Assignment - 2

1) $x_{n+1} = p x_n + \left(\frac{qa}{x_n^2}\right) + \left(\frac{ra^2}{x_n^5}\right)$

$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(p x_n + \frac{qa}{x_n^2} + \frac{ra^2}{x_n^5}\right)$

$\Rightarrow \quad l = pl + \frac{qa}{l^2} + \frac{ra^2}{l^5}$

Given, $l = a^{1/3}$ ~~$=\alpha$~~

$a^{1/3} = p a^{1/3} + \frac{qa}{(a^{1/3})^2} + \frac{ra^2}{(a^{1/3})^5}$

$\Rightarrow (1-p)\cancel{a^{1/3}} = q\cancel{a^{1/3}} + r\cancel{a^{1/3}}$

$\therefore \quad p+q+r = 1 \quad - (1)$

$x - a^{1/3} = 0$

$\Rightarrow x^3 - a = 0$

$\Rightarrow \alpha^3 - a = 0$

$\Rightarrow a = \alpha^3$ [$\because \alpha$ is exact root]

Lets, $x_n - \alpha = e_n, \quad x_{n+1} - \alpha = e_{n+1}$

$\Rightarrow x_n = \alpha + e_n, \quad x_{n+1} = \alpha + e_{n+1}$

$\alpha + e_{n+1} = p(\alpha + e_n) + qa(\alpha + e_n)^{-2} + ra^2(\alpha + e_n)^{-5}$

$\Rightarrow \alpha + e_{n+1} = p(\alpha + e_n) + \frac{qa}{\alpha^2}\left(1 + \frac{e_n}{\alpha}\right)^{-2} + \frac{ra^2}{\alpha^5}\left(1 + \frac{e_n}{\alpha}\right)^{-5}$

$\Rightarrow \alpha + e_{n+1} = p\alpha + pe_n + \left(\frac{qa}{\alpha^2}\right)\left(1 - 2\left(\frac{e_n}{\alpha}\right) + 3\left(\frac{e_n}{\alpha}\right)^2 - \dots\right)$

$+ \left(\frac{ra^2}{\alpha^5}\right)\left(1 - 5\left(\frac{e_n}{\alpha}\right) + 15\left(\frac{e_n}{\alpha}\right)^2 - \dots\right)$

Put $a = \alpha^3$

$\Rightarrow \alpha + e_{n+1} = p\alpha + pe_n + q\alpha\left(1 - 2\left(\frac{e_n}{\alpha}\right) + 3\left(\frac{e_n}{\alpha}\right)^2 - \dots\right)$

$+ (r\alpha)\left(1 - 5\frac{e_n}{\alpha} + 15\left(\frac{e_n}{\alpha}\right)^2 - \dots\right)$

$\Rightarrow \alpha + e_{n+1} = (p\alpha + q\alpha + r\alpha) + \left(p - \frac{2}{\cancel{\alpha}}q\cancel{\alpha} - \frac{5r\cancel{\alpha}}{\cancel{\alpha}}\right)e_n$

$+ \left(q\cancel{\alpha}\cdot\frac{3}{\alpha^2} + r\cancel{\alpha}\cdot\frac{15}{\alpha^2}\right)e_n^2 + \text{term}(e_n^3, \dots)$

$\Rightarrow \alpha + e_{n+1} = (p+q+r)\alpha + (p - 2q - 5r)e_n$

$+ \left(\frac{3q}{\alpha} + \frac{15r}{\alpha}\right)e_n^2 + \text{term}(e_n^3, e_n^4 \dots)$

Comparing both side
We get $\quad p + q + r = 1$

$p - 2q - 5r = 0 \quad - (\text{II})$

$\frac{3q}{\alpha} + \frac{15r}{\alpha} = 0 \Rightarrow q = -5r \quad - (\text{III})$

~~Sum~~ (I) − (II)

$3q + 6r = 1$

Put (III) in this

$-15r + 6r = 1$

$\Rightarrow \quad r = -1/9$

$q = 5/9$

$p = 5/9$

$\phi'(4) = 0, \ \phi''(4) = 0, \ \phi'''(4) = 3$

$\phi''''(4) = 1.5, \ \phi'''''(4) = 0$

Coefficient of $e_n^3 = -q\alpha \cdot \frac{4}{\alpha^{2}} - r\alpha \cdot \frac{35}{\alpha^{2}}$

$= \frac{-4q}{\alpha^2} - \frac{35r}{\alpha^2}$

So, $\frac{e_{n+1}}{e_n^3} = \frac{1}{\alpha^2}(-4q - 35r)$.

