Visvesvaraya National Institute of Technology, Nagpur
Department of Mathematics
Assignment: Probability Theory(PT-3) **Topics: Multiple RV's**

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1. Let $F(x,y) = \begin{cases} 1 & \text{if } x+2y \geq 1 \\ 0 & \text{if } x+2y < 1 \end{cases}$. Does F defines the DF in the plane?

2. Let (X,Y) have the joint PDF $f(x,y) = \dfrac{1}{2}$ inside the square with the corners at the points $(1,0),(0,1),(-1,0)$ and $(0,-1)$ in the $xy-$plane, and $=0$ otherwise. Find the marginal PDFs of X and Y, and the two conditional PDF's.

3. Let (X,Y) have the joint PDF $f(x,y) = \begin{cases} \dfrac{4}{3}\left(xy + \dfrac{x^2}{2}\right) & \text{if } \begin{matrix} 0<x<1, \\ 0<y<2 \end{matrix} \\ 0 & \text{otherwise} \end{cases}$, find $P\left\{Y < 1 \mid X < \dfrac{1}{2}\right\}$.

4. For the bi-variate (Cauchy) RV (X,Y) with PDF
$$f(x,y) = \frac{c}{2\pi}\left(c^2 + x^2 + y^2\right)^{-3/2}, -\infty < x, y < \infty, c > 0.$$
Find the marginal PDF's of X and Y. Find the conditional PDF Y given $X = x$.

5. Consider a sample of size 2 drawn with replacement (without replacement) from an urn containing two white, one black, and two red balls. Let the RV's X_1 and X_2 be defined as: for $k = 1, 2, X_k = 1$ or 0, depending upon whether the ball drawn on the k^{th} draw is a white or non white. Find the (i) joint PMF of (X_1, X_2) (ii) marginal PMF's of X_1 and X_2 and (iii) conditional PMF's.

6. An urn contains 3 red and 2 green balls. A random sample of two balls are drawn with replacement (without replacement). Let $X = 0$ if the first ball drawn is green, $= 1$ if the first ball drawn is red, and let $Y = 0$, if second ball drawn is green, $= 1$if the second ball drawn is red. Find (i) joint PMF's of X and Y. (ii) conditional PMF's.

7. Suppose that at two points in a room (or on a city or in the ocean) one measures the intensity of sound caused by general background noise. Let X_1 and X_2 be RV's representing the intensity of sound at two points. Suppose the RV's X_1 and X_2 are contentious with joint PDF given by
$$f(x_1, x_2) = \begin{cases} x_1 x_2 e^{-\frac{1}{2}(x_1^2 + x_2^2)} & \text{if } x_1 > 0, x_2 > 0 \\ 0 & \text{otherwise} \end{cases}.$$
Find the marginal PDF's of $X_1$and X_2. Further, find $P\{X_1 \leq 1, X_2 \leq 1\}$ and $P\{X_1 + X_2 \leq 1\}$.

8. For the RV (X,Y) the joint PDF is given as
(a) $f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{4} & \text{if } 0 \leq x, y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$ (b) $f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$,
Find for (a) and (b), (i) $P\{X \leq 1, Y \leq 1\}$ (ii) $P\{X + Y \leq 1\}$ (iii) $P\{X + Y > 2\}$ (iv) $P\{X \leq 2Y\}$ (v) $P\{X > 1\}$ (vi) $P\{X = Y\}$ (vii) $P\{Y > 1 \mid X \leq 1\}$ (viii) $P\{X > Y \mid Y > 1\}$.

9. Consider the joint PDF of the RV (X,Y), given by $f(x,y) = \begin{cases} xe^{-x(1+y)} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$.
Find the $E[X]$ and $E[Y]$.

10. Two points are selected randomly on a line of length a so as to be on opposite sides of the midpoint of the line. Find the probability that the distance between them is less than $\frac{1}{3}a$.

11. Suppose that two buses A and B, operate on a route. A person arrives at a certain bus stop on this route at time 0. Let X and Y be the arrival times of buses A and B, respectively, at this bus stop. Suppose that X and Y are independent and have the density functions given respectively, by $f_1(x) = \begin{cases} \frac{1}{a} & 0 \leq x \leq a \\ 0 & \text{elsewhere} \end{cases}$ and $f_2(y) = \begin{cases} \frac{1}{b} & 0 \leq y \leq b \\ 0 & \text{elsewhere} \end{cases}$. What is the probability that bus A will arrive before bus B?

12. Let X and Y be jointly distributed with PDF $f(x, y) = \begin{cases} \frac{1+xy}{4} & \text{if } |x| < 1, |y| < 1 \\ 0 & \text{otherwise} \end{cases}$. Show that the RV's X^2 and Y^2 are independent but X and Y are not independent.

13. Let X and Y be independent RV's with common PDF $f(x) = \begin{cases} \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}(\log x)^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$.

    Find the PDF of $Z = XY$.

14. Let X and MY be independent RV's with common PDF $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$. Find the joint PDF of U and V in the following cases:

    (i) $U = \frac{X+Y}{2},\ V = \frac{(X-Y)^2}{2}$ (ii) $U = \sqrt{X^2+Y^2},\ V = \tan^{-1}\left(\frac{X}{Y}\right),\ -\frac{\pi}{2} < V \leq \frac{\pi}{2}$

    (iii) $U = X^2 + Y^2$.