5/10/24

Probability Theory

Sample space: The set of all possible outcomes (or) results of a random experiment is called sample space and is denoted by S.

Ex: Tossing a coin $S = \{H, T\}$

Tossing two coins $S = \{HH, HT, TH, TT\}$

Throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$

Event of a random experiment:

A subset of a sample space is called event of and is denoted by E.

Ex: Tossing two coins $S = \{HH, HT, TH, TT\}$

Let E be the event of getting one head

$\therefore E = \{HT, TH\} \subset S$

Mutually exclusive events:

The events E_1 and E_2 of a sample space are said to be mutually exclusive events if

$E_1 \cap E_2 = \phi$.

Probability of an event:

Let E be an event of a sample space S. The probability of the event E is denoted by P(E) and is defined by

$$P(E) = \frac{\text{The number of elements in } E}{\text{The number of elements in } S} = \frac{n(E)}{n(S)}$$

Clearly $P(E) \geq 0$ $\qquad n(E) \leq n(S)$ $[\because E \subseteq S]$

$\Rightarrow 0 \leq P(E) \leq 1$

Result: Let E_1 and E_2 be events of a sample space S Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Result: Let E_1 be an event of a sample space S & let E_1^c be the event that denotes non-occurance of E_1

$$\therefore E_1 \cup E_1^c = S$$

$$P(E_1 \cup E_1^c) = P(E_1) + P(E_1^c) - P(E_1 \cap E_1^c)$$

$$P(S) = P(E_1) + P(E_1^c) - P(\phi)$$

$$1 = P(E_1) + P(E_1^c) - 0$$

$$\Rightarrow \boxed{P(E_1^c) = 1 - P(E_1)}$$

The above definition of probability is applicable only if the sample space contains finite number of elements. That is this definition is NOT applicable for the sample spaces which are containing infinite number of elements. Therefore, we need a new approach to study the probability with respect to sample spaces which are containing finite number elements or infinite number of elements. That new approach is nothing but Random Variable (RV).

Ex: speed of a car at the given time. $S = [0, 200]$

Let E be the event of car speed lies between 0 and 50 km/hr

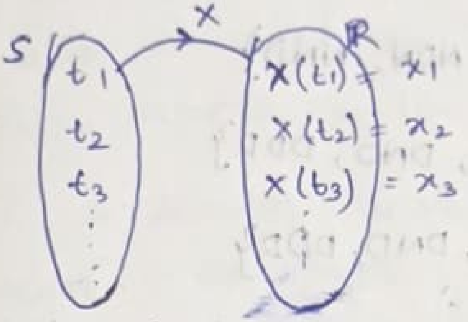
$$P(E) = \frac{n(E)}{n(S)} = \frac{\infty}{\infty}$$

Random Variable

Random Variable is nothing but a function from a sample space S to set of real numbers $\mathbb{R}$.

Usually Random variables are denoted by X, Y, Z.

That is $X : S \to \mathbb{R}$ is a function.



In general $X(t) = x$

Let $\{X = x_1\} = \{t \in S \,/\, X(t) = x_1\}$

↳ The set of all elements in sample S which are mapped to the real number x_1 under the random variable ~~real~~ ~~value~~ (X).

Let $\{X \leq x_2\} = \{t \in S \,/\, X(t) \leq x_2\}$

↳ The set of all elements in S which are having images less than or equal to x_2

<u>Example</u>: When we test three electronic devices, the sample space

$S = \{NNN, DNN, NDN, NND, DDN, NDD, DND, DDD\}$

N - non-defective

D - defective.

Let X be the number of defective devices

$X(NNN) = 0$

$X(DNN) = 1$

$X(NDN) = 1$

$X(NND) = 1$

$X(DDN) = 2$

$X(NDD) = 2$

$X(DND) = 2$

$X(DDD) = 3$

Clearly X is a function from S to $\mathbb{R}$.

$\therefore$ X is a Random Variable

$\therefore \{x=0\} = \{NNN\} \subset S$

$\{x=1\} = \{DNN, NDN, NND\} \subset S$

$\{x=2\} = \{DDN, NDD, DND\} \subset S$

$\{x=3\} = \{DDD\} \subset S$

$\{x \leq 1\} = \{NNN, DNN, NDN, NND\}$

$\{2 \leq x \leq 3\} = \{DDN, NDD, DND, DDD\}$

$\{x \geq 2\} = \{DDN, NDD, DND, DDD\}$

* Let Y be the number of non-defective devices

$Y(NNN) = 3$ $\quad Y(DDN) = 1$ $\quad$ Clearly Y is a function
$Y(DNN) = 2$ $\quad Y(DND) = 1$ $\quad$ from S to $\mathbb{R}$
$Y(NDN) = 2$ $\quad Y(NDD) = 1$ $\quad \therefore Y$ is a RV.
$Y(NND) = 2$ $\quad Y(DDD) = 0$ $\quad$ (random variable)

* We have $S = \{NNN, DNN, NDN, NND, DDN, NDD, DND, DDD\}$

$\{x=0\} = \{NNN\} \subset S \Rightarrow P(x=0) = \frac{1}{8} = f(0)$

$\{x=1\} = \{NDN, NND, DNN\} \Rightarrow P(x=1) = \frac{3}{8} = f(1)$

$\{x=2\} = \{\cancel{NND}\ DDN, NDD, DND\} \Rightarrow P(x=2) = \frac{3}{8} = f(2)$

$\{x=3\} = \{DDD\} \Rightarrow P(x=3) = \frac{1}{8} = f(3)$

Clearly, $P(x=0) + P(x=1) + P(x=2) + P(x=3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$
$= 1$

x	0	1	2	3
$P(X=x) = f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Discrete Random Variable:

A random variable which takes finite number of values or countably infinite values, is called discrete Random Variable.

* If a random variable x takes either finite number of values or countably infinite values, then x is called discrete random variable.

Ex: If x denotes the number of students secured AA grade in MAL205, then x takes 0, 1, 2, 3, ... 140

Ex: If x denotes the number of accidents in Nagpur in 2026, then

$x = 0, 1, 2, 3, 4, 5, \ldots$

Continuous Random Variable:

If a random variable x takes numbers from an interval, then x is called continuous Random Variable.

Ex: If x denotes speed of a car at a given time then x takes the values from the interval $[0, 200]$

Definition:

Let x be a discrete RV. then a function $f(x)$ is said to be probability ~~dis~~ density function or probability mass function or probability function if

1) $f(x) \geq 0$

2) $f(x) = P(X = x)$

3) $\sum_{x} f(x) = 1.$

Definition:

Let X be a ~~RR~~ RV which takes the values $x_1, x_2, x_3, \ldots x_{10}$. A function $f(x)$ is said to be probability density function or probability mass function or probability function if

1) $f(x_1) \geq 0, f(x_2) \geq 0, \ldots f(x_{10}) \geq 0$

2) $f(x_1) = P(\{X = x_1\}), f(x_2) = P(\{X = x_2\}), \ldots f(x_{10}) = P(\{X = x_{10}\})$

3) $f(x_1) + f(x_2) + \cdots + f(x_{10}) = 1$

$E_1 = \{X = x_1\} = \{t \in S / X(t) = x_1\} \subset S$

$E_2 = \{X = x_2\} = \{t \in S / X(t) = x_2\} \subset S$

$\vdots$

$E_{10} = \{X = x_{10}\} = \{t \in S / X(t) = x_{10}\} \subset S$

$\{X = x_1\} \cup \{X = x_2\} \cup \{X = x_3\} \cdots \cup \{X = x_{10}\} = S$

$P[\{X = x_1\} \cup \{X = x_2\} \cup \{X = x_3\} \cdots \cup \{X = x_{10}\}] = P(S)$

$(\because P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2))$

$P[\{X = x_1\}] + P(\{X = x_2\}) + \cdots + P(\{X = x_{10}\}) = 1$

$\Rightarrow P(X = x_1) + P(X = x_2) + \cdots + P(X = x_{10}) = 1$

$\Rightarrow P(X = x_1) + P(X = x_2) = 1 - [P(X = x_3) + P(X = x_4) + \cdots + P(X = x_{10})]$

$\Rightarrow P(X \leq x_2) = 1 - P(X > x_2)$

In general, $P(X \leq x_k) = 1 - P(X > x_k)$

Example :

A box contains 25 items which includes 5 defective items. We have to choose 4 items from the box. If the RV x denotes the defective items, then find the probability mass function of x, when the items are choosen without replacement.

Sol: clearly 'x' takes 0, 1, 2, 3, 4

$P(x=0) = P(\text{No item is defective})$

$= \dfrac{{}^{20}C_4}{{}^{25}C_4} = 0.383 = \dfrac{969}{2530}$

$P(x=1) = P(1 \text{ defective and } 3 \text{ non-defective})$

$= \dfrac{{}^{5}C_1 * {}^{20}C_3}{{}^{25}C_4} = 0.45$

$P(x=2) = P(2 \text{ defective and } 2 \text{ non-defective})$

$= \dfrac{{}^{5}C_2 * {}^{20}C_2}{{}^{25}C_4} = 0.15$

$P(x=3) = P(3 \text{ defective and } 1 \text{ non-defective})$

$= \dfrac{{}^{5}C_3 * {}^{20}C_1}{{}^{25}C_4} = \cancel{0.15}\ 0.0158$

$P(x=4) = P(4 \text{ defective}) = \dfrac{{}^{5}C_4}{{}^{25}C_4} = 0.000395$

$P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) = 0.999 \simeq 1$

Probability that getting atmost two defective items

$= P(\{x=0\} \cup \{x=1\} \cup \{x=2\})$

$= P(x=0) + P(x=1) + P(x=2)$ $\left[\because \{x=0\}, \{x=1\}, \{x=2\} \text{ are mutually disjoint}\right]$

$= 0.983$

Probability that getting more than 3 defective items

$= P(X > 3)$

$= P(X = 4)$

$= 0.00034$

Probability that getting atleast one defective item

$= P(X \geq 1)$

$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 1 - P(X=0)$

$= 1 - \frac{969}{2530} = 0.61699$

Let f be a function such that

$f(0) = P(X=0)$

$f(1) = P(X=1)$

$f(2) = P(X=2)$

$f(3) = P(X=3)$

$f(4) = P(X=4)$

$f(x) \geq 0 \ \forall \ x = 0, 1, 2, 3, 4$

$f(x) = P(X = x)$

$\therefore$ f is a probability density function.

1b: Above problem with replacement:

5 - defective
20 - non defective

$P(X=0) = \frac{20^4}{25^4} = 0.4096$

$P(X=1) = \frac{5 \times 20^3 \times 4}{25^4} = 0.1024 \times 4 = 0.4096$

$P(X=2) = \frac{5^2 \times 20^2 \times {}^4C_2}{25^4} = 0.0256 \times 6 = 0.1536$

$P(X=3) = \frac{5^3 \times 20 \times {}^4C_3}{25^4} = 0.0064 \times 4 = 0.0256$

$P(X=4) = \frac{5^4}{25^4} = 0.0016$

Cummulative Probability Density Function (CDF):

Let X be a distrete RV which takes the values $x_1, x_2, x_3, \ldots x_{10}$. Where $x_1 < x_2 < x_3 \ldots < x_{10}$

Let 'f' be the PDF (probability density function) of X.

The CDF 'F' of the RV is defined by

$$F(x_1) = f(x_1)$$

$$F(x_2) = f(x_1) + f(x_2)$$

$$F(x_3) = f(x_1) + f(x_2) + f(x_3)$$

$$\vdots$$

$$F(x_{10}) = f(x_1) + f(x_2) + \cdots + f(x_{10})$$

In general, $F(x) = \sum_{t \le x} f(t)$

Pb: In the previous problem, the RV X takes 0, 1, 2, 3, 4. The PDF of the RV X is given by

X	0	1	2	3	4
$f(x) = P(X = x)$	0.383	0.45	0.15	0.0158	0.00039

CDF = ?

