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# Numerical Methods and Probability Theory

**Root of an equation:** If there exists a $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$, then $\alpha$ is called root of the equation $f(x) = 0$.

If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation. Otherwise, $f(x) = 0$ is called trancedental equation.

Ex: $\sin x - e^x + 1 = 0 \rightarrow$ trancedental eq$^n$

→ How to find an approximation for root of the equation $f(x) = 0$

Let $\alpha$ be an exact root of $f(x) = 0$

Let $x_0$ be an initial approximation for $\alpha$

Using a numerical method with $x_0$, we get next approximation $x_1$ for $\alpha$

Similarly, Using a numerical method with $x_1$, we get next approximation $x_2$ for $\alpha$

⋮

We repeat this procedure, to get a sequence $\{x_n\}_{n=0}^{\infty}$ of approximation for $\alpha$

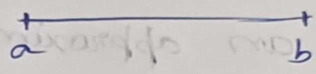
If $\underset{n \to \infty}{Lt}\, x_n = \alpha$, then we say that sequence of approximations for $\alpha$, converges to $\alpha$.

$f(x) = 0 \qquad x \in [a, b]$

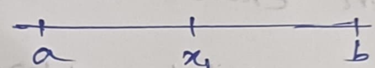
**Result:** Let $f$ be a continuous function on $[a, b]$, such that $f(a)\, f(b) < 0$, then the equation $f(x) = 0$ possesses atleast one root in $(a, b)$

**Bisection Method:**

Let $f(x) = 0$, $x \in [a, b]$ suppose that $f(x) = 0$ has a root in $[a, b]$

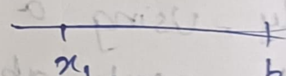



Let $x_1 = \dfrac{a+b}{2}$

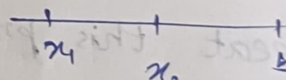



Suppose $f(a)\, f(x_1) > 0$ and $f(x_1)\, f(b) < 0$, then root of the equation $f(x) = 0$ lies between $x_1$ and $b$

Then, take $x_2 = \dfrac{x_1 + b}{2}$




If $f(x_1)\, f(x_2) < 0$ and
$f(x_2)\, f(b) > 0$, then




root lies between $x_1$ and $x_2$

$\therefore$ We take $x_3 = \dfrac{x_1 + x_2}{2}$

We continue this procedure

Then, we get a sequence $\{x_n\}_{n=0}^{\infty}$ of approximation for a root of the equation $f(x) = 0$.

Pb: Find an approximation for the root of $x^2 + 3x + 2 = 0$ that lies between 0 and $-1.5$

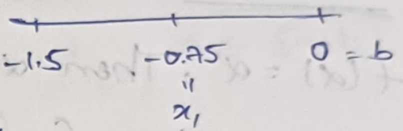
Sol: $f(x) = 0$, $f(x) = x^2 + 3x + 2$

$a = -1.5$, $b = 0$

$[-1.5, 0]$

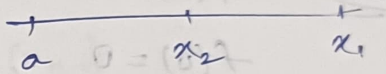
$$x_1 = \frac{a+b}{2} = \frac{-1.5}{2} = -0.75$$

$f(a) \cdot f(x_1) < 0$ and $f(x_1) f(b) > 0$




∴ Root lies between a and $x_1$

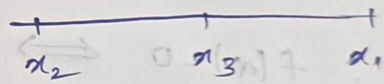
$$x_2 = \frac{a + x_1}{2} = -1.125$$




$f(a) f(x_2) > 0$, $f(x_2) f(x_1) < 0$

∴ Root lies between $x_2$ and $x_1$

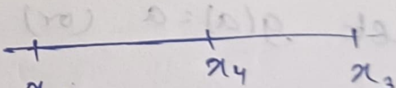
$$x_3 = \frac{x_2 + x_1}{2} = \frac{-1.875}{2} = -0.937$$




$f(x_2) f(x_3) < 0$, $f(x_3) f(x_1) > 0$

∴ Root lies between $x_2$ and $x_3$

$$x_4 = \frac{x_2 + x_3}{2} = \frac{-2.062}{2} = -1.031$$




$f(x_2) f(x_4) > 0$, $f(x_4) f(x_3) < 0$

$$\therefore \quad x_5 = \frac{x_4 + x_3}{2} = \frac{-1.968}{2} = -0.984$$

Pb: Each of the following equations has a root in $[0,1]$. Find an approximation for a root of the following equations. Perform six iterations

1) $x^5 + 2x - 1 = 0$  2) $e^{-x} - x = 0$

3) $\cos x - x = 0$  4) $\log(1+x) - \cos x = 0$

Fixed point: Let $f$ be a function such that $f(\alpha) = \alpha$. Then $\alpha$ is called a fixed point of the function $f$.

Ex: $f(x) = x^2$

$f(0) = 0$, $f(1) = 1$

$\therefore$ 0 and 1 are fixed points of $f(x) = x^2$

$f(x) = 0$ suppose that $f(\alpha) = 0$. Assume that

$f(x) = g(x) - x$

$\therefore f(\alpha) = 0 \iff g(\alpha) - \alpha = 0$

$\iff \alpha$ is fixed point of $g$.

Existence of fixed points:

If $g : [a,b] \to [a,b]$ is a continuous function, then $g$ possesses atleast one fixed point in $[a,b]$

Let $g(a) = a$ (or) $g(b) = b$

$\Rightarrow a$ or $b$ is fixed point of $g$

Let $g(a) \neq a$ and $g(b) \neq b$

Define $h(x) = g(x) - x$

$h$ is a continuous function on $[a,b]$

$h(a) = g(a) - a > 0$

$h(b) = g(b) - b < 0$

$\therefore \exists\ \alpha \in (a,b) \ni h(\alpha) = 0$

$\Rightarrow g(\alpha) = \alpha$

$\Rightarrow \alpha$ is fixed point of $g$

<u>Uniqueness of fixed point:</u>

Let $g : [a,b] \to [a,b]$ be a differentiable function such that $|g'(x)| \leq K < 1$ for all $x \in [a,b]$. Then $g$ possesses a unique fixed point $\alpha \in [a,b]$

Let $x_0 \in [a,b]$. Define $x_1 = g(x_0)$

$x_2 = g(x_1)$

$\vdots$

$x_n = g(x_{n-1})$

Then the sequence $\{x_n\}$ converges to $\alpha$ $\Leftarrow$ $n = 1, 2, 3, \ldots$

Clearly $g$ has atleast one fixed point in $[a,b]$

Suppose that $\alpha_1$ and $\alpha_2$ are fixed points of $g$ in $[a,b]$

$\Rightarrow g(\alpha_1) = \alpha_1$ and $g(\alpha_2) = \alpha_2$

Let $\alpha_1 < \alpha_2$

$\therefore$ By Lagrange Mean Value Theorem, $\exists\ c \in (\alpha_1, \alpha_2)$

$$\ni g'(c) = \frac{g(\alpha_2) - g(\alpha_1)}{\alpha_2 - \alpha_1}$$

$$\left| \frac{g(\alpha_2) - g(\alpha_1)}{\alpha_2 - \alpha_1} \right| = |g'(c)|$$

$$\Rightarrow \left| \frac{g(\alpha_2) - g(\alpha_1)}{\alpha_2 - \alpha_1} \right| \leq K$$

$$\Rightarrow |g(\alpha_2) - g(\alpha_1)| \leq K(\alpha_2 - \alpha_1)$$

$\Rightarrow |\alpha_2 - \alpha_1| \le k(\alpha_2 - \alpha_1)$

$\Rightarrow k \ge 1$

$\therefore$ Our assumption is incorrect.

$\therefore \alpha_1 = \alpha_2$

Start with $|x_n - \alpha| = |g(x_{n-1}) - g(\alpha)|$ — ①

Applying LMVT on $[x_{n-1}, \alpha]$, we get

$$\frac{g(\alpha) - g(x_{n-1})}{\alpha - x_{n-1}} = g'(c) \qquad c \in (x_{n-1}, \alpha)$$

$\Rightarrow |g(\alpha) - g(x_{n-1})| = |g'(c)| \cdot |\alpha - x_{n-1}|$

$\le k|\alpha - x_{n-1}|$ — ②

Using ② in ①, we get

$|x_n - \alpha| \le k|x_{n-1} - \alpha|$ — ③

Similarly, $|x_{n-1} - \alpha| \le k|x_{n-2} - \alpha|$ — ④

$\vdots$

$|x_1 - \alpha| \le k|x_0 - \alpha|$

Using ④ in ③, we get

$|x_n - \alpha| \le k^2 |x_{n-2} - \alpha|$

$\le k^2 [k|x_{n-3} - \alpha|]$

$\vdots$

$\therefore |x_n - \alpha| \le k^n |x_0 - \alpha|$

Since $0 < k < 1$, $k^n \to 0$ as $n \to \infty$.

$\therefore \lim_{n\to\infty} |x_n - \alpha| = 0$

$\Rightarrow \boxed{\lim_{n\to\infty} x_n = \alpha}$

$\boxed{* \; |g'(x)| \leq k \leq 1 \quad \forall\, x \in [a, b]}$

## Iterative Method :

Let $f(x) = 0$ be the given equation

Let $f(x) = g(x) - x$, where $g$ is a differentiable on $[a, b]$ such that $|g'(x)| \leq k < 1 \;\forall\, x \in [a, b]$

$\alpha$ is a root of $f(x) = 0 \iff \alpha$ is a fixed point of $g$ on $[a, b]$

choose $x_0 \in [a, b]$

Defined $x_n = g(x_{n-1})$, $n = 1, 2, \ldots$

According to the previous result, $\lim_{n\to\infty} x_n = \alpha$

Pb: Find a root of the equation $3x = \cos x + 1$ in $[0, 1]$

Sol:

$\Rightarrow x = \dfrac{\cos x + 1}{3}$

$\Rightarrow x + \left(-\dfrac{\cos x - 1}{3}\right) = 0 \Rightarrow \dfrac{\cos x + 1}{3} - x = 0$

$g(x) = \dfrac{+\cos x + 1}{3} \Rightarrow g'(x) = \dfrac{-\sin x}{3}$

$|g'(x)| = \dfrac{|\sin x|}{3} \leq \underbrace{\dfrac{1}{3}}_{k} < 1$

Let $x_0 = 0.5$

$x_1 = g(x_0) = g(0.5) = 0.626$

$x_2 = g(x_1) = g(0.626) = 0.603$

$x_3 = g(x_2) = g(0.603) \approx 0.607$

$\boxed{x_4 = g(x_3) = 0.607}$

$f(x) = 3x - \cos x - 1 = 0$

$\boxed{f(x_4) \simeq 0}$

Pb: Find a ~~positive~~ root of $x^4 - x = 10$ in $[1, 2]$.

Sol:
$x^4 - 10 - x = 0$ $\qquad f(x) = g(x) - x$

$\Rightarrow g(x) = x^4 - 10$

$g'(x) = 4x^3 > 1 \quad \forall\, x \in [1, 2]$

$\therefore g(x) = x^4 - 10$ — does not work

$\cancel{x(x^3 - 1) - 10 = 0}$ $\qquad x^4 = 10 + x$

$\cancel{x = \frac{10}{x^3 - 1}}$ $\quad \Rightarrow \quad x = (10 + x)^{1/4}$

$(10 + x)^{1/4} - x = 0$

$\Rightarrow g(x) = (10 + x)^{1/4}$

$g'(x) = \frac{1}{4}(x + 10)^{-3/4} = \frac{1}{4(x+10)^{3/4}} < 1$

$\forall\, x \in [1, 2]$

Let $x_0 = 1.2$

$x_1 = g(x_0) = g(1.2) \approx 1.829$

$x_2 = g(x_1) = g(1.829) \approx 1.854$

$x_3 = g(x_2) = g(1.854) \approx 1.855$

$x_4 = g(x_3) = 1.855$

$\boxed{f(x_4) \simeq 0}$

Pb: Using the iterative method find an approximation for roots of the following equations in $(0, 1)$

1) $x^5 + 2x - 1 = 0$ 2) $e^{-x} - x = 0$

3) $\cos x - x = 0$ 4) $\log(1+x) - \cos x = 0$

$\log(1+x) = \cos x$

$1 + x = e^{\cos x}$

$\underbrace{e^{\cos x} - 1}_{g(x)} - x = 0$

Sol:

1) $x(x^4 + 2) = 1$

$\frac{1}{x^4+2} - x = 0$ $g(x) = \frac{1}{x^4+2} = (x^4+2)^{-1}$ $\frac{1}{2}$

$g'(x) = -(x^4+2)^{-2}(4x^3)$ $-\frac{1}{x^2}$

$x = (1-2x)^{1/5}$

$\rightarrow (1-2x)^{1/5} - x = 0$ $\rightarrow$ $g(x) = (1-2x)^{1/5}$

$g'(x) = \frac{1}{5} \cdot (1-2x)^{-4/5}$

$|g'(x)| = \frac{1}{5} \frac{1}{|(1-2x)^{4/5}|}$

$\frac{1}{x^4+2} - x = 0 \rightarrow g(x) = \frac{1}{x^4+2}$ $g'(x) = \frac{-4x^3}{(x^4+2)^2}$

