

$$C \xrightarrow{\Phi} [0, 1]$$

$P(A_1)$

Tuj kabi samzat nahi?

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$$C \longrightarrow \{\tfrac{1}{2}, 1, 0\}$$

$S = \{H, T\}$ $\{P(H)\}$

$$S = \{\underset{1/4}{(H,H)}, \underset{1/4}{(H,T)}, \underset{1/4}{(T,H)}, \underset{1/4}{(T,T)}\}$$

$0 \leq P(A) \leq 1$

Axioms of Probability

To each event A in the class C of the events we associate a real no. P(A) and P(A) is called probability of A provided the following axioms are satisfied.

AI : For ~~event~~ every event A in
$P(A) \geq 0$

AII : For sure / certain event C,
$P(S) = 1$

A3 : For any no. of mutually exclusive events $A_1, A_2 \ldots$
$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + P(A_3)$

Theorems : i) If $A_1 \subset A_2$ then $P(A_1) \leq P(A_2)$
$P(A_1/A_2) = P(A_2) - P(A_1)$

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2) For every event A, $0 \le P(A) \le 1$
3) For impossible event ϕ $P(\phi) = 0$
4) If A^c is complement of A then $P(A^c) = 1 - P(A)$

5) If

$$P(A \cup B) = \begin{cases} P(A) + P(B) & A \& B \text{ mutually exclusive} \\ P(A) + P(B) - P(A \cap B) & A \& B \text{ are not mutually exclusive} \end{cases}$$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$$

For any two events A & B
$$P(A) = P(A \cap B) + P(A \cap B^c)$$

Q) A single die is tossed once. Find the probability of 1 or 4 turn up.

$P(1) = 1/6$
$P(4) = 1/6$
$P(1) + P(4) = 1/3$ //

Conditional Probability

let A and B be two events such that $P(A) > 0$
then $P(B/A)$ is

the probability of B given that A has occurred Since A is known to have occurred it becomes the new sample space replacing the originals

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

<u>example</u> : Find the probability that a single toss of a dice will result in a number less than 4 if
(a) No prior information is given
(b) an odd number

$P(a) = 3/6 = 1/2$
$P(b) = 2/3$

Baye's Theorem

Suppose $A_1 \dots A_n$ are mutually exclusive events whose union is sample space S. Then if A is any event we have:

$$P(A_k|A) = \frac{P(A_k)\,P(A|A_k)}{\sum_{j=1}^{n} P(A_j)\,P(A|A_j)}$$

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eg A coin is tossed twice. Possible outcomes are -
1 → H or T — ①
2 → T or H

S.S = {HH, HT, TH, TT}
Sample space
1 2 3 4

each element is called event

$P(1) = 1/4$
$P(2) = 1/4$
$P(3) = 1/4$
$P(4) = 1/4$

Random Variable

Let E be a random experiment and S be the corresponding sample space. A random variable function is : $X : S \to R$

Suppose a coin is tossed twice sol^n the possible sample spaces are :

$S = \{HH, HT, TH, TT\}$

Let X represent the number of heads that can come up with

$X(HH) = 2$
$X(HT) = 1$
$X(TH) = 1$
$X(TT) = 0$

Discrete Random Variable

A random variable that takes on a finite or countably infinite number of values is called discrete random variable.

Discrete Probability Distribution

Let X be a discrete random variable and let assume that if it takes the value $x_1, x_2 \dots x_n$. Define

$$P(x = x_i) = f(x_i)$$

Then f is called as discrete ~~function~~ probability.

$$P(HH) = \begin{cases} f(HH) = 1/4 \\ f(HT) = 1/4 \\ f(TH) = 1/4 \\ f(TT) = 1/4 \end{cases}$$

refer to 1

$$\Sigma f(x_i) = 1$$

Distribution of functions

$$x_i \geq 0$$

$$\Sigma f(x_i) = 1$$

$$P(\underset{\text{heads}}{X} \geq 1) = 3/4$$ refer to 1

$$P(\underset{\text{tails}}{Y} \leq 1) = 3/4$$

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Distribution ~~Discrete~~ functions

Let X be a random variable. Then a distribution function F on X is defined as

$$F(x) = P(X \leq x) \quad \text{when } x \in \mathbb{R}$$

$$F(x_2) = 1/2$$

Properties:

1) $F(x)$ is non decreasing
2) $\lim_{x \to -\infty} F(x) = 0 \qquad \lim_{x \to \infty} F(x) = 1$
3) $F(x)$ is continious from right
$$\lim_{h \to 0^+} F(x+h) = F(x)$$

Let X be a discrete random variable. and f denotes the probability function. Then the distribution function is determined by —

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u)$$

If f takes only finitely many number of value $x_1, x_2, x_3 \cdots$ (in some order) then distribution function $F(x)$ is defined as —

$$F(x) = \begin{cases} 0 & x < x_1 \\ f(x_1) & x_1 \leq x < x_2 \\ f(x_1) + f(x_2) & x_2 \leq x < x_3 \\ \vdots & \\ f(x_1) + f(x_2) \cdots f(x_n) & x_n < x < \infty \end{cases}$$

According to (1)

$$F(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ 1/4 & 0 \le x < 1 \\ 3/4 & 1 \le x < 2 \\ 1 & 2 \le x < \infty \end{cases}$$

Continious Random variable

A random variable X is said to be continious random variable if it takes all possible values in a given interval of R.

Example: Age, height in a large population

Probability function

$f(x) = P(X = x)$

if (i) $f(x) \ge 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$$P(a \le x \le b) = \int_{a}^{b} f(x)\,dx$$

Q. Find the constant c such that the function

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is a probability function. Also compute $P(1 < x < 2)$

$$\int_0^3 cx^2\,dx = 1$$

$$\frac{cx^3}{3}\Big|_0^3 = 1$$

$$c = \frac{1}{\frac{3\times3\times3}{3}} = \frac{1}{9}$$

$$\int_1^2 cx^2\,dx = \frac{c}{3}\left(x^3\right)\Big|_1^2$$

$$= \left(\frac{8-1}{3}\right)\cdot\frac{1}{9}$$

$$= \frac{7}{27}$$

$$F(x) = \int_{-\infty}^{x} cx^2\,dx = \frac{1}{9}\,\frac{x^3}{3}\Big|_0^x$$

$$= \frac{x^3}{27}$$

Suppose that a pair of fair dice are thrown and x denotes the random variable of the sum of the outcome. Find the probability function and the distribution function.

Sol^n:

$$P(x) = \begin{cases} 0 & x = 1 \\ 1/36 & x = 2 \\ 2/36 & x = 3 \\ 3/36 & x = 4 \\ 4/36 & x = 5 \\ 5/36 & x = 6 \\ 6/36 & x = 7 \\ 5/36 & x = 8 \\ 4/36 & x = 9 \\ 3/36 & x = 10 \\ 2/36 & x = 11 \\ 1/36 & x = 12 \end{cases}$$

$$P(X) = \begin{cases} 0 & x = 1 \\ \frac{1}{36} & x = 2, 12 \\ \frac{2}{36} & x = 3, 11 \\ 3/36 & x = 4, 10 \\ 4/36 & x = 5, 9 \\ 5/36 & x = 6, 8 \\ 6/36 & x = 7 \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 2 \\ 1/36 & x \geq 2 \ \& \ x < 3 \\ 3/36 & x \geq 3 \ \& \ x < 4 \\ 6/36 & x \geq 4 \ \& \ x < 5 \\ 10/36 & x \geq 5 \ \& \ x < 6 \\ 15/36 & x \geq 6 \ \& \ x < 7 \\ 21/36 & x \geq 7 \ \& \ x < 8 \\ 26/36 & x \geq 8 \ \& \ x < 9 \\ 30/36 & x \geq 9 \ \& \ x < 10 \\ 33/36 & x \geq 10 \ \& \ x < 11 \\ 35/36 & x \geq 11 \ \& \ x < 12 \\ 1 & x \geq 12 \end{cases}$$

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Eg: A random variable x has the following probability function

x:	0	1	2	3	4	5	6	7
$f(x)$:	0	$2k$	$2k$	$2k$	$3k^2$	$2k^2$	$2k^2$	$7k^2+k$

(i) Find k

(ii) Evaluate $P(x \geq 6)$, $P(x < 6)$
$P(x \leq c) > \frac{1}{2}$, $P(0 < x < 5)$

$\Sigma f(x) = 1$

$8k + 3k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0$

~~$10k^2 + 5k + 2k =$~~

$10k^2 + 10k - k - 1 = 0$

$10k(k+1) - 1(k+1) = 0$

$k = -1$ or $k = 0.1$ //

$P(x < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= 0 + 0.1 + 0.2 + 0.2 + 0.3 + 0.01$

$= 0.81$ //

$P(x \geq 6) = 1 - P(x < 6)$

$= 0.19$ //

$P(x \leq c) > 1/2$

$P(0) + P(1) \cdots P(c) > 1/2$

~~$0 + 0.1 + 0.2 + 0.$~~

$$F(x) \begin{cases} 0 & x < 0 \\ 0.1 & 0 < x < 2 \\ 0.3 & 2 < x < 3 \\ 0.5 & 3 < x \leq 4 \\ & = 1/2 \end{cases}$$

$\boxed{c = 4}$

Joint Distribution

Discrete Case

discrete

Let X and Y be 2 random variables, we define the joint probability on X and Y by

$$f(x, y) = P(X = x, Y = y)$$, where $f(x,y)$ satisfies the condition

i) $f(x,y) \geq 0$

ii) $\sum_x \sum_y f(x,y) = 1$

Let X can assume $x_1, x_2, \ldots, x_n$ and Y can assume $y_1, y_2, \ldots y_n$.