Sol: ∴ The CDF 'F' is given by

$$F(0) = f(0) = 0.383$$

$$F(1) = f(0) + f(1) = 0.833$$

$$F(2) = f(0) + f(1) + f(2) = 0.983$$

$$F(3) = f(0) + f(1) + f(2) + f(3) = 0.9988$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 0.99919 \simeq 1$$

Probability that getting atmost three defective items

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= f(0) + f(1) + f(2) + f(3)$$

$$= F(3) = 0.9988$$

Pb. Let X be a RV which takes 1, 2, 3, 4

The CDF of X is given by

$F(1) = \frac{1}{3}$, $F(2) = \frac{4}{9}$, $F(3) = \frac{13}{27}$, $F(4) = 1$

Find PDF of X, $P(x \le 2)$, $P(x > 3)$

Sol:

$F(1) = \frac{1}{3} \Rightarrow f(1) = \frac{1}{3}$

$F(2) = \frac{4}{9} \Rightarrow f(2) = \frac{4}{9} - \frac{1}{3} = \frac{1}{9}$

$F(3) = \frac{13}{27} \rightarrow f(1) + f(2) + f(3) = \frac{13}{27}$

$\Rightarrow f(3) = \frac{1}{27}$

$F(4) = 1 \Rightarrow f(1) + f(2) + f(3) + f(4) = 1$

$\Rightarrow f(4) = \frac{14}{27}$

x	1	2	3	4
$f(x) = P(X = x)$	$\frac{1}{3}$	$\frac{3}{27}$	$\frac{1}{27}$	$\frac{14}{27}$

$P(x \le 2) = F(2) = \frac{4}{9}$

$P(x > 3) = P(x = 4) = \frac{14}{27}$

Pb: If a RV x takes 1, 2, 3, 4 such that

$2P(x=1) = 3P(x=2) = P(x=3) = 5P(x=4)$

Then find the probability distribution of X

Sol:

$P(x=2) = \frac{2}{3} P(x=1)$

$P(x=3) = 2P(x=1)$

$P(x=4) = \frac{2}{5} P(x=1)$

But $P(x=1) + P(x=2) + P(x=3) + P(x=4) = 1$

$\rightarrow P(x=1) + \frac{2}{3} P(x=1) + 2P(x=1) + \frac{2}{5} P(x=1) = 1$

$\rightarrow \frac{61}{15} P(x=1) = 1 \longrightarrow \boxed{P(x=1) = \frac{15}{61}}$ $= \frac{15}{61}$

$\boxed{P(x=2) = \frac{10}{61}}$ $\boxed{P(x=3) = \frac{30}{61}}$ $\boxed{P(x=4) = \frac{6}{61}}$

Pb: If a discrete RV x $1, 2, 3, 4, \ldots$

If $P(x=r) = k(1-a)^{r-1}$, $0 < a < 1$, $r = 1, 2, 3, \ldots$

Then Find k

Sol: $P(x=1) + P(x=2) + P(x=3) + \ldots = 1$

$k(1) + k(1-a) + k(1-a)^2 + k(1-a)^3 + \ldots = 1$

$\frac{k}{1-(1-a)} = 1 \Rightarrow \frac{k}{a} = 1$

$\boxed{k = a}$

Joint Probability Distribution Function :

In a chemical reaction, what is the probability that temperature lies b/w 20°c and 30°c and pressure lies between 40 and 50 pascals?

Let x be the RV that denotes temperature

Let y be the RV that denotes pressure

$P(20 \le x \le 30, 40 \le y \le 50)$

Definition : ~~Let~~

Let x and y be discrete RV. A function $f(x,y)$ is said to be joint PDF of the RV x and y if

1) $f(x,y) \geq 0 \ \forall (x,y)$

2) $f(x,y) = P(x=x, y=y)$

3) $\sum_x \sum_y f(x,y) = 1$

Definition:

Let X and Y be discrete R.Vs. Let $f(x,y)$ be the joint PDF of X and Y. The marginal PDF of X is denoted by $g(x)$ and is defined by $g(x) = \sum_{y} f(x,y)$

Similarly, the marginal PDF of Y is denoted by $h(y)$ and is defined by $h(y) = \sum_{x} f(x,y)$

Pb: Two pens are selected at random from a box that contains 3 blue pens, 2 red pens and 3 green pens. If 'X' denotes no. of blue pens selected, Y denotes no. of red pens selected. Then find the joint PDF of X and Y. Also find

1) $P(X+Y \le 1)$ 2) Marginal PDF of X

3) Marginal PDF of Y.

Sol: Clearly X can take $0, 1, 2$

Y can take $0, 1, 2$

$$P(X=0, Y=0) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28} = 0.107$$

$$P(X=1, Y=0) = \frac{{}^3C_1 \cdot {}^3C_1}{{}^8C_2} = \frac{9}{28} = 0.3214$$

$$P(X=2, Y=0) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28} = 0.107$$

$$P(X=0, Y=1) = \frac{{}^3C_1 \cdot {}^2C_1}{{}^8C_2} = \frac{6}{28} = 0.214$$

$$P(X=1, Y=1) = \frac{{}^3C_1 \cdot {}^2C_1}{{}^8C_2} = \frac{6}{28} = 0.214$$

$$P(X=0, Y=2) = \frac{{}^2C_2}{{}^8C_2} = \frac{1}{28} = 0.0357$$

$$P(X=1, Y=2) = P(X=2, Y=1) = 0 = P(X=2, Y=2)$$

Y \ X	0	1	2
0	3/28	9/28	3/28
1	6/28	6/28	0
2	1/28	0	0

Joint PDF of X and Y

$$P(X+Y \le 1) = P(X=0, Y=0) + P(X=1, Y=0) + P(X=0, Y=1)$$

$$= \frac{3}{28} + \frac{9}{28} + \frac{6}{28} = \boxed{\frac{18}{28}}$$

Let $g(x)$ be the marginal PDF of X. Then

$$g(x) = \sum_y f(x,y)$$

$$g(0) = \sum_y f(0,y) = f(0,1) + f(0,2) + f(0,0)$$

$$= \frac{6}{28} + \frac{1}{28} + \frac{3}{28} = \frac{10}{28}$$

$$g(1) = \sum f(1,y) = f(1,0) + f(1,1) + f(1,2) = \frac{9}{28} + \frac{6}{28} + 0$$

$$= \frac{15}{28}$$

$$g(2) = \sum f(2,y) = f(2,0) + f(2,1) + f(2,2) = \frac{3}{28} + 0 + 0$$

$$= \frac{3}{28}$$

x	0	1	2
g(x)	10/28	15/28	3/28

Marginal PDF of X

Let $h(y)$ be the marginal PDF of Y

Then $h(y) = \sum_x f(x,y)$

$h(0) = \sum_x f(x,0) = f(0,0) + f(1,0) + f(2,0) = \frac{3}{28} + \frac{9}{28} + \frac{3}{28}$

$= \frac{15}{28}$

$h(1) = \sum_x f(x,1) = f(0,1) + f(1,1) + f(2,1)$

$= \frac{6}{28} + \frac{6}{28} + 0 = \frac{12}{28}$

$h(2) = \sum_x f(x,2) = f(0,2) + f(1,2) + f(2,2)$

$= \frac{1}{28} + 0 + 0 = \frac{1}{28}$.

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$

Marginal PDF of Y

12/10/24

Probability density function of a continuous RV

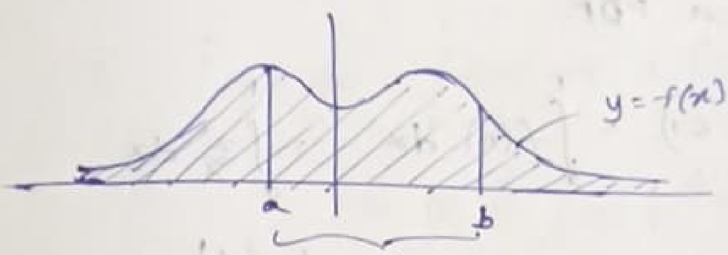
Let X be a continuous RV that takes values from $(-\infty, \infty)$. A function $f: \mathbb{R} \to \mathbb{R}$ is said to be PDF (or) Probability mass function of the RV X if

1) $f(x) \geq 0 \ \forall \ x \in (-\infty, \infty)$

2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

3) $P(a < X < b) = \int_{a}^{b} f(x)\, dx$

*



$\searrow P(a < X < b)$

1) $P(X = \alpha) = 0 \ \forall \ \alpha \in (-\infty, \infty)$

2) $P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b)$
$= P(a < X \leq b)$

3) $\int_{-\infty}^{\infty} f(x)\, dx = 1 \Rightarrow \int_{-\infty}^{k} f(x)\, dx + \int_{k}^{\infty} f(x)\, dx = 1$

$\Rightarrow P(X < k) + P(X > k) = 1$

$\Rightarrow \boxed{P(X < k) = 1 - P(X > k)}$

Pb: Suppose that the error in the reaction temperature for a controlled laboratory experiment is a continuous RV X having the PDF.

$$f(x) = \begin{cases} x^2/3 & \text{if } -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

1) Verify if f is a PDF

2) Find $P(0 \le x \le 1)$

Sol:

1) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= 0 + \int_{-1}^{2} \frac{x^2}{3}\,dx + 0$$

$$= \left[\frac{x^3}{9}\right]_{-1}^{2} = \frac{9}{9} = 1$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

$\therefore f$ is a PDF.

2) $$P(0 \le x \le 1) = \int_{0}^{1} f(x)\,dx = \int_{0}^{1} \frac{x^2}{3}\,dx$$

$$= \left[\frac{x^3}{9}\right]_{0}^{1}$$

$$\boxed{P(0 \le x \le 1) = 1/9}$$

<u>Cummulative Distribution function of a Continuous RV:</u>
(CDF)

Let x be a continuous RV that takes values from $-\infty$ to ∞. Let f be a PDF of X. The cummulative distribution function F of X is defined by

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad ; x \in (-\infty, \infty)$$

$\therefore F'(x) = f(x)$ provided F is differentiable at x.

Let $a < b$

$$F(b) - F(a) = \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$-\infty \quad a \quad b$

$$F(b) - F(a) = \int_{-\infty}^{a} f(t)\,dt + \int_{a}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$$\int_{a}^{b} f(t)\,dt = \boxed{P(a \le X \le b)}$$

Pb. If $f(x) = \begin{cases} x e^{-x^2/2} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ then

1) Show that f is a PDF of a continuous RV X.

2) Find CDF of X

Sol.

1) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} x e^{-x^2/2}\,dx$$

Put $-\frac{x^2}{2} = t \Rightarrow -x\,dx = dt$

$$= \int_{0}^{-\infty} (-e^{t})\,dt = \underset{t \to -\infty}{Lt} (-e^{t}) - (-1)$$

$$= 1$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

19/10/24

2) for $x \le 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt \quad \left(\because f(t) = 0 \ \forall\ t \le 0\right)$$

$$F(x) = 0$$

for $x > 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

Given $f(t) = \begin{cases} t e^{-t^2/2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

$$\therefore F(x) = 0 + \int_{0}^{x} t e^{-t^2/2}\,dt$$

from

$$F(x) = \int_0^x t e^{-t^2/2} dt$$

$$-\frac{t^2}{2} = k \Rightarrow t\,dt = -dk$$

$$F(x) = \int_0^{-x^2/2} e^k (-dk) = \left[-e^k\right]_0^{-x^2/2}$$

$$\boxed{F(x) = 1 - e^{-x^2/2}} \text{ for } x > 0$$

$$\therefore \boxed{F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-x^2/2} & \text{for } x > 0 \end{cases}} : \text{CDF of } X$$

Pb: The density function of a continuous RV X is given by

$$f(x) = \begin{cases} ax & \text{for } 0 \le x \le 1 \\ a & \text{for } 1 \le x \le 2 \\ 3a - ax & \text{for } 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

a) Find 'a'

b) Find CDF F of x

Sol: a) W.K.T

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

So $$\int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx + 0 = 1$$

$\Rightarrow \frac{a}{2}(1) + a + 3a - \frac{a}{2}(9-4) = 1$

$\Rightarrow \frac{9a}{2} - \frac{5a}{2} = 1 \Rightarrow \boxed{a = \frac{1}{2}}$

b) $F(x) = \int_{-\infty}^{x} f(t)\, dt$

for $x \le 0$: $F(x) = \int_{-\infty}^{x} f(t)\, dt = \boxed{0}$

for $0 \le x \le 1$: $F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{0} f(t)\, dt + \int_{0}^{x} f(t)\, dt$

$= 0 + a\int_{0}^{x} t\, dt = \frac{ax^2}{2}$

$= \boxed{\frac{x^2}{4}}$

for $1 \le x \le 2$: $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$= \int_{-\infty}^{0} f(t)\, dt + \int_{0}^{1} f(t)\, dt + \int_{1}^{x} f(t)\, dt$

$= 0 + \int_{0}^{1} at\, dt + \int_{1}^{x} a\, dt$

$= \frac{a}{2} + a(x-1) = ax - \frac{a}{2} = \boxed{\frac{2x-1}{4}}$

for $2 \le x \le 3$: $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$= \int_{-\infty}^{0} f(t)\, dt + \int_{0}^{1} f(t)\, dt + \int_{1}^{2} f(t)\, dt + \int_{2}^{x} f(t)\, dt$

$= 0 + \int_{0}^{1} at\, dt + \int_{1}^{2} a\, dt + \int_{2}^{x} (3a - at)\, dt$

$= \frac{a}{2} + a + 3a(x-2) - \frac{a}{2}(x^2 - 4)$

$= \frac{3a}{2} - 6a + 3ax - \frac{a}{2}x^2 + 2a$

$= -\frac{a}{2}x^2 + 3ax - \frac{5a}{2}$

$= \boxed{-\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4}}$

for $x \ge 3$: $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$= 0 + \frac{a}{2} + a + 3a - \frac{a}{2}(9-4) + 0$

$= 2a = \boxed{1}$

CDF of X:

$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{x^2}{4} & \text{for } 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4} & \text{for } 1 \le x \le 2 \\ -\frac{x^2}{4} + \frac{3x}{2} - \frac{5}{4} & \text{for } 2 \le x \le 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$

Pb: The CDF F of a continuous RV X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1/2 \\ 1 - \frac{3}{25}(3-x)^2 & \text{if } \frac{1}{2} \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$

Find the PDF (f) of X and evaluate $P(|X| \le 1)$ and $P(\frac{1}{3} \le X \le 4)$ by using CDF and PDF.