$|g'(x)| = \frac{4x^3}{(x^4+2)^2} < 1 \quad \forall x \in (0, 1)$

Let $x_0 = 0.5$

$x_1 = g(x_0) = g(0.5) = 0.485$

$x_2 = g(x_1) = g(0.485) = 0.486$

$x_3 = g(x_2) = \underline{0.486}$

2) $g(x) = e^{-x}$

$|g'(x)| = |e^{-x}| < 1 \quad \forall x \in (0,1)$

let $x_0 = 0.5$

$x_1 = g(x_0) = 0.606$

$x_2 = g(x_1) = 0.545$

$x_3 = g(x_2) = 0.58$

$x_4 = g(x_3) = 0.56$

$x_5 = g(x_4) = 0.571$

$x_6 = g(x_5) = 0.565$

$x_7 = g(x_6) = 0.568$

3) $g(x) = \cos x < 1 \quad \forall x \in (0,1)$

$|g'(x)| = |\sin x| < 1 \quad \forall x \in (0,1)$

let $x_0 = 0.5$

$x_1 = g(0.5) = 0.877$

$x_2 = g(x_1) = 0.639$

$x_3 = g(x_2) = 0.803$

$x_4 = g(x_3) = 0.694$

$x_5 = g(x_4) = 0.769$

$x_6 = g(x_5) = 0.7189$

$x_7 = g(x_6) = 0.752$

$x_8 = g(x_7) = 0.730$

$x_9 = g(x_8) = 0.745$

$x_{10} = 0.735$

$x_{11} = 0.741$

$x_{12} = 0.738$

$x_{13} = 0.74$

4) $\log(1+x) - \cos x = 0$

$1 + x = e^{\cos x}$

$\underbrace{(e^{\cos x} - 1) - x}_{g(x)} = 0$

$\rightarrow g'(x) = -\sin x \, e^{\cos x}$ ✗

for $x \in (0,1)$

$\cos^{-1}(\log(1+x)) - x = 0$

$$\frac{-1}{\sqrt{1-(\log(1+x))^2}} \cdot \frac{1}{1+x} = g'(x) < 1$$

$\Rightarrow \quad g(x) = \cos^{-1}(\log(1+x))$

~~Let $x_0 = 0.5$~~

~~$x_1 = g(x_0) = 1.394$~~

~~$x_2 = g(x_1) = 1.426$~~

~~$x_3 = g(x_2) = 1.416$~~

~~$x_4 = g(x_3) = 1.419$~~

~~$x_5 = g(x_4) = 1.418$~~

Let $x_0 = 0.9$

$x_1 = g(x_0) = 0.874$

$x_2 = g(x_1) = 0.892$

$x_3 = g(x_2) = 0.879$

$x_4 = g(x_3) = 0.888$

$x_5 = g(x_4) = 0.882$

$x_6 = 0.886$

$x_7 = 0.883$

$x_8 = 0.885$

$x_9 = 0.884$

$\boxed{x_{10} = 0.885}$

## By bisection method:

1) $[0, 1]$   $a$   $b$     $x^4 + 2x - 1 = 0$

$a \quad x_1 \quad b$

$x_1 = \frac{1}{2}$

$\underset{<0}{f(a)}\, f(x_1) = (-1)\left(\frac{1}{16}\right) < 0$     $f(x_1)\, f(b) > 0$

$\Rightarrow$ root lies between $a$ and $x_1$

$a \quad x_2 \quad x_1$

$x_2 = \frac{a + x_1}{2} = \frac{1}{4}$

$f(a)\, \underset{<0}{f(x_2)} > 0$     $f(x_2)\, \underset{>0}{f(x_1)} < 0$

$x_2 \quad x_3 \quad x_1$

$\Rightarrow$ root lies between $x_2$ and $x_1$

$x_3 = \frac{x_2 + x_1}{2} = \frac{3}{8}$

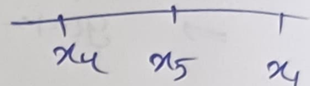
$f(x_2)\, \underset{<0}{f(x_3)} > 0$     $f(x_3)\, f(x_1) < 0$

$x_3 \quad x_4 \quad x_1$

$x_4 = \frac{x_3 + x_1}{2} = \frac{7}{16}$

$f(x_3) f(x_4) > 0 \quad f(x_4) f(x_1) < 0$
$\quad\quad\quad < 0$

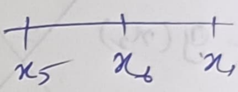
$\therefore$ root lies b/w $x_4$ & $x_1$




$x_5 = \frac{x_4 + x_1}{2} = \frac{15}{32}$

$f(x_4) f(x_5) > 0 \quad f(x_5) f(x_1) < 0$
$\quad\quad\quad < 0$

$x_6 = \frac{x_5 + x_1}{2} = \frac{31}{64}$

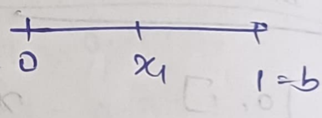



$\boxed{= 0.484}$ approximate root

2) $e^{-x} - x$

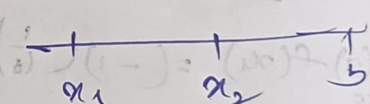
$[0, 1] \quad x_1 = \frac{1}{2}$

$f(a) f(x_1) > 0 \quad f(x_1) f(b) < 0$
$> 0 \quad > 0 \quad\quad\quad < 0$




$\therefore$ root lies b/w $x_1$ and b

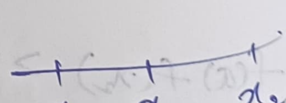
$x_2 = \frac{x_1 + b}{2} = \frac{3}{4}$




$f(x_1) f(x_2) < 0 \quad f(x_2) f(b) > 0$
$> 0 \quad < 0$

$\therefore$ root lies b/w $x_1$ and $x_2$

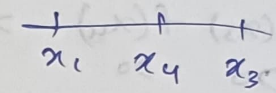
$x_3 = \frac{x_1 + x_2}{2} = \frac{5}{8}$




$f(x_1) f(x_3) < 0 \quad f(x_3) f(x_2) > 0$
$> 0 \quad < 0$

$\therefore$ root lies b/w $x_1$ & $x_3$

$x_4 = \frac{x_1 + x_3}{2} = \frac{7}{16} = 0.4375$

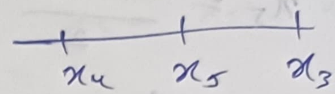



$f(x_1)\, f(x_4) > 0$     $f(x_4)\, f(x_3) < 0$

$>0 \quad >0 \qquad\qquad >0 \quad <0$

$\therefore$ root lies b/w $x_4$ & $x_3$

$x_5 = \frac{x_4 + x_3}{2} = \frac{17}{32}$




$f(x_4)\, f(x_5) > 0$     $f(x_5)\, f(x_3) < 0$

$>0$

$\therefore$ root lies b/w $x_5$ & $x_3$

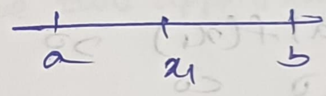
$x_6 = \frac{x_5 + x_3}{2} = \frac{37}{64}$

$\boxed{x_6 = 0.578}$

3) $\cos x - x$

$[0, 1]$     $x_1 = \frac{1}{2}$

a b

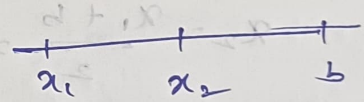



$f(a)\, f(x_1) > 0$     $f(x_1)\, f(b) < 0$

$>0 \quad >0 \qquad\qquad <0$

$\therefore$ Root lies between $x_1$ and b

$x_2 = \frac{x_1 + b}{2} = \frac{3}{4}$

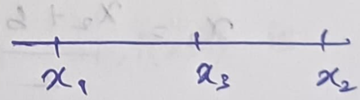



$f(x_1)\, f(x_2) < 0$     $f(x_2)\, f(b) > 0$

$>0 \quad <0 \qquad\qquad <0 \quad <0$

$\therefore$ Root lies b/w $x_1$ & $x_2$

$x_3 = \frac{x_1 + x_2}{2} = \frac{5}{8}$

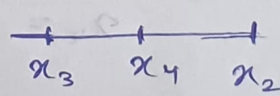



$f(x_1)\, f(x_3) > 0$     $f(x_3)\, f(x_2) < 0$

$>0 \quad >0 \qquad\qquad >0 \quad <0$

$\therefore$ Root lies b/w $x_3$ and $x_2$

$x_4 = \frac{x_3 + x_2}{2} = \frac{11}{16}$

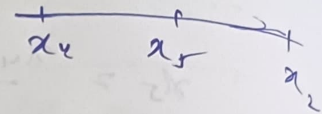


$f(x_3)\,f(x_4) > 0$ $\qquad$ $f(x_4)\,f(x_2) < 0$

$f(x_3) > 0$, $f(x_4) > 0$; $f(x_2) < 0$

$\therefore$ Root lies b/w $x_4$ & $x_2$

$$x_5 = \frac{x_4 + x_2}{2} = \frac{23}{32}$$




$f(x_4)\,f(x_5) > 0$ $\qquad$ $f(x_5)\,f(x_2) < 0$

$f(x_4) > 0$, $f(x_5) > 0$

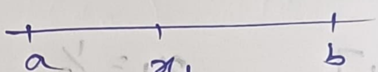
$\therefore$ Root lies b/w $x_5$ & $x_2$

$$x_6 = \frac{x_5 + x_2}{2} = \frac{47}{64} = \boxed{0.734}$$

4) $\log(1+x) - \cos x$

$[0, 1]$ ($a = 0$, $b = 1$)

$x_1 = \frac{1}{2}$

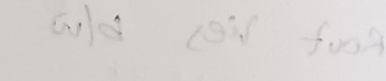



$f(a)\,f(x_1) > 0$ $\qquad$ $f(x_1)\,f(b) < 0$

$f(a) < 0$, $f(x_1) < 0$; $f(x_1) < 0$, $f(b) > 0$

$\therefore$ Root lies b/w $x_1$ & $b$

$$x_2 = \frac{x_1 + b}{2} = 3/4$$

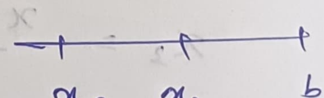



$f(x_1)\,f(x_2) > 0$ $\qquad$ $f(x_2)\,f(b) < 0$

$f(x_1) < 0$, $f(x_2) < 0$

$\therefore$ Root lies b/w $x_2$ & $b$

$$x_3 = \frac{x_2 + b}{2} = 7/8$$

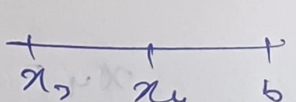



$f(x_2)\,f(x_3) > 0$ $\qquad$ $f(x_3)\,f(b) < 0$

$f(x_2) < 0$, $f(x_3) < 0$

$\therefore$ Root lies b/w $x_3$ & $b$

$$x_4 = \frac{x_3 + b}{2} = \frac{15}{16}$$



$f(x_4)\, f(x_3) < 0$ $\quad$ $f(x_4)\, f(b) > 0$

$>0 \quad <0$

$\therefore$ Root lies b/w $x_4$ & $x_3$

$$x_5 = \frac{x_4 + x_3}{2} = \frac{29}{32}$$

$x_4 \quad x_5 \quad x_3$

$f(x_4)\, f(x_5) > 0$ $\quad$ $f(x_5)\, f(x_3) < 0$

$>0 \quad >0$

$$x_6 = \frac{x_5 + x_3}{2}$$

$$x_6 = \frac{57}{64} = \boxed{0.89}$$

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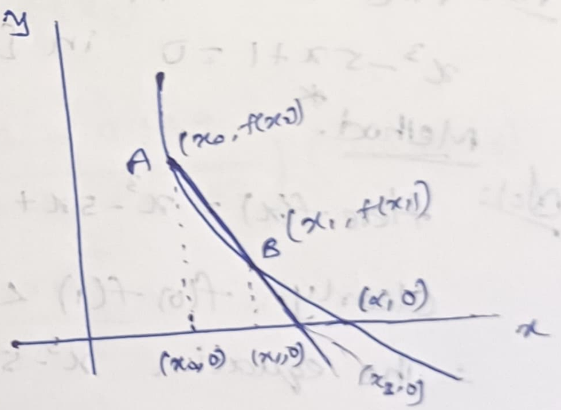
## Secant (or) ~~Regula Falsi Method~~ Method :

Let $\alpha$ be the root of $f(x) = 0$. Let $(x_0, f(x_0))$ and $(x_1, f(x_1))$ be two points lying on the curve $y = f(x)$.

Let us suppose that $x_0$ and $x_1$ be two approximations for '$\alpha$'.