$\Rightarrow \frac{e_{n+1}}{e_n^3} = \frac{1}{\alpha^2}\left(\frac{-20}{9} + \frac{35}{9}\right) = \frac{5}{3\alpha^2}$

$\therefore e_{n+1} = \left(\frac{5}{3\alpha^2}\right) e_n^3$ (order = 3)

2) $\{x_n\}_1^\infty$

$x_0 = 5$

$x_{n+1} = \frac{1}{16} x_n^4 - \frac{1}{2} x_n^3 + 8x_n - 12$

$x_{k+1} - \phi(\alpha) = \phi(x_k) - \phi(\alpha)$

$x_1 = 4.5625$ $\quad = \phi(x_k - \alpha + \alpha) - \phi(\alpha)$

$x_2 = 4.0952$ $\quad = \cancel{\phi(\alpha)} + ((x_k - \alpha)\phi'(\alpha) \dots) - \cancel{\phi(\alpha)}$

$x_3 = 4.0004$ $\quad = (x_k - \alpha)\phi'(\alpha) + \frac{(x_k - \alpha)^2}{2!}\phi''(\alpha) + \dots$

$x_4 = 4$ (Convergence to $\alpha = 4$)

$\left\{ \frac{x_{k+1} - \alpha}{(x_k - \alpha)^3} = \frac{3}{3!} = \frac{1}{2} \right\}$

3) $x^2 y + y^3 = 10$, $\quad xy^2 - x^2 = 3 \Rightarrow y = \sqrt{\frac{3}{x} - x}$

$$\begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} & \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} \\ \left.\frac{\partial g}{\partial x}\right|_{(x_0, y_0)} & \left.\frac{\partial g}{\partial y}\right|_{(x_0, y_0)} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = -\begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 + h \\ y_0 + k \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

$f(x,y) \equiv x^2y + y^3 - 10 = 0$
$g(x,y) \equiv xy^2 - x^2 - 3 = 0$

$\frac{\partial f}{\partial x} = 2xy \quad , \quad \frac{\partial f}{\partial y} = x^2 + 3y^2$

$\frac{\partial g}{\partial x} = y^2 - 2x \quad , \quad \frac{\partial g}{\partial y} = 2xy$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} 2xy & x^2+3y^2 \\ y^2-2x & 2xy \end{bmatrix}^{-1}_{(x_0,y_0)} \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} 2xy & x^2+3y^2 \\ y^2-2x & 2xy \end{bmatrix}^{-1}_{(x_0,y_0)} \begin{bmatrix} x^2y+y^3-10 \\ xy^2-x^2-3 \end{bmatrix}_{(x_0,y_0)}$$

$(x_0, y_0) \equiv (0.8, 2.2)$

$(x_1, y_1) \equiv (0.901, 2.041)$

$(x_2, y_2) \equiv (0.977, 2.008)$

$\therefore \quad x = 1$ and $y = 2$

4) $f(x,y) \equiv \ln(x^2+y) - 1 + y = 0$
$g(x,y) \equiv \sqrt{x} + xy = 0$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} & \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \\ \left.\frac{\partial g}{\partial x}\right|_{(x_0,y_0)} & \left.\frac{\partial g}{\partial y}\right|_{(x_0,y_0)} \end{bmatrix}^{-1} \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y}, \quad \frac{\partial f}{\partial y} = \cancel{2x}\ \frac{1}{x^2+y} + 1$$

$$\frac{\partial g}{\partial x} = \frac{1}{2\sqrt{x}} + y, \quad \frac{\partial g}{\partial y} = x$$

$$(x_0, y_0) \equiv (2.4, -0.6)$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{2x}{x^2+y} & \frac{1}{x^2+y}+1 \\ \frac{1}{2\sqrt{x}}+y & x \end{bmatrix}\Bigg|_{(x_0,y_0)} \begin{bmatrix} \ln(x^2+y)-1+y \\ \sqrt{x}+xy \end{bmatrix}_{(x_0,y_0)}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.4125 \\ -0.644 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.4122 \\ -0.643 \end{bmatrix}$$