Then

$$P(X = x_j, Y = y_j) = f(x_j, y_k)$$

$$P(X = x_j) = P(X = x_j, y = y_1) + P(X = x_j, y = y_2) = \sum \ldots$$

$$P(Y = y_j) = P(X = x_1, y = y_j) + P(X = x_2, y = y_j) \ldots$$

$$= \sum_{k=1} P(x_k, y_j)$$

Joint - distribution

~~Let X and Y be two discrete random variables. Then~~

X \ Y	y_1	y_2	y_3	...	y_n	Total
x_1	$f(x_1 y_1)$	$f(x_1 y_2)$	. . . . .		$f(x_1, y_n)$	$\sum f(x_1 \ldots$
x_2	$f(x_2 y_1)$	$f(x_2 y_2)$	$f(x_2 y_3)$		$f(x_2 y_n)$	$\sum f(x_2 \ldots$
x_3	$f(x_3 y_1)$	$f(x_3 y_2)$	$f(x_3 y_3)$	-	$f(x_3 y_n)$	$\sum f(x_3 \ldots$
⋮						
x_n	$f(x_n y_1)$	$f(x_n y_2)$	$f(x_n y_3)$	.	$f(x_n y_n)$	$\sum f(x_n \ldots$
	$\sum f(x_k y_1)$ +	$\sum f(x_k y_2)$ +	$\sum f(x_k y_3)$ +		+ $\sum f(x_k y_n)$	1 =

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Eg// Let the joint probability function f of two discrete random variables X and Y be:

$$f(x,y) = \begin{cases} c(2x+y) & \text{if } 0 \le x \le 2 \\ & 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

when $x, y \in \mathbb{Z}$

① Find c

② Find $P(X=2, Y=1)$, $P(X \geq 1, y \leq 2)$

$$\boxed{\cancel{\int_0^3 \int_0^1 c(2x+y)\, dx\, dy} \qquad \cancel{c \int \left[x^2 + xy\right]_0^1 dy}}$$

$$\sum_{y=0}^{3} \sum_{x=0}^{1} c(2x+y)$$

	0	1	2	3
0	0	c	2c	3c
1	2c	3c	4c	5c
2	4c	5c	6c	7c
	6c	9c	12c	15c

$42c = 1$

$\{c = 1/42\}$

$\rightarrow \sum_{j=1}^{n} \sum_{k=1}^{n} f(x_j, y_k)$

$$P(X=2, Y=1) = f(2,1) = 5c = 5/42$$

$$P(X \geq 1, Y \leq 2)$$

$= f(1,0) + f(1,1) + f(1,2) + f(2,0) + f(2,1) + f(2,2)$

$= 2c + 3c + 4c + 4c + 5c + 6c$

$= 5c + 8c + 11c$

$= 24c$

$= 24/42$

$= 4/7$

The joint distribution function ~~for~~ / cumulative joint distribution function is

$$F(x,y) = P(X \leq x, Y \leq y) = \sum_{u \leq x} \sum_{v \leq y} f(u,v)$$

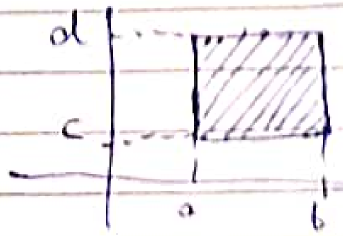
Continious Case

let X and Y be two continious function random variables. Then the probability that X lies in between a and b, Y lies between c and d. Then the probabil is given by $P(a < x < b,\ c \le y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Here $f(x,y)$ satisfies :

(i) $f(x,y) \geq 0$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\cancel{dy}\,dx\,dy = 1$



<u>Note</u> : If A represents any event there will be a region R_A in x-y plane such that

$$P(A) = \iint_R f(x,y)\,dy\,dx$$

<u>Example 2</u> :

Let the joint distribution function of random variables X and Y be :

$$f(x,y) = \begin{cases} \frac{1}{96} xy & 0 < x < 4 \\ & 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X+Y < 3)$

$$\int_0^4 \int_1^{3-x} \frac{1}{96} xy \, dy\, dx$$

$$\frac{1}{96}\int_0^4 \left(\frac{xy^2}{2}\right)\Big|_1^{3-x} dx$$

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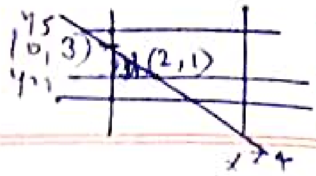
$y = -x+3$
$1 = -x+3$
$x = 2$



$$\frac{1}{96\times 2}\int_0^4 x\left[(3-x)^2-1\right]dx$$

$$\frac{1}{96\times 2}\int_0^4 x(9+x^2-6x)-x\;dx$$

$$\frac{1}{96\times 2}\int_0^4 8x+x^3-6x^2\;dx$$

$$\frac{1}{96\times 2}\left(\frac{8x^2}{2}+\frac{x^4}{4}-\frac{6x^3}{3}\right)\Bigg|_0^4$$

$$\frac{1}{96\times 2}\left[4(4)^2+\frac{4^4}{4}-2(4)^3\right]$$

$$64+64-128$$

$$\int_0^2\int_1^{3-x}\frac{1}{96}xy\,dy\,dx$$

$$\frac{1}{96}\int_0^2 x\left(\frac{y^2}{2}\right)\Bigg|_1^{3-x}dx$$

$$\frac{1}{96\times 2}\int_0^2 x\left((3-x)^2-1\right)dx$$

$$\frac{1}{96\times 2}\int_0^2 8x+x^3-6x^2\,dx$$

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$$\frac{1}{2}\times\frac{1}{96}\left(4x^2+\frac{x^4}{4}-\cancel{6}2x^3\right)_0^2$$

$$\frac{1}{2}\times\frac{1}{96}\left[4\times4+\cancel{8}\frac{4\times4}{\cancel{4}}-2\times4\times2\right]$$

$$\frac{1}{2}\times\frac{1}{96}\left[\cancel{16}+4-\cancel{16}\right]$$

$$\frac{1}{2}\times\frac{1}{24}=\frac{1}{48}$$

<u>Note</u> : $\frac{\partial^2 F}{\partial x\,\partial y}=f(x,y)$

$$P(X\le x)=\int_{-\infty}^{x}\int_{-\infty}^{\infty}f(u,v)\,dv\,du$$

$$P(Y\le y)=\int_{-\infty}^{y}\int_{-\infty}^{\infty}f(u,v)\,du\,dv$$

Then $F_1(x)$ and $F_2(y)$ are called marginal distribution function or distribution function of X and Y.

Eg. Find distribution function and marginal distribution function.

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$x<0 \quad y<1 \quad F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx = 0$

$x<0 \quad 1<y<5 \quad F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx = 0$

$x<0 \quad y>5 \quad F(x,y) = 0$

$0<x<4 \quad y<1 \quad F(x,y) = 0$

$0<x<4 \quad 1<y<5 \quad F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{xy}{96}\,dy\,dx$

$$\int_{0}^{x}\int_{1}^{y} \frac{xy}{96}\,dy\,dx$$

$$\frac{1}{2\times 96}\int_{0}^{x} u\,(y^2-1)\,du$$

$$\frac{1}{4\times 96}\, x^2(y^2-1)$$

$$\frac{1}{384}\, x^2(y^2-1)$$

$0<x<4 \quad y>5 \quad \int_{0}^{4}\int_{1}^{5} \frac{xy}{96}\,dy\,dx$

$\Rightarrow x^2/16$

$x > 4, \; y < 1$

$F(x,y) = \iint xy/96 = 0$

$x > 4 \quad 1 < y < 5$

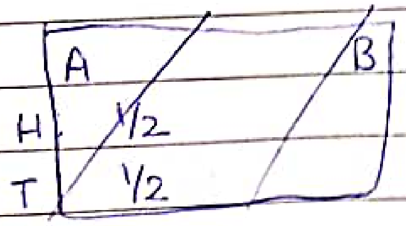
$$\int_0^4 \int_0^y xy/96 \; dy\,dx$$

$$= \int_0^4 \frac{xy^2}{2\times 96}\, dx$$

$$= \frac{x^2 y^2}{4\times 96} = \frac{16\, y^2}{4\times 96} = y^2/24$$

$x > 4 \quad y > 5$

$F(x,y) = 1$



Mathematical Expectation

Let X be a discrete random variable and let $f(x)$ be the corresponding probability mass function.

$f(x) = P(X = x)$

or $f(x) = P(X = x_k)$ where $k = 1, 2, 3 \dots$

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and if $\sum |x| f(x) < \infty$ we say that the expected value/ mean/ mathematical expression of x exists and is defined as

$$E(x) = \sum_{k=1}^{\infty} x_k f(x_k)$$

Note that the series $\sum_{k=1}^{\infty} |x_k| f(x_k)$ may not converge but this series $\sum_{k=1}^{\infty} x_k f(x_k)$ converges. In this case we can say $E(x)$ does not exists.

eg. Two players A and B, play a coin tossing game. A gives B one dollar if a head turns up otherwise B gives A one dollar. If probability of coin showing head is p then find expected gain of A.

	x_i	$p(x)$	$p_i x_i$
H :	-1	p	$-p$
T :	1	$(1-p)$	$(1-p)$

$$\sum p_i x_i = 1-2p$$

let X denote the random variable of amount paid to A

$$x \begin{cases} 1 & p \\ -1 & (1-p) \end{cases}$$

$1-2p > 0$ when $p > 1/2$ & $1-2p < 0$ when $p < 1/2$

at $p = 1/2$

$1 - 2p = 0$

[so unbiased coin is used]

Example 2:

The probability density function for a random variable is given by

$$f(x) \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Then the expected value of x is $\int_{-\infty}^{\infty} x f(x) dx$

$$\int_0^2 x \cdot \frac{1}{2} x \, dx = \cancel{\left.\frac{x^2}{4}\right|_0^2} \quad \cancel{2}$$

$$\left.\frac{1}{2} \frac{x^3}{3}\right|_0^2$$

$$8/6 = 4/3$$

Theorem: Let x and y be two random variables. Then:

1. $E(cx) = c E(x)$ for any constant c
2. $E(x+y) = E(x) + E(y)$
3. $E(xy) = E(x) \cdot E(y)$ provided x and y are independent.

Exercise

Suppose that a game is to be played with a single fair die. In this game a player win ₹20 if 2 turns up, ₹40 if 4 turns up, loses ₹30 if 6 turns up and neither win / loses if any other face turns up.