Sol: We have $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$\Rightarrow F'(x) = f(x)$, provided F is diff at x.

Case I: Let $x < 0 \rightarrow F(x) = 0$

$\Rightarrow F'(x) = 0$ $\quad \therefore f(x) = 0$

Case II: $0 < x < 1/2 \rightarrow F(x) = x^2$

$F'(x) = 2x$. $\quad \therefore f(x) = 2x$

Case III: $\frac{1}{2} < x < 3 \rightarrow F(x) = 1 - \frac{3}{25}(3-x)^2$

$F'(x) = \frac{6}{25}(3-x) = f(x)$

Case IV: $x > 3 \rightarrow F(x) = 1$

$$F'(x) = 0 = f(x)$$

$$f(x) = \begin{cases} 0 & x \le 0 \\ 2x & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3-x) & \frac{1}{2} < x \le 3 \\ 0 & x \geq 3 \end{cases} \quad : \text{PDF of } X$$

$$F'(0^-) = \underset{x \to 0^-}{Lt} \frac{F(x) - F(0)}{x - 0} = \underset{x \to 0^-}{Lt} \frac{0-0}{x} = 0$$

$$F'(0^+) = \underset{x \to 0^+}{Lt} \frac{F(x) - F(0)}{x - 0} = \underset{x \to 0^+}{Lt} \frac{x^2 - 0}{x} = \underset{x \to 0^+}{Lt} x = 0$$

$$\therefore F'(0^+) = F'(0^-) = F'(0) = 0$$

$$F'(\tfrac{1}{2}^-) = \underset{x \to \frac{1}{2}^-}{Lt} \frac{F(x) - F(\frac{1}{2})}{x - \frac{1}{2}} = \underset{x \to \frac{1}{2}^-}{Lt} \frac{x^2 - \frac{1}{4}}{x - \frac{1}{2}} = \underset{x \to \frac{1}{2}^-}{Lt} x + \frac{1}{2}$$

$$= 1$$

$$F'(\tfrac{1}{2}^+) = \underset{x \to \frac{1}{2}^+}{Lt} \frac{F(x) - F(\frac{1}{2})}{x - \frac{1}{2}} = \underset{x \to \frac{1}{2}^+}{Lt} \frac{1 - \frac{3}{25}(3-x)^2 - \frac{1}{4}}{x - \frac{1}{2}}$$

$$= \underset{x \to \frac{1}{2}^+}{Lt} \frac{\frac{3}{100}\left(25 - (6-2x)^2\right)}{x - \frac{1}{2}}$$

$$= \underset{x \to \frac{1}{2}^+}{Lt} \frac{3}{100} \cdot \frac{(11 - 2x)(-1 + 2x)}{x - \frac{1}{2}}$$

$$= \frac{6}{100} \cdot (10) = 0.6$$

$F'(\frac{1}{2}^-) \neq F'(\frac{1}{2}^+) \Rightarrow$ Not differentiable at $x = \frac{1}{2}$

$$F'(3^+) = F'(3^-) = 0 = F'(3)$$

$$P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1) \quad \text{- using CDF}$$

$$= 1 - \frac{12}{25} - 0 = \frac{13}{25}$$

Using PDF

$$P(-1 \le X \le 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1/2} f(x)\,dx + \int_{1/2}^{1} f(x)\,dx$$

$$= 0 + \int_{0}^{1/2} 2x\,dx + \int_{1/2}^{1} \frac{6}{25}(3-x)\,dx$$

$$= \frac{1}{4} + \frac{6}{25}\left(\frac{3}{2} - \frac{1}{2}\cdot\frac{3}{4}\right)$$

$$= \frac{1}{4} + \frac{6}{25}\cdot\frac{9}{8} = \frac{1}{4} + \frac{27}{100} = \frac{52}{100}$$

$$= \frac{13}{25}$$

$$P\left(\frac{1}{3} \le X \le 4\right) = F(4) - F\left(\frac{1}{3}\right) \quad \text{: using CDF}$$

$$= 1 - \frac{1}{9} = \boxed{\frac{8}{9}}$$

$$P\left(\frac{1}{3} \le X \le 4\right) = \int_{1/3}^{4} f(x)\,dx = \int_{1/3}^{1/2} f(x)\,dx + \int_{1/2}^{3} f(x)\,dx + \int_{3}^{4} f(x)\,dx$$

$$= \int_{1/3}^{1/2} 2x^2\,dx + \int_{1/2}^{3} \frac{6}{25}(3-x) + \int_{3}^{4} 0\,dx$$

$$= \left(\frac{1}{4} - \frac{1}{9}\right) + \frac{6}{25}\left(3\cdot\frac{5}{2} - \frac{1}{2}\cdot\frac{35}{4}\right) + 0$$

$$= \frac{5}{36} + \frac{6}{25}\cdot\frac{25}{8} = \frac{5}{36} + \frac{3}{4} = \frac{32}{36}$$

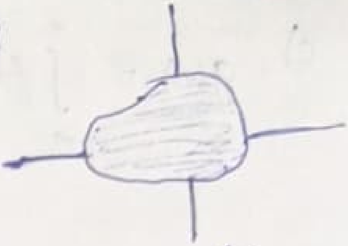
$$= \boxed{\frac{8}{9}}$$

Joint PDF of continuous RV's :

Let X and Y be continuous RV's. A function $f(x,y)$ is said to be joint PDF of the RV's X and Y if

1) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

2) $P((x,y) \in A) = \iint_A f(x,y)\, dx\, dy$



where A is a region in $\mathbb{R}^2$

The marginal PDF $g(x)$ of X is given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Similarly, the marginal PDF $h(y)$ of Y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Pb: Consider $f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

a) Is f a joint PDF?

b) If so, find $P((x,y) \in A)$, where

$$A = \{(x,y) \in \mathbb{R}^2 \,/\, 0 \le x \le \tfrac{1}{2},\ \tfrac{1}{4} \le y \le \tfrac{1}{2}\}$$

c) Find the marginal PDF's of X and Y.

Sol:

a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\, dx\, dy$$

$$= \frac{2}{5}\int_0^1 (1+3y)\, dy$$

$$= \frac{2}{5}\left(1+\frac{3}{2}\right) = 1$$

$\therefore$ Given f is a joint PDF.

b) $$P((x,y) \in A) = \iint_A f(x,y)\, dx\, dy = \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\, dx\, dy$$

$$= \frac{2}{5}\int_{1/4}^{1/2}\left(\frac{1}{4}+\frac{3}{2}y\right) dy$$

$$= \frac{2}{5}\left(\frac{1}{4}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{3}{16}\right) = \frac{9+4}{160} = \boxed{\frac{13}{160}}$$

c) $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 f(x,y)\,dy$

$$= \int_0^1 \frac{2}{5}(2x+3y)\,dy$$

$$= \frac{2}{5}\left(2x+\frac{3}{2}\right)$$

$$\boxed{g(x) = \begin{cases} \frac{2}{5}\left(2x+\frac{3}{2}\right) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

Similarly, $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{2}{5}(2x+3y)\,dx$

$$h(y) = \frac{2}{5}(1+3y)$$

$$\boxed{h(y) = \begin{cases} \frac{2}{5}(1+3y) & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

Pb: The joint PDF of the continuous RV's X and Y is given by

$$f(x,y) = \begin{cases} 1/y & \text{if } 0 < x < y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

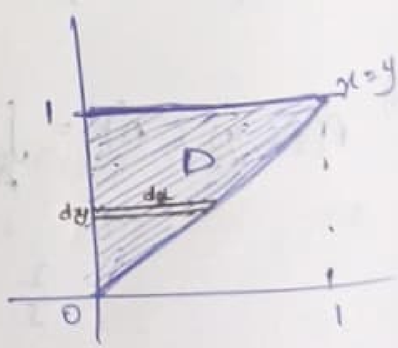
1) S.T $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

2) Find $P(X+Y > \frac{1}{2})$

Sol:

1) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dx\,dy$

$$+ \iint_{D^*} \underbrace{f(x,y)}_{\to 0}\,dx\,dy$$

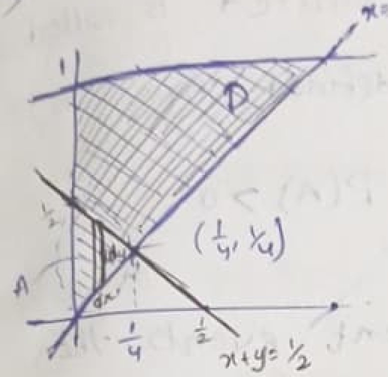


$$= \iint_D \frac{1}{y}\,dx\,dy + 0$$

D^* - region outside D

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^y \frac{1}{y}\,dx\,dy = \int_0^1 \frac{1}{y}\cdot y\,dy = \boxed{1}$$

2) $P(X+Y > \frac{1}{2}) = ?$



required region is

D − A

total region

$$P(X+Y>\tfrac{1}{2}) = 1 - P(X+Y\le \tfrac{1}{2})$$

$$= 1 - \iint_A f(x,y)\,dy\,dx$$

$$= 1 - \int_0^{1/4}\int_x^{\frac{1}{2}-x} \frac{1}{y}\,dy\,dx$$

$$= 1 - \int_0^{1/4} \left[\ln y\right]_x^{\frac{1}{2}-x} dx$$

$$= 1 - \int_0^{1/4} \ln\left(\tfrac{1}{2}-x\right) - \ln x \, dx$$

$$= 1 - \int_0^{1/4} \ln\left(\tfrac{1}{2}-x\right) dx + \int_0^{1/4} \ln x\,dx$$

$$\tfrac{1}{2} - x = t \Rightarrow dx = -dt$$

$$= 1 + \int_{1/2}^{1/4} \ln t\,dt + \left[x\ln x - x\right]_0^{1/4}$$

$$= 1 + \left[t\ln t - t\right]_{1/2}^{1/4} + \tfrac{1}{4}\ln\tfrac{1}{4} - \tfrac{1}{4}$$

$$= 1 + \tfrac{1}{2}\ln\tfrac{1}{4} - \cancel{\tfrac{1}{2}} - \tfrac{1}{2}\ln\tfrac{1}{2} + \cancel{\tfrac{1}{2}}$$

0.301

$$= 1 + \tfrac{1}{2}\ln\tfrac{1}{2} = 1 - \frac{\ln 2}{2}$$

$$= \boxed{\approx 0.65}$$

1/10/24

Conditional Probability :

Let A and B be two events.