5) $f(x,y) \equiv y\cos(xy) + 1 = 0$
$g(x,y) \equiv \sin(xy) + x - y = 0$

$$\frac{\partial f}{\partial x} = -y^2\sin(xy), \quad \frac{\partial f}{\partial y} = -xy\sin(xy) + \cos xy$$

$$\frac{\partial g}{\partial x} = y\cos(xy)+1, \quad \frac{\partial g}{\partial y} = x\cos(xy) - 1$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \\ \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} \end{bmatrix}^{-1} \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix}$$

$$(x_0, y_0) \equiv (1, 2)$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} -y^2\sin(xy) & \cos(xy) - xy\sin(xy) \\ y\cos(xy)+1 & x\cos(xy) - 1 \end{bmatrix}^{-1}\Bigg|_{(x_0,y_0)} \times \begin{bmatrix} y\cos(xy)+1 \\ \sin(xy)+x-y \end{bmatrix}\Bigg|_{(x_0,y_0)}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.0796 \\ 1.9453 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.0861 \\ 1.9437 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1.0861 \\ 1.9436 \end{bmatrix}$$

6) $f(x,y) \equiv x^2 + y^2 - 1.12 = 0$

$g(x,y) \equiv xy - 0.23 = 0$

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$

$\frac{\partial g}{\partial x} = y$, $\frac{\partial g}{\partial y} = x$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} 2x_0 & 2y_0 \\ y_0 & x_0 \end{bmatrix}^{-1} \begin{bmatrix} x_0^2 + y_0^2 - 1.12 \\ x_0 y_0 - 0.23 \end{bmatrix}$$

$$(x_0, y_0) \equiv (1, 1)$$

7) $f(x, y, z) \equiv x^3 + y^3 - 53 = 0$

$g(x, y, z) \equiv 2y^3 + z^4 - 69 = 0$

$p(x, y, z) \equiv 3x^5 + 10z^2 - 770 = 0$

$\frac{\partial f}{\partial x} = 3x^2$, $\frac{\partial f}{\partial y} = 3y^2$, $\frac{\partial f}{\partial z} = 0$

$\frac{\partial g}{\partial x} = 0$, $\frac{\partial g}{\partial y} = 6y^2$, $\frac{\partial g}{\partial z} = 4z^3$

$\frac{\partial p}{\partial x} = 15x^4$, $\frac{\partial p}{\partial y} = 0$, $\frac{\partial p}{\partial z} = 20z$

$$X_1 = X_0 - J^{-1} Y$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - \left.\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \end{bmatrix}^{-1}\right|_{(x_0, y_0, z_0)} \cdot \begin{bmatrix} f(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) \\ p(x_0, y_0, z_0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} 3x_0^2 & 3y_0^2 & 0 \\ 0 & 6y_0^2 & 4z_0^3 \\ 15x_0^4 & 0 & 20z_0 \end{bmatrix} \cdot \begin{bmatrix} x_0^3 + y_0^3 - 53 \\ 2y_0^3 + z_0^4 - 69 \\ 3x_0^5 + 10z_0^2 - 770 \end{bmatrix}$$

$(x_0, y_0, z_0) \equiv (3, 3, 2)$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 2.9998 \\ 2.9631 \\ 2.0309 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2.998 \\ 2.9626 \\ 2.0302 \end{bmatrix}$$

$\therefore$ $x = 2.998$

$y = 2.963$

$z = 2.03$

8) (i) $8x_2 + 2x_3 = -7$

$3x_1 + 5x_2 + 2x_3 = 8$

$6x_1 + 2x_2 + 8x_3 = 26$

$$\begin{bmatrix} 0 & 8 & 2 \\ 3 & 5 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \\ 26 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$= \left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$= \left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & 6 & 3 \end{array}\right]$$

$6x_3 = 3 \Rightarrow x_3 = \frac{1}{2}$

$8x_2 + 2x_3 = -7 \Rightarrow 8x_2 = -8 \Rightarrow x_2 = -1$

$3x_1 - 5 + 1 = 8 \Rightarrow x_1 = 4$

(ii) $6x_2 + 13x_3 = 61$
$6x_1 - 8x_3 = -38$
$13x_1 - 8x_2 = 79$

$$\begin{bmatrix} 0 & 6 & 13 \\ 6 & 0 & -8 \\ 13 & -8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 61 \\ -38 \\ 79 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 6 & 13 & 61 \\ 6 & 0 & -8 & -38 \\ 13 & -8 & 0 & 79 \end{array}\right]$$