Let X be the random variable for which player wins money

$$X \begin{cases} 20 & x = 2 \\ \cancel{30}\ 40 & x = 4 \\ -30 & x = 6 \\ 0 & x = 1, 3, 5 \end{cases}$$

$P(1)$

$$P(x) = 1/6 = P(2) = P(3) = P(4) = P(5) = P(6)$$

$$E(x) = \sum x_i P(x_i)$$

$$E(X) = \frac{1}{6} \times 20 + \frac{1}{6} \times 40 - \frac{1}{6} \times 30$$

$$\frac{\cancel{60}\ 40}{\cancel{6}} - \cancel{0}\,5 = ₹5$$

Definition: Let μ denote the mean / or mathe expectation of a random variable X. Then variance of X is defined and denoted by

$$V(X) = \left[E(X-\mu)^2\right]$$

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$\cancel{E}\, E(x)^2 - E(x)$

and it's positive root is called standard deviation & denoted by

$$\sigma_x = \sqrt{\underset{Var(x)}{\text{Variance}}} = \sqrt{E\left[(x-\mu)^2\right]}$$

Q) Find the variance and standard deviation of the pdf:

$$f(x) \begin{cases} x/2 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^2 x \cdot \frac{x}{2}\,dx$$

$$= \frac{1}{2}\int_0^2 x^2\,dx$$

$$= \frac{x^3}{6}\Big|_0^2 = 8/6 = 4/3$$

$$\mu = 4/3$$

$\cancel{E(x^2)} =$

$$V(x) = \sqrt{E(x-4/3)^2}$$

$$= \int_{-\infty}^{\infty} (x-4/3)\, f(x)\,dx$$

$$= \int_0^2 (x-4/3)^2\, x/2\,dx$$

$$= \cancel{\int_0^2 \frac{x^2}{2} - \frac{2x}{3}\,dx}$$

$$\cancel{= x^3/6 - \frac{2x^2}{?}\Big|_0^2 = \frac{8}{96} - \frac{404}{6}}$$

$$\int \left(x^2 + \frac{16}{9} - \frac{8x}{3}\right)\frac{x}{2}\,dx$$

$$\int_0^2 \frac{x^3}{2} + \frac{8x}{9} - \frac{4x^2}{3}\,dx$$

$$\left.\frac{x^4}{8} + \frac{4x^2}{9} - \frac{4x^3}{9}\right|_0^2$$

$$= \frac{16}{8} + \frac{4\cdot 4}{9} - \frac{4\cdot 4\cdot 2}{9}$$

$$2 + \frac{16}{9} - \frac{32}{9}$$

$$2 - \frac{16}{9}$$

$$Var(x) = 2/9$$

$$\sigma = \sqrt{2/9} = \sqrt{2}/3$$

Theorem: let x be a random variable with μ as mathematical expectation mean. Then -

① $\sigma_x^2 = [E(x-\mu)^2] = E(x^2) - (\mu)^2 = E(x^2) - [E(x)]^2$

② $Var(cx) = c^2 Var(x)$ for any constant c

③ $\min[E(x-a)^2] = [E(x-\mu)^2]$

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(4) $Var(X+Y) = Var(X) + Var(Y)$

$\sigma^2_{X+Y} = \sigma_x^2 + \sigma_y^2$

(5) $Var(X-Y) = Var(X) + Var(Y)$

$= \sigma_x^2 + \sigma_Y^2$

provided X & Y are independent

$P(\cancel{X=x})$

Independent random Variable

Suppose that X and Y are two discrete random variables. If the events $X=x$ and $Y=y$ are independent for all x and y we say that X and Y are independent random variables. In this case,

$$P(X=x, Y=y) = P(X=x)\,P(Y=y)$$

$$f(x,y) = f_1(x) \cdot f_2(y)$$

Conversely, if for all x & y, $f(x,y)$ can be written as $f(x,y) = f_1(x) \cdot f_2(y)$ for some f_1 and f_2, X and Y are independent, otherwise dependent.

In case of X and Y $\rightarrow$ independent

$$P(X \le x, Y \le y) = P(X \le x) \cdot P(Y \le y)$$

$$F(x,y) = F(x) \cdot F(y)$$

Theorem: For 2 independent random variables X and Y

$$E(XY) = E(X) \cdot E(Y)$$

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Proof: Let $f(x,y) = P(X \leq x, Y \leq y)$ be the corresponding pdf and pmf. Since X and Y are independent

$$f(x,y) \equiv f(x) \cdot f(y) \quad \text{when}$$

$f(x) \Rightarrow$ pdf for x
$f(y) \rightarrow$ pdf for y

$$E(xy) = \sum_x \sum_y xy f(x,y)$$

$$= \sum_x \sum_y xy\, f(x) \cdot f(y)$$

$$= \sum_x x f(x) \sum_y y f(y)$$

Hence Proved

$$E(xy) = E(x) \cdot E(y)$$

Theorem: let $\mu = E(x)$ then prove that

$$E(x-\mu) = 0$$

Proof

$$E(x-\mu) = E(x) - E(\mu)$$

$$= \mu - \sum f(x_i) \cdot \mu$$

$$= \mu - \mu$$

$$= 0$$

Hence Proved

$E(\cancel{x}) + E(x) + E(\mu) \to 0$

① To prove: $E(x^2) - (E(x))^2 = E(x-\mu)^2$

$$E(x-\mu)^2 = E(x^2 + \mu^2 - 2x\mu)$$

$$= E(x^2) + E(\mu^2) - \cancel{E}(2x\mu)$$

$$= E(x^2) + 1 - E(2\mu x)$$

$$= E(x^2) + \cancel{1} - E(x) - \cancel{1}$$

$$= E(x^2) - E(x)$$

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2) $Var(cx) = c^2 Var(x)$

Proof:

$$Var(cx) = \sqrt{E[(x-\mu)^2]}$$

$E(x)$ ~~$E(x^2)$~~ $E(x)$

$$= c^2 Var(x)$$

3) $\min_{a \in E} E[(x-a)^2] = E[(x-\mu)^2]$

Proof:

$$E[(x-a)^2] = [E(x^2+a^2-2ax)]$$

$$= E(x^2) + E(a^2) - E(2ax)$$

$$= E(x^2) + E(a^2) - [\cancel{E(2) \cdot E(x) \cdot E(a)}]$$

$$[E(x-\mu+\mu-a)^2]$$

$$= E[(x-\mu)+(\mu-a)]^2$$

$$= E(x-\mu)^2 + E(\mu-a)^2 + E(2(\mu-a)(x-\mu))$$

$$= E(x-\mu)^2 + 2(\mu-a)\underbrace{(\cancel{\quad})}_{E(x-\mu)} + (\mu-a)^2$$

$$= E(x-\mu)^2 + (\mu-a)^2$$

$$\min_{a \in \mathbb{R}} E[(x-\mu)^2]$$

Hence Proved

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$Var(X+Y) = Var(X) + Var(Y)$ provided X & Y are independent

~~E(X+Y)~~ Proof: $E[((X+Y) - \mu_{XY})^2] = Var(X+Y)$

$$Var(X+Y) = E[[(X+Y) - (\mu_X + \mu_Y)]^2]$$

$$= E[(X+Y)^2 + (\mu_X + \mu_Y)^2 - 2(X+Y)(\mu_X+\mu_Y)]$$

$$= E[(X+Y)^2] + E[(\mu_X + \mu_Y)^2] - 2(\mu_X+\mu_Y)E(X+Y)$$

$$= E[(X+Y)^2] + (\mu_X+\mu_Y)^2 - 2(\mu_X+\mu_Y)E(X+Y)$$

$$E[[(X-\mu_X) + (Y-\mu_Y)]^2]$$

$$= \cancel{E[(X-\mu_X)]^2 + E[(Y-\mu_Y)^2]}$$

$$= E[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)]$$

$$= E[(X-\mu_X)]^2 + E[(Y-\mu_Y)^2] + 2[\cancel{E(X-\mu_X)}E(Y-\mu_Y)]$$

$$= E[(X-\mu_X)]^2 + E[(Y-\mu_Y)]^2$$

$$= Var(X) + Var(Y)$$

Suppose $X \to$ random variable

$\mu \to E(X) \to$ mean, $\sigma \to$ standard deviation

[Standardized random variable] $\leftarrow X^* = \dfrac{X-\mu}{\sigma}$

$$E(X^*) = 0$$

$$\sigma_{X^*} = \sqrt{E((X^*-\mu))} = 1$$

g $\longrightarrow$ $X : S \to R \xrightarrow{g} R$

$g \circ X$ is also random variable.

Probability (contd.)

Function of random variable

let X be a discrete random variable with pmf f and g $R : R$ be a given function. Then $y = g \circ x = g(x)$ is also a random variable and its pmf is given by

$$h(y) = P(Y = y) = \sum_{(x): g(x) = y} P(X = x) = \sum_{\{x : g(x) = y\}} f(x) \geq 0$$

$$\sum_{y} h(y) = 1$$

Proof: $\sum_{y} h(y) = \sum P(Y = y) = \sum_{Y} \sum_{(x \circ g(x) = y)} f(x) = 1$

$$E(Y) = \sum_{i=1}^{\infty} y_i h(y_i) = \sum_{i=1}^{\infty} g(x_i) h(y_i)$$

$$h(y_i) = \sum_{\{x : g(x) = y_i\}} f(x_i)$$

Repeat the same for continious random variable

Q) Compute $E(3x^2 - 2x)$ for

$$f(x) \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E(3x^2-2x^0) = \int_{-\infty}^{x} (3x^2-2x) \cancel{\tfrac{1}{2}} f(x)\, dx$$

$$= \int_{0}^{x} (3x^2-2x) \frac{1}{2}x\, dx$$

$$\frac{1}{2}\int_{0}^{x} 3x^3-2x^2\, dx$$

$$\frac{1}{2}\left(\frac{3x^4}{4}-\frac{2x^3}{3}\right)\Big|_0^x$$

$$E(3x^2-2x) = \frac{3x^4}{8} - \frac{x^3}{3}$$

$$E(3x^2-2x) \begin{cases} 0 & x < 1 \\ 1/24 & 1 \le x < 2 \\ 10/3 & 2 \le x \end{cases}$$

Let X be a continuous random variable with pdf f and $g : R \to \mathbb{R}$

$$E(Y) = E(g(x)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$$

In case of joint distribution

$$E(g(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dy\, dx$$

Moments

Let X be a random variable and μ is the mean then r^{th} central moment is defined as

$$\mu_r = E[(X-\mu)^r] \qquad r = 0, 1, 2 \ldots$$

$$= \begin{cases} \sum (x-\mu)^r f(x) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx & \text{(continious case)} \end{cases}$$

The r-th moment of X about origin also called r^{th} raw moment, is defined by -

$$\mu'_r = E(X^r) \quad , \quad r = 0, 1, 2, 3 \ldots$$

$\mu_0 = 1$
$\mu_1 = 0$
$\mu_2 =$ variance

Prove that :

$$\mu_r = \mu'_r - \binom{r}{1}\mu'_{r-1}\mu + \ldots + (-1)^j \binom{r}{j} \mu'_{r-j}\mu^j \ldots + (-1)^r \mu'_0 \mu^r$$