The slope of the line joining $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is given by

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$




$\therefore$ The equation of the line passing through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is given by

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1).$$

Since the above line intersects the x-axis at $(x_2, 0)$ we get

$$0 - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1)$$

$$\Rightarrow \cancel{x_3 = x_1 - f(x_1)\left[\frac{f(x_1)-f(x_0)}{x_1-x_0}\right]}$$

$$\Rightarrow x_2 = x_1 - \frac{(x_1-x_0)}{f(x_1)-f(x_0)}\left[f(x_1)\right]$$

Similarly, $$x_3 = x_2 - \frac{(x_2-x_1)}{f(x_2)-f(x_1)} f(x_2)$$

$$x_4 = x_3 - \frac{x_3-x_2}{f(x_3)-f(x_2)} f(x_3)$$

$\vdots$

$$x_n = x_{n-1} - \frac{x_{n-1}-x_{n-2}}{f(x_{n-1})-f(x_{n-2})} f(x_{n-1})$$

$n = 2, 3, 4, \ldots$

Pb: Find an approximation for root of $x^3 - 5x + 1 = 0$ in $[0,1]$ by using *Regula Falsi Method.*

Sol: Here $f(x) = x^3 - 5x + 1$

Clearly $f(0)\, f(1) < 0$

$\therefore$ The equation $x^3 - 5x + 1 = 0$ has a root in $[0,1]$

Let $x_0 = 0.4$ and $x_1 = 0.6$

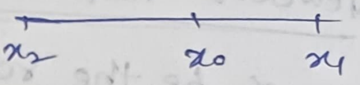
$$x_2 = x_1 - \frac{(x_1-x_0)}{f(x_1)-f(x_0)} f(x_1)$$

$$= 0.6 - \frac{0.2}{-1.784+0.936}(-1.784)$$

$$x_2 = 0.6 - \frac{0.3568}{0.848} = 0.179$$

$f(x_2)\, f(x_0) < 0 \qquad f(x_0)\, f(x_1) > 0$

$>0 \quad <0$




$$x_3 = x_2 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_2)$$

$$= 0.179 - \frac{0.179 - 0.4}{0.1107 + 0.936} (0.1107)$$

$$= 0.179 + \frac{0.0245}{1.0467}$$

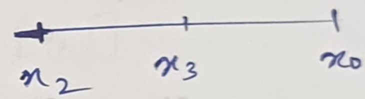
$$x_3 = 0.202$$

$$\boxed{x_{Next} = x_{new} - \frac{(x_{new} - x_{old})}{f(x_{new}) - f(x_{old})} f(x_{new})}$$

$f(x_2)\, f(x_3) < 0 \qquad f(x_2)\, f(x_0) > 0$

$>0 \quad <0$

$-0.0017$




$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.202 - \left(\frac{0.202 - 0.179}{-0.00176 - 0.1107}\right) \cdot (-0.00176)$$

$$= 0.202 - \frac{4.048 \times 10^{-5}}{0.11246}$$

$$= 0.202 - 0.36 \times 10^{-3}$$

$$= 0.202 - 0.00036$$

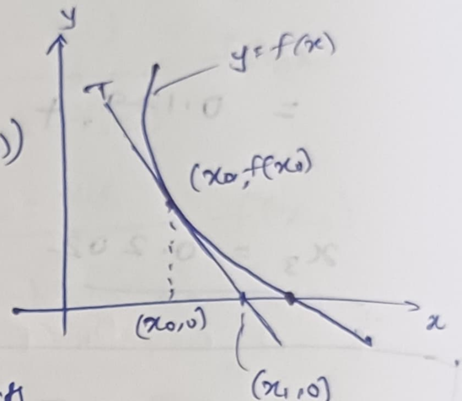
$$\boxed{x_4 = 0.201} \qquad x_4 \simeq x_3$$

∴ The approximation for the root of $x^3 - 5x + 1 = 0$ is $x_4 = 0.201$

## Newton - Raphson Method :

Let $\alpha$ be the root of $f(x)=0$. Assume that $f$ is a differentiable function such that $f'$ is continuous.

Let $x_0$ be an approximation for $\alpha$.




$\therefore$ the equation of tangent line to $y=f(x)$ at $(x_0, f(x_0))$ is given by

$$y - f(x_0) = f'(x_0)(x-x_0)$$

Since the above line intersects the x-axis at $(x_1, 0)$, we get

$$-f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$\vdots$

In general, $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$ ; $n=1,2,3,\ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; n=0,1,2,\ldots$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of approximations obtained by Newton - Raphson Method with respect to $f(x)=0$ in the interval $[a,b]$

Suppose that $\underset{n\to\infty}{Lt}\, x_n = \beta$

$$\underset{n\to\infty}{Lt}\, x_{n+1} = \underset{n\to\infty}{Lt}\, x_n - \underset{n\to\infty}{Lt}\, \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \beta = \beta - \frac{f(\underset{n\to\infty}{Lt}\, x_n)}{f'(\underset{n\to\infty}{Lt}\, x_n)}$$

$f$ is continuous at $y_0 \in [a,b]$ $\Longleftrightarrow$ for any sequence in $[a,b]$, $\underset{n\to\infty}{Lt}\, f(y_n) = f(\underset{n\to\infty}{Lt}\, y_n)$

$$\beta = \beta - \frac{f(\beta)}{f'(\beta)}$$

$$\Rightarrow f(\beta) = 0$$

That means, if $\{x_n\}_{n=1}^{\infty}$ converges, then it converges to the root of $f(x)=0$.

Pb: Perform four iterations of Newton - Raphson method to find an approximation for a +ve root of $x^3 - 5x + 1 = 0$ in $[0,1]$

Sol: Here $f(x) = x^3 - 5x + 1$

$f(0)\, f(1) < 0$

$\therefore$ Given equation has a root in $[0,1]$

$f'(x) = 3x^2 - 5$.

Let $x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{\frac{11}{8}}{\frac{17}{4}}$$

$$= 0.5 - \frac{11}{34} = 0.176$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.201$$

$$x_3 = x_2 - \frac{f(x_1)}{f'(x_1)} = 0.201$$

Pb: Find an approximation for $(17)^{1/3}$ using Newton Raphson method.

Sol: Let $(17)^{1/3} = x$

$\Rightarrow x^3 = 17 \Rightarrow x^3 - 17 = 0$

Let us choose $x_0 = 2.3$ ; $f(x) = x^3 - 17$

$f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.604$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.572$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.571$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.571$$

$\therefore (17)^{1/3} \simeq 2.571$

Pb: Find an approximation for root of the equation $27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1 = 0$ which lies b/w $-1$ and $0$, Use N-R method

Sol: Choose $x_0 = -0.4$

$f(x) = 27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1$

$f'(x) = 135x^4 + 108x^3 + 108x^2 + 56x + 9$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -0.378$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.363$$

$x_3 = -0.353$

$x_4 = -0.347$ $\qquad\qquad$ $x_9 = -0.335$

$x_5 = -0.342$

$x_6 = -0.339$

$x_7 = -0.337$

$x_8 = -0.336$

Pb Find an approximation for a root of $x^3 - 3x^2 + 4 = 0$ in $[1, 2.5]$

Ans $\cancel{f(1)\ f(2.5)}$ $\cancel{>0 \quad >0}$ $\qquad$ Here $f(1)\ f(2.5) > 0$ but root exists

Let $x_0 = 1.1$

$f(x) = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1.673$

$x_2 = 1.847$

$x_3 = 1.926$

$x_4 = 1.963$

$x_5 = 1.982$

$x_6 = 1.991$

$x_7 = 1.995$

$x_8 = 1.998$

$x_9 = 1.999$

$x_{10} = 1.999$

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→ Newton - Raphson Method

$f(x) = 0$

$\alpha$ is an exact root of $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , n = 0, 1, 2, 3, \ldots$$

If $\alpha$ is a repeated root, then $f(\alpha) = 0$ and $f'(\alpha) = 0$

In this case, the sequence $\{x_n\}_{n=0}^{\infty}$ of approximations for $\alpha$ may not converge to $\alpha$

Consider $\underbrace{(x-1)^2 + 0.512}_{f(x)} = 0$

$$f'(x) = 2(x-1)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{[(x_n-1)^2 + 0.512]}{2(x_n-1)} \quad ; n = 0, 1, 2, 3 \ldots$$

$x_0 = 0.5$

$x_1 = 1.262$

$x_2 = 0.153$

$x_3 = 0.879$

$x_4 = 3.06$

$x_5 = 1.9$

$x_6 = 1.17$

$x_7 = -0.39$

$x_8 = 0.48$

$x_9 = 1.23$

$x_{10} = 0.03$

$x_{11} = 0.78$

$x_{12} = 2.07$

$x_{13} = 1.29$

$x_{14} = 0.29$

$x_{15} = 1$

$x_{16} = -29.7$

$x_{17} = -14.37$

It is diverging.

→ This method is not working everytime.

## Newton - Raphson method for approximation of a root with multiplicity 'm':

Let us suppose that $\alpha$ is a root of $f(x)=0$ having multiplicity 'm'.
That is $f(\alpha)=0$, $f'(\alpha)=0$, .. but $f^{m}(\alpha)=0$ and

$f^{(m+1)}(\alpha) \neq 0$.

In this case $f(x)$ can be written as

$$f(x) = (x-\alpha)^m \phi_1(x) \quad , \text{where } \phi_1(\alpha) \neq 0$$

Similarly $f'(x)$ can be written as ~~$f(x) = (x-\alpha)^m \phi_1(x)$~~

$$f'(x) = (x-\alpha)^{m-1} \phi_2(x) \quad , \text{where } \phi_2(\alpha) \neq 0$$

Define $g(x) = \dfrac{f(x)}{f'(x)} = (x-\alpha) \dfrac{\phi_1(x)}{\phi_2(x)}$

Consider $g(x) = 0$

Clearly $\alpha$ is a root of $g(x)=0$. Now, apply Newton - Raphson method with respect to $g(x)=0$

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \quad , n = 0, 1, 2, \ldots$$

### Order of Convergence :

Let $\alpha$ be a root of $f(x)=0$

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of approximations obtained by a numerical method ⊛ for 'α' which are

Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to 'α'.

→

If there exists largest positive real number $p \in \mathbb{R}^+$ such that $\underset{n\to\infty}{Lt} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = \lambda$, then $p$ is called the order of convergence of the numerical method ⊛, where $\lambda$ is a constant.

For large $n$, $\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} \approx \lambda$

$$\Rightarrow |x_{n+1}-\alpha| \approx \lambda |x_n-\alpha|^p$$

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* $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

Suppose $f$ is infinitely differentiable function, then

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \cdots$$

**Order of convergence of Newton - Raphson method:**

Let $\alpha$ be the exact root of $f(x) = 0$

Suppose that $f$ is sufficiently differentiable function

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \cdots \quad \text{--- (1)}$$

Let $x_n - \alpha = e_n$

$$\Rightarrow x_n = \alpha + e_n$$

Substituting $x_n = \alpha + e_n$ in (1), we get

$$\alpha + e_{n+1} = \alpha + e_n - \frac{f(\alpha+e_n)}{f'(\alpha+e_n)}$$

$$\Rightarrow e_{n+1} = e_n - \left[\frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) \cdots}{f'(\alpha) + \frac{e_n}{1!} f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) \cdots}\right]$$

$$e_{n+1} = e_n - \frac{\left[0 + \frac{e_n}{1!} f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \cdots\right]}{\left[f'(\alpha) + \frac{e_n}{1!} f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) \cdots\right]}$$

$$e_{n+1} = e_n - \left[e_n + \frac{e_n^2}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n^3}{3!}\frac{f'''(\alpha)}{f'(\alpha)} \cdots\right]$$

$$\left[1 + \frac{e_n}{1!}\cdot\frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n^2}{2!}\frac{f'''(\alpha)}{f'(\alpha)} \cdots\right]^{-1}$$

$$e_{n+1} = e_n - \left[e_n + \frac{e_n^2}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n^3}{3!}\frac{f'''(\alpha)}{f'(\alpha)} \cdots\right]$$

$$\left[1 - \frac{e_n}{1!}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^2}{2!}\frac{f'''(\alpha)}{f'(\alpha)} - \frac{e_n^3}{3!}\frac{f^{iv}(\alpha)}{f'(\alpha)} \cdots\right]$$

$$e_{n+1} = e_n - \left[e_n - \frac{e_n^2}{1!}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^3}{2!}\left\{\frac{f'''(\alpha)}{f'(\alpha)}\right\} \cdots\right]$$

$$- \left[\frac{e_n^2 f''(\alpha)}{2!\, f'(\alpha)} - \frac{e_n^3}{2!}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 \cdots\right]$$

$\pm$ terms that contain $e_n^3, e_n^4 \cdots$

$$e_{n+1} = \left[e_n^2 \frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n^3}{2!}\frac{f'''(\alpha)}{f'(\alpha)} + \frac{e_n^4}{3!}\frac{f^{iv}(\alpha)}{f'(\alpha)} \cdots\right]$$

$$- \left[\frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n^3}{2}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 \cdots\right]$$

$\pm$ terms that contain $e_n^3, e_n^4 \cdots$

$$e_{n+1} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\cdot e_n^2 \pm \text{Terms of } e_n^3, e_n^4, e_n^5, \cdots$$

$$\lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty}\left[\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} e_n^2 \pm \text{Terms of } e_n^3, e_n^4, \cdots\right]$$

$$\lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty} \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} e_n^2$$

$$\Rightarrow \lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$x_n - \alpha = e_n$

$$\Rightarrow \lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^2} = \lambda$$

$$\Rightarrow \lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^2} = |\lambda|.$$

If $f(\alpha)=0,\ f'(\alpha)=0,\ f''(\alpha)\neq 0$ then the order of convergence of Newton-Raphson method is one.