$R_3 \leftrightarrow R_1$

$$= \left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 6 & 0 & -8 & -38 \\ 0 & 6 & 13 & 61 \end{array}\right]$$

$R_2 \rightarrow R_2 - \frac{6}{13} R_1$

$$= \left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 0 & 8\times\frac{6}{13} & -8 & -38-\frac{6\times 79}{13} \\ 0 & 6 & 13 & 61 \end{array}\right]$$

$R_2 \rightarrow 13R_2$

$$= \left[\begin{array}{ccc|c} 13 & -8 & 0 & 79 \\ 0 & 48 & -104 & -968 \\ 0 & 6 & 13 & 61 \end{array}\right]$$

$R_3 \rightarrow R_3 - \frac{1}{8} R_2$

$$= \begin{bmatrix} 13 & -8 & 0 & 79 \\ 0 & 48 & -104 & -968 \\ 0 & 0 & 26 & 182 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{8} R_2$

$$= \begin{bmatrix} 13 & -8 & 0 & 79 \\ 0 & 6 & -13 & -121 \\ 0 & 0 & 26 & 182 \end{bmatrix}$$

$$26x_3 = 182 \Rightarrow x_3 = 7$$

$$6x_2 = -121 + 13\times 7$$

$$\therefore \quad x_2 = -5$$

$$13x_1 + 8\times 5 = 79$$

$$\Rightarrow \quad x_1 = 3$$

(iii)

$$10x_1 - x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 8$$
$$2x_1 + 3x_2 + 20x_3 = 7$$

$$\begin{bmatrix} 10 & -1 & 2 \\ 1 & 10 & -1 \\ 2 & 3 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 1 & 10 & -1 & 8 \\ 2 & 3 & 20 & 7 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$= \left[\begin{array}{ccc|c} 1 & 10 & -1 & 8 \\ 10 & -1 & 2 & 4 \\ 2 & 3 & 20 & 7 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 10R_1$

$R_3 \longrightarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 10 & -1 & 8 \\ 0 & -101 & 12 & -76 \\ 0 & -17 & 22 & -9 \end{array}\right]$$

Doolittle's method $A = LU$ with L's diagonal as 1

Crout's method $A = LU$ with U's diagonal as 1.

$$\begin{bmatrix} 1 & 10 & -1 & 8 \\ 0 & -101 & 12 & -76 \\ 0 & 0 & 22-12\times\frac{17}{101} & \end{bmatrix}$$

$$-9 + 76 \times \frac{17}{101}$$

$$19.98x_3 = 3.792$$

$$\Rightarrow \underbrace{x_3 = 0.1898}$$

$$-101x_2 + 2.2776 = -76$$

$$\Rightarrow \underbrace{x_2 = 0.7750}$$

$$x_1 + 7.775 - 0.1898 = 8$$

$$\Rightarrow \underbrace{x_1 = 0.4148}$$

9) (i)
$$5x_1 + 4x_2 + x_3 = 3.4$$
$$10x_1 + 9x_2 + 4x_3 = 8.8$$
$$10x_1 + 13x_2 + 15x_3 = 19.2$$

$$\underbrace{\begin{bmatrix} 5 & 4 & 1 \\ 10 & 9 & 4 \\ 10 & 13 & 15 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 3.4 \\ 8.8 \\ 19.2 \end{bmatrix}}_{B}$$

$$A = LU$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 4 & 1 \\ 10 & 9 & 4 \\ 10 & 13 & 15 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$u_{11}=5$, $u_{12}=4$, $u_{13}=1$

$l_{21}=2$, $u_{22}=1$, $u_{23}=2$

$l_{31}=2$, $l_{32}=5$, $u_{33}=3$

$$\therefore \quad A = \begin{bmatrix} 5 & 4 & 1 \\ 10 & 9 & 4 \\ 10 & 13 & 15 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 5 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{U}$$

$$AX = B$$

$$\Rightarrow \underbrace{LUX}_{Y} = B$$

Lets, UX be $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$LY = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 8.8 \\ 19.2 \end{bmatrix}$$