The probability that the event B occurs when it is given that the event A had occured is called conditional probability $P(B|A)$ is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{where } P(A) > 0 \quad -①$$

If A and B are independent events, then

$P(B|A) = P(B)$

$\therefore$ From ①, $P(B) = \dfrac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(A)\,P(B)$

Let X and Y be RV's corresponding to the events A and B respectively. Let $f(x,y)$ be the joint PDF of X and Y.

Let $g(x)$ be the marginal PDF of X.

Let $h(y)$ be the marginal PDF of Y.

Suppose that X and Y are distinct

$$P(Y=y|X=x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0;$$

Similarly we define,

$$P(X=x|Y=y) = \frac{f(x,y)}{h(y)}, \quad h(y) > 0$$

Definition :

Let X and Y be RV's. The PDF of the RV X when it is given that $Y=y$ is denoted by $f(x/y)$ and is defined by

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

Similarly, the PDF of the RV Y when it is given that $X = x$ is denoted by $f(y|x)$ and is defined by

$$f(y|x) = \frac{f(x,y)}{g(x)}$$

Next

$$P(a<X<b \mid Y=y) = \sum_{a<x<b} f(x|y)$$

provided X and Y are discrete

$$P(a<X<b \mid Y=y) = \int_a^b f(x|y)\,dx$$

provided X and Y are continuous.

Q: Two pens are selected at Random from a box that contains 3 blue pens, 2 red pens, 3 green pens. If X denotes no. of blue pens, and Y denotes no. of red pens selected, then Find the conditional probability distribution of X, when it is given that $Y=1$, Further find $P(X=0|Y=1)$

Sol:

$Y \backslash X$	0	1	2
0	$3/28$	$9/28$	$3/28$
1	$6/28$	$\frac{6}{28}$	0
2	$\frac{1}{28}$	0	0

x	0	1	2
$g(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$

$$f(x|1) = \frac{f(x,1)}{h(1)}, \quad 0, 1, 2$$

$$\therefore f(0|1) = \frac{f(0,1)}{h(1)} = \frac{\frac{6}{28}}{\frac{12}{28}} = \frac{1}{2}$$

$$f(1|1) = \frac{f(1,1)}{h(1)} = \frac{6/28}{12/28} = 1/2$$

$$f(2|1) = \frac{f(2,1)}{h(1)} = 0$$

The conditional PDF of X when $y=1$, is given by

x	0	1	2
$f(x|1)$	$\frac{1}{2}$	$\frac{1}{2}$	0

21/10/24

Pb: The joint PDF of the continuous RVs X and Y is given by

$$f(x,y) = \begin{cases} 10xy^2 & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

1) Find the marginal PDF's of X and Y

2) Find ~~f(y|x)~~ the conditional PDFs of X & Y

3) Find $P(Y > \frac{1}{2} \mid X = 0.25)$

4) Find $P(\frac{1}{2} < X < 1 \mid Y = 0.4)$

Sol:

1) Marginal PDF of X is $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$g(x) = \int_x^1 (10xy^2)\,dy$$

$$g(x) = \begin{cases} \frac{10x}{3}(1-x^3) & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal PDF of Y is $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$h(y) = \int_0^y 10xy^2\,dx$$

$$= \frac{10y^2}{2}(y^2) = \begin{cases} 5y^4 & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

2) $$P(y|x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10x}{3}(1-x^3)} = \begin{cases} \frac{3y^2}{1-x^3} & \text{if } 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(x|y) = \frac{f(x,y)}{h(y)} = \frac{10xy^2}{5y^4} = \begin{cases} \frac{2x}{y^2} & \text{if } 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

3) $$P(Y > \tfrac{1}{2} \mid X = 0.25) = \int_{\frac{1}{2}}^{1} \frac{3y^2}{1-x^3}\,dy = \frac{64}{21}\cdot\frac{1}{3}\left[1-\frac{1}{8}\right]$$

$= f(y|0.25)$

$$= \frac{7(64)}{21(3)(8)}$$

$$= \frac{56}{63} = \boxed{\frac{8}{9}}$$

iv) $P\left(\frac{1}{2} < X < 1 \mid Y = 0.4\right)$

$$= f(x \mid 0.4) = \int_{1/2}^{1} \frac{2x}{(0.4)^2}\, dx$$

$$= \frac{2(25)}{4(2)} (3/4) = \frac{75}{16}$$

$$\boxed{= \frac{75}{16}}$$

Definition:

The RV's X and Y are said to be statistically independent if

$$f(x,y) = g(x)\, h(y) \quad \forall\, (x,y)$$

where f is joint PDF of X and Y and g and h are marginal PDFs of X and Y respectively.

Pb: Two pens are selected at random from a box that contains 3 blue pens, 2 red pens, 3 green pens. If X denotes no. of blue pens selected, Y denotes no. of red pens selected. then we have

Y \ X	0	1	2
0	3/28	9/28	3/28
1	6/28	6/28	0
2	1/28	0	0

x	0	1	2
g(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
h(y)	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$

$f(0,1) = \frac{6}{28}$ $\quad g(0) = \frac{10}{28}$, $\quad h(1) = \frac{12}{28}$

$f(0,1) \neq g(0) \cdot h(1)$

$\therefore$ X and Y are statistically dependent.

* Suppose that a RV X takes 2, 3, 3, 3, 4, 4, 10

Average (or) Mean of X $= \dfrac{2+3+3+3+4+4+10}{7}$

$= 2\cdot\frac{1}{7} + 3\cdot\frac{3}{7} + 4\cdot\frac{2}{7} + 10\cdot\frac{1}{7}$

$= \sum_x x \cdot f(x)$

x	2	3	4	10
f(x)	1/7	3/7	2/7	1/7

Definition:

Let X be a RV. Let f(x) be a PDF of X. Then the mean (or) average (or) expected value of X is defined by

$E(X) = \mu = \sum_x x \cdot f(x)$, provided X is discrete RV

$E(X) = \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$, provided X is continuous RV.

Pb: Let X be the RV that denotes the life of a certain electronic device in hours. The PDF of X is given by $f(x) = \begin{cases} \dfrac{20000}{x^3} & x > 100 \\ 0 & \text{otherwise} \end{cases}$

Find the expected (or) average life of the device.

Ans:
$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{100} x f(x)\,dx + \int_{100}^{\infty} x f(x)\,dx$

$= 0 + \int_{100}^{\infty} x \cdot \frac{20000}{x^3}\,dx$

$= (20000)\int_{100}^{\infty} x^{-2}\,dx$

$= 20000\left[-\frac{1}{x}\right]_{100}^{\infty} = 200$ hrs

24/10/24

x_1	x_2	x_3	x_4	x_5
-10	0	10	20	30

Average = 10

$$\text{Variance} = \frac{(x_1-10)^2+(x_2-10)^2+(x_3-10)^2+(x_4-10)^2+(x_5-10)^2}{5}$$

$$= (x_1-10)^2\cdot\frac{1}{5}+(x_2-10)^2\cdot\frac{1}{5}+(x_3-10)^2\frac{1}{5}+(x_4-10)^2\frac{1}{5}+(x_5-10)^2\cdot\frac{1}{5}$$

$= 200$

x	-10	0	10	20	30
$f(x)$	1/5	1/5	1/5	1/5	1/5

$$\text{Variance} = \sum_x (x-\mu)^2 f(x)$$

y_1	y_2	y_3	y_4	y_5
8	9	10	11	12

Average = 10

$$\text{Variance} = \frac{(y_1-10)^2+(y_2-10)^2+(y_3-10)^2+(y_4-10)^2+(y_5-10)^2}{5}$$

$= 2$

Variance:

Let X be a RV whose PDF is $f(x)$. The variance of the RV X is denoted by σ^2 and is defined by

$$\sigma^2 = \sum_x (x-\mu)^2 f(x) \text{ provided X is discrete.}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx; \text{ provided X is continuous.}$$

$$\sigma^2 = E((X-\mu)^2) = \sum_x (x-\mu)^2 f(x)$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Standard Deviation:

The positive square root of Variance is called standard deviation and is denoted by 'σ'.

We have $\sigma^2 = E((X-\mu)^2)$

$$= E(X^2 + \mu^2 - 2\mu X)$$

$$= E(X^2) + E(\mu^2) - E(2\mu X)$$

$$\sigma^2 = E(X^2) + \mu^2 E(1) - 2\mu E(X)$$

$$\sigma^2 = E(X^2) + \mu^2 \cdot 1 - 2\mu(\mu)$$

$$\sigma^2 = E(X^2) - \mu^2 = E(X^2) - (E(X))^2$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

Pb: Let the RV X represent the number of defective parts of a machine when 3 parts are sampled from a production line and tested. The following is the PDF of X

X	0	1	2	3
f(x)	0.51	0.38	0.1	0.01

Find the variance of X.

Sol:

$$\sigma^2 = \sum_x (x-\mu)^2 f(x) = E(X^2) - (E(X))^2$$

$$E(X) = \mu = \sum_x x \cdot f(x) = 0.38 + 0.2 + 0.03$$

$$= 0.61$$

$$E(X^2) = \sum_x x^2 f(x) = 0.38 + 0.4 + 0.09$$

$$= 0.87$$

$$\sigma^2 = (0.87) - (0.61)^2 = 0.497$$

Q: The weekly demand for a drinking water product, in thousand of liters from a local chain of efficiency stores is a continuous RV X having the PDF

$$f(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

Sol: mean: $\mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = E(X)$

$$= \int_{-\infty}^{1} 0 + \int_{1}^{2} x(2)(x-1)\,dx + \int_{2}^{\infty} 0$$

$$= 2\int_{1}^{2} x^2 - x\,dx = 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= 2\left[\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right]$$

$$\text{mean} = 2\left[\frac{7}{3} - \frac{3}{2}\right] = \boxed{\frac{5}{3}}$$

Variance: $\sigma^2 = E(X^2) - (E(X))^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 2\int_{1}^{2} x^3 - x^2\,dx$$

$$= 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= 2\left[4 - \frac{8}{3} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= 2\left[\frac{17}{12}\right] = \boxed{\frac{17}{6}}$$

$$\sigma^2 = \frac{17}{6} - \frac{25}{9} = \frac{51-50}{18} = \boxed{\frac{1}{18}}$$: Variance

r^{th} moment of a RV: (or) r^{th} central moment

Let X be a RV whose PDF is $f(x)$. The r^{th} moment (or) r^{th} central moment of the RV X is denoted by μ_r and is defined by $\mu_r = E((X-\mu)^r)$, $r = 0,1,2,3,\ldots$

$$\mu_r = \sum_x (x-\mu)^r f(x), \text{ when } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx, \text{ when } x \text{ is continuous.}$$

r^{th} raw moment of a RV:

Let X be a RV whose PDF is $f(x)$. The r^{th} raw moment of the RV X is denoted by μ_r' and is defined by

$$\mu_r' = E(X^r), \ r = 0,1,2,3,\ldots$$

$$\mu_r' = \sum_x x^r f(x), \text{ when } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} x^r f(x), \text{ when } x \text{ is continuous}$$

* $\mu_0 = E((X-\mu)^0) = E(1) = 1$

$\mu_1 = E(X-\mu) = E(X) - E(\mu) = \mu - \mu E(1) = 0$

7/11/24

Relation between moments and Raw moments:

By definition, $\mu_r = E((X-\mu)^r)$, $r = 0,1,2,3,\ldots$

$$\mu_0 = E(X^0) = E(1) = 1$$

$$\mu_1 = E(X-\mu) = E(X) - E(\mu) = \mu - E(1)\cdot\mu$$
$$= \mu - \mu = 0$$

$$\mu_2 = E((x-\mu)^2) = E(x^2+\mu^2-2\mu x)$$

$$= E(x^2) + E(\mu^2) - E(2\mu x)$$

$$= \mu_2' + \mu^2 E(1) - 2\mu E(x)$$

$$= \mu_2' + \mu^2 - 2\mu^2$$

$$\mu_2 = \mu_2' - \mu^2 = \mu_2' - (\mu_1')^2 \quad [\because \mu = E(x) = \mu_1']$$

$$\mu_3 = E((x-\mu)^3)$$

$$= E(x^3 - \mu^3 - 3x^2\mu + 3x\mu^2)$$

$$= E(x^3) - E(\mu^3) - 3\mu E(x^2) + 3\mu^2 E(x)$$

$$= \mu_3' - \mu^3 - 3\mu\mu_2' + 3\mu^2(\mu)$$

$$= \mu_3' - 3\mu\mu_2' + 2\mu^3$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