Central Moment | Row Moment | moment round

$\mu_r = E((x-\mu)^r)$ | $\mu'_r = E(x^r)$ | $\mu^{\circ}_r = E((x-0)^r)$

$\mu_0 = \cancel{0}\ 1$ | $\mu'_0 = \cancel{0}\ 1$ | $\mu^{\circ}_0 = 1$

$\mu_1 = \cancel{\mu}\ 0$ | $\mu'_1 = \mu$ | $\mu^{\circ}_1 = \mu - E(x)$

$$\mu_r = E((x-\mu)^r) = \mu'_r - {}^rC_1 \mu'_{r-1}\mu + {}^rC_2 \mu'_{r-2}\mu^2$$

eg Find the first four moment
(a) about the origin
(b) about mean
for a random variable X with density

$$f(x) \begin{cases} \frac{4}{81} x(9-x^2) & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) $\mu'_0 = 1$

$$\mu'_1 = \int_0^3 (x-\mu)\frac{4}{81} x(9-x^2)\,dx \qquad \mu = 0$$

$$= \int_0^3 \frac{4}{81} x^2(9-x^2)\,dx$$

$$= \frac{4}{81}\left(3x^3 - \frac{x^5}{5}\right)\Big|_0^3 = \frac{4}{81}\left(81 - \frac{3^5}{5}\right)$$

$$\frac{4}{81}\cancel{(3-1/5)}$$

$$\frac{4}{81} \times \frac{162}{5}$$

$$\mu_3' = \int_0^3 x^3 \frac{4}{81} x(9-x^2)\,dx$$

$$= \frac{4}{81}\int_0^3 x^4(9-x^2)\,dx$$

$$= \frac{4}{81}\int_0^3 (9x^4 - x^6)\,dx$$

$$\frac{4}{81}\left(\frac{9x^5}{5} - \frac{x^7}{7}\right)_0^3$$

$$\frac{4}{81}\left(\frac{9 \cdot 243}{5} - \frac{3^7}{7}\right)$$

$$4 \cdot 27\left(\frac{2}{35}\right) = 216/35$$

$$\mu_2' = \frac{4}{81}\int_0^3 x^3(9-x^2)\,dx$$

$$= \frac{4}{81}\int_0^3 (9x^3 - x^5)\,dx$$

$$\frac{4}{81}\left(\frac{9x^4}{4} - \frac{x^6}{6}\right)_0^3$$

$$\frac{4}{\cancel{81}}\left(\frac{9 \cdot \cancel{81}}{4} - \frac{\cancel{81} \cdot 9}{6}\right)$$

$$4\left(9/4 - 9/6\right)$$

$$\cancel{4}\left(\frac{54-36}{\cancel{24}\ 6}\right) = \frac{18}{6} = 3$$

$\mu_4' = 27/2$

$\mu_0 = 1$

$\mu_1 = 0$

$\mu_2 = 11/25 \quad \cancel{\mu_2' - 2\mu_1' \mu_1}$

$\mu_2 = \mu_2' - 2\mu_1' \cdot \mu_1$ // { Formula }

Moment Generating Function

The moment generating function on a random variable x is defined by

$$M_x(t) = E(e^{xt}) = \begin{cases} \sum_{i=1}^{\infty} e^{tx_i} f(x_i) \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \end{cases}$$

Theorem: $M_x(t) = 1 + \mu t + \mu' \frac{t^2}{2!} + \dots \mu_r' \frac{t^r}{r!}$

$$\mu_r' = \frac{d^r}{dt^r} M_x(t) \Big|_{t=0}$$

$$M_x(t) = E(e^{tx})$$
$$= E\left(1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} \dots\right)$$
$$= 1 + tE(x) + \frac{t^2}{2!} E(x^2) + \frac{t^3}{3!} E(x^3) \dots$$
$$= 1 + \mu_1' + \frac{t^2}{2!} \mu_2'$$

Q) Prove that if X and Y are independent random variable.

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$M_{X+Y}(t) = E(e^{tX} e^{tY})$$
$$= E(e^{tX}) \cdot E(e^{tY})$$

Q) The random variable X can assume the value 1 and -1 with probability 1/2 each.

① Find MGF

② Find first four moments about origin

$$X \{-1, 1\}$$

$$P(X = -1) = 1/2$$
$$P(X = 1) = 1/2$$

$$M_x(t) = e^{tx_1} f(x_1) + e^{tx_2} f(x_2)$$

$$= \frac{1}{2} e^{tx_1} + e^{tx_2}$$

$$= \frac{1}{2} \left(e^{-tx} + e^{tx}\right)$$

$$\mu_0' = E(X-0) = \int x \cdot \frac{1}{2}\left(e^{-tx} + e^{tx}\right) dx$$

$$= 1$$

$$\frac{1}{2}\left(e^{-t} + e^{t}\right)$$

$$= \frac{1}{2}\left(1 - \cancel{t} + \frac{t^2}{2} - \cancel{\frac{t^3}{3!}} \dots + 1 + \cancel{t} + \frac{t^2}{2!} + \cancel{\frac{t^3}{3!}}\right)$$

$$M_x(t) = \frac{1}{2}\left(\cancel{2} + \cancel{2} \cdot \frac{t^2}{2!} + \cancel{2} \cdot \frac{t^4}{4!} + \cancel{2}\,\frac{t^6}{6!} \dots\right)$$

$\mu_1' = 0 = \mu_3' = \mu_5' = \ldots$

$\mu_2' = 1 = \mu_4 = \mu_6 \ldots$

Same question with

$$f(x) \begin{cases} 2e^{-2x} & x>0 \\ 0 & x<0 \end{cases}$$

Find MGF & first four moments about origin

$$\cancel{M_x(t) = \int_0^\infty e^{tx} \cdot 2e^{-2x}\, dt}$$

$$\cancel{2e^{-2x}}$$

Theorem: (a) If a and b are constants the MGF of $\frac{x+a}{b}$ is

$$M_{\frac{x+a}{b}}(t) = e^{at/b} M_x(t/b)$$

(b) Suppose X and Y are random variable having moment generating function $M_x(t)$ and $M_y(t)$. Then X and Y have the same probability distribution

$$M_x(t) = M_y(t)$$

$$M_x(t) = \int_0^\infty 2e^{-2x} \cdot e^{tx} \, dx \qquad (t<2)$$

$$\cancel{\boxed{2e^{-2x} \Big/ \frac{e^{tx}}{x} \Big|_0^\infty}}$$

$$2\int_0^\infty e^{x(t-2)} \, dx$$

$$2 \left. \frac{e^{x(t-2)}}{(t-2)} \right|_0^\infty \qquad \frac{1}{(t-2)} 2(0-1)$$

-ve

$$\cancel{\boxed{2\left(\frac{0+1}{t-2}\right)}} \qquad \frac{-2}{t-2}$$

$$= \frac{1}{1-(t/2)}$$

$$= (1 - t/2)^{-1}$$

$$M_x(t) = 1 + t/2 + t^2/4! + t^3/8 \ldots$$

$$\left\{ \begin{array}{l} \mu_0 = 1 \\ \mu_1 = 1/2 \\ \mu_2 = 1/2 \\ \mu_3 = 3/2 \end{array} \right\}$$

Recall :

$$P(X, Y) = f(x, y)$$

$$P(X = x_k) = \sum_{i=1}^{n} f(x_k, y_i) \qquad \sum_{i=1}^{n} f(x_k, y_i)$$

$k = 1, 2, 3 \ldots$

$$P(Y = y_k) = \sum_{i=1}^{n} f(x_i, y_k) \qquad \sum_{i=1}^{m} f(x_i, y_k)$$

$$P(a < x < b,\ c < y < d) = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

$$A \subseteq \mathbb{R}^2 \qquad P(A) = \iint_A f(x, y)\, dx\, dy$$

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dy\, dx$$

$$\frac{d^2 F}{dx\, dy} = f(x, y)$$

Marginal distribution

Marginal density function
$$f_1(x) = \int_{-\infty}^{\infty} f(x, v)\, dv$$

$$f_2(y) = \int_{-\infty}^{\infty} f(u, y)\, du$$

Verience and expectation of joint distribution

Let X and Y be two random variables with pmf/pdf $f(x,y)$ mean w.r.t x (or Y) is given by

$$\mu_x = E(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$$

$$\mu_y = E(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

Verience

$$\sigma_x^2 = Var(x) = E((x-\mu_x)^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)^2 f(x,y)\, dx\, dy$$

$$\sigma_y^2 = Var(y) = E((y-\mu)^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\mu_y)^2 f_x f_y\, dx\, dy$$

Co-verience

$$\sigma_{xy} = \text{co-var}(x,y)$$

$$= E((x-\mu_x)(y-\mu_y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y)\, dx\, dy$$

For discrete case:

$$\mu_x = \sum\sum x f(x,y) \qquad \mu_y = \sum\sum y f(x,y)$$

$$\sigma_{xy} = \sum\sum (x-\mu_x)(y-\mu_y) f(x,y)$$

Theorem: $\sigma_{xy} = E(XY) - E(X)\,E(Y)$

$$\boxed{\begin{aligned} & E\big((X-\mu_X)(Y-\mu_Y)\big) \\ & E(X-\mu_X)\cdot E(Y-\mu_Y) \\ & [E(X) - E(\mu_X)]\,[E(Y) - E(\mu_Y)] \end{aligned}}$$

$$\sigma_{xy} = \sum_x \sum_y (x-\mu_x)(y-\mu_y)\, f(x,y)$$

$$\sum\sum \big(xy - x\mu_y - y\mu_x + \mu_x\mu_y\big) f(x,y)$$

$$E\big(xy - x\mu_y - y\mu_x + \mu_x\mu_y\big)$$

$$= E(XY) - \mu_x E(Y) - \mu_y E(X) + \mu_x\mu_y$$

$$= E(XY) - E(X)\cdot E(Y) - \cancel{E(Y)\cdot E(X)} \cancel{+ E(X)E(Y)}$$

$$= E(XY) - E(X)\cdot E(Y)$$

Theorem: $Var(X \pm Y) = Var(X) + Var(Y) \pm 2\, Cov(X\,Y)$

$$Var(X \pm Y) = E\big((X \pm Y) - (\mu_X \pm \mu_Y)\big)^2$$

$$= E\big((X - \mu_X) \pm (Y - \mu_Y)\big)^2$$

$$= E\big((X-\mu_X)^2 + (Y-\mu_Y)^2 \pm 2(X-\mu_X)(\ldots$$

$$= E(X-\mu_X)^2 + E(Y-\mu_Y)^2 \pm 2\, E\big((X-\mu_X)(Y\ldots$$

$$Var(X) = Var(X) + Var(Y) \pm 2\,cov(X,Y)$$

Q) what happens to σ_{xy} when $X=Y$,

$$\sigma_{xy} = \begin{cases} 0 & \text{if } X \,\&\, Y \text{ are independent} \\ \sigma_x \cdot \sigma_y & X = Y \end{cases}$$