If $f(\alpha)=0,\ f'(\alpha)\neq 0$, then the order of convergence of Newton-Raphson method is two.

* Now we will see if $f(\alpha)=0,\ f'(\alpha)=0,\ f''(\alpha)\neq 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $x_n - \alpha = e_n \Rightarrow x_n = \alpha + e_n$

$$\alpha + e_{n+1} = \alpha + e_n - \frac{f(\alpha+e_n)}{f'(\alpha+e_n)}$$

$$\Rightarrow e_{n+1} = e_n - \left\{\frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!}f''(\alpha) + \frac{e_n^3}{3!}f'''(\alpha) - \cdots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!}f'''(\alpha) + \frac{e_n^3}{3!}f^{iv}(\alpha) - \cdots}\right.$$

Here $f(\alpha)=0$ and $f'(\alpha)=0$

$$\Rightarrow e_{n+1} = e_n - \frac{\left[\frac{e_n^2}{2!} + \frac{e_n^3}{3!}\frac{f'''(\alpha)}{f''(\alpha)} \cdots\right]}{\left[e_n + \frac{e_n^2}{2!}\frac{f'''(\alpha)}{f''(\alpha)} + \frac{e_n^3}{3!}\frac{f^{iv}(\alpha)}{f''(\alpha)} \cdots\right]}$$

$$e_{n+1} = e_n - \left[\frac{e_n}{2!} + \frac{e_n^2}{3!}\frac{f'''(\alpha)}{f''(\alpha)} \cdots\right]\left[1 + \frac{e_n}{2!}\frac{f'''(\alpha)}{f''(\alpha)} + \frac{e_n^2}{3!}\frac{f^{iv}(\alpha)}{f''(\alpha)} + \cdots\right]^{-1}$$

$$e_{n+1} = e_n - \left[\frac{e_n}{2!} + \frac{e_n^2}{3!}\frac{f'''(\alpha)}{f''(\alpha)} \cdots\right]\left[1 - \frac{e_n}{2!}\frac{f'''(\alpha)}{f''(\alpha)} - \frac{e_n^2}{3!}\frac{f^{iv}(\alpha)}{f''(\alpha)} \cdots\right]$$

$$e_{n+1} = e_n - \left[\frac{e_n}{2!} - \frac{e_n^2}{4}\frac{f'''(\alpha)}{f''(\alpha)} - \frac{e_n^3}{2!3!}\frac{f^{iv}(\alpha)}{f'(\alpha)} \cdots\right]$$

$$- \left[\frac{e_n^2}{3!}\frac{f'''(\alpha)}{f''(\alpha)} - \frac{e_n^3}{3!2!}\left(\frac{f'''(\alpha)}{f''(\alpha)}\right)^2 - \cdots\right]$$

$$e_{n+1} = e_n - \frac{e_n}{2} \pm \text{terms of } e_n^2, e_n^3, \cdots$$

$$\underset{n\to\infty}{Lt}\, e_{n+1} = \underset{n\to\infty}{Lt} \frac{e_n}{2} \pm \text{Terms of } e_n^2, e_n^3 \cdots$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{e_{n+1}}{e_n} = \frac{1}{2}$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{x_{n+1} - \alpha}{x_n - \alpha} = \lambda$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = |\lambda|$$

---

$\therefore$ If $f(\alpha) = 0$, $f'(\alpha) = 0$, $f''(\alpha) \neq 0$ then the order of convergence of Newton - Raphson method is one.

Order of Convergence of iterative method.

Let $\alpha$ be the exact root of $f(x)=0$. Then $\alpha$ be the fixed point of the function $g$, where $g$ satisfies $|g'(x)|<1$, $f(x)=x-g(x)=0$

Then $x_{n+1}=g(x_n)$, $n=0,1,2,3,\dots$

Let $x_n-\alpha=e_n$

$$\Rightarrow e_{n+1}+\alpha = g(e_n+\alpha) = g(\alpha+e_n)$$

$$\Rightarrow e_{n+1}+\alpha = \underbrace{g(\alpha)}_{\alpha} + e_n g'(\alpha) + \frac{e_n^2}{2!} g''(\alpha) + \frac{e_n^3}{3!} g'''(\alpha)\dots$$

$$\Rightarrow e_{n+1} = e_n g'(\alpha) + \frac{e_n^2}{2!} g''(\alpha) + \frac{e_n^3}{3!} g'''(\alpha) - \dots$$

If $g'(\alpha)\neq 0$, then $e_{n+1} = g'(\alpha) e_n$ + terms of $e_n^2, e_n^3, \dots$

$$\lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty} (g'(\alpha) e_n + \text{terms of } e_n^2, e_n^3, \dots)$$

$$\Rightarrow \lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty} g'(\alpha) e_n$$

$$\Rightarrow \lim_{n\to\infty} \frac{e_{n+1}}{e_n} = g'(\alpha)$$

$\Rightarrow$ the order of convergence $=1$

If $g'(\alpha)=0$, $g''(\alpha)\neq 0$, then order of convergence $=2$

Solution of Non-linear Equations:

$$\left.\begin{aligned} x^2-y&=2 \\ x+y^2&=5 \end{aligned}\right\}$$ System of non-linear equations

Newton Raphson method for solving system of Non-linear equations.

Let us consider $f(x)=0$. Let $\alpha$ be the root of $f(x)=0$

Let $x_0$ be an approximation for '$\alpha$'.

$\therefore \alpha - x_0 = h$

$\Rightarrow \alpha = x_0 + h$

$\therefore f(\alpha)=0 \Rightarrow f(x_0+h)=0$

$\Rightarrow f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots = 0$

$\Rightarrow f(x_0) + hf'(x_0) \approx 0$

$\Rightarrow h \approx \frac{-f(x_0)}{f'(x_0)}$

$\therefore \alpha \approx \underbrace{x_0 - \frac{f(x_0)}{f'(x_0)}} = x_1$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Ingeneral, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $n = 0, 1, 2, \dots$

Let $(\alpha, \beta)$ be a solution of the system of non linear equations $f(x, y)=0$ and $g(x,y)=0$

Let $(x_0, y_0)$ be an approximation for $(\alpha, \beta)$.

$\therefore \left.\begin{array}{l} \alpha - x_0 = h \\ \beta - y_0 = k \end{array}\right\} \Rightarrow \begin{array}{l} \alpha = x_0 + h \\ \beta = y_0 + k \end{array}$

Since $(\alpha, \beta)$ is a solution of $f(x,y)=0$ and $g(x,y)=0$,

We get $f(\alpha, \beta)=0$ and $g(\alpha, \beta)=0$

$$\Rightarrow f(x_0 + h, y_0 + k) = 0 \quad \text{and} \quad g(x_0 + h, y_0 + k) = 0$$

$$\Rightarrow f(x_0, y_0) + h \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + k \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} + \frac{1}{2!}\left(2hk \frac{\partial^2 f}{\partial x \partial y} + h^2 \frac{\partial^2 f}{\partial x^2} + k^2 \frac{\partial^2 f}{\partial y^2}\right)_{(x_0,y_0)} + \dots = 0$$

$$\Rightarrow g(x_0, y_0) + h \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} + k \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} + \frac{1}{2!}\left(2hk \frac{\partial^2 g}{\partial x \partial y} + h^2 \frac{\partial^2 g}{\partial x^2} + k^2 \frac{\partial^2 g}{\partial y^2}\right)_{(x_0,y_0)} + \dots = 0$$

By neglecting second order and higher order terms, we get

$$f(x_0, y_0) + h \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + k \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \approx 0$$

$$g(x_0, y_0) + h \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} + k \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} \approx 0$$

$$\Downarrow$$

$$h \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + k \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \approx -f(x_0, y_0)$$

$$h \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} + k \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} \approx -g(x_0, y_0)$$

$$\Rightarrow \begin{bmatrix} \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \\ \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \approx -\begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}$$

$$J_0(x_0, y_0) \begin{bmatrix} h \\ k \end{bmatrix} \approx -F(x_0, y_0)$$

where $J_0(x_0, y_0) = \begin{bmatrix} \left.\dfrac{\partial f}{\partial x}\right|_{(x_0, y_0)} & \left.\dfrac{\partial f}{\partial y}\right|_{(x_0, y_0)} \\ \left.\dfrac{\partial g}{\partial x}\right|_{(x_0, y_0)} & \left.\dfrac{\partial g}{\partial y}\right|_{(x_0, y_0)} \end{bmatrix}$

and $F(x_0, y_0) = \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}$

$\therefore \begin{bmatrix} h \\ k \end{bmatrix} \approx - J_0^{-1}(x_0, y_0)\, F(x_0, y_0)$

$\Rightarrow \begin{bmatrix} h \\ k \end{bmatrix} \approx - \left.\begin{bmatrix} \dfrac{\partial g}{\partial y} & -\dfrac{\partial f}{\partial y} \\ -\dfrac{\partial g}{\partial x} & \dfrac{\partial f}{\partial x} \end{bmatrix}\right|_{(x_0, y_0)} \cdot \dfrac{1}{|J_0(x_0, y_0)|} \cdot F(x_0, y_0)$ — (1)

We have $\alpha = x_0 + h, \ \beta = y_0 + k$

$\therefore \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x_0 + h \\ y_0 + k \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$

$\approx \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - J_0^{-1}(x_0, y_0) \cdot F(x_0, y_0)$

$\therefore \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - J_0^{-1}(x_0, y_0)\, F(x_0, y_0)$

Similarly, $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - J_1^{-1}(x_1, y_1)\, F(x_1, y_1)$

$\vdots$

Ingeneral, $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J_n^{-1}(x_n, y_n) \cdot F(x_n, y_n)$

$n = 0, 1, 2, 3, \ldots$

$$X_{n+1} = X_n - J_n^{-1} F(X_n), \quad n = 0, 1, 2, 3, \ldots$$

where $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$

$J_n = J_n(x_n, y_n)$

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* Let $f(x, y) = 0$, $g(x, y) = 0$ be a system of non-linear equations

Let $(\alpha, \beta)$ be a solution of the above system

Suppose that $(x_0, y_0)$ be an approximation for $(\alpha, \beta)$

By Newton-Raphson method, we have

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J_n^{-1} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$

where, $J_n = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$

**Pb:** Find an approximation for the solution of the system $x^2 + xy + y^2 = 7$

$x^3 + y^3 = 9$

Take the initial approximation $(1.5, 0.5)$

**Sol:** The given system can be written as

$x^2 + xy + y^2 - 7 = 0$

$x^3 + y^3 - 9 = 0$

$f(x, y) = x^2 + xy + y^2 - 7$

$g(x, y) = x^3 + y^3 - 9$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x + y & x + 2y \\ 3x^2 & 3y^2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{|J|} \cdot \mathrm{Adj}(J)$$

$$= \frac{1}{6xy^2 + 3y^3 - 3x^3 - 6x^2y} \begin{bmatrix} 3y^2 & -(x+2y) \\ -3x^2 & 2x+y \end{bmatrix}$$

$$= \frac{1}{3(y^3 - x^3 + 2xy(y-x))} \begin{bmatrix} 3y^2 & -(x+2y) \\ -3x^2 & 2x+y \end{bmatrix}$$

$$J^{-1} = \frac{1}{3(y-x)[y^2+x^2+3xy]} \begin{bmatrix} 3y^2 & -(x+2y) \\ -3x^2 & 2x+y \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{1}{3(y_n - x_n)(y_n^2 + x_n^2 + 3x_ny_n)} \begin{bmatrix} 3y_n^2 & -(x_n+2y_n) \\ -3x_n^2 & (2x_n+y_n) \end{bmatrix} \begin{bmatrix} x_n^2 + x_ny_n + y_n^2 - 7 \\ x_n^3 + y_n^3 - 9 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} + \frac{4}{57} \begin{bmatrix} 0.75 & -2.5 \\ -6.75 & 3.5 \end{bmatrix} \begin{bmatrix} -3.75 \\ -5.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} + \frac{4}{57} \begin{bmatrix} 10.9\ldots \\ 6.06\ldots \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.2675 \\ 0.9254 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + 0.0202 \begin{bmatrix} -11.39567 \\ 1.934 \end{bmatrix} = \begin{bmatrix} 2.0373 \\ 0.9645 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + (0.02831) \begin{bmatrix} -1.2731 \\ 1.2093 \end{bmatrix}$$

$$= \begin{bmatrix} 2.0373 \\ 0.9645 \end{bmatrix} + \begin{bmatrix} -0.036 \\ 0.0342 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2.0013 \\ 0.9987 \end{bmatrix}}$$

Pb: Solve the following system of linear equations by Gauss-elimination method

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

Sol:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}}_{b}$$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$ ; $R_3 \to R_3 - 2R_1$

$$[A|b] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$[A|b] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} r_1 = 0 \\ r_2 = 1 \\ r_3 = 2 \end{array} \quad r_1 < r_2 < r_3$$

$$x_1 + x_2 + x_3 = 6$$
$$-x_2 + x_3 = 1$$
$$x_3 = 2 \Rightarrow x_2 = 1 \Rightarrow x_1 = 3$$

Pb: Using Gauss-elimination method solve

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}}_{b}$$

Sol:
$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 4 & 0 & 2 & 1 & 8 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array}\right]$$

$R_4 \to 2R_4 - R_1$, $R_3 \to 2R_3 - 3R_1$, $R_2 \to R_2 - 2R_1$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 0 & -2 & 0 & 5 & 28 \\ 0 & 1 & 1 & 6 & 44 \\ 0 & 5 & 3 & 0 & 0 \end{array}\right]$$

$R_4 \to R_4 - 5R_3$ ; $R_3 \to 2R_3 + R_2$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 0 & -2 & 0 & 5 & 28 \\ 0 & 0 & 2 & 17 & 116 \\ 0 & 0 & -2 & -30 & -220 \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 0 & -2 & 0 & 5 & 28 \\ 0 & 0 & 2 & 17 & 116 \\ 0 & 0 & 0 & 13 & -104 \end{array}\right]$$

$\Rightarrow 2x_1 + x_2 + x_3 - 2x_4 = -10$

$-2x_2 + 5x_4 = 28$

$2x_3 + 17x_4 = 116$

$13x_4 = -104 \Rightarrow \boxed{x_4 = -8} \to \boxed{x_3 = 126}$

$\boxed{x_2 = -34}$

$$x_1 = \frac{2x_4 - x_2 - x_3 - 10}{2} = \frac{-16 + 34 - 126 - 10}{2}$$

$\boxed{x_1 = -59}$

## Lower triangular matrix :

A square matrix of the form $\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 \cdots & 0 \\ \vdots & & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$ is called lower triangular matrix.