Moment Generating Function :

The moment generating function (MGF) of a RV x is denoted by $M_x(t)$, and is defined by

$$M_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f(x)\,dx \quad \text{when } x \text{ is conti}$$

$$= \sum_x e^{xt} f(x) \quad \text{when } x \text{ is descrete}$$

We have $M_x(t) = E(e^{xt})$

$$= E\left(1 + \frac{(xt)}{1!} + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} + \frac{(xt)^4}{4!} + \dots\right)$$

$$M_x(t) = E(1) + E(xt) + \frac{E((xt)^2)}{2!} + \frac{E((xt)^3)}{3!} + \frac{E((xt)^4)}{4!} + \dots + \frac{E((xt)^r)}{r!} + \dots$$

$$M_x(t) = 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \frac{t^4}{4!}E(x^4) + \dots + \frac{t^r}{r!}E(x^r) + \dots$$

$$M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \frac{t^4}{4!}\mu_4' + \dots + \frac{t^r}{r!}\mu_r' + \dots \quad -①$$

That is, μ_r' = the coefficient of $\frac{t^r}{r!}$ in the moment ~~gathe~~ generating function $M_X(t)$

Differentiating ① two times w.r.t 't', we get

$$\frac{d^2 M_X(t)}{dt^2} = 0 + 0 + \frac{2!}{2!}\mu_2' + \frac{3!\,t}{3!}\mu_3' + \text{terms containing } t$$

Taking $t = 0$, we get

$$\left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = \mu_2' + 0 = \mu_2'$$

Differentiating ① r-times with respect to 't', and taking $t = 0$, we get

$$\boxed{\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu_r'}$$

Pb. The moment generating function of a RV 'X' is given by $M_X(t) = \frac{3}{4-t}$. Find Mean and Variance of the RV X and μ_3 of RV X

Sol.
$E(X)$ = Mean $= \mu = \mu_1'$

Variance $= E(X^2) - (E(X))^2$

$= \mu_2' - (\mu_1')^2$

$$\mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{3}{(4-t)^2}\right|_{t=0} = \frac{3}{16}$$

$$\mu_2' = \left.\frac{6}{(4-t)^3}\right|_{t=0} = \frac{3}{32}$$

$$\mu_3' = \left.\frac{18}{(4-t)^4}\right|_{t=0} = \frac{9}{128}$$

$$\text{Mean} = \mu_1' = \frac{3}{16}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{3}{32} - \frac{9}{256} = \frac{15}{256}$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$~~= \frac{9}{128} - \frac{135}{16(256)} + \frac{3}{(256)8}~~$$

$$~~= \frac{159}{4096} = 0.0388~~$$

$$\mu_3 = \frac{9}{128} - \frac{27}{512} + \frac{27}{2048} = \frac{63}{2048}$$

Pb: ~~Let x be the~~ A dice is thrown. Let x be the corresponding RV. Find the moment generating function of the RV X. Hence, find Mean and Variance of X.

Sol: Clearly RV takes the values 1, 2, 3, 4, 5, 6

$$f(1) = P(X=1) = \frac{1}{6}$$

Similarly, $f(2) = f(3) = f(4) = f(5) = f(6) = \frac{1}{6}$

$$M_x(t) = E(e^{xt}) = \sum_x e^{xt} f(x)$$

$$M_x(t) = e^t.f(1) + e^{2t}f(2) + e^{3t}f(3) + e^{4t}f(4) + e^{5t}f(5) + e^{6t}f(6)$$

$$M_x(t) = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$$\mu_1' = \mu = \frac{d}{dt}(M_x(t))\Big|_{t=0} = \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{6(7)}{2(6)} = \boxed{3.5}$$

$$\sigma^2 = -\mu^2 + (\mu_2')^2 = -(\mu_1')^2 + (\mu_2')^2$$

$$\sigma^2 = -\left(\frac{7}{2}\right)^2 + \frac{1}{6}\cdot\frac{6(7)(13)}{6} = \boxed{\frac{35}{12}}$$

Standard Probability Distributions :

1. Binomial Distribution (Discrete : Finite)
2. Poisson Distribution (Discrete : Countably Infinite)
3. Geometric Distribution (Discrete : Finite)
4. Normal Distribution (Continuous)

Binomial Distribution :

1) Let the random experiment contains n number of ~~traits~~ trials.
2) Suppose that each trial has two outcomes, namely, success and failure
3) The probability of getting success is p and the probability of getting failure is q, where $p+q=1$
4) The probabilities p and q remain constant from trail to trial.

Let x be the RV that denotes the number of success

Therefore x takes $0, 1, 2, 3, \ldots\ldots n$

It is said to be that x follows the binomial distribution if the PDF f of x is given by

$$f(x) = P(x=x) = {}^nC_x \cdot p^x q^{n-x} \quad ; \; x = 0, 1, 2, 3, \ldots n.$$

$$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} {}^nC_x \, p^x q^{n-x} . \qquad f(x) \geq 0 \;\forall x$$

$$= {}^nC_0 \, p^0 \cdot q^n + {}^nC_1 \, p \, q^{n-1} + \ldots\ldots + {}^nC_n \, p^n q^0$$

$$= (p+q)^n = \underline{1}$$

Also $f(x) \geq 0 \;\forall x$

$\therefore f$ is a PDF

Mean $(x) = \mu = E(x) = \sum_{x=0}^{n} x \cdot f(x)$

$$= \sum_{x=0}^{n} x \cdot {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} p \cdot n \cdot \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x}$$

$$= pn \sum_{x=1}^{n} {}^{n-1}C_{x-1} p^{x-1} q^{n-x}$$

$$= pn \left[{}^{n-1}C_0 p^0 q^n + {}^{n-1}C_1 p^1 q^{n-2} + \dots + {}^{n-1}C_{n-1} p^{n-1} q^0 \right]$$

$$= np(p+q)^{n-1} = \boxed{np} : \text{Mean}$$

$$E(x^2) = \sum_{x=0}^{n} x^2 f(x) = \sum_{x=0}^{n} x\,(x f(x))$$

$$= \sum_{x=0}^{n} (x-1)\,x\,f(x) + \sum_{x=0}^{n} x f(x)$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!\,(n-x)!} p^x \cdot q^{n-x} + np$$

$$= n(n-1)\,p^2\,(p+q)^{n-2} + np$$

$$\boxed{E(x^2) = n(n-1)p^2 + np}$$

Variance $= E(x^2) - (E(x))^2$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p)$$

$$\boxed{\text{Variance} = npq}$$

Example : The probability that a patient recovers from a rare blood cancer is 0.4. If 15 people are known to be suffering from the rare blood cancer, then what is the probability that

1) At least one survive.
2) From 3 to 8 people survive
3) Exactly 5 survive.

Sol:

Clearly $n = 15$, $p = 0.4$, $q = 1 - p = 0.6$

Let x be the number of people survived from the cancer

$\therefore x = 0, 1, 2, 3 \ldots 15$

Suppose that x follows binomial distribution

$$P(X = x) = f(x) = {}^{15}C_x \cdot (0.4)^x \cdot (0.6)^{15-x} \qquad x = 0, 1, 2, 3, \ldots 15$$

1) $P(\text{atleast one survive}) = P(x \geq 1) = 1 - P(x = 0)$

$$= 1 - {}^{15}C_0 (0.4)^0 (0.6)^{15}$$

$$= 0.9995$$

2) $P(3 \text{ to } 8 \text{ people survive}) = P(x = 3) + P(x = 4) + P(x = 5) + P(x = 6) + P(x = 7) + P(x = 8)$

$$= 0.0633 + 0.1267 + 0.1859 + 0.2065 + 0.177 + 0.118$$

$$= 0.8774$$

3) $P(x = 5) = 0.1859$

Q: An irregular 6-faced dice is manufactured such that the probability that it gives 3 even numbers in 5 throws is twice the probability that it gives 2 even numbers in 5 throws. How many sets of exactly 5 throws can be expected to give no even number out of 2500 sets.

Sol:

Let us focus on one set. Here one set means 5 throws

Here random experiment is contains 5 throws

Here success is getting even number

$\therefore n=5$; Here success is getting even number

Let p be the getting even number

Let x be the number of even numbers in five throws.

$\therefore x$ takes 0, 1, 2, 3, 4, 5

Assume that x follows binomial distribution

$P(X=x) = f(x) = {}^5C_x\, p^x q^{n-x}$ $x = 0, 1, 2, 3, 4, 5$

It is given that

$$P(X=3) = 2\, P(X=2)$$

$$\cancel{{}^5C_3}\, p^3 q^2 = 2 \cdot \cancel{{}^5C_2}\, p^2 q^3 \Rightarrow p = 2q$$

W.K.T $p+q=1 \Rightarrow 3q=1 \Rightarrow q=\frac{1}{3}$

$p = \frac{2}{3}$

P(getting no even number in one set) $= P(X=0)$

$$= {}^5C_0 \left(\frac{2}{3}\right)^0 \cdot \left(\frac{1}{3}\right)^5 = \frac{1}{3^5}$$

The number of sets that give no even number out of 2500 sets $= 2500 \times \frac{1}{3^5} \approx 10$

Pb: In sampling a large number of items manufactured by a company, the mean number of defective items in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain atleast 3 defective items?

sol:

Let us focus on one sample of 20 items

$\therefore n = 20$

Let x be the number of defective items in one sample of 20 items

$\therefore x$ takes $0, 1, 2, 3, \ldots 20$

Assume that x follows binomial distribution

$$P(X = x) = f(x) = {}^{20}C_x \, p^x \, q^{n-x}$$

It is given that mean $= E(X) = 2$

$\Rightarrow np = 2 \Rightarrow 20p = 2$

$p = 1/10 \Rightarrow q = 9/10$

Here success = getting defective items

P(getting atleast 3 defective items in a sample) $= P(X \geq 3)$

$= 1 - P(X \leq 2)$

$$= 1 - \left({}^{20}C_0 \, p^0 q^{20} + {}^{20}C_1 \, p \, q^{19} + {}^{20}C_2 \, p^2 q^{18} \right)$$

$= 0.323$

The number of sample that contain atleast 3 defective items $\approx 1000 \times 0.323$

≈ 323

Poisson Distribution:

i) Suppose that the random experiment consisting of n number of trials, where n is large.

ii) In each trial there are only two outcomes, namely success and failure.

iii) Let p and q be the probability of getting success and failure respectively.

iv) Suppose that p and q remain constant trial to trial.

Let X be the RV that denotes the number of success. Then X takes $0, 1, 2, 3, \ldots$

It is said to be that X follows poisson distribution if the PDF of X is given by

$$f(x) = P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

where $\lambda > 0$

$$\sum_x f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right] = e^{-\lambda} \cdot e^{\lambda}$$

$$\sum_x f(x) = 1$$

mean (μ or $E(X)$) $= \sum_x x f(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = \lambda \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!}$

$$E(X) = \lambda e^{-\lambda}\left[0 + 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= \lambda e^{-\lambda} (e^{\lambda})$$

$$E(X) = \lambda$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 f(x) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} (x-1)x \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} x\frac{\lambda^x}{x!}\right]$$

$$= e^{-\lambda}\left[\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}\lambda^2 + \lambda\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}\right]$$

$$E(X^2) = \lambda^2 + \lambda$$

$$\text{Variance} = E(X^2) - (E(X))^2 = \lambda^2 + \lambda - \lambda^2$$

$$\boxed{= \lambda}$$

14/11/24

Pb: The number of emergency admissions each day to a hospital is found to be have poisson distribution with mean 4. Find the probability that on a particular day there will be no emergency admission.

Sol:

Let X be the number of emergency admissions on a day

Mean $(X) = 4 = \lambda$

∴ The poisson distribution of X is given by

$$P(X=x) = f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \; ; \; x = 0, 1, 2, 3, \ldots$$

$$= \frac{e^{-4}4^x}{x!}$$

Probability of getting no emergency admission on a day $= P(X=0)$

$$= \frac{e^{-4}4^0}{0!}$$

$$\boxed{= e^{-4}}$$

Q: In a book of 600 pages, there are 60 ~~typo~~ typographical errors. Assuming poisson distribution for the number of errors per page, find the probability that ~~a randomly choosen 4 pages will not contain 2 errors.~~ each of 4 randomly chosen pages will have 2 errors.