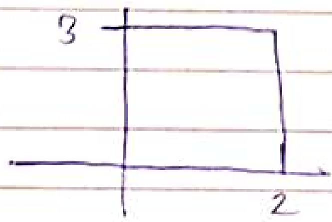
Correlation coefficients

$$\rho = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y} = \begin{cases} 0 & \text{if } X \,\&\, Y \text{ are independent} \\ 1 & \text{if } X = Y \end{cases}$$

Example: The joint p.m.f of X and Y

$$f(x,y) = \begin{cases} \frac{1}{42}(2x+y) & 0 \le x \le 2,\ 0 \le y \le 3,\ x,y \in \mathbb{Z} \\ 0 & \end{cases}$$

Find: $E(x)$, $E(y)$, $E(xy)$
$E(x^2)$, $E(y^2)$, $\sigma_x \sigma_y$
$Var(x)$, $Var(y)$, ρ



$$\rightarrow \quad E(x) = \sum_{y=0}^{3}\sum_{x=0}^{2} x \cdot f(x)$$

$$= \sum_{y=0}^{3}\sum_{x=0}^{2} \frac{2x^2 + xy}{42}$$

$$= \frac{1}{42}\sum_{y=0}^{3} 2\cdot(0^2+1^2+2^2) + (0+y+2y)$$

$$= \frac{1}{42}\sum_{y=0}^{3} 10 + 3y \Rightarrow \frac{1}{42}\left[10(4) + 3(0+1+2+3)\right]$$

$$= 40 + 18/42 = 58/42$$

$$E(y) = \sum\sum y \cdot f(x, y)$$

$$= \frac{1}{42} \sum_{x=1}^{2} \sum_{y=0}^{3} 2xy + y^2$$

$$= \frac{1}{42} \sum_{y=0}^{3} 2y(1+2) + 3y^2$$

$$= \frac{1}{42} \sum_{y=0}^{3} 6y + 3y^2$$

$$= \frac{1}{42} \left[6(0+1+2+3) + 3(0+1^2+2^2+3^2) \right]$$

$$= \frac{1}{42} [36 + 42]$$

$$= 78/42 = 13/7$$

$$E(xy) = \sum\sum xy \, f(x,y)$$

$$= \frac{1}{42} \sum\sum 2x^2 y + xy^2$$

$$= \frac{1}{42} \sum \cancel{2x^2(0} \; 2y(0+1+4) + y^2(0+1+2)$$

$$= \frac{1}{42} \sum_{y=0}^{3} 10y + 3y^2$$

$$= \frac{1}{42} \left[10(0+1+2+3) + 3(0+1+4+9) \right]$$

$$\frac{1}{42}\left(60 + 3 \times 14 \right) = 102/42$$

$E(x^2) = 17/7$

$Var(x) = 17/7 - 58/42$

~~$E(y^2)$~~ $E(y^2) = 33/7$

$\sigma_{xy} = E(xy) - E(x) \cdot E(y)$

eg: If X and Y are continious random variable with pdf:

H.W

$$f(x,y) \begin{cases} \frac{1}{210}(2x+y), & 2 < x < 6 \\ 0, & 0 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

$E(x)$, $E(y)$, $E(xy)$

$E(x^2)$, $E(y^2)$, σ_{xy}

$Var(x)$, $Var(y)$, ρ

Conditional Distribution

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

$$f(y|x) = P(Y=y \mid X=x)$$

$$= \frac{P(Y=y, X=x)}{P(X=x)}$$

$$= \frac{f(x,y)}{f(x)}$$

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

If X and Y are two discrete random variables and $\{A : X = x\}$ and $\{B : Y = y\}$ be two events is —

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{f(x,y)}{f_1(x)}$$

let $f(x,y)$ denote p.m.f. we can write the above inequality as

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$, where $f_1(x)$ is marginal density/mass function of X

We call $f(Y|X)$ as conditional mass function of Y for a given X.

Similarly we can define

$$f(x|y) = \frac{f(x,y)}{f_2(y)}$$

$f_2(y) \rightarrow$ marginal probability mass function

$$P((c<y<d) \mid x_1 < x < x_2)) = \int_c^d f(y|x)\,dy$$

Conditional expectation :-

$$f(x,y) = \begin{cases} 1/42\,(2x+y) & 0 \le x \le 2,\ 0 \le y \le 3 \quad x, y \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(x) = \sum_{k=0}^{3} f(x_1, y_k)$$

$$= \frac{1}{42}\left[(2x+0) + (2x+1) + (2x+2) + (2x+3)\right]$$

$$= \frac{8x+6}{42}$$

Find $f(y|x)$, $f(x|y)$.

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$\frac{\cancel{\sum_{x=0}^{2}\sum_{y=0}^{3} f(x,y)} \quad \cancel{1/42}(2x+y)}{\frac{8x+6}{\cancel{42}}}$$

$$= \frac{2x+y}{8x+6}$$

$$f_2(y) = \sum_{x=0}^{2} f(x_k, y_1)$$

$$= \frac{1}{42}\left[(2(0)+y) + (2(1)+y) + (2(2)+y)\right]$$

$$\frac{1}{42}\left[y+2+y+4+y\right]$$

$$\frac{3y+6}{42}$$

$$f(x|y) = \frac{2x+y}{3y+6}$$

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Conditional Expectation

$$E(Y|X=x) = \int_{-\infty}^{\infty} y\, f^{Y|X}(y)\,dy = \int_{-\infty}^{\infty} y\,\frac{f(x,y)}{f_1(x)}\,dy$$

Calculate expectation of Y given $X=2$

$$E(Y|X=2) = \sum_{y=-\infty}^{\infty} y\, f(y|x)$$

$$= \sum_{y=0}^{3} y \left(\frac{2(2)+y}{8(2)+6}\right)$$

$$= \sum_{y=0}^{3} y\left(\frac{4+y}{16+6}\right)$$

$$\frac{1}{22}\sum_{y=0}^{3} 4y+y^2$$

$$\frac{1}{22}\left[4(1+2+3)+(0+1+4+9)\right]$$

$$\frac{1}{22}(24+14)$$

$$38/22 = 19/11$$

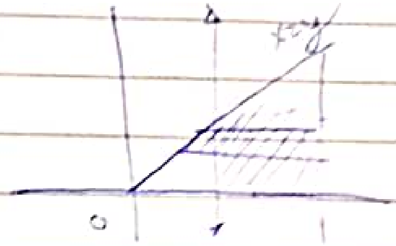
Example: The joint density function of random variables X and Y is given by:

$$f(x,y) = \begin{cases} 8xy & 0 \le x \le 1 \\ & 0 < y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Q) Find marginal density function of X and Y
Find conditional density function of X & Y
Find conditional expectation of X and Y to sum Y.

$$f_1(x) = \int_0^x 8xy\,dy$$

$$= 8x \cdot \frac{x^2}{2} = 4x^3$$



$$f_2(y) = \int_0^1 8xy\,dx$$

$$8y\,\frac{x^2}{2} = 4y$$

$$f_2(y) = \int_y^1 8xy\,dx$$

$$= 8y\left(\frac{1-y^2}{2}\right)$$

$$= 4y - 4y^3$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} \dfrac{8xy}{4x^3} = \dfrac{2y}{x^2} & 0 < y \le x \\ 0 & \text{otherwise} \end{cases}$$

$$f(x|y) = \frac{f(x,y)}{f_2(x)} = \begin{cases} \dfrac{2x}{1-y^2} & y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(y|x) = \int_{-\infty}^{\infty} y\, f(y|x)\, dy$$

$$= \int_{0}^{x} y \frac{2y}{x^2} dy$$

$$= \frac{2}{x^2} \left. \frac{y^3}{3} \right|_0^x$$

$$= \frac{2}{3} x$$

$$E(x|y) = \int_{-\infty}^{\infty} x\, f(x|y)\, dx$$

$$= \int_{y}^{1} x \cdot \frac{2x}{1-y^2} dx$$

$$\frac{1}{1-y^2} \left. \frac{2x^3}{3} \right|_y^1$$

$$\frac{2}{3(1-y^2)} (1-y^3)$$

eg//

The number of items produced in a factory during a week is a minimum variable with mean 50 and standard deviation 5. What can be said about the probability that the weeks production will be between 40 and 60

Chebyshev's Inequality

Suppose X is a random variable with mean μ and variance σ^2. Then if $\epsilon > 0$ be any number then

$$P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$ or if we take $\epsilon = k\sigma$

$$P(|X-\mu| \geq k\sigma) \leq 1/k^2$$

$\mu = 50$

$\sigma = 5$

$$P(40 \leq X \leq 60) = P(-10 < X-50 < 10)$$

$$= P(|X-50| < 10)$$

$$= P(|X-\mu| < 10)$$

$$= 1 - P(|X-\mu| \geq 10)$$

$\epsilon = 10$

$$\therefore \quad P(|X-\mu| \geq 10) \leq \sigma^2/\epsilon^2$$

$$\leq 25/100$$

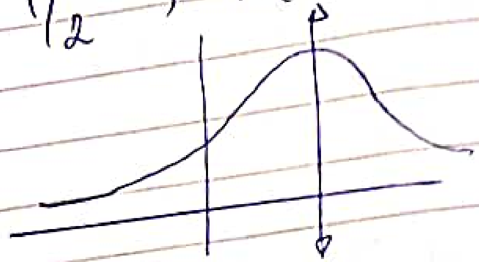
$$\leq 0.25$$

$$\therefore \quad 1 - P(|X-\mu| \geq 10)$$

$$= 1 - 0.25$$

$$P(40 < X < 60) = 0.75$$

i.e. Median → x such that
$P(X \le x) = 1/2$, $P(X \ge x) = 1/2$



$\alpha = \beta/c$

Median

The median is that values of x such that
$$P(X \le x) = 1/2$$
$$P(X \ge x) = 1/2$$

In particular median divides the density curve two parts having area $= 1/2$.

eg: Find median of
$$p.d.f = \frac{1}{(x+1)(x+2)} \qquad x = 0,1,2\ldots$$

$$\int_0^X \frac{1}{(x+1)(x+2)} = \frac{1}{2}$$

$$\int_0^X \left(\frac{1}{x+1} - \frac{1}{x+2}\right) = \frac{1}{2}$$

$$\ln\left(\frac{x+1}{x+2}\right)\Big|_0^X = \frac{1}{2}$$

$$\ln\left(\frac{x+1}{x+2}\right) - \ln\left(\frac{1}{2}\right)$$

$$P(X) = \sum_{x=0}^{X} \frac{1}{(x+1)(x+2)}$$

$X = 0$

$P(X) = P(0) = 1/2$

Skewness

Often distribution function is symmetric about any number. Instead it may have ~~talks~~ tails longer than other side. If longer tails occurs on right side, The density function is called right skew,

If ~~take~~ longer tail occurs on left the density function is called left skew.