$\therefore \quad y_1 = 3.4$

$2y_1 + y_2 = 8.8 \Rightarrow y_2 = 2$

$2y_1 + 5y_2 + y_3 = 19.2 \Rightarrow y_3 = 2.4$

$$UX = Y$$

$$\Rightarrow \begin{bmatrix} 5 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 2 \\ 2.4 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_3 &= 0.8 \\ x_2 &= 0.4 \\ x_1 &= 0.2 \end{aligned}\right\}$$

(ii) $x_1 + x_2 + x_3 = 1$

$4x_1 + 3x_2 - x_3 = 6$

$3x_1 + 5x_2 + 3x_3 = 4$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}}_{B}$$

$$AX = B$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{U}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$u_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$
$l_{21} = 4$, $u_{22} = -1$, $u_{23} = -5$
$l_{31} = 3$, $l_{32} = -2$, $u_{33} = -10$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

$$A = LU$$

$$AX = B$$
$$\Rightarrow LUX = B$$

Lets, $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$y_1 = 1$, $y_2 = 2$, $y_3 = 5$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\left.\begin{aligned} x_3 &= -0.5 \\ x_2 &= 0.5 \\ x_1 &= 1 \end{aligned}\right\}$$

10) (i) $x_1 - 4x_2 + 2x_3 = 81$
$-4x_1 + 25x_2 + 4x_3 = -153$
$2x_1 + 4x_2 + 15x_3 = 324$

$$\underbrace{\begin{bmatrix} 1 & -4 & 2 \\ -4 & 25 & 4 \\ 2 & 4 & 15 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 81 \\ -153 \\ 324 \end{bmatrix}}_{B}$$

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 25 & 4 \\ 2 & 4 & 15 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$\Rightarrow \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 25 & 4 \\ 2 & 4 & 15 \end{bmatrix}$$

$l_{11} = 1$, $u_{12} = -4$, $u_{13} = 2$
$l_{21} = -4$, $l_{22} = 9$, $l_{23} = 4/3$
$l_{31} = 2$, $l_{32} = 12$, $l_{33} = -5$

So,
$$\begin{bmatrix} 1 & -4 & 2 \\ -4 & 25 & 4 \\ 2 & 4 & 15 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 9 & 0 \\ 2 & 12 & -5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$A = LU$

$$AX = B$$
$$\Rightarrow LUX = B$$
Lets, ~~UX~~ UX be Y

$LY = B$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 9 & 0 \\ 2 & 12 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 81 \\ -153 \\ 324 \end{bmatrix}$$

$y_1 = 81$
$y_2 = 19$
$y_3 = 13.2$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 81 \\ 19 \\ 13.2 \end{bmatrix}$$

$$\left.\begin{aligned} x_3 &= 13.2 \\ x_2 &= 1.4 \\ x_1 &= 60.2 \end{aligned}\right\}$$

(ii) - $x_1 + x_2 + x_3 = 1$
$4x_1 + 3x_2 - x_3 = 6$
$3x_1 + 5x_2 + 3x_3 = 4$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}}_{B}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$l_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$

$l_{21} = 4$, $l_{22} = -1$, $u_{23} = 5$

$l_{31} = 3$, $l_{32} = 2$, $l_{33} = -10$

So, $$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$AX = B$

$\Rightarrow LUX = B$

Lets, UX be Y

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$y_1 = 1$; $y_2 = -2$, $y_3 = -0.5$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -0.5 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_3 &= -0.5 \\ x_2 &= 0.5 \\ x_1 &= 1 \end{aligned}\right\}$$

11) $x_1 + x_2 - x_3 = 2$
$2x_1 + 2x_2 + 5x_3 = -3$
$3x_1 + 2x_2 - 3x_3 = 6$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}}_{B}$$

$$\cancel{\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ 0 & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$l_{11} = 1$, $u_{12} = 1$, $u_{13} = -1$

$l_{21} = 2$, $l_{22} = 0$,

$x = \frac{1}{2}(y+1)$, $y = \frac{1}{2}(x+z)$, $z = \frac{1}{2}(y+w)$

$w = \frac{1}{2}(1+z)$

$x^{(0)} = (0,0,0,0)^T$

$x^{(1)} = 0.5$
$y^{(1)} = 0.25$
$z^{(1)} = 0.125$
$w^{(1)} = 0.5625$

$x^{(2)} = 0.625$
$y^{(2)} = 0.375$
$z^{(2)} = 0.46875$
$w^{(2)} = 0.734375$

$x^{(3)} = 0.6875$
$y^{(3)} = 0.578125$
$z^{(3)} = 0.65625$
$w^{(3)} = 0.828125$

16) (i) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

$B_1 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right)$

$\Rightarrow \theta = \frac{\pi}{4}$

$S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$$E_1 = S_1^T A S_1$$

$$= \begin{bmatrix} 5 & 2.8284 & 0 \\ 2.8284 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 5 & 2.8284 \\ 2.8284 & 5 \end{bmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right) = \frac{\pi}{4}$$