## Upper triangular matrix :

A square matrix of the form $\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} \cdots & u_{2n} \\ \vdots & & \cdot & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ is called upper triangular matrix

## LU-Decomposition for solving system of linear equations :

Consider the system $A_{n\times n} X_{n\times 1} = b_{n\times 1}$, where A is invertible matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

A

The no. of unknowns in L = 1+2+3 = 6

The no. of unknowns in U = 3+2+1 = 6

∴ The total no. of unknowns in L and U = 6+6
= 12

The no. of equations we have = 9 (from A = LU)

The no. of unknowns to be choosen arbitarly
= 12 - 9 = 3

* If $l_{11} = l_{22} = l_{33} = 1$, then it is called Doolitt's method.

* If $u_{11} = u_{22} = u_{33} = 1$, then this method is called Crout's method.

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**Gauss-Seidel iterative method for solving system of linear equations:**

Suppose that the following system of linear equations has a unique solution.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}$$

Suppose that the matrix $A$ is diagonally dominant matrix i.e,

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

Let $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ be an initial approximation for $(x_1, x_2, x_3)$

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right]$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right]$$

Then, $x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right]$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right]$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right]$$

In general, $x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right]$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right]$$

$$k = 0, 1, 2, 3, \ldots$$

Then the sequence $\{(x_1^{(k)}, x_2^{(k)}, x_3^{(k)})\}_{k=0}^{\infty}$ which converges to the exact solution $(x_1, x_2, x_3)$ of the system $Ax = b$, provided $A$ is diagonally dominant matrix.

Pb: Solve the following system by Gauss-Seidel method.

$$20x_1 + x_2 - 2x_3 = 17$$

$$2x_1 - 3x_2 - 20x_3 = 25$$

$$3x_1 + 20x_2 - x_3 = -18$$

Sol:

$|3| < |2| + |20|$

Clearly the coefficient matrix is NOT diagonally dominant.

* for matrix $A$ to be diagonally matrix. We write the above system as follows:

$$20x_1 + x_2 - 2x_3 = 17$$

$$\cancel{2x_1 + 3x_2 - 20x_3 = 2}$$

$$3x_1 + 20x_2 - x_3 = -18$$

$$2x_1 - 3x_2 - 20x_3 = 25$$

Now clearly the coefficient matrix is diagonally dominant.

$$\therefore\ x_1 = \frac{1}{20}\left[17 - x_2 + 2x_3\right]$$

$$x_2 = \frac{1}{20}\left[-18 - 3x_1 + x_3\right]$$

$$x_3 = \frac{1}{20}\left[2x_1 - 3x_2 - 25\right]$$

$$= -\frac{1}{20}\left[25 - 2x_1 + 3x_2\right]$$

In general, $x_1^{(k+1)} = \frac{1}{20}\left[17 - x_2^{(k)} + 2x_3^{(k)}\right]$

$$x_2^{(k+1)} = \frac{1}{20}\left[-18 - 3x_1^{(k+1)} + x_3^{(k)}\right]$$

$$x_3^{(k+1)} = -\frac{1}{20}\left[25 - 2x_1^{(k+1)} + 3x_2^{(k+1)}\right]$$

$$k = 0, 1, 2, 3, \ldots$$

Let $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0, 0, 0)$

$$x_1^{(1)} = \frac{1}{20}[17] = \underline{0.85}$$

$$\cancel{x_2^{(1)} = \frac{1}{20}[-18] = -0.9}$$

$$x_2^{(1)} = \frac{1}{20}\left[-18 - 3x_1^{(1)} + x_3^{(0)}\right]$$

$$= \frac{1}{20}\left[-18 - 3(0.85)\right] = \underline{-1.0275}$$

$$x_3^{(1)} = -\frac{1}{20}[25 - 2x_1^{(1)} + 3x_2^{(1)}]$$

$$x_3^{(1)} = -1.00875$$

$$x_1^{(2)} = \frac{1}{20}[17 - x_2^{(1)} + 2x_3^{(1)}]$$

$$x_1^{(2)} = 0.8$$

$$x_2^{(2)} = \frac{1}{20}[-18 - 3x_1^{(2)} + x_3^{(1)}]$$

$$x_2^{(2)} = -1.0705$$

$$x_3^{(2)} = -\frac{1}{20}[25 - 2x_1^{(2)} + 3x_2^{(2)}]$$

$$x_3^{(2)} = \cancel{-1.0154} \; -1.0094$$

$$\cancel{x_2^{(3)} = \frac{1}{20}[-18}$$

$$x_1^{(3)} = \frac{1}{20}[17 - x_2^{(2)} + 2x_3^{(2)}] = \boxed{0.80258}$$

$$x_2^{(3)} = \frac{1}{20}[-18 - 3x_1^{(3)} + x_3^{(2)}] = \boxed{-1.0708}$$

$$x_3^{(3)} = -\frac{1}{20}[25 - 2x_1^{(3)} + 3x_2^{(3)}] = \boxed{-1.0091}$$

$$x_1^{(4)} = \frac{1}{20}[17 - x_2^{(3)} + 2x_3^{(3)}] = \boxed{0.80263}$$

$$x_2^{(4)} = \frac{1}{20}[-18 - 3x_1^{(4)} + x_3^{(3)}] = \boxed{-1.0708}$$

$$x_3^{(4)} = -\frac{1}{20}[25 - 2x_1^{(4)} + 3x_2^{(4)}] = \boxed{-1.0091}$$

$$(x_1^{(4)}, x_2^{(4)}, x_3^{(4)}) = (0.8026, -1.0708, -1.0091)$$

Pb: Using Gauss seidel method find an approximate solution of the following system. Perform three iterations.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27.$$

Sol: For matrix A to be diagonally dominant matrix.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

$$\therefore \quad x_1 = \frac{1}{10}[3 + 2x_2 + x_3 + x_4]$$
$$x_2 = \frac{1}{10}[15 + 2x_1 + x_3 + x_4]$$
$$x_3 = \frac{1}{10}[27 + x_1 + x_2 + 2x_4]$$
$$x_4 = \frac{1}{10}[-9 + x_1 + x_2 + 2x_3]$$

Let $x_1^{(0)}, x_2^{(0)}, x_3^{0} = (0, 0, 0)$

$$x_1^{(1)} = 0.3$$
$$x_2^{(1)} = \frac{1}{10}[15 + 2x_1^{(1)} + x_3^{(0)} + x_4^{(0)}] = 1.56$$
$$x_3^{(1)} = \frac{1}{10}[27 + x_1^{(1)} + x_2^{(1)} + 2x_4^{(0)}] = 2.886$$
$$x_4^{(1)} = \frac{1}{10}[-9 + x_1^{(1)} + x_2^{(1)} + 2x_3^{(1)}] = -0.1368$$

$$x_1^{(2)} = \frac{1}{10}(3 + 2x_2^{(1)} + x_3^{(1)} + x_4^{(1)}) = 0.8869$$

$$x_2^{(2)} = \frac{1}{10}(15 + 2x_1^{(2)} + x_3^{(1)} + x_4^{(1)}) = 1.9521$$

$$x_3^{(2)} = \frac{1}{10}(27 + x_1^{(2)} + x_2^{(2)} + 2x_4^{(1)}) = 2.9565$$

$$x_4^{(2)} = \frac{1}{10}(-9 + x_1^{(2)} + x_2^{(2)} + 2x_3^{(2)}) = -0.0248$$

$$x_1^{(3)} = \frac{1}{10}(3 + 2x_2^{(2)} + x_3^{(2)} + x_4^{(2)}) = 0.9836$$

$$x_2^{(3)} = \frac{1}{10}(15 + 2x_1^{(3)} + x_3^{(2)} + x_4^{(2)}) = 1.9899$$

$$x_3^{(3)} = \frac{1}{10}(27 + x_1^{(3)} + x_2^{(3)} + 2x_4^{(2)}) = 2.9924$$

$$x_4^{(3)} = \frac{1}{10}(-9 + x_1^{(3)} + x_2^{(3)} + 2x_3^{(3)}) = -0.00417$$

$$(x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}) = (0.9836, 1.9899, 2.9924, -0.0042)$$

Pb: Find the LU-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

by taking the diagonal elements of the upper triangular matrix $U$ as 1. Hence solve the system $AX = b$, where

$$b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}.$$

Sol: Let $A = LU$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$\Rightarrow l_{11} = 1$ , $l_{21} = 4$ , $l_{31} = 3$

$l_{11} u_{12} = 1$ $\quad l_{11} u_{13} = 1$ $\quad l_{21} u_{12} + l_{22} = 3$

$\Rightarrow u_{12} = 1$ $\quad u_{13} = 1$ $\quad \Rightarrow l_{22} = -1$

$l_{21} u_{13} + l_{22} u_{23} = -1$ $\quad l_{31} u_{12} + l_{32} = 5$

$\Rightarrow 4 - u_{23} = -1$ , $\quad l_{32} = 2$

$u_{23} = 5$

$l_{31} u_{13} + l_{32} u_{23} + l_{33} = 3$

$3 + 10 + l_{33} = 3 \Rightarrow l_{33} = -10$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

We have the system $AX = b$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\Rightarrow LUX = b$ — (1)

Let $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ — (2)

Using (2) in (1) we get

$$LY = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ 4y_1 - y_2 \\ 3y_1 + 2y_2 - 10y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$\Rightarrow y_1 = 1$ ; $y_2 = -2$ ; $y_3 = -\frac{1}{2}$

from (2)

$$UX = Y \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -\frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + 5x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -\frac{1}{2} \end{bmatrix}$$

$\rightarrow$ $x_3 = -\frac{1}{2}$ $\quad x_2 = \frac{1}{2}$ $\quad , x_1 = 1$

Pb: Find the LU-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix}$$ by taking $l_{11} = l_{22} = l_{33} = 1$

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**Sufficient condition for existance of LU-decomposition**

Positive definite matrix: A matrix $A$ is said to be positive definite if the ~~ee~~ eigenvalues of the matrix $A$ are positive.

If the matrix $A$ is ~~de~~ positive definite then LU-decomposition exists for $A$.

Pb: Find the inverse of $A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{bmatrix}$

Sol: Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(W.K.T $A = [Ae_1 \; Ae_2 \; Ae_3]$)

Let $A^{-1} = [c_1 \; c_2 \; c_3]$

$A^{-1} = [A^{-1}e_1 \; A^{-1}e_2 \; A^{-1}e_3]$

~~where~~ $\therefore A^{-1}e_1 = c_1$ ; $A^{-1}e_2 = c_2$ ; $A^{-1}e_3 = c_3$

where $c_1, c_2, c_3$ are columns of the matrix $A^{-1}$.