If $\alpha_3 = \dfrac{E[(X-\mu)^3]}{\sigma^3} = \begin{cases} >0 & \text{right skew} \\ <0 & \text{left skew} \\ =0 & \text{symmetric} \end{cases}$

$$\% A = \frac{A_1}{\int_{-\infty}^{\infty} f(x)dx} \times 100 = 100 A_1 \rightarrow \text{percentile}$$

Let $\alpha \in [0,1]$ then $x_\alpha \in \mathbb{R}$ for which

$$\alpha = \int_{-\infty}^{\infty} f(x)dx$$

Then the area left to $x = x_0$ is $100 \times \alpha\%$ is called 100^{th} percentile

Median = 50^{th} percentile.

Example: We toss a coin 6 - times. What is the probability of getting exactly 2 heads.

$${}^6C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^4$$

{ Binomial Distribution

$$\frac{6 \times 5}{2} \times \frac{1}{2^6} = \frac{15}{64}$$

$$\left[(p+q)^n = \sum_{r=0}^{n} {}^nC_r \, p^r q^{n-r} \right]$$

$$E(e^{tx}) = 1 + \mu t + \frac{\mu' t^2}{2!} + \frac{\mu_3' t^3}{3!} + \ldots$$

$\mu_2 \rightarrow$ central

$\mu_2' \rightarrow$ raw moment

replace t by iw

Characteristic eqn

Replace 't' in Moment generating function by iw the function we obtain is Characteristic function

$$\phi_x(w) = M_x \, e^{(iwx)}$$

$$= \sum_{\infty} e^{iwx} f(x) \quad \text{discrete}$$

$$\int_{-\infty}^{\infty} e^{iwx} f(x)\,dx \quad \text{Continious}$$

$$\phi_x = \begin{cases} \sum |f(x)| dx = \sum f(x) = 1 \\ \int_{-\infty}^{\infty} |f(x)| dx = 1 \end{cases}$$

$$\phi_x = 1 + i\mu_1' w - \mu_2' \frac{w^2}{2!} + \cdots + i^r \mu_r' \frac{w^r}{r!}$$

$$\mu_n' = (-1)^r i^r \frac{d^r}{jw^r} \phi_x(w) \Big|_{w=0}$$

<u>Exer:</u> Find characteristic functions

$$f(x) \begin{cases} 1/2a & |x| < a \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_x(w) = \int_{-a}^{a} e^{iwx} f(x)\, dx$$

$$= \frac{1}{2a} \int_{-a}^{a} e^{iwx} dx$$

$$\frac{1}{2a} \frac{e^{iwa} - e^{-iwa}}{iw}$$

$$= \frac{\sin(wa)}{aw}$$

$$E(x-\mu) = \sum (x-\mu) f(x) = \sum x f(x) - \mu f(x)$$
$$= \mu - \mu = 0$$

<u>Mean deviation</u>: $\sum |x-\mu| f(x)$
$\int |x-\mu| f(x) \neq 0$

Q) Find the probability that in five tosses of a fair die a 4 will appear : (i) exactly twice (ii) atleast 2.

$P(X=2) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$

$P(X \geq 2) = 1 - P(X<2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - {}^5C_0 \left(\frac{5}{6}\right)^5 - {}^5C_1 \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4$

$\sum f(x) = (p+q)^n$
$= 1$

Note for

Prope

Binomial Distribution

Let p be the probability that an event will happen then $q = 1-p$ is the probability that the event won't happen.

Suppose, we are having n-trials. The probability that it will happen exactly x-times in n trials is given by,

$$f(x) = P(X=x) = {}^nC_x \, p^x q^{n-x}$$

$$f(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

The discrete probability function $f(x)$ in ① is called Binomial distribution since for $x = 1, 2, 3 \ldots n$ it corresponds to successive terms in the Binomial expansion.

$\sum f(x) = (p+q)^n = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 \cdots + {}^nC_x q^{n-x} p^x \cdots + p^n$
$= 1$

Note for $n=1 \rightarrow$ distribution is called ~~Binomial~~ Bernoulli's distribution

Properties :

1) Mean : $\mu = \sum_{x=0}^{n} x f(x) = np$

when $x=1$

$\mu = 1$

$$= \sum \frac{x\, n!}{(n-x)!\, x!} p^x q^{n-x}$$

$$= \sum \frac{n!}{(x-1)!\,(n-x)!} p^x q^{n-x}$$

$$= \sum \frac{n\,(n-1)!}{(x-1)!\,(n-1-(x-1))!} p^x q^{n-x}$$

$$= np \sum \frac{(n-1)!}{(x-1)!\,(n-1-(x-1))!} p^{x-1} q^{((n-1)-(x-1))}$$

$$= np \underbrace{\sum {}^{n-1}C_{x-1}\, p^{x-1} q^{[(n-1)-(x-1)]}}_{\boxed{(p+q)^{n-1} = 1}}$$

$= \underline{np}$

2) Variance

$$\sigma^2 = E(x-\mu)^2 = \sum (x-np)^2 \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum \frac{(x-np)^2 (n-1)}{}$$

$$\sum (x^2 + n^2p^2 - 2xnp) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$\sum x^2 \frac{n!}{x!(n-x)!} p^x q^{n-x} + n^2p^2 - 2n^2p^2$$

$$\sum_{x=0}^{n} \frac{x\, n!}{(x-1)!(n-x-(x-1))!} p^x q^{n-x} - n^2p^2$$

$$\sum \frac{x\, n!}{(x-1)!(n-x)!} p^x q^{n-x} - n^2p^2$$

$$\sum \frac{((x-1)+1)\, n!}{(x-1)!(n-x)!} p^x q^{n-x} - n^2p^2$$

$$\sum \frac{(x-1)\, n!\, p^x q^{n-x}}{(x-1)!(n-x)!} + \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x} - n^2p^2$$

$$\sum \frac{n!}{(x-2)!(n-x)!} p^x q^{n-x} + np - n^2p^2$$

$$p^2 n(n-1) \sum \frac{(n-2)!}{(x-2)!((n-2)-(x-2))!} p^{x-2} q^{[(n-2)-(x-2)]} + np -$$

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$$p^2(n^2-n) + np - n^2p^2$$

$$\cancel{p^2n^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np - np^2$$

$$= np(1-p)$$

$$\sigma^2 = npq$$

Standard Deviation

$\therefore$ 3) $\sigma = \sqrt{npq}$

4) * coefficient of skewness ☆

$$\frac{E(x-\mu)^3}{\sigma^3} = \frac{1}{\sigma^3}\sum_{x=0}^{n}(x-np)^3 \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \frac{q-p}{\sqrt{npq}}$$

5) Coefficient of kurtosis

$$\alpha_4 = \frac{E((x-\mu)^4)}{\sigma^4}$$ Definition

measures peakness of a density/ ~~mass~~ mean/distribution function

$$\alpha_4 = \frac{1}{\sigma^4} \sum (x-np)^4 \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= 3 + \frac{1 - 6pq}{npq}$$

Moment Generating Function

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} \, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

Characteristics function

$$\phi(w) = E(e^{iwx}) = \sum_{x=0}^{n} e^{iwx} \, {}^nC_x p^x q^{n-x}$$

$$= (q + pe^{iw})^n$$

eg// If 20% of the bolts produced by a machine are defective. Determine the probability that out of 4 chosen at random less than 2 bolts are defective.

p = 0.8
q = 0.2

$$P(X \leq 2) = 4C_2 (0.2)^2 (0.8)^2$$

$$\frac{4 \times 3}{2} \; 0.04 \, (0.64)$$

$$(0.24)(0.64) = 0.153$$

$$P = {}^4C_0\, p^0 q^4 + {}^4C_1\, p^1 q^3$$
$$= 0.8192$$

eg : If the probability that an individual will suffer a bad reaction from an injection of a given serum is 0.001. Determine the probability that out of 2000 individuals

(a) exactly 3 (b) more than 2

individual will suffer a bad reaction.

$$P(X=3) = {}^{2000}C_3\,(0.001)^3\,(0.999)^{1997} \simeq 0.1805$$

$$P(X>2) = 1 - P(X \leq 2)$$

$$= 1 - {}^{2000}C_0\,(0.001)^0(0.999)^{2000} - {}^{2000}C_1\,(0.001)^1(0.999)^{1999} - {}^{2000}C_2\,(0.001)^2\,(0.999)^{1998}$$

Poisson's Distribution

Let X be a discrete random variable such that values $0, 1, 2, 3 \ldots$ such that the p.m.f

$$f(x) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, 3 \ldots$$

where λ is a given constant. Then the distribution is called Poisson's Distribution.

