Department of Mathematics
Probability Theory (MAL-205)
Assignment on
Chebyshev's Inequality and Two Dimensional RVs

1. The number of items produced in a factory during a week is a random variable with mean 50 and standard deviation 5. What can be said about the probability that this week's production will be between 40 and 60?

2. The distribution of scores on an IQ test has mean 100 and standard deviation 16. Show that the probability of a student having an IQ above 148 or below 52 is at most 1/9.

3. Use Chebyshevs inequality to determine how many times a fair coin be tossed in order that the probability will be at least 0.9 that the ratio of the number of heads to the number of tosses will be between 0.4 and 0.6.

4. Let $F_{X,Y}(x,y) = \begin{cases} 1, & \text{if } x+2y \geq 1 \\ 0, & \text{if } x+2y \leq 1. \end{cases}$ Does it define a joint CDF in the plane?

5. Let $F_{X,Y}(x,y) = \begin{cases} 1-e^{-(x+y)}, & \text{if } x \geq 0,\ y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$
   Can this function be a joint CDF of a bivariate rv (X,Y)?

6. Let X be a rv that has the value {-1, 0, 1} depending upon whether a child in school is performing below, at, or above grade level, respectively. Let Y be rv that is equal to zero if a child comes from an impoverished family and equal to one otherwise. In a particular class, it is observed that 20% of the children come from impoverished families and performing below grade level, that 20% are from impoverished families and performing at grade level, and that 6% are from impoverished family and performing above grade level. Of the remaining children, half are performing at grade level and one third are performing above grade level. Construct a table of the joint pmf of X and Y. Compute marginal pmfs of both rvs. Are X and Y independent?

7. Rvs X and Y have joint pdf given by $f_{X,Y}(x,y) = \begin{cases} 2(x+y), & \text{if } 0 \leq x \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

   Find the conditional density functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

8. A bivariate rvs (X,Y) has the value $(x_i, y_j) = (i,j)$; $(i = 0,1,2,3;\ j = 1,2,3,4)$ and the joint probabilities are given by $p_{X,Y}(i,j) = P(X=i,\ Y=j) = C(3i+4j)$, where C is a constant. Find (i) the value of C, (ii) the marginal distributions of X and Y, (iii) $P(x \geq 2,\ Y \leq 3)$, $P(Y=2|X=3)$.

9. The joint pdf of bivariate rvs (X,Y) is given by $f_{X,Y}(x,y) = \begin{cases} K, & \text{if } 0 < y \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$

   Determine the constant K. Find the marginal density functions X and Y. Also compute $P(0 < X < 1/2,\ 0 < Y < 1/2)$.

10. A candy company distributes boxes of chocolates with a mixture of creams, toffees, and cordials. Suppose that the weight of each box is 1 kilogram, but the individual weights of the creams, toffees and cordials vary from box to box. For a randomly selected box, let X and Y represent the weights of the creams and the toffees, respectively and the joint pdf of bivariate rvs (X,Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} 24xy, & \text{if } 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1, \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the probability that in a given box the cordials account for more than 1/2 of the weight.
(ii) Find the marginal density of the weight of the creams.
(iii) Find the probability that the weight of the toffees in a box is less than 1/8 of a kilogram if it is known that creams constitute 3/4 of the weights.

11. Suppose that at two points in a room (or on a city or in the ocean) one measures the intensity of sound caused by general background noise. Let X and Y be rvs representing the intensity of sound at two points. Suppose the joint CDF of bivariate rvs (X, Y) is given by

$$F_{X,Y}(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)}, & \text{if } x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the marginal densities of X and Y. (ii) Are X and Y independent?
(iii) Compute $P(X \le 1,\ Y \le 1)$ and $P(X + Y \le 1)$.

12. Let X and Y be two rvs, each having spectrum $(-\infty,\ \infty)$. If the conditional density function of X on the hypothesis $Y = y$ is $\frac{|y|e^{-x^2y^2}}{\sqrt{\pi}}$ and density function of Y is $\frac{\lambda e^{-\lambda^2 y^2}}{\sqrt{\pi}}$, then prove that the density function of X is $\frac{\lambda}{\pi(x^2+\lambda^2)}$.