$\Rightarrow Ac_1 = e_1$, $Ac_2 = e_2$, $Ac_3 = e_3$

Let $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

(by taking $l_{11} = l_{22} = l_{33} = 1$)

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$\Rightarrow u_{11} = 3$ $\quad u_{12} = 0$ $\quad u_{13} = 3$

$l_{21} = 0$ $\quad u_{22} = 1$ $\quad u_{23} = -2$

$l_{31} = 1$ $\quad l_{32} = -2$ $\quad u_{33} = 1$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A c_1 = e_1$$

$$c_1 = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$$

$$LUc_1 = e_1$$

let $Uc_1 = Y_1 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$LY_1 = e_1 \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_1 - 2y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} y_1 = 1 \\ y_2 = 0 \\ y_3 = -1 \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = \begin{bmatrix} 3c_{11} + 3c_{31} \\ c_{21} - 2c_{31} \\ c_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\rightarrow c_{31} = -1, \quad c_{21} = -2, \quad c_{11} = 4/3$$

---

$$A c_2 = e_2 \quad ; \quad c_2 = \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix}$$

$$LUc_2 = e_2$$

$$Uc_2 = Y_2 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow LY_2 = e_2 \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_1 - 2y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 = 0, \quad y_2 = 1, \quad y_3 = 2$$

$$\Rightarrow \begin{bmatrix} 3c_{12} + 3c_{32} \\ c_{22} - 2c_{32} \\ c_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{matrix} c_{32} = 2 \\ c_{22} = 5 \end{matrix}, \quad c_{12} = -2$$

---

$$A c_3 = e_3 \quad ; \quad c_3 = \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} \qquad y_1 = 0, \; y_2 = 0, \; y_3 = 1$$

$$LUc_3 = e_3$$

$$Uc_3 = Y_3$$

$$\begin{bmatrix} 3c_{13} + 3c_{33} \\ c_{23} - 2c_{33} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} c_{33} = 1 \\ c_{23} = 2 \\ c_{13} = -1 \end{matrix}$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 4/3 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 1 \end{bmatrix}}$$

Q. Let $A = \begin{bmatrix} 2 & -6 & 10 \\ 1 & 5 & 1 \\ -1 & 15 & -5 \end{bmatrix}$

Solve the following systems

$Ax_1 = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$ $\quad Ax_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ $\quad Ax_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$Ax_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\quad Ax_5 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ $\quad Ax_6 = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

**Note**: If $A$ is a sparse matrix, then Gauss-Seidel iterative method is a suitable method for solving $Ax = b$ compare to Gauss elimination and LU-decomposition method.

* Sparse matrix $\Rightarrow$ No of ~~elem~~ 0's in matrix $>$ No. of non-zeros.

* Let $A_{n\times n}$ be a matrix. A scalar $\lambda$ ~~such~~ is said to be eigenvalue of the matrix $A$ if there exists a non-zero vector $X \in \mathbb{R}^n$ such that

$$AX = \lambda X$$ , and $X$ is called the eigenvector of the matrix $A$ corresponding to the eigen value $\lambda$.

$$(A - \lambda I)X = 0_{n\times 1}$$

$$\therefore X \neq 0 \Leftrightarrow |A - \lambda I| = 0$$

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* Let $A_{n\times n}$ be a matrix

The vectors $x_1, x_2, \dots x_n$ are L.I eigenvectors of the matrix $A_{n\times n}$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$ respectively
$\Leftrightarrow \exists$ matrix $P$ such that

$$P^{-1}AP = D, \text{ where } P = [x_1, x_2, \dots x_n]$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & 0 \cdots & \lambda_n \end{bmatrix}$$

* A matrix $B_{n\times n}$ is said to be orthogonal matrix if $BB^T = B^T B = I_{n\times n} \Leftrightarrow |B||B^T| = 1 \Leftrightarrow |B| = \pm 1$

Also, $B^T = B^{-1}$

$B_{n\times n}$ is orthogonal matrix $\Leftrightarrow$ The columns of $B_{n\times n}$ are orthonormal vectors.

Ex: $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

→ orthogonal matrix

---

Let $A_{n\times n}$ be a symmetric matrix
($\therefore A^T = A$)

Then the eigenvectors of $A_{n\times n}$ are orthonormal vectors. The columns of $P$ are orthogonormal.

$P$ is orthogonal matrix

$\therefore PP^T = P^TP = I$

$\Leftrightarrow \boxed{P^{-1} = P^T}$

* Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Let $T_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

clearly $T_1$ is orthogonal matrix

That is $T_1 T_1^T = T_1^T T_1 = I$

$\Rightarrow \underline{T_1^{-1} = T_1^T}$

$$T_1^T A T_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos\theta + b\sin\theta & b\cos\theta + c\sin\theta \\ -a\sin\theta + b\cos\theta & -b\sin\theta + c\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta & -a\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) + c\sin\theta\cos\theta \\ -a\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) + c\sin\theta\cos\theta & a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta \end{bmatrix}$$

$$T_1^T A T_1 = \begin{bmatrix} \overbrace{a\cos^2\theta + c\sin^2\theta + b\sin 2\theta}^{\lambda_1} & b\cos 2\theta + (c-a)\sin\theta\cos\theta \\ b\cos 2\theta + (c-a)\sin\theta\cos\theta & \underbrace{a\sin^2\theta + c\cos^2\theta - b\sin 2\theta}_{\lambda_2} \end{bmatrix}$$

$\downarrow$

$T_1^{-1} A T_1$

$= B_1$

$B_1$ is a diagonal matrix $\Leftrightarrow$ $b\cos 2\theta + (c-a)\sin\theta\cos\theta = 0$

$\Rightarrow b\cos 2\theta + \frac{c-a}{2}\sin 2\theta = 0$

$\rightarrow \tan 2\theta = \frac{2b}{a-c}$

$\rightarrow \boxed{\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right)}$

Here the remaining elements of $B_1$ are eigen values

eg: Let $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$\theta = \frac{\tan^{-1}\left(\frac{4}{0}\right)}{2} = \pi/4$

$\lambda_1 = 0 + 0 + 2\sin(\pi/2) = 2$

$\lambda_2 = 0 + 0 - 2\sin(\pi/2) = -2$

Pb: Using the Jacobi method find an approximation for all the eigen values of the matrix.

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

sol:

Clearly $a_{13}$ is the largest off-diagonal element numerically in the matrix A.

Form a 2x2 matrix $B_1 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$

$B_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Let $T_1 = \begin{bmatrix} \cos\theta, & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$T_1^T B_1 T_1$ is a diagonal matrix $\Leftrightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right)$

$\Longleftrightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{4}{0}\right) = \pi/4$

$$S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

taking $\theta = \pi/4$, we get

$$S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$S_1^T A S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{2} & 2 & 3/\sqrt{2} \\ \sqrt{2} & 3 & \sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = E_1.$$

Clearly $a_{12} = 2$ is the largest off-diagonal element in the matrix $E_1$ numerically.

Form the $2\times2$ matrix, $B_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

Let $T_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$T_1^T B_2 T_1$ is diagonal matrix $\Leftrightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right)$

$\Rightarrow \theta = \pi/4$

$\therefore S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(for $S_2$ ~~or~~ $S_1$ here we are writing first Identity matrix and replacing the elements of $B_2 \rightarrow a_{11}\ a_{12}\ a_{21}\ a_{22}$ as $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$)

Take $\theta = \pi/4$

$S_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$S_2^T E_1 S_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = E_2$

We have $S_2^T E_1 S_2 = E_2$

$S_2^T S_1^T A S_1 S_2 = E_2$

$(S_1 S_2)^T A (S_1 S_2) = E_2$

Let $P = S_1 S_2$

$\Rightarrow P^T A P = E_2$

$\Rightarrow P^{-1} A P = E_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\therefore$ eigenvalues $\to 5, 1, -1$

eigenvectors are the coloumns of 'P'

$$P = S_1 S_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

## Disadvantages of Jacobi Method :

1) It is NOT applicable for non-symmetric matrices.
2) The off-signal elements which became zero in an iteration may be non-zero in the subsequent iterations.

## Power Method :

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of a matrix $A_{n\times n}$ such that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$.

Let $x_1, x_2, \ldots x_n$ be the eigenvectors of $A_{n\times n}$ corresponding to $\lambda_1, \lambda_2, \ldots \lambda_n$ respectively.

$\therefore Ax_1 = \lambda_1 x_1, \; Ax_2 = \lambda_2 x_2, \ldots \; Ax_n = \lambda_n x_n$

Let $x \in R^n$

$\therefore \exists$ scalars $c_1, c_2, c_3, \ldots c_n \in R \ni$

$$x = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

$$\Rightarrow Ax = c_1 A x_1 + c_2 A x_2 + \ldots + c_n A x_n$$

$$\Rightarrow Ax = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \ldots + c_n \lambda_n x_n$$

$$\Rightarrow A(Ax) = c_1 \lambda_1 A x_1 + c_2 \lambda_2 A x_2 + \ldots + c_n \lambda_n A x_n$$

$$\Rightarrow A^2 x = c_1 \lambda_1^2 x_1 + c_2 \lambda_2^2 x_2 + \ldots + c_n \lambda_n^2 x_n$$

In general, $A^k x = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \ldots + c_n \lambda_n^k x_n$

$k \in N$

$$\Rightarrow A^k x = \lambda_1^k \left[ c_1 x_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k x_n \right]$$

$$\underset{k\to\infty}{Lt}\, A^k x = \underset{k\to\infty}{Lt}\, \lambda_1^k \left[ c_1 x_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k x_n \right] \quad \text{--- (1)}$$

But $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_{n-1}| \geq |\lambda_n|$

$$\Rightarrow \left|\frac{\lambda_2}{\lambda_1}\right| < 1 \ ; \ \left|\frac{\lambda_3}{\lambda_1}\right| < 1 \ ; \cdots \left|\frac{\lambda_n}{\lambda_1}\right| < 1$$

$\therefore$ from (1), we get

$$\underset{k\to\infty}{Lt}\, A^k x = \underset{k\to\infty}{Lt}\, \lambda_1^k c_1 x_1$$

Similarly, $\underset{k\to\infty}{Lt}\, A^{k+1} x = \underset{k\to\infty}{Lt}\, \lambda_1^{k+1} c_1 x_1$

Let $x_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$\therefore \underset{k\to\infty}{Lt}\, A^k x = \underset{k\to\infty}{Lt}\, \lambda_1^k c_1 \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \underset{k\to\infty}{Lt} \begin{bmatrix} \lambda_1^k c_1 a_1 \\ \lambda_1^k c_2 a_2 \\ \vdots \\ \lambda_1^k c_1 a_n \end{bmatrix}$$

Let $(A^k x)_r$ be the $r^{th}$ component of the vector $A^k x$

$$\therefore \underset{k\to\infty}{Lt}\, (A^k x)_r = \underset{k\to\infty}{Lt}\, \lambda_1^k c_1 a_r \ ; r = 1, 2, 3, \ldots n \quad \text{--- } (*)$$

Similarly, $\underset{k\to\infty}{Lt}\, (A^{k+1} x)_r = \underset{k\to\infty}{Lt}\, \lambda_1^{k+1} . c_1 . a_r \ ; r = 1, 2, 3, \ldots n \quad \text{--- } (**)$

$$\frac{(**)}{(*)} \Rightarrow \underset{k\to\infty}{Lt} \frac{(A^{k+1} x)_r}{(A^k x)_r} = \underset{k\to\infty}{Lt} \frac{\lambda_1^{k+1} c_1 a_r}{\lambda^k c_1 a_r} = \lambda_1$$

$$\Rightarrow \lambda_1 = \underset{k\to\infty}{Lt} \frac{(A (A^k x))_r}{(A^k x)_r} \ ; r = 1, 2, 3, \ldots n$$

input vector

$\Rightarrow \lambda_1 = \underset{k\to\infty}{Lt} \frac{(A(IVP))_r}{(IVP)_r}$, where $IVP = A^k x$

$k \to \infty$ means, after a large number of iterations.

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$\lambda_1 =$ After sufficien no. of iterations $\frac{(A(IVP))_r}{(IVP)_r}$, $r = 1, 2, 3, \ldots n$

$$\lambda_1 = \max \left\{ \left|\frac{(A(IVP))_1}{(IVP)_1}\right|, \left|\frac{(A(IVP))_2}{(IVP)_2}\right|, \ldots \left|\frac{(A(IVP))_n}{(IVP)_n}\right| \right\}.$$

Let $V_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Iteration-1: Let $AV_0 = y_1$

Let $m_1$ be the largest component of $y_1$ numerically

Define $V_1 = \frac{1}{m_1} y_1$

Iteration-2: Let $AV_1 = y_2$

Let $m_2$ be the largest component of the vector $y_2$ numerically

Define $V_2 = \frac{1}{m_2} y_2$.

Iteration-3: Let $AV_2 = y_3$

Let $m_3$ be the largest component of the vector $y_3$ numerically

Define $V_3 = \frac{1}{m_3} y_3$

$\vdots$

Iteration-k: Let $AV_{k-1} = y_k$, where $V_{k-1}$ is the vector obtained from $(k-1)^{th}$ iteration

~~Let $m_1$~~

$$|\lambda_1| = \max \left\{ \frac{|(y_k)_1|}{|(V_{k-1})_1|}, \frac{|(y_k)_2|}{|(V_{k-1})_2|}, \ldots \frac{|(y_k)_n|}{|(V_{k-1})_n|} \right\}$$

Also, $V_{k-1}$ is the eigenvector corresponding to $\lambda_1$.