Sol: average number of errors per page $= \frac{60}{600} = 0.1$

$\Rightarrow$ mean $= \lambda = 0.1$

Let X be the no. of errors per page.

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Probability that a page contain 2 error $= P(X=2)$

$$= \frac{e^{-0.1}(0.1)^2}{2!}$$

The required probability $= (P(X=2))^4$

$$= 4.174 \times 10^{-10}$$

* Poisson Distribution as a limiting case of Binomial Distribution.

We know the binomial distribution, $P(X=x) = {}^{n}C_{x} \cdot p^{x} q^{n-x}$

$x = 0, 1, 2, 3, \ldots n$

Take $\lambda = np$

$\Rightarrow p = \lambda/n$; $q = 1-p$

$$\therefore P(X=x) = {}^{n}C_{x} \cdot \left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-(x-1))}{x!} \cdot \frac{\lambda^{x}}{n^{x}}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$P(X=x) = \frac{\lambda^{x}}{x!}\left[\frac{n(n-1)(n-2)\ldots(n-(x-1))}{n.n.n.\ldots n}\right]\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$P(X=x) = \frac{\lambda^x}{x!}\left[\frac{n}{n}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)\right]$$

$$\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \underset{n\to\infty}{Lt}\frac{\lambda^x}{x!}\left[1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\left[1\cdot1\cdot1\cdots1\right]\underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^n\underbrace{\underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^{-x}}_{1}$$

$$= \frac{\lambda^x}{x!}\underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^n\cdot 1$$

$$\underset{n\to\infty}{Lt}\ \boxed{P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}}$$

Geometric Distribution:

1) Let a random experiment consisting of n trials, where n is large.
2) Suppose that each trial has only two outcomes namely success and failure.
3) Let p and q be the probability of getting success and failure respectively in each trial.
4) The probability p and q remain constant from traid to trial.

Let X be the number of trials required to get first success. Then X takes $1, 2, 3, \ldots$

The PDF of X is given by

$$f(x) = P(X=x) = p\cdot q^{x-1};\quad x=1,2,3,\ldots$$

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} p\,q^{x-1} = p\cdot\left[q^0+q^1+q^2+\cdots\right]$$

$$= \frac{p}{1-q} = \boxed{1}$$

$$\text{Mean}(x) = E(x) = \sum_{x=1}^{\infty} x\, f(x)$$

$$= \sum_{x=1}^{\infty} x \cdot p \cdot q^{x-1}$$

$$= p[\cancel{q} + q^0 + 2q + 3q^2 + \dots\dots]$$

W.K.T $\quad \frac{1}{1-q} = q^0 + q + q^2 + q^3 + \dots$

By diff ; $\quad \frac{1}{(1-q)^2} = q^0 + 2q + 3q^2 + \dots\dots$

$$\rightarrow \text{Mean}(x) = E(x) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = 1/p$$

$$\boxed{\text{mean} = 1/p}$$

$$E(x^2) = \sum_{x=1}^{\infty} x^2 f(x) = \sum_{x=1}^{\infty} x^2 p q^{x-1}$$

$$= p[1^2 + 2^2 \cdot q + 3^2 q^2 + 4^2 q^3 \dots]$$

$$\frac{q}{(1-q)^2} = q + 2q^2 + 3q^3 + \dots$$

diff ; $\quad \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} = 1 + 2^2 \cdot q + 3^2 q^2 + \dots$

$E \qquad \rightarrow \frac{1+q}{(1-q)^3}$

$$E(x^2) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$$

$$\text{Variance}(x) = E(x^2) - (E(x))^2 = \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$\boxed{= \frac{q}{p^2}}$$

18/10/24

Pb: If the probability that an applicant for a driving license will pass the road test on any given trial is 0.8, then what is the probability that he will finally pass the test

i) on the 4^{th} trial.

ii) in fewer than 4 trials.

Sol: Let x denote the trials required to pass road test

$\therefore$ x takes $1, 2, 3, 4, \ldots$

$\therefore$ x follows geometric distribution.

The probability of getting success (passing road test) is given by : $p = 0.8$

$\therefore q = 0.2$

The PDF of x is given by:

$$P(X=x) = f(x) = pq^{x-1} = (0.8)(0.2)^{x-1}$$

i) $P(X=4) = (0.8)(0.2)^3 = \boxed{0.0064}$

ii) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= (0.8)(1 + 0.2 + 0.04) = (1.24)(0.8)$

$= \boxed{0.992}$

Gaussian Distibution (or) Normal distribution :

A countinuous RV x follows the normal distribution if its PDF of is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2}, \quad -\infty < x < \infty$$

We denote it by $X \sim N(\mu, \sigma)$ (or) $X \sim N(\mu, \sigma^2)$

$\sigma > 0, \mu > 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

let $\frac{x-\mu}{\sqrt{2}\sigma} = t \Rightarrow dx = \sqrt{2}\sigma\, dt$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-t^2} dt\,(\sqrt{2}\sigma)$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-t^2} dt$$

$$\cancel{= \frac{1}{\sqrt{\pi}}\left[t e^{-t^2} + 2\int_{-\infty}^{\infty} t^2 e^{-t^2}\right]}$$

let $t^2 = s \Rightarrow 2t\,dt = ds \Rightarrow dt = \frac{ds}{2\sqrt{s}}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} \frac{e^{-s}}{\sqrt{s}}\,ds$$

$$= \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-s} s^{\frac{1}{2}-1}\,ds$$

$$\boxed{\Gamma n = \int_{0}^{\infty} e^{-x} x^{n-1}\,dx}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \Gamma(1/2)$$

$$= \frac{1}{\sqrt{\pi}}(\sqrt{\pi}) \quad (\because \Gamma(1/2) = \sqrt{\pi})$$

$$\boxed{= 1}$$

$\therefore$ f is PDF.

Mean $(\mu) = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x\, e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

let $\frac{x-\mu}{\sqrt{2}\sigma} = t \Rightarrow dx = \sqrt{2}\sigma\,dt$

$$\mu = \frac{\sqrt{2}\sigma}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t)\, e^{-t^2} dt$$

$$= \frac{1}{\pi}\left[\mu\int_{-\infty}^{\infty} e^{-t^2} dt + \sqrt{2}\sigma\int_{-\infty}^{\infty} t\, e^{-t^2} dt\right]$$

$$\mu = \frac{1}{\sqrt{\pi}}\left[2\mu\int_0^\infty e^{-t^2}\,dt + 0\right] \quad \left(\because te^{-t^2} \text{ is odd function}\right)$$

$$= \frac{2\mu}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \mu$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x^2 e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\,dx$$

let $\frac{x-\mu}{\sqrt{2}\sigma} = t \Rightarrow dx = \sqrt{2}\sigma\,dt$

$$E(x^2) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\mu+\sqrt{2}\sigma t)^2 e^{-t^2}\,dt$$

$$= \frac{1}{\sqrt{\pi}}\left[\mu^2\int_{-\infty}^{\infty} e^{-t^2}\,dt + 2\sigma^2\int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt + 2\sqrt{2}\mu\sigma\underbrace{\cancel{\int_{-\infty}^{\infty} te^{-t^2}\,dt}}_{\text{odd function}}\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[2\mu^2\int_0^\infty e^{-t^2}\,dt + 4\sigma^2\int_0^\infty t^2 e^{-t^2}\,dt\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[2\mu^2\left(\frac{\sqrt{\pi}}{2}\right) + 4\sigma^2\left(\frac{\sqrt{\pi}}{4}\right)\right]$$

$$\boxed{\begin{aligned} &t^2 = s \\ &2t\,dt = ds \\ &\int_0^\infty \frac{e^{-s}}{2}\cdot s^{1/2}\,ds \\ &= \frac{1}{2}\Gamma\tfrac{3}{2} = \frac{\sqrt{\pi}}{4}\end{aligned}}$$

$$E(x^2) = \mu^2 + \sigma^2$$

$$\therefore \text{Variance}(x) = E(x^2) - (E(x))^2$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$= \sigma^2$$

We have $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}; \; -\infty < x < \infty$

$$f'(x) = -2\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\frac{1}{\sqrt{2}\sigma}\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

$f'(x) = -\left(\frac{x-\mu}{\sigma^2}\right) f(x)$

for critical points, $f'(x) = 0$

$\Rightarrow -f(x)\left(\frac{x-\mu}{\sigma^2}\right) = 0$

$\Rightarrow x = \mu \qquad (\because f(x) \neq 0)$

$f''(x) = -\frac{1}{\sigma^2}\left[f(x) + f'(x)(x-\mu)\right]$

$f''(\mu) = -\frac{1}{\sigma^2} f(\mu)$

$= -\frac{1}{\sqrt{2\pi}\,\sigma^3} < 0$

$\Rightarrow f$ has local maxima at $x = \mu$

and $f(\mu) = \frac{1}{\sqrt{2\pi}\cdot\sigma}$

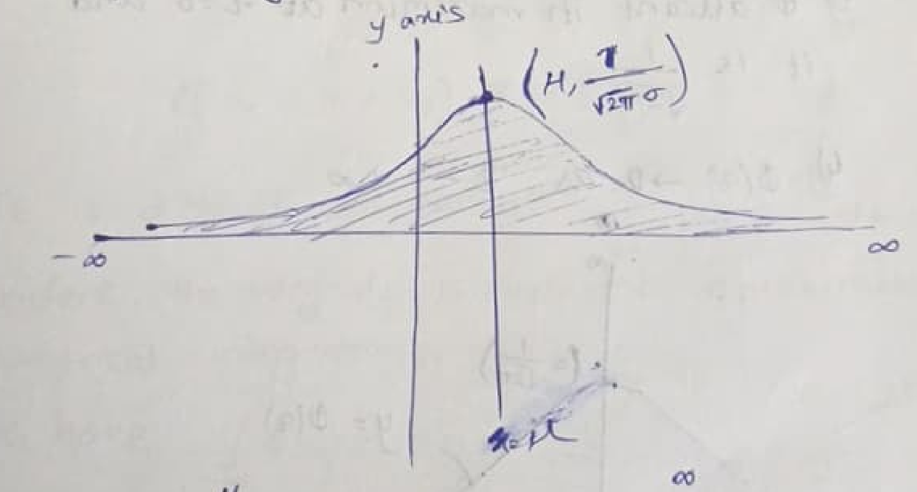
Further $f(x) \to 0$ as $x \to \pm\infty$

Let $t = \frac{x-\mu}{\sqrt{2}\,\sigma}$

Then $f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-t^2} \qquad -\infty < t < \infty$

Clearly $f(-t) = f(t)$. Therefore, the graph of f is symmetric about $t = 0$

$\Rightarrow f$ is symmetric about $\boxed{x = \mu}$



Clearly, $\int_{-\infty}^{\mu} f(x)\,dx = \frac{1}{2}$ and $\int_{\mu}^{\infty} f(x)\,dx = \frac{1}{2}$

21/11/24

Standard Normal Distribution:

X is a continuous RV

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}, \quad -\infty < x < \infty$$

Mean $(X) = \mu$ and Variance $(X) = \sigma^2$

In this case, we write it as $X \sim N(\mu; \sigma^2)$

If $\sigma = 1$ and $\mu = 0$, then normal distribution is called Standard Normal distribution and is denoted by $X \sim N(0,1)$

That is, we say that a RV Z follows standard normal distribution if its PDF is given by

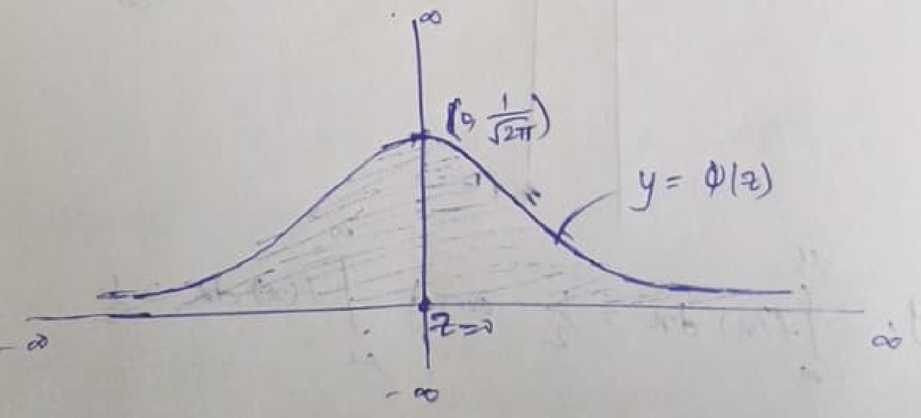
$$\phi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}, \quad z \in (-\infty, \infty)$$

Clearly 1) $\int_{-\infty}^{\infty} \phi(z)\, dz = 1$

2) $\phi(z) = \phi(-z)$, that is ϕ is symmetric about $z = 0$

3) ϕ attains its maximum at $z = 0$ and it is $\frac{1}{\sqrt{2\pi}}$.

