Visvesvaraya National Institute of Technology, Nagpur
Department of Mathematics

Assignment-4 **Subject: Probability Theory**

1. The joint PMF of a bivariate RV (X, Y) is given by
$$p(x,y) = \begin{cases} k(2x+y), & x = 1,2; y = 1,2 \\ 0, & \text{otherwise} \end{cases}$$
where k is a constant.
Find (a) the value of k, (b) the marginal PMF's of X and Y. (c) Are X and Y independent? (d) Find the conditional PMF's $P_{Y|X}(y|x)$ and $P_{X|Y}(x|y)$ (e) Find $P(Y = 2|X = 2)$ and $P(X = 2|Y = 2)$.

2. Consider an experiment of drawing randomly three balls from an urn containing two red, three white, and four blue balls. Let (X, Y) be a bivariate RV where X and Y denote, respectively, the number of red and white balls chosen.
(a) Find the joint PMF's of (X, Y). (b) Find the marginaI PMF's of X and Y. Are X and Y independent?
(c) Find the conditional PMF's $P_{Y|X}(y|x)$ and $P_{X|Y}(x|y)$
(d) Find $P(Y = 2|X = 2)$ and $P(X = 2|Y = 2)$.

3. The joint pmf of a bivariate RV (X, Y) is given by
$$p(x,y) = \begin{cases} kx^2y, & x = 1,2; y = 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$
where k is a constant.
Find (a) the value of k, (b) the marginal PMF's of X and Y. (c) Are X and Y independent?

4. Consider an experiment of tossing two coins three times. Coin A is fair, but coin B is not fair, with $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$. Consider a RV (X, Y), where X denotes the number of heads resulting from coin A and Y denotes the number of heads resulting from coin B.
(a) Find the joint PMF's of (X, Y). (b) Find $P(X = Y), P(X > Y)$, and $P(X + Y \leq 4)$.

5. Suppose we select one point at random from within the circle with radius R. If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen in this circle, then (X, Y) is a uniform bivariate RV with joint PDF given by
$$f(x,y) = \begin{cases} k, & x^2 + y^2 \leq R^2 \\ 0, & \text{otherwise} \end{cases}$$
where k is a constant.
(a) Determine the value of k. (b) Find the marginal PDF's of X and Y. (c) Find the probability that the distance from the origin of the point selected is not greater than a.

6. A manufacturer has been using two different manufacturing processes to make computer memory chips. Let (X, Y) be a bivariate RV, where X denotes the time to failure of chips made by process A and Y denotes the time to failure of chips rnade by process B. Assuming that the joint PDF of (X, Y) is
$$f(x,y) = \begin{cases} abe^{-(ax+by)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$
where $a = 10^{-4}$ and b = 1.2×10^{-4}, determine $P(X > Y)$.

7. A smooth-surface table is ruled with equidistant parallel lines a distance D apart. A needle of length L, where $L \leq D$, is randomly dropped onto this table. What is the probability that the needle will intersect one of the lines? (This is known as Buffon's needle problem.)

8. The joint PDF of a bivariate RV (X, Y) is given by
$$f(x,y) = \begin{cases} \frac{1}{y} e^{-\frac{x}{y}} e^{-y}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$
(a) Verify that $f(x, y)$ is PDF. (b) Find $P(X > 1 | Y = y)$.

9. Suppose the joint PMF of (X, Y) is given by
$$p(x,y) = \begin{cases} \frac{1}{3}, & (0,1), (1,0), (2,1) \\ 0, & \text{otherwise} \end{cases}$$
(a) Are X and Y independent? (b) Are X and Y uncorrelated?

10. Let (X, Y) be a $2D$ RV with the joint PDF given by
$$f(x,y) = \frac{x^2 + y^2}{4\pi} \exp\left(-\frac{x^2 + y^2}{2}\right), \; -\infty < x, y < \infty.$$
Show that X and Y are not independent but are uncorrelated.

11. Let X and Y be independent RV's, each uniformly distributed over $(0, 1)$. Let $Z = X + Y, W = X - Y$. Find the marginal PDF's of Z and W.

12. Let $Y = \dfrac{(X - \lambda)}{\sqrt{\lambda}}$, where X is a Poisson RV, with parameter λ. Show that $Y \sim N(0, 1)$ when λ is sufficiently large. (Hint: Find the moment generating function of Y and let $\lambda \to \infty$.)