Pb: Using the Power method, find the largest eigenvalue numerically and the corresponding eigenvector of the matrix

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$ Use 9 iterations

Sol:

Let $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Iteration 1: $A V_0 = \begin{bmatrix} -8 \\ 4 \\ 18 \end{bmatrix} = y_1 \Rightarrow m_1 = 18$

$V_1 = \frac{1}{m_1} y_1 = \begin{bmatrix} -4/9 \\ 2/9 \\ 1 \end{bmatrix}$

Iteration 2: $A V_1 = \begin{bmatrix} 10.55 \\ -1.11 \\ -7.77 \end{bmatrix} = y_2 \Rightarrow m_2 = 10.55$

$V_2 = \frac{y_2}{m_2} = \begin{bmatrix} 1 \\ -0.105 \\ -0.736 \end{bmatrix}$

Iteration 3: $A V_2 = \begin{bmatrix} -17.631 \\ 6.842 \\ 18.947 \end{bmatrix} = y_3 \quad m_3 = 18.947$

$V_3 = \frac{y_3}{m_3} = \begin{bmatrix} -0.93 \\ 0.3611 \\ 1 \end{bmatrix}$

Iteration 4: $A V_3 = \begin{bmatrix} 18.403 \\ -7.639 \\ -18.05 \end{bmatrix} = y_4 \quad m_4 = 18.403$

$V_4 = \frac{y_4}{m_4} = \begin{bmatrix} 1 \\ -0.415 \\ -0.981 \end{bmatrix}$

Iteration 5 : $AV_4 = \begin{bmatrix} -19.6 \\ 9.094 \\ 19.698 \end{bmatrix}$ $m_5 = 19.698$

$V_5 = \frac{y_5}{m_5} = \begin{bmatrix} -0.995 \\ 0.461 \\ 1 \end{bmatrix}$

Iteration 6 : $AV_5 = \begin{bmatrix} 19.775 \\ -9.492 \\ -19.75 \end{bmatrix} = y_6$ $m_6 = 19.775$

$V_6 = \frac{y_6}{m_6} = \begin{bmatrix} 1 \\ -0.48 \\ -0.998 \end{bmatrix}$

Iteration 7 : $AV_6 = \begin{bmatrix} -19.91 \\ 9.7675 \\ 19.922 \end{bmatrix} = y_7$ $m_7 = 19.922$

$V_7 = \frac{y_7}{m_7} = \begin{bmatrix} -0.9997 \\ 0.4902 \\ 1 \end{bmatrix}$

Iteration 8 : $AV_7 = \begin{bmatrix} 19.957 \\ -9.88 \\ -19.955 \end{bmatrix} = y_8$ $m_8 = 19.957$

$V_8 = \frac{y_8}{m_8} = \begin{bmatrix} 1 \\ -0.495 \\ -0.999 \end{bmatrix}$

Iteration 9 : $AV_8 = \begin{bmatrix} -19.9803 \\ 9.9416 \\ 19.9807 \end{bmatrix} = y_9$ $m_9 = 19.9807$

$\cancel{V_9 = \frac{y_9}{m_9} = \begin{bmatrix} -0.999 \\ 0.4975 \\ 1 \end{bmatrix}}$

$|\lambda_1| = \max \left\{ \frac{|(y_9)_1|}{|(V_8)_1|}, \frac{|(y_9)_2|}{|(V_8)_2|}, \frac{|(y_9)_3|}{|(V_9)_3|} \right\}$

$= \max \left\{ \frac{19.9803}{1}, \frac{9.9416}{0.495}, \frac{19.9807}{0.999} \right\}$

$= \max \{ 19.9803, 20.084, 20.00 \}$

$|\lambda_1| = 20.084 \Rightarrow \boxed{\lambda_1 = \pm 20.084}$

The exact eigenvalues of A are $-20, 5, -10$

$V_8$ is the corresponding eigen vector for the numerically largest eigen value.

Pb: Find the smallest eigenvalue numerically of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, Use four iterations of power method.

Sol:

- eigen values → matrix

$\lambda_1, \lambda_2, \lambda_3 \longrightarrow A$

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \Leftarrow A^{-1}$

→ Largest eigen value of $A^{-1}$ we can get.

Its reciprocal will be the smallest eigen value for A.

First Find $A^{-1}$, and find approximation for the largest eigenvalue numerically of $A^{-1}$.

$$A^{-1} = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}$$

Let $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Iteration 1: $A^{-1} V_0 = \begin{bmatrix} 1.5 \\ 2 \\ 1.5 \end{bmatrix} = y_1$ $\quad m_1 = 2$

$$V_1 = \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix}$$

Iteration 2 : $A^{-1} V_1 = \begin{bmatrix} 1.25 \\ 1.75 \\ 1.25 \end{bmatrix} = y_2$ $m_2 = 1.75$

$$V_2 = \frac{y_2}{m_2} = \begin{bmatrix} 0.71428 \\ 1 \\ 0.71428 \end{bmatrix}$$

Iteration 3 : $A^{-1} V_2 = \begin{bmatrix} 1.2142 \\ 1.7142 \\ 1.2142 \end{bmatrix} = y_3$ , $m_3 = 1.7142$

$$V_3 = \frac{y_3}{m_3} = \begin{bmatrix} 0.7083 \\ 1 \\ 0.7083 \end{bmatrix}$$

Iteration 4 : $A^{-1} V_3 = \begin{bmatrix} 1.2083 \\ 1.7084 \\ 1.2083 \end{bmatrix} = y_4$

$$|\lambda_0| = \max\left\{ \frac{|(y_4)_1|}{|(V_3)_1|}, \frac{|(y_4)_2|}{|(V_3)_2|}, \frac{|(y_4)_3|}{|(V_4)_3|} \right\}$$

$$= \max\left\{ \frac{1.2083}{0.7083}, \frac{1.7084}{1}, \frac{1.2083}{0.7083} \right\}$$

$$= \max\{1.7059, 1.7084, 1.7059\}$$

$\boxed{\lambda_0 = \pm 1.7084}$ → largest eigenvalue for $A^{-1}$

⇒ smallest eigenevalue numericall of A is

$$\pm \frac{1}{1.7084} = \pm 0.5853$$

$$\boxed{\pm 0.5853}$$

The exact eigen values of A are $2, 2+\sqrt{2}, 2-\sqrt{2}$
↓ 3.414, ↓ 0.5857

For $\lambda = \pm 0.5853$ Check $|A - \lambda I|$

- $|A - 0.5853 I| = 0.0019$ ✓ ⇒ $\boxed{\lambda_1 = 0.5853}$ is the eigenvalue.

$|A + 0.5853 I| = 1.210$ ✗

* We got the smallest eigen value ($\lambda_1$). We can also find the largest eigen value. Now, for the ($\lambda_2$)

third eigenvalue of A we know that

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace}(A)$$

$$\rightarrow \boxed{\lambda_2 = \text{Trace}(A) - \lambda_1 - \lambda_3}$$

For $\lambda_3$: $v_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Iteration 1:

$$A v_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = y_1 \qquad m_1 = 1$$

$$v_1 = \frac{y_1}{m_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration 2:

$$A v_1 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = y_2 \qquad m_2 = 2$$

$$v_2 = \frac{y_2}{m_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Iteration 3:

$$A v_2 = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = y_3 \qquad m_3 = 4$$

$$v_3 = \frac{y_3}{m_3} = \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix}$$

Iteration 4:

$$A v_3 = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = y_4 \qquad m_4 = 3.5$$

$$v_4 = \begin{bmatrix} 0.7143 \\ -1 \\ 0.7143 \end{bmatrix}$$

Iteration 5:

$$A v_4 = \begin{bmatrix} 2.428 \\ -3.428 \\ 2.428 \end{bmatrix} = y_5$$

$$|\lambda_3| = \max \left\{ \frac{|2.428|}{|0.7143|}, 3.428, \frac{2.428}{0.7143} \right\}$$

$$|\lambda_3| = 3.428$$

$$\lambda_3 = \pm 3.428$$

for $\lambda_3 = 3.428 \rightarrow |A - \lambda I| = -0.05 \simeq 0$ ✓

$$\boxed{\lambda_3 = 3.428}$$

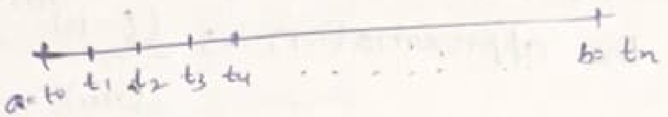
$\lambda_2 = 6 - (0.5853) - (3.428)$

$\boxed{\lambda_2 = 1.9867}$

The eigen values are $0.5853$, $1.9867$, $3.428$.

For differential equations:

* Consider $\dfrac{dy}{dt} = f(t, y)$, $y(t_0) = \alpha$, $t \in [a, b]$




$t_0 < t_1 < t_2 < t_3 \ldots t_{n-1} < t_n$

Let $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 \ldots = t_n - t_{n-1} = h$

where $h$ is called step size (or) step length.

we find $y(t_1), y(t_2), y(t_3) \ldots$ approximately by a numerical method

$t_1 - t_0 = h$
$t_2 - t_1 = h$
$t_3 - t_2 = h$
$\vdots$
$t_n - t_{n-1} = h$

---

$t_n - t_0 = nh \Rightarrow \boxed{h = \dfrac{t_n - t_0}{n}}$

$\Rightarrow h = \dfrac{b-a}{n}$, where $n$ is the no. of subintervals.

$\Rightarrow n = \dfrac{b-a}{h}$.

Taylor method for solving First Order ODE:

Consider $\dfrac{dy}{dt} = f(t, y)$, $y(t_0) = \alpha$

By taylor theorem, we get

$$y(t) = y(t_0) + \frac{t - t_0}{1!} y'(t_0) + \frac{(t-t_0)^2}{2!} y''(t_0) + \ldots + \frac{(t-t_0)^n}{n!} y^{(n)}(t_0) + \frac{(t-t_0)^{n+1}}{(n+1)!} y^{(n+1)}(c)$$

$c \in (t_0, t)$

Neglecting $\frac{(t-t_0)^{n+1}}{(n+1)!} y^{(n+1)}(c)$, we get,

$$y(t) \approx \underbrace{\boxed{y(t_0) + \frac{(t-t_0)^1}{1!} y'(t_0) + \frac{(t-t_0)^2}{2!} y''(t_0) + \dots + \frac{(t-t_0)^n}{n!} y^{n}(t_0)}}_{P_n(t)}$$

$P_n(t)$ approximates the function $y(t)$.

The error in the approximation is $\frac{(t-t_0)^{n+1}}{(n+1)!} y^{(n+1)}(c)$.

Pb: Let $\frac{dy}{dt} = y^2 + t^2$, $y(0) = 1$. Find the second degree Taylor polynomial which approximates the solution of the above ODE. Hence find an approximation for $y(0.4)$

Sol:

$$P_2(t) = y(t_0) + \frac{(t-t_0)}{1!} y'(t_0) + \frac{(t-t_0)^2}{2!} y''(t_0)$$

Here $t_0 = 0$, $y(t_0) = 1$

$$y'(t) = y^2 + t^2 = (y(t))^2 + t^2$$

$$y'(0) = (y(0))^2 + 0^2 = 1$$

$$y''(t) = 2yy' + 2t$$

$$y''(0) = 2\, y(0)\, y'(0) = 2$$

$$\therefore P_2(t) = 1 + \frac{(t-0)}{1!}(1) + \frac{(t-0)^2}{2!}(2)$$

$$\Rightarrow \boxed{P_2(t) = 1 + t + t^2}$$

$$P_2(t) \approx y(t)$$

$$y(0.4) \approx P_2(0.4) = \boxed{1.56}$$

Pr: Consider $y''' + yy'' = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.
Find the 7th degree Taylor polynomial $P_7(t)$ which approximates $y(t)$. Hence, find $y(0.2)$

Sol:

$$P_7(t) = y(t_0) + \frac{(t-t_0)}{1!} y'(t_0) + \frac{(t-t_0)^2}{2!} y''(t_0) + \frac{(t-t_0)^3}{3!} y'''(t_0) + \cdots + \frac{(t-t_0)^7}{7!} y^{(7)}(t_0)$$

Clearly $t_0 = 0$

$\therefore y(0) = 0 \Rightarrow y(t_0) = 0$

$y'(0) = 0 \Rightarrow y'(t_0) = 0$

$y''(0) = 1 \Rightarrow y''(t_0) = 1$

From the given equation $y''' + yy'' = 0$

$$y'''(t) = -y(t)\, y''(t) \quad - \circledast$$

$$y'''(t_0) = -y(t_0)\, y''(t_0) = 0(1) = 0$$

Diff $\circledast$ w.r.t 't':

$$y''''(t) = -y'(t)\, y''(t) - y(t)\, y'''(t) \quad -①$$

$$y''''(t_0) = -y'(t_0)\, y''(t_0) - y(t_0)\, y'''(t_0)$$

$$y^{(4)}(t_0) = 0 - 0 = 0$$

Diff ① w.r.t 't';

$$y^{(5)}(t) = -(y'')^2 - y'y''' - y'y''' - y\, y^{(4)} \quad -②$$

$$y^5(t_0) = -1 - 0 = -1$$

Diff ② w.r.t 't';

$$y^{(6)}(t) = -2(y'')(y''') - 2(y''y''' + y'y^{(4)}) - y'y^{(4)} - y\, y^{(5)} \quad -③$$

$$y^{(6)}(t_0) = -(0)(0) = 0$$

Diff ③ w.r.t 't'

$$y^{(7)}(t_0) = \cancel{-2\left((y''')^2 + y''y^{(4)}\right)}$$

$$= -4\left((y^{(4)})^2 + y''y^{(4)}\right) - 2\left(y''y^{(5)} + y'y^{(7)}\right)$$

$$- y'y^{(5)} - y''y^{(6)} - y'y^{(5)} - y'y^{(6)}$$

$$y^{7}(0) = \cancel{-(1)(-1) - (1)(-1)} \quad \boxed{0}$$

$$= -2$$

$$P_5 = \frac{t^2}{2} - \frac{t^5}{5!}$$

---

* $y(t) = P_n(t) + \frac{(t-t_0)^{n+1}}{(n+1)!} y^{(n+1)}(c)$ , $c \in (t_0, t)$

$$\text{Error} = \frac{(t-t_0)^{n+1}}{(n+1)!} y^{(n+1)}(c)$$

Clearly, Error in the approximation is very small, if $t$ is close to $t_0$ and '$n$' is large.