4) $\phi(z) \to 0$ as $z \to \pm\infty$



Suppose that the RV X follows Normal distribution that is $X \sim N(\mu, \sigma)$. Then its PDF is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}, \quad -\infty < x < \infty$$

then $P(x_1 < X < x_2) = P(x_1 - \mu < X - \mu < x_2 - \mu)$

$$= P\left(\frac{x_1-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{x_2-\mu}{\sigma}\right)$$

Take $Z = \frac{X-\mu}{\sigma}$, we get

$P(x_1 < X < x_2) = P(z_1 < Z < z_2)$. where

$$z_1 = \frac{x_1-\mu}{\sigma}, \quad z_2 = \frac{x_2-\mu}{\sigma}$$

Also, $P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)\,dx$

$$= \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

Take $\frac{x-\mu}{\sigma} = z$

$dx = \sigma dz$

$$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2}{2}} \cdot \sigma\, dz$$

$$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{z_1}^{z_2} \phi(z)\,dz$$

$$P(x_1 < X < x_2) = \underline{P(z_1 < Z < z_2)}$$

It is difficult to evaluate $\int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = P(z_1 < Z < z_2)$

therefore, the integral is evaluated approximately by numerical integration.

We have $\int_{-\infty}^{\infty} \phi(z)\,dz = 1 = P(-\infty < Z < \infty)$

$P(-\infty < Z < 0) = \frac{1}{2}$ and $P(0 < Z < \infty) = \frac{1}{2}$

$P(-3 < Z < 0) = P(0 < Z < 3)$

In general, if we have $P(0 < z < k)$ for every $k > 0$, then we get $P(-k < z < 0)$ by default

Also, $P(0 < z < k) = \int_0^k \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_0^k \phi(z)\, dz$

z	.00	.01	.02
0.0	.0000	.004	0.008
0.1	0.0398	0.0438	
0.2	0.0793	0.0832	

0.201

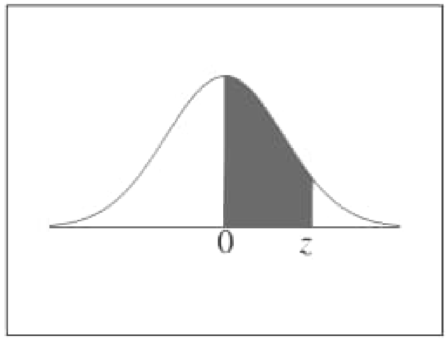
$P(0 < z < 0.1) = 0.0398$

$P(0 < z < 0.201) = 0.0832$

$P(0 < z < \underbrace{0.31}_{0.3+0.01}) = 0.1217$

$P(0 < z < \underbrace{0.74}_{0.7+0.04}) = 0.2704$

Standard Normal Distribution Table



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

22/11/24

Pb: It is given that a continuous RV x follows normal distribution with $\sigma = 0.05$ and $P(x < 6) = 0.03$. Then find $P(0 < x < 100)$.

Sol:

$$P(0 < x < 100) = P(0-\mu < x-\mu < 100-\mu)$$

$$= P\left(\frac{0-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{100-\mu}{\sigma}\right)$$

$$= P\left(\frac{-\mu}{\sigma} < Z < \frac{100-\mu}{\sigma}\right)$$

we need μ

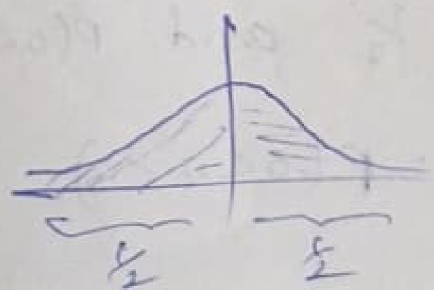
It is given that $P(x < 6) = 0.03$

$$\Rightarrow P(-\infty < x < 6) = 0.03$$

$$\Rightarrow P\left(\frac{-\infty-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right) = 0.03$$

$$\Rightarrow P\left(-\infty < Z < \frac{6-\mu}{\sigma}\right) = 0.03$$

Generally -



Here probability is less than $\frac{1}{2}$

$\Rightarrow \frac{6-\mu}{\sigma}$ is negative

$$P\left(-\infty < z < \frac{6-\mu}{\sigma}\right) = P\left(\frac{\mu-6}{\sigma} < z < \infty\right) = 0.03$$

$$\frac{1}{2} - P\left(0 < z < \frac{\mu-6}{\sigma}\right) = 0.03$$

$$\Rightarrow P\left(0 < z < \frac{\mu-6}{\sigma}\right) = 0.47$$

$$\Rightarrow \frac{\mu-6}{\sigma} = 1.88$$

$$\underline{\underline{\mu = 6.094}}$$

$$P(0 < x < 100) = P\left(\frac{-\mu}{\sigma} < z < \frac{100-\mu}{\sigma}\right)$$

$$= P(-121.88 < z < 1878.12)$$

$$= P(0 < z < 121.88) + P(0 < z < 1878.12)$$

$$\approx \frac{1}{2} + \frac{1}{2} \approx \boxed{1}$$

Q6. The marks obtained by students in a exam are approximately normally distributed with mean 65 and standard deviation 5. If 3 students are selected randomly, then what is the probability that atleast one of them would scored above 75 marks in the exam.

Sol. $\sigma = 5$, $\mu = 65$

Let x denote the marks obtained by students. Then, it is given that the RV x follows normal distribution.

$$P\left(\begin{array}{l}\text{A student scores more} \\ \text{than 75 marks}\end{array}\right) = P(x > 75)$$

$$= P\left(\frac{x-\mu}{\sigma} > \frac{75-\mu}{\sigma}\right)$$

$$= P\left(z > \frac{10}{5}\right) \quad \left(\because \begin{array}{l}\mu = 65 \\ \sigma = 5\end{array}\right)$$

$$= P(z > 2)$$

$$= \frac{1}{2} - P(0 < z < 2)$$

$$P\begin{bmatrix} \text{A student scores} \\ \text{more than} \\ \text{75 marks} \end{bmatrix} = \frac{1}{2} - 0.4772 = 0.0228$$

Let Y denote the number of students scored more than 75 marks out of 3 students

$\therefore$ Y takes 0, 1, 2, 3. Also $n = 3$

$\therefore P(Y = y) = {}^3C_y \cdot p^y \cdot q^{3-y}$, $p = 0.0228$, $q = 0.9772$

$$P(Y \geq 1) = 1 - P(Y = 0)$$

$$= 1 - ({}^3C_0\, p^0 \cdot q^3)$$

$$= 1 - (0.9772)^3$$

$$\underline{P(Y \geq 1) = 0.0668}$$

Pb: In an engineering exam, a student is considered to have failed, secured second class, first class, and distinction according to he/she scores, less than 45%, between 45% and 60%, between 60% and 75%, and above 75% respectively. In a particular year 10% of students failed in the exam and 5% of students ~~who have~~ got distinction. Find the % of students who have got first class and second class. Assume that marks follows normal distribution.

Sol:

Let x denote the percentage of marks obtained by students

$P(X < 45) = 0.1$ and $P(X > 75) = 0.05$

$\Rightarrow P\left(\frac{X-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right) = 0.1$ & $P\left(\frac{X-\mu}{\sigma} > \frac{75-\mu}{\sigma}\right) = 0.05$

$\Rightarrow P\left(z < \frac{45-\mu}{\sigma}\right) = 0.1$ & $P\left(z > \frac{75-\mu}{\sigma}\right) = 0.05$

$\Rightarrow P\left(-\infty < z < \frac{45-\mu}{\sigma}\right)$ $< \frac{1}{2}$ $\Downarrow$ $\frac{45-\mu}{\sigma} \to -ve$; $P\left(\frac{75-\mu}{\sigma} < z < \infty\right)$ $< \frac{1}{2}$ $\Downarrow$ $\frac{75-\mu}{\sigma} \to +ve$

$\Rightarrow \underbrace{P\left(\frac{\mu-45}{\sigma} < z < \infty\right)}_{\frac{1}{2} - P\left(0 < z < \frac{\mu-45}{\sigma}\right)} = 0.1$ & $\frac{1}{2} - P\left(0 < z < \frac{75-\mu}{\sigma}\right) = 0.05$

$P\left(0 < z < \frac{\mu-45}{\sigma}\right) = 0.4$ & $P\left(0 < z - \frac{75-\mu}{\sigma}\right) = 0.45$

$\frac{\mu-45}{\sigma} = 1.28$ and $\frac{75-\mu}{\sigma} = 1.65$.

$\mu - 45 = 1.28\sigma$ and $75 - \mu = 1.65\sigma$

$\Rightarrow \sigma = \frac{30}{293} = 10.24$; $\mu = 58.1$

$P(45 < x < 60) = \cancel{P(0<x<60)} - \cancel{P(0<x<40)}$

second class $= P\left(\frac{45-\mu}{\sigma} < z < \frac{60-\mu}{\sigma}\right)$

$= P(-1.279 < z < 0.185)$

$= P(0 < z < 0.185) + P(0 < z < 1.279)$

$= 0.0714 + 0.3997$

$= 0.4711$

$P(60 < x < 75) = P\left(\frac{60-\mu}{\sigma} < z < \frac{75-\mu}{\sigma}\right)$

first class

$= P(0.185 < z < 1.65)$

$= P(0 < z < 1.65) - P(0 < z < 0.185)$

$= 0.4505 - 0.0714 = 0.3791$

∴ The % of students scored first class = 37.9%.

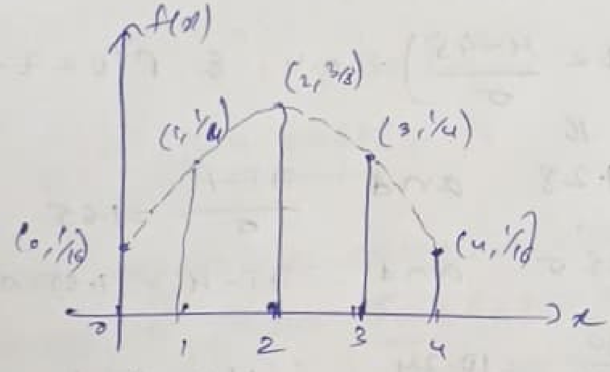
% of students scored second class = 47.1%.

Graphical Representation of a probability Distribution

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$\text{Mean }(\bar{x}) = 0\times\frac{1}{16} + 1\times\frac{1}{4} + 2\times\frac{3}{8} + 3\times\frac{1}{4} + 4\times\frac{1}{16}$$

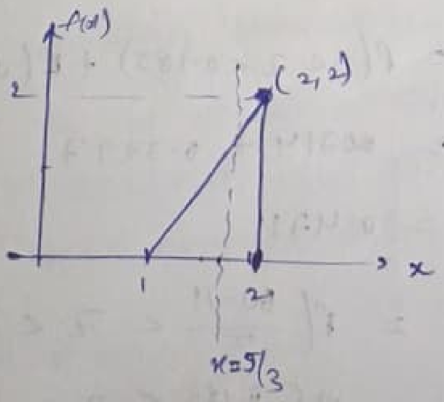
$$= 2$$



symmetric

*Let $f(x) = \begin{cases} 2(x-1) & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

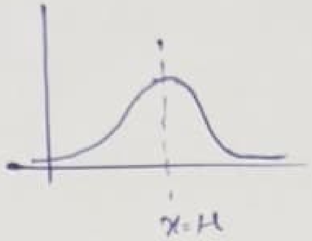
$$\text{Mean} = 2\int_1^2 (x^2 - x)\,dx = 2\left(\frac{7}{3} - \frac{3}{2}\right) = \frac{14}{3} - 3 = \frac{5}{3}$$



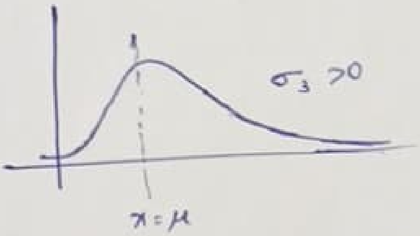
Asymmetric

Skewness : The coefficient of skewness of a PDF $f(x)$ is given by $\sigma_3 = \dfrac{E(x-\mu)^3}{\sigma^3}$

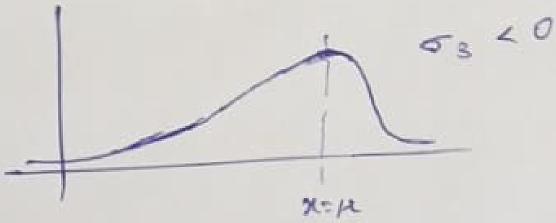
If $\sigma_3 = 0$; then the PDF is symmetric about the mean i.e



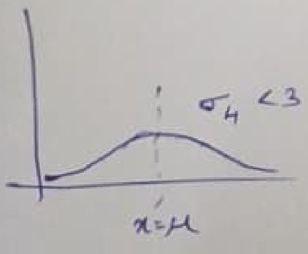
If $\sigma_3 > 0$, then PDF is skewed right side with respect to mean μ.