$$S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = S_2^T E_1 S_2$$

$$= \begin{bmatrix} 7.8284 & 0 & 0 \\ 0 & 2.1715 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigen values are 7.8284, 2.1715, 1

$$S_2 = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

(ii)

$$D = S_2^T E_1 S_2$$

$$\Rightarrow D = S_2^T (S_1^T A S_1) S_2$$

$$\Rightarrow D = (S_1 S_2)^T A (S_1 S_2)$$

$$P = S_1 S_2 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.5 & -0.7071 \\ 0.7071 & 0.7071 & 0 \\ 0.5 & -0.5 & 0.7071 \end{bmatrix}$$

$$X_1 \qquad X_2 \qquad X_3$$

(ii)

$$\begin{bmatrix} 2 & \sqrt{2} & 4 \\ \sqrt{2} & 6 & \sqrt{2} \\ 4 & \sqrt{2} & 2 \end{bmatrix} = A$$

$$B_1 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right) = \frac{\pi}{4}$$

$$S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$E_1 = S_1^T A S_1$$

$$= \begin{bmatrix} 6 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}, \quad S_2 = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta = \frac{\pi}{4}$$

$$E_2 = S_2^T E_1 S_2 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues $\rightarrow$ 8, 4, -2.

$$D = S_2^T E_1 S_2$$
$$\Rightarrow D = S_2^T S_1^T A S_1 S_2$$
$$\Rightarrow D = (S_1 S_2)^T A (S_1 S_2) \qquad \{\because (S_1 S_2)^T = S_2^T S_1^T\}$$

$$P = S_1 S_2 = \begin{bmatrix} 0.5 & -0.5 & -0.7071 \\ 0.7071 & 0.7071 & 0 \\ 0.5 & -0.5 & 0.7071 \end{bmatrix}$$
$$\qquad\qquad \underbrace{\quad}_{X_1} \quad \underbrace{\quad}_{X_2} \quad \underbrace{\quad}_{X_3}$$

(iii)

$A =$

$B_1 =$ [0 … a

$T =$

$\theta =$

$S_1 =$

$E_1 =$

$=$

$B_2 =$ [… 2.

$S_2 =$

(iii)

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = \frac{\pi}{4}$$

$$S_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = S_1^T A S_1$$

$$\cancel{= \begin{bmatrix} 0.8535 & 0.1464 & -0.5 \\ 0.1464 & 0.8535 & 0.5 \end{bmatrix}}$$

$$= \begin{bmatrix} 5 & 0 & 2.1213 \\ 0 & 1 & 0.7071 \\ 2.1213 & 0.7071 & 3 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 5 & 2.1213 \\ 2.1213 & 3 \end{bmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2 \times 2.1213}{5-3}\right) = 0.56514$$

$$S_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

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$$E_2 = S_2^T E_1 S_2$$

$$= \begin{bmatrix} 6.3452 & 0.3786 & 1.3\times10^{-5} \\ 0.3786 & 1 & 0.5971 \\ 1.3\times10^{-5} & 0.5971 & 1.6547 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 0.5971 \\ 0.5971 & 1.6547 \end{bmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\times0.5971}{1-1.6547}\right) = -0.534655$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$E_3 = S_3^T E_2 S_3$$

$$= \begin{bmatrix} 6.3452 & 0.3258 & 0.1929 \\ 0.3258 & 0.6463 & -1\times10^{-5} \\ 0.1929 & -1\times10^{-5} & 2.0084 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 6.3452 & 0.3258 \\ 0.3258 & 0.6463 \end{bmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\times0.3258}{6.3452-0.6463}\right) = 0.056921$$

$$E_4 = S_4^T E_3 S_4 = \begin{bmatrix} 6.3637 & 3.4\times10^{-5} & 0.1926 \\ 3.4\times10^{-5} & 0.6278 & -0.01 \\ 0.1926 & -0.01 & 2.0084 \end{bmatrix}$$