Property

① Mean: $\mu = E(x) = \sum_{x=0}^{\infty} \frac{x \lambda^x e^{-\lambda}}{x!}$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$y = x-1$

$$= e^{-\lambda} \sum_{y=-1}^{\infty} \frac{\lambda^{y+1}}{y!}$$

$$= e^{-\lambda} \lambda \sum \frac{\lambda^y}{y!} \to e^{\lambda}$$

$$= \lambda$$

In the previous problem

Mean: $np = (0.001)(2000) = 2$

$\therefore \lambda = 2$

$P(x=3) = \frac{\lambda^3 e^{-\lambda}}{3!} = \frac{\cancel{8}^{4} \times e^{-2}}{\cancel{6}^{3}} = 0.18045$

(2) Variance $\sigma^2 = E((x-\lambda)^2) = \sum (x-\lambda)^2 \frac{\lambda^x e^{-\lambda}}{x!}$

$$\sum (x^2+\lambda^2-2x\lambda)\frac{\lambda^x e^{-\lambda}}{x!}$$

$$\sum_{x=0}^{\infty} x^2\frac{\lambda^x e^{-\lambda}}{x!} + \lambda^2 \sum_{x=0}^{\infty} \cancel{\frac{\lambda^x e^{-\lambda}}{x!}}^{\,1} - 2\lambda \sum_{x=0}^{\infty} x\frac{\lambda^x e^{-\lambda}}{x!}$$

$$\sum \frac{x\lambda^x e^{-\lambda}}{(x-1)!} + \lambda^2 - 2\lambda(\lambda)$$

$x-1=y$

$-\lambda^2$

$$\sum \frac{(y+1)\lambda^{y+1}\cdot e^{-\lambda}}{y!} - \lambda^2$$

$$\sum \frac{y\lambda^{y+1}e^{-\lambda}}{y!} + \sum \frac{\lambda^{y+1}e^{-\lambda}}{y!} - \lambda^2$$

$$\lambda \sum \frac{y\lambda^y e^{-\lambda}}{y!} + \lambda \sum \frac{\lambda^y e^{-\lambda}}{y!} - \lambda^2$$

$$\cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$$

(3) Standard deviation : $\sigma = \sqrt{\lambda}$

(4) Skewness : $\alpha_3 = \frac{E(x-\mu)^3}{\sigma^3} = \frac{1}{\sqrt{\lambda}}$

(5) Kurtosis : $\alpha_4 = \frac{E(x-\mu)^4}{\sigma^4} = 3 + \frac{1}{\lambda}$

6) MGF

$$M(t) = E(e^{tx}) = \frac{\sum e^{tx} \lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum \frac{e^{tx} \lambda^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

$$= e^{\lambda(e^t-1)}$$

7) Characteristics : $\phi(\omega) = E(e^{j\omega x}) = e^{\lambda(e^{j\omega}-1)}$

Relation b/w poission and Binomial Distribution

Binomial Distribution

$$P(X=x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

let $\lambda = np \rightarrow$ mean

$$P(X=x) = \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\frac{\lambda^x}{x!} \frac{n!}{(n-x)!} \frac{1}{n^x} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\left[\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\right]$$

$$\lim_{n\to\infty} \left[\frac{n(n-1)\cdots(n-x-1)}{n^x} \cdot \left(1-\frac{\lambda}{n}\right)\right]$$

$$\left[\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\right]\left[1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\lim_{n\to\infty}$$

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!} \quad (1)$$

$$\therefore \boxed{P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!}}$$

Example: let us toss a coin until head appears. what is the probability that first head will appear in the 5^{th} attempt.

Sol^n: $P(X=5)=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$$=\frac{1}{32}$$

Example: let us toss a die until a 4 appears. What is the probability that a 4 will appear in 6th attempt.

$$Pf(x)=\left(\frac{5}{6}\right)^{5}\left(\frac{1}{6}\right)^{1}$$

if probability that 4 will appear $=p$

$$P(x)=(1-p)^{5}\,p^{1}$$

Sum of these 2 numbers is no. of attempts

Geometric Distribution

Suppose that a p.m.f for a discrete random variable X has the following

$$P(X=k) = (1-p)^{k-1} \cdot p \qquad k=1,2,3$$

Then X is called to have Geometric distribution with parameter p (possibility of success in each trial)

Properties

Mean

$$\mu = E(x) = \sum_{k=0}^{\infty} k\,(1-p)^{k-1}\,p$$

$$= p\left[0 + (1-p)^{0} + 2(1-p) + 3(1-p)^2 \ldots\right.$$

$$S = (1-p)^{0} + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 \ldots$$

$$S(1-p) = (1-p) + 2(1-p)^2 + 3(1-p)^3 \ldots$$

$$S - S(1-p) = 1 + (1-p) + (1-p)^2 + (1-p)^3 \ldots$$

$$Sp = \frac{1}{1-(1-p)}$$

$$Sp = \frac{1}{p}$$

$$S = \frac{1}{p^2}$$

$$\therefore \mu = pS = \frac{1}{p}$$

2) Variance $E(x - 1/p)^2 = \sum_{k=1}^{\infty} (k-\frac{1}{p})^2 (1-p)^{k-1} \cdot p$

$$= \frac{1-p}{p^2}$$

3) S.D: $\sigma = \frac{\sqrt{1-p}}{p}$

4) Skewness: $\alpha_3 = \frac{2-p}{\sqrt{1-p}}$

5) Kurtosis: $\alpha_4 = 6 + \frac{p^2}{1-p}$

6) ~~~~ MGF $M(t) = \sum_{k=1}^{\infty} e^{tk} (1-p)^{k-1} p$

$$= pe^{t} \left(1 + (1-p)e^{t} + (1-p)^2 e^{2t} \cdots\right)$$

$$= \frac{pe^{t}}{1-(1-p)e^{t}}$$

7) Characteristic function

$$\phi(\omega) = \frac{pe^{i\omega}}{1-(1-p)e^{i\omega}}$$

Normal Distribution

The most important continious probability normal distribution & the density function is given by

$$f(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2s^2}}, \quad -\infty < x < \infty$$

where $m = \mu$ is the mean
$s = \sigma$ is standard deviation

Expectation: $xf(x) = \frac{1}{s\sqrt{2\pi}} \int x e^{-\frac{(x-m)^2}{2s^2}} dx$

$$y = \frac{x-m}{s}$$

$$\frac{1}{\cancel{s}\sqrt{2\pi}} \int_{-\infty}^{\infty} (sy+m) e^{-y^2/2} \cancel{s}dy$$

$$\frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} sy e^{-y^2/2} dy + \int_{-\infty}^{\infty} m e^{-y^2/2} dy \right]$$

$\int f(x)dx = 1$

$$y^2/2 = p$$
$$y\,dy = dp$$

$$= s\int e^{-p} dp \quad + m$$

$$= \frac{se^{-p}}{p} \quad + m$$

$$= \frac{2se^{-y^2/2}}{y^2}\Bigg|_{-\infty}^{\infty} \quad + m$$

$$= m$$

$$\boxed{\mu = E(x) = m}$$

Variance :

$$\sigma^2 = E(x)-\mu)^2 = \int \frac{1}{s\sqrt{2\pi}} (x-\mu)^2 e^{\frac{(x-\mu)^2}{2s^2}} dx$$

$$\frac{1}{s\sqrt{2\pi}} \int (x^2+\mu^2 - 2\mu x) e^{\frac{(x-\mu)^2}{2s^2}} dx$$

$$\left\{ y = \frac{x-\mu}{s} \right\}$$

$$\int \frac{x^2}{s\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2s^2}} dx + \int \frac{\mu^2 e^{-(x-\mu)^2/2s^2}}{s\sqrt{2\pi}} dx - 2\mu \int \frac{x e^{\frac{-(x-\mu)^2}{2s^2}}}{s\sqrt{2\pi}} dx$$

$$\frac{1}{\cancel{s}\sqrt{2\pi}} \int (sy+\mu)^2 e^{-\frac{y^2}{2}} \cancel{s}\, dy + \mu^2 - 2\mu^2$$

$$\frac{1}{\sqrt{2\pi}} \int (sy+\mu)^2 e^{-y^2/2} dy \qquad -\mu^2$$

$$\frac{1}{s\sqrt{2\pi}} \int (s^2y^2 + \mu^2 + 2\mu s y) e^{-y^2/2} dy - \mu^2$$

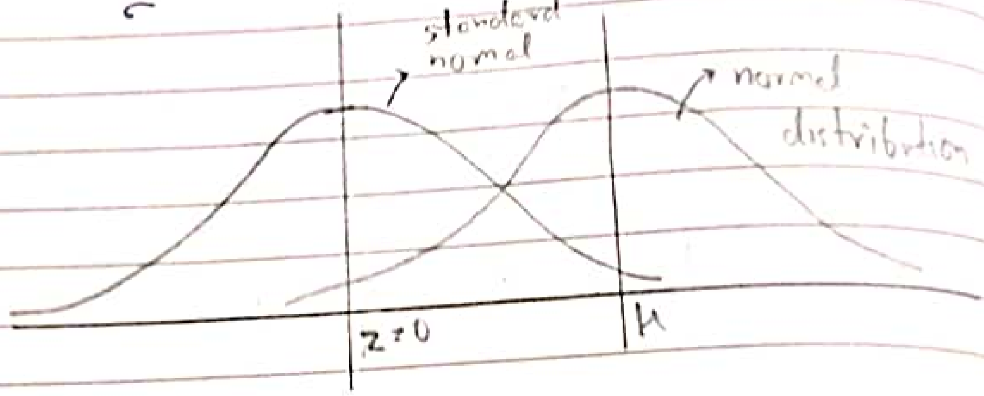
$$\mu^2 + 2\mu s^2 - \mu^2$$

$$s^2 - 2\mu^2 + 2\mu^2$$

$$= s^2$$

Let

$Z = \frac{X-\mu}{\sigma}$ for Z, $E(Z)=0$, $\sigma_Z = 1$



$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}$$

(1) $P(-x \le X \le x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-x}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$

$P(a \le Z \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_{a}^{b} e^{-z^2/2} dz$

$P(-1 \le Z \le 1) = 0.6827$
$P(-2 \le Z \le 2) = 0.9541$
$P(-3 \le Z \le 3) = 0.9973$

MGF:

$M_X(t) = \int \frac{1}{s\sqrt{2\pi}} e^{xt} e^{-\frac{(x-\mu)^2}{2s^2}} dx$

$y = \frac{x-\mu}{s}$

$\frac{1}{\cancel{s}\sqrt{2\pi}} \int e^{(ys+\mu)t} e^{-y^2/2} \cancel{s}\, dy$

$$\frac{1}{\sqrt{2\pi}} \int e^{ys+\mu-y^2/2} dy$$

$$\frac{1}{\sqrt{2\pi}} e^{\mu} \int_{-\infty}^{\infty} e^{ys-y^2/2} dy$$

$$\int e^{ys-y^2/2} = \left[y \cdot e^{ys-y^2/2}\right]_{-\infty}^{\infty} - \int y(s-y) e^{ys-y^2/2} dy$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(y\sigma+\mu)t} \cdot e^{-y^2/2} dy$$

$$\frac{e^{\mu t}}{\sqrt{2\pi}} \int e^{y\sigma t-y^2/2} dy$$

$$\frac{e^{\mu t}}{\sqrt{2\pi}} \int e^{y\sigma t-y^2/2} dy$$

$$\frac{e^{\mu t}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(y^2-2y\sigma t+\sigma^2t^2) - \sigma^2 t^2)} dy$$

$$\frac{e^{\mu t}}{\sqrt{2\pi}} \int e^{-1/2 (y-\sigma t)^2 + \frac{\sigma^2 t^2}{2}} dy$$

$$\frac{e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int e^{-1/2 (y-\sigma t)^2} dy$$

$$\left[M_x(t) = e^{\mu t + \sigma^2 t^2/2} \right]$$

Coeff of skewness : $\alpha_3 = 0$
Coeff of kurtosis : $\alpha_4 = 3$
Characteristic function :
$$\varphi(t) = e^{i\mu w - \frac{1}{2}\sigma^2 w^2}$$