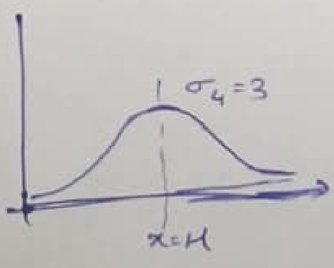
If $\sigma_3 < 0$ then PDF is skewed left side with respect to the mean μ.



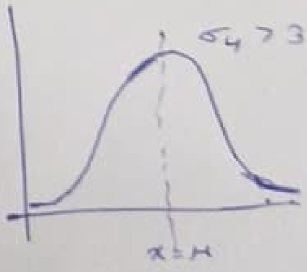
Kurtosis : The coefficient of kurtosis of a PDF $f(x)$ is given by $\sigma_4 = \dfrac{E(x-\mu)^4}{\sigma^4}$



Platykurtic



Mesokurtic



Leptokurtic

25/11/24

Pb: The probability function of a discrete RV x is given by

$$f(x) = \begin{cases} 2^{-x} & \text{if } x = 1, 2, 3, 4, \dots \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of the RV $U = x^4 + 1$

Sol: Suppose that the new RV U takes the value u

$\therefore u = x^4 + 1 \Rightarrow x = (u-1)^{1/4}$.

The PDF $g(u)$ of the RV U is given by

$$g(u) = \begin{cases} 2^{-(u-1)^{1/4}} & \text{if } u = 2, 17, 82, \dots \\ 0 & \text{otherwise} \end{cases}$$

Pb: The PDF of the continuous RV X is given by

$$f(x) = \begin{cases} \frac{x^2}{81} & -3 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Find the PDF of the RV $U = \frac{1}{3}(12 - x)$

Sol: Suppose that u and x are values taken by U and X respectively. Then, we get

$$u = \frac{1}{3}(12 - x)$$

$$\Rightarrow x = 12 - 3u$$

$\therefore$ The PDF $g(u)$ of U is given by

$$g(u) = f(x(u)) \left| \frac{dx}{du} \right|$$

$$\Rightarrow g(u) = \frac{(12-3u)^2}{81} |(-3)|$$

$$= \frac{(4-u)^2}{3} \quad \therefore$$

Since $-3 < x < 6$, we get $-3 < 12 - 3u < 6$

$$\Rightarrow 2 < u < 5$$

$$\therefore\ g(u) = \begin{cases} \dfrac{(4-u)^2}{3} & \text{if } 2 < u < 5 \\ 0 & \text{otherwise} \end{cases}$$

clearly $\int_{-\infty}^{\infty} g(u)\,du = \int_2^5 \frac{(4-u)^2}{3}\,du = \frac{1}{3}\left[\frac{(4-u)^3}{-3}\right]_2^5$

$$= \frac{1}{9}(9) = 1$$

Pb: The joint PDF of the RV's X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{xy}{96} & \text{if } 0 < x < 4,\ 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint PDF of the RV's $U = XY^2$, $V = X^2Y$

Sol: Let x and y be the values taken by the RV's X and Y respectively

Let u and v be the values taken by the RV's U and V respectively

$\therefore\ u = xy^2,\ v = x^2y$

$\Rightarrow x = \frac{u}{y^2} \Rightarrow v = \frac{u^2}{y^4}y \Rightarrow y = \left(\frac{u^2}{v}\right)^{1/3} = u^{2/3}v^{-1/3}$

$x^2 = \frac{v}{y} = \frac{v^{4/3}}{u^{2/3}} = \left(\frac{v^2}{u}\right)^{2/3}$

$\Rightarrow x = \left(\frac{v^2}{u}\right)^{1/3} = v^{2/3}u^{-1/3}$

$u = \phi_1(x,y)$ $\quad v = \phi_2(x,y)$

$x = \psi_1(u,v)$ $\quad y = \psi_2(u,v)$

$g(u,v) = f(x(u,v), y(u,v))\,|J(x,y)|$

$$= \frac{\left(\left(\frac{v^2}{u}\right)\left(\frac{u^2}{v}\right)\right)^{1/3}}{96} \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right|$$

$$= \frac{(uv)^{1/3}}{96} \left|\begin{vmatrix} -\frac{1}{3}u^{-4/3}v^{2/3} & \frac{2}{3}u^{-1/3}v^{-1/3} \\ \frac{2}{3}u^{-1/3}v^{-1/3} & -\frac{1}{3}u^{2/3}v^{-4/3} \end{vmatrix}\right|$$

$$g(u,v) = \frac{(uv)^{1/3}}{96} \left| \frac{1}{9}(uv)^{-2/3} - \frac{4}{9}(uv)^{-2/3} \right|$$

$$= \frac{(uv)^{1/3}}{96} \quad \frac{(uv)^{-2/3}}{3}$$

$$g(u,v) = \frac{(uv)^{-1/3}}{288}$$

Since $0 < x < 4$ and $1 < y < 5$ we get

$0 < v^{2/3} u^{-1/3} < 4$ and $1 < u^{2/3} v^{-1/3} < 5$

$v^{2/3} < 4u^{1/3}$ and $v^{1/3} < u^{2/3} < 5v^{1/3}$

$\Rightarrow$ $v^2 < 64u$ and $v < u^2 < 125v$

The joint PDF of U and V is given by

$$g(u,v) = \begin{cases} \frac{(uv)^{-1/3}}{288} & \text{if } v^2 < 64u \text{ and } v < u^2 < 125v \\ 0 & \text{otherwise} \end{cases}$$

Pb: If the joint PDF of the RVs X and Y is given by $f(x,y) = \begin{cases} \frac{xy}{96} & \text{for } 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$, then

Find the PDF of $U = X + 2Y$

Sol: Let us define a new RV V as $V = X$

$\therefore$ we get $u = x + 2y$, $v = x$

$\Rightarrow y = \frac{u-v}{2}$, $x = v$

$u = \phi_1(x,y)$ $\quad v = \phi_2(x,y)$

$x = \psi_1(u,v)$ $\quad y = \psi_2(u,v)$

$$g(u,v) = f(x(u,v), y(u,v)) \, |J(x,y)|$$

$$= \frac{v(u-v)}{2(96)} \left| \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| \right|$$

$$g(u,v) = \frac{v(u-v)}{192} \left| \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \right|$$

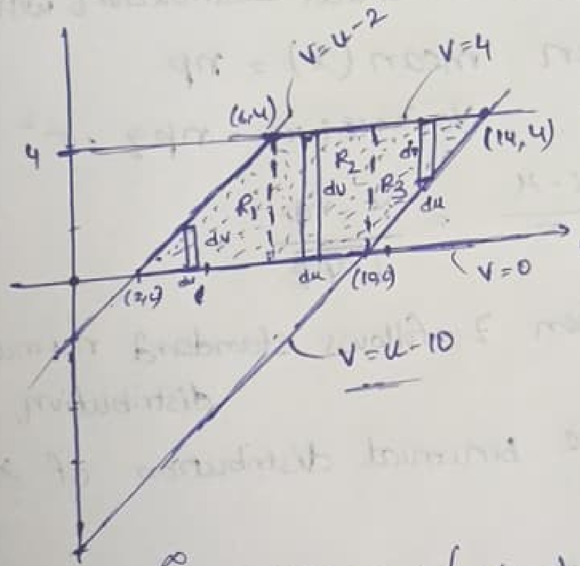
$$g(u,v) = \frac{v(u-v)}{384}$$

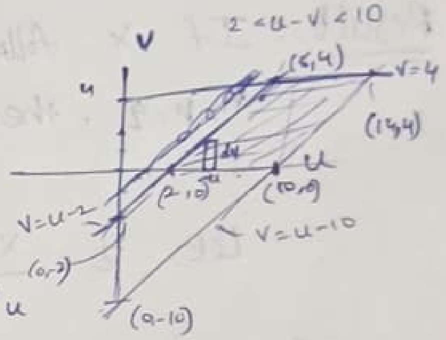
since $0 < x < 4$ and $1 < y < 5$ we get

$0 < v < 4$ and $1 < \frac{u-v}{2} < 5$

$\Rightarrow 2 < u - v < 10$

$$\therefore g(u,v) = \begin{cases} \frac{(u-v)v}{384} & \text{if } 2 < u-v < 10 \text{ and } 0 < v < 4 \\ 0 & \text{otherwise} \end{cases}$$





$$h(u) = \int_{-\infty}^{\infty} g(u,v)\,dv = \int_{R_1} g(u,v)\,dv + \int_{R_2} g(u,v)\,dv + \int_{R_3} g(u,v)\,dv$$

$$= \int_0^{u-2} \frac{uv - v^2}{384}\,dv + \int_0^4 \frac{uv - v^2}{384}\,dv + \int_{u-10}^{4} \frac{uv-v^2}{384}\,dv$$

$$= \frac{1}{384}\left[\left[\frac{uv^2}{2} - \frac{v^3}{3}\right]_0^{u-2} + \left[\frac{uv^2}{2} - \frac{v^3}{3}\right]_0^4 + \left[\frac{uv^2}{2} - \frac{v^3}{3}\right]_{u-10}^4 \right]$$

$$= \frac{1}{384}\left[\frac{u(u-2)^2}{2} - \frac{(u-2)^3}{3} + 8u - \frac{64}{3} + 8u - \frac{64}{3} - \frac{u(u-10)^2}{2} + \frac{(u-10)^3}{3} \right]$$

$$= \frac{1}{384}\left[\frac{u}{2}(2u-12)(8) + \frac{(u-10)^3-(u-2)^3}{3} + 16u - \frac{128}{3} \right]$$

$$= \frac{1}{384}\left[8u^2 - 48u - \frac{26}{3}u^2 + \frac{292}{3}u + 336 + 16u - \frac{128}{3} \right]$$

$$= \frac{1}{384}\left[-\frac{2}{3}u^2 + \frac{196}{3}u + \frac{880}{3} \right]$$

$$h(u) = \frac{-u^2+98u+440}{576}$$

Result: If x follows binomial distribution with n, p, q then mean$(x) = np$

Variance$(x) = npq = \sigma^2$

Let $z = \frac{x-\mu}{\sigma} = \frac{x-np}{\sqrt{npq}}$

If $n \to \infty$, then z follows standard normal distribution

Pb: Suppose that the binomial distribution of x is given by

$$P(x=x) = {}^{15}C_x (0.4)^x (0.6)^{15-x}, \quad x = 0,1,2,3,\ldots 15$$

$$P(x=4) = {}^{15}C_4 \cdot (0.4)^4 (0.6)^{11}$$
$$= 0.1268$$

$$P(3.5 < x < 4.5) = P\left(\frac{3.5-np}{\sqrt{npq}} < \frac{x-np}{\sqrt{npq}} < \frac{4.5-np}{\sqrt{npq}}\right)$$

$$= P\left(\frac{3.5-6}{1.897} < z < \frac{4.5-6}{1.897}\right)$$

$$= P(-1.32 < z < -0.79)$$

$$= P(0.79 < z < 1.32)$$

$$= P(0<z<1.32) - P(0<z<0.79)$$

$$= 0.1214$$