Relation b/w Binomial & Normal Characteristics

When $n \to$ large and $p \to$ small

Binomial distribution
$$\mu = np$$
$$\sigma = \sqrt{np(1-p)}$$

$$Z = \frac{x - np}{\sqrt{np(1-p)}} \qquad \left[\frac{x-\mu}{\sigma}\right]$$

Suppose we take np & nq greater than 5

$$\lim_{n\to\infty} Z = \frac{x - np}{\sqrt{np(1-p)}}$$

$$\therefore \lim_{n\to\infty} \varphi \; P\left(a \leq \frac{x-np}{\sqrt{np(1-p)}} \leq b\right)$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-z^2/2}\, dz$$

Assignment 8

Q12) $f(x|Y=y) = \frac{|y| e^{-x^2y^2}}{\sqrt{\pi}}$

$f(y) = \frac{\lambda e^{-\lambda^2 y^2}}{\sqrt{\pi}}$

$f(x|y) = \frac{f(x,y)}{f(y)}$

$f(x,y) = f(x|Y=y) \cdot f(y)$

$= \frac{|y| e^{-x^2y^2}}{\sqrt{\pi}} \cdot \frac{\lambda e^{-\lambda^2 y^2}}{\sqrt{\pi}}$

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \frac{\lambda}{\pi} \int_{-\infty}^{\infty} |y| e^{-x^2y^2} \cdot e^{-\lambda^2y^2}\,dy$

$\frac{2\lambda}{\pi} \int_0^{\infty} y\, e^{-x^2y^2} \cdot e^{-\lambda^2y^2}\,dy$

let $(x^2+\lambda^2)y^2 = t$

$2(x^2+\lambda^2)y\,dy = dt$

$\frac{\lambda}{(x^2+\lambda^2)\pi} \int_0^{\infty} e^{-t}\,dt = \frac{\lambda}{\pi(x^2+\lambda^2)} e^{-t}\Big|_{\infty}^{0}$

$= \frac{\lambda}{\pi(x^2+\lambda^2)}$ Hence Proved

Stocastic Process

Let T be a index/ ordered set/ any set of ... a stocastic process is a collection of random variables $\{X(t) : t \in T\}$ or $\{ \ldots \}$

If T is countable → we say that $\{X(t) : t \in T\}$ is a discrete time stocastic process.

If T is uncountable → we say that $\{X(t)\}$ is a continious time stocastic process and ~~of~~

Note: The space S = Possible outcome is state space.

1.) Discrete time & discrete space

A motor insurance company reviews customer yearly. Three levels of discount are possible $\{0\%, 20\%, 40\%\}$ depending on the number of accident by the customer.

Here $T = \{0, 1, 2, 3 \ldots\}$

$S = \{0, 20, 40\}$

2) Discrete Time and continious time space

Let X_n denotes the temperature of on n^{th} day of calender year of moon.

$n = 1, 2, 3 \ldots 365$.

$S = (10, 60)$

3) Continuous time & Continuous space

modify example 2 by removing the ...

4) Continuous time & discrete space

Number of persons entering in a room at a certain time.

Def (Mean)

For stochastic process, $\{X(t) : t \in T\}$ the mean is calculated

$$\mu(t) = E(X(t)) \quad \forall \ t \in T$$

Def (Auto-correlation)

Autocorrelation $\{X(t) : t \in T\}$ denoted by $R(t_1, t_2)$ for any $t_1, t_2 \in T$ as expectation of product of $X(t_1)$ & $X(t_2)$ is $R(t_1, t_2) = E(X(t_1), X(t_2))$

Def (Auto-covariance)

Autocovariance of $\{X(t) : T\}$ denoted by $C(t_1, t_2)$ is

$$C(t_1, t_2) = E[(X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2))]$$

$$= E[X(t_1) X(t_2)] - X(t_1)\mu(t_2) - \mu(t_1)X(t_2) - \mu(t_1)\mu(t_2)$$

$$= R(t_1, t_2) - \mu(t_1)\mu(t_2)$$

Correlation

$$\rho(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{Var(X_{t_1}) \cdot Var(X_{t_2})}}$$

where $Var(X_{t_i}) = \{E[(X_{t_i}) - \mu(t_i)]^2$

Def (Counting process)

Counting process is a stocastic process with state space $S = \{0, 1, 2 \ldots\}$ such that $X(t)$ is a non decreasing in t.

Example:

$X(t) \rightarrow$ number of students present at time t.

$x(t) \rightarrow$ number of students entering class at t

↓

not non-decreasing

Note: A counting process $X(t)$ must satisfy

(i) $X(t) \geq 0$.

(ii) $X(t)$ is integered value

(iii) if $s \leq t$, $X(s) \leq X(t)$

Increment in stocastic process in (t_1, t_2) is computed as $X(t_2) - X(t_1)$

Increment
- independent increments
- strictly stationary increment
- weakly

Independent Increments

if for m $t_1 \le t_2 \le t_3 \le t_4$.
$X(t_2) - X(t_1)$ and $Y(t_4) - Y(t_3)$ are independent to each other.

$X(t) \rightarrow$ number of babies born at time t,
T → small
→ very large.

Strictly stationary process if joint distribution of $X_{t_1}, X_{t_2} \dots X_{t_n}$ and X_{h+t_1}, Y_{h+t_2} + X_{h+t_n} are identical for any h if t ~~+~~ $h \in T$ and for any number n.

Weakly stationary process

A stocastic process of $\{X(t) : T\}$ with $E(X_t^2) < \infty$ $\forall$ $t \in T$ is called w.s.p.
if ① $E(X_t)$ is constant $\forall$ $t \in T$
② $Cov(X_t, X_{t+k})$ depends only on k.

example : Let X_n $n \geq 1$ be independent random variable with $E(X_n) = 0$ $\forall$ $n \in N$
$\sigma_{X_n}^2 = 1$ $\forall$ $n \in N$

Then ~~Covar~~ Cov $(X_n, X_{n+m}) = E(X_n, X_{m+n}) - E(X_n) \cdot E(X_{m+n})$
$= E(X_n) \cdot E(X_{m+n}) - E(X_n) \cdot E(X_{m+n})$
$= \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$

It is a W.S.S.P.

7)

The speed of cars at a point is normally distributed with a mean speed 90 km/hr and standard deviation 10 km/hr. Probability that car is selected is running with speed more than 100 km/hr.

$\cancel{P(X > 100)} = P$

$P\left(\cancel{\begin{matrix} X > 100 \\ X-90 > 10 \end{matrix}}\right) = \cancel{e^{\,S^2/S^2}}$

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(x-\mu)^2/2\sigma^2}$

$\frac{x-\mu}{\sigma} = p$

$dx = \sigma\,dp$

$P(X > 100) = \cancel{1 - P(X < 100)}$

$\int_{1}^{\infty} \frac{1}{\cancel{10}\sqrt{2\pi}} e^{-p^2/2} \cancel{10}\,dp$

$\frac{1}{\sqrt{2\pi}} \int_{1}^{\infty} e^{-p^2/2}\,dp$

$n-1 = 1/2$

$p^2/2 = m$

$2p\,dp = 2\,dm$

$dp = \frac{m\,dm}{p}$

$dp = \frac{m}{\sqrt{2m}}$

$dp = \frac{\sqrt{m}}{\sqrt{2}}\,dm$

$\frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} e^{-m} (m)^{1/2}\,dm$

$= \frac{1}{2\sqrt{\pi}} \sqrt{3/2}$

$= \frac{\sqrt{3/2}}{2\sqrt{\pi}}$

Note

$$\frac{1}{\sigma\sqrt{2\pi}}\int e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$P(x > 100)$

$P\left(\frac{x-90}{10} > 1\right)$ convert in terms of z

$P(z > 1)$

$$= \frac{1}{\sqrt{2\pi}}\int_{1}^{\infty} e^{-z^2/2}\,dz \quad — (1)$$

Similarly,

$P(60 < x < 70)$

$P\left(\frac{-30}{10} < \frac{x-90}{10} < \frac{-20}{10}\right)$

$P(-3 < z < -2)$

$$\frac{1}{\sqrt{2\pi}}\int_{-3}^{-2} e^{-z^2/2}\,dz$$

Using (1)

$$\cancel{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2}\,dz} + \cancel{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1} e^{-z^2/2}\,dz}$$

$$\cancel{1 - \frac{1}{\sqrt{2\pi}}}$$

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-z^2/2}\,dz - \frac{1}{\sqrt{2\pi}}\int_{0}^{1} e^{-z^2/2}\,dz = 0.5 - \underbrace{\boxed{\frac{0.6827}{2}}}_{P(0<z<1)}$$

given

Q) ~~Find number of~~

Let X be number of times that a fair coin flipped 40 times lands heads. Find $P(X=10)$ using normal distribution & compare with exact soln.

Exact soln: $P = {}^{40}C_{10} \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^{30}$

$\mu = np = 20$

$\left\{ P = {}^{40}C_{10} \frac{1}{2^{40}} \right\}$

$\sigma = \sqrt{npq}$

$= \sqrt{40 \cdot \frac{1}{4}}$

$= \sqrt{10}$

$$\boxed{\cancel{f(x) \quad \frac{1}{\sigma\sqrt{2\pi}} \sum_{x=1}^{40} x e^{-(x-20)^2/20} \qquad P(X=10) = \frac{1}{\sqrt{20\pi}} 10 e^{-(-10)^2/20} = \frac{10e^{-5}}{\sqrt{20\pi}}}}$$

$X = 10$

$$P(9.5 < x < 10.5) = \int_{9.5}^{10.5} = \frac{1}{\sqrt{20\pi}} \int_{9.5}^{10.5} e^{-(x-20)^2/20} dx$$

$$\frac{-10.5}{\sqrt{10}} \leq \frac{x-20}{\sqrt{10}} \leq \frac{-9.5}{\sqrt{10}}$$

$$P\left(\frac{-10.5}{\sqrt{10}} \leq z \leq -9.5/\sqrt{10}\right) = \frac{1}{\sqrt{2\pi}} \int_{-10.5/\sqrt{10}}^{-9.5/\sqrt{10}} e^{-z^2/2} dz$$

2.334×10^{-3}