Consider $y'(t) = f(t, y(t))$ , $y(t_0) = y_0$ ; $t \in [a, b]$

Let $h = \frac{b-a}{n}$

$a = t_0 \quad t_1 \quad t_2 \quad t_3 \quad \cdots \quad t_{n-1} \quad t_n$

$$t_1 - t_0 = t_2 - t_1 = \cdots = t_n - t_{n-1} = h$$

$$y(t) = y(t_{i-1}) + \frac{(t-t_{i-1})}{1!} y'(t_{i-1}) + \frac{(t-t_{i-1})^2}{2!} y''(t_{i-1}) + \cdots + \frac{(t-t_{i-1})^n}{n!} y^{(n)}(t_{i-1}) + \frac{(t-t_{i-1})^{n+1}}{(n+1)!} y^{(n+1)}(c)$$

$c \in (t_{i-1}, t_i)$

Taking $t = t_i$

$$y(t_i) = y(t_{i-1}) + \frac{(t_i - t_{i-1})^1}{1!} y'(t_{i-1}) + \frac{(t_i - t_{i-1})^2}{2!} y''(t_{i-1})$$
$$+ \cdots + \frac{(t_i - t_{i-1})^n}{n!} y^{n}(t_{i-1}) + \frac{(t_i - t_{i-1})^{n+1}}{(n+1)!} y^{(n+1)}(c)$$

$c \in (t_{i-1}, t_i)$

Let $y(t_i) = y_i$, $y^{(n)}(t_i) = y_i^{(n)}$

$$\Rightarrow y_i = y_{i-1} + \frac{h}{1!} y^{(1)}_{i-1} + \frac{h^2}{2!} y^{(2)}_{i-1} + \cdots + \frac{h^n}{n!} y^{(n)}_{i-1}$$
$$+ \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(c)$$

Neglecting $\frac{h^{n+1}}{(n+1)!} y^{(n+1)}(c)$, we get

Taylor method of order 'n'

$$\boxed{y_i \approx y_{i-1} + h y^{(1)}_{i-1} + \frac{h^2}{2!} y^{(2)}_{i-1} + \cdots + \frac{h^n}{n!} y^{(n)}_{i-1}} \; ; \quad -(1)$$

$j = 1, 2, 3, \ldots n$

$j=1 \Rightarrow y_1 \approx y_0 + h y_0' + \frac{h^2}{2!} y_0^{(2)} + \cdots + \frac{h^n}{n!} y_0^{(n)}$

$j=2 \Rightarrow y_2 \approx y_1 + h y_1' + \frac{h^2}{2!} y_1^{(2)} + \cdots + \frac{h^n}{n!} y_1^{(n)}$

Taking $n = 1$ in (1), we get

$$\boxed{y_i \approx y_{0\,i-1} + h y^{(1)}_{i-1} \; ; \; i = 1, 2, 3, \ldots n} \rightarrow \text{Euler method.}$$

Pb: Consider $y' = e^{-(t^2+y^2)}$, $y(0) = -1$. Find $y(0.1)$, $y(0.2)$, $y(0.3)$ using Euler method.

Sol:

$t_0 = 0$ $\quad 0.1 = t_1$ $\quad 0.2 = t_2$ $\quad 0.3 = t_3$ $\quad \Rightarrow h = 0.1$

By Euler method,

$$y_i = y_{i-1} + h\, y_{i-1}^{(1)} \quad ; \quad i = 1, 2, 3$$

$i=1$ ; $\quad y_1 \approx y_0 + h\, y_0^{(1)}$

$y(0) = -1, \quad y'(0) = e^{-(-1)} = 0.3679$

$$y(t_1) \approx y(t_0) + h\, y'(t_0)$$

$$\approx -1 + (0.1)(0.3679)$$

$$y(t_1) \approx -0.96321$$

$i=2$, $\quad y(t_2) \approx y(t_1) + h\, y'(t_1)$

$$\approx y(t_1) + h\left(e^{-(t_1^2 + (y(t_1))^2)}\right)$$

$$\approx -0.924$$

$$y(t_3) \approx -0.883$$

**Pb:** Find all the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

Use Jacobi method. Iterate the Jacobi method till the off-diagonal elements in magnitude are less than 0.0005.

**Sol:** Clearly $a_{12} = 2$ is the largest off-diagonal element numerically.

$$B_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\theta = \frac{\tan^{-1}\left(\frac{2(2)}{1-1}\right)}{2} = \pi/4$$

$$S_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Ans:

$$\begin{bmatrix} 3.3728 \\ -2.37221 \\ 2 \end{bmatrix}$$

$\theta = \pi/4$

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1^{-1} = S_1^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1^{-1} A S_1 = \begin{bmatrix} 3 & 0 & 0.7071 \\ 0 & -1 & 2.12132 \\ 0.7071 & 2.12132 & 1 \end{bmatrix} = E_1$$

clearly $a_{23} = 2.12132$ is the largest off-diagonal element in $E_1$

$$B_2 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -1 & 2.12132 \\ 2.12132 & 1 \end{bmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right) = -0.5651 \text{ rad}$$

$$S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.84451 & +0.5355 \\ 0 & -0.5355 & 0.84451 \end{bmatrix}$$

$$S_2^{-1} = S_2^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.84451 & -0.5355 \\ 0 & +0.5355 & 0.84451 \end{bmatrix}$$

$$S_2^{-1} E_1 S_2 = \begin{bmatrix} 3 & -0.3786 & 0.5971 \\ -0.3786 & -2.3451 & 0.00013 \\ 0.5971 & 0.00013 & 2.3451 \end{bmatrix}$$

$$= E_2'$$

Clearly $a_{31} = 0.5971$ is the largest off diagonal element in $E_2$

$$B_3 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0.5971 \\ 0.5971 & 2.3451 \end{bmatrix}$$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2b}{a-c}\right) = 0.53459 \text{ rad}$$

$$S_3 = \begin{bmatrix} 0.8604 & 0 & -0.5094 \\ 0 & 1 & 0 \\ 0.5094 & 0 & 0.8604 \end{bmatrix}$$

$$S_3^{-1} E_2 S_3 = \begin{bmatrix} 3.3527 & -0.325 & 5\times10^{-5} \\ -0.325 & -2.345 & 0.19 \\ 5\times10^{-5} & 0.1929 & 1.9911 \end{bmatrix}$$

After solving 6 iterations we get the eigen values as $3.3728, -2.37221, 2$.

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## Modified Euler method

Consider $\frac{dy}{dt} = f(t, y)$, $y(t_0) = \alpha$, $t \in [a, b]$

Let $h$ be the step size

$t_0 = a \quad t_1 \quad t_2 \quad t_3 \quad \cdots \quad t_{i-1} \quad t_i \quad \cdots \quad t_{n-1} \quad t_n = b$

Euler method : $y_i = y_{i-1} + h\, y'_{i-1}$ ; $i = 1, 2, 3 \ldots n$

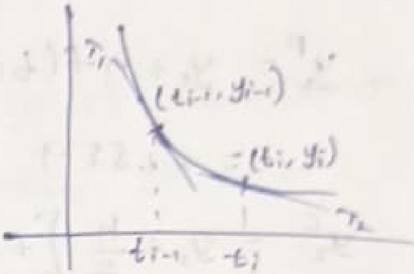
$$\Rightarrow y_i = y_{i-1} + h\, y'(t_{i-1}) \quad — (*)$$

$$y'(t) = \frac{dy}{dt} = f(t, y(t)) \rightarrow y'(t_{i-1}) = f(t_{i-1}, y(t_{i-1}))$$

$$\therefore (*) \Rightarrow y_i = y_{i-1} + h\, f(t_{i-1}, y_{i-1})$$

$\therefore y_i = y_{i-1} + h.$ slope of the tangent at $(t_{i-1}, y_{i-1})$

$$y_i = y_{i-1} + h\left[\frac{y'(t_{i-1}) + y'(t_i)}{2}\right]$$




$$\Rightarrow y_i = y_{i-1} + \frac{h}{2}\left[y'_{i-1} + y'_i\right]$$

$$y_i = y_{i-1} + \frac{h}{2}\left[f(t_{i-1}, y_{i-1}) + f(t_i, y_i)\right]$$

Modified Euler method

$$y_i^p = y_{i-1} + h\, f(t_{i-1}, y_{i-1}) \quad , i = 1, 2, 3, \ldots n$$

$$y_i^c = y_{i-1} + \frac{h}{2}\left[f(t_{i-1}, y_{i-1}) + f(t_i, y_i^p)\right] \quad ; i = 1, 2, \ldots n$$

p - predictor

c - collector.

Pb: Consider $\frac{dy}{dt} = \sin(e^t + e^{-t} + y^2)$ , $y(1) = -2$

Find an approximation for, $y(1.1)$, $y(1.2)$ and $y(1.3)$ by Modified Euler method.

Sol: we have, $f(t, y) = \sin(e^t + e^{-t} + y^2)$

$$t_0 = 1,\ t_1 = 1.1,\ t_2 = 1.2,\ t_3 = 1.3$$

$$\therefore h = 0.1, \quad y_0 = -2$$

We have to compute $y_1$, $y_2$ and $y_3$

$$y_1^p = y_0 + h\, f(t_0, y_0)$$

$$= -1.928$$

$y_1^c = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f(t_1, y_1^p)\right]$

$= -2 + \frac{0.1}{2}\left[f(1, -2) + f(1.1, -1.928)\right]$

$\boxed{y_1^c = -1.9291}$

$\therefore \boxed{y_1 = -1.929}$

$y_2^p = y_1 + h f(t_1, y_1)$

$= -1.859$

$y_2^c = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f(t_2, y_2^p)\right]$

$= -1.858$

$\therefore \boxed{y_2 = -1.858}$

$y_3^p = y_2 + h f(t_2, y_2) = -1.787$

$y_3^c = y_2 + \frac{h}{2}\left[f(t_2, y_2) + f(t_3, y_3^p)\right] = \boxed{-1.784}$

Taylor Method of order 4 :

$\frac{dy}{dt} = f(t, y), \quad y(t_0) = \alpha$

$y_i = y_{i-1} + h y'_{i-1} + \frac{h^2}{2!} y''_{i-1} + \frac{h^3}{3!} y'''_{i-1} + \frac{h^4}{4!} y^{iv}_{i-1}$

Runge-kutta (RK) Method of order 4 :

Consider $\frac{dy}{dt} = f(t, y), \; y(t_0) = \alpha, \; t \in [a, b]$

Let $h$ be the step size

$a = t_0 \quad t_1 \quad t_2 \quad \cdots \quad t_{n-1} \quad t_n = b$

Let $k_1 = hf(t_{i-1}, y_{i-1})$

$k_2 = hf(t_{i-1} + \frac{h}{2}, y_{i-1} + \frac{k_1}{2})$

$k_3 = hf(t_{i-1} + \frac{h}{2}, y_{i-1} + \frac{k_2}{2})$

$k_4 = hf(t_{i-1} + h, y_{i-1} + k_3)$

$y_i = y_{i-1} + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$, $i = 1, 2, \ldots n$

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P: $\frac{dy}{dx} = \log_e(t^2 + y^2 + 1)$, $y(0) = 3$

Find an approximation for $y(0.2)$, $y(0.4)$ and $y(0.6)$ by Runge-kutta method of order four.

S(o): $h = 0.2$, $f(t, y) = \log_e(t^2 + y^2 + 1)$

$t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$

$y(0) = 3 \Rightarrow y(t_0) = 3 \Rightarrow y_0 = 3$

we have to find $y_1, y_2, y_3$

i=1 (for $y_1$) Finding $y_1$:

$k_1 = hf(t_0, y_0) = 0.2 \log_e((0.0)^2 + 9 + 1) = 0.460$

$k_2 = hf(t_0 + 0.1, y_0 + 0.23) = 0.487$

$k_3 = hf(t_0 + 0.1, y_0 + 0.243) = 0.489$

$k_4 = hf(t_0 + 0.2, y_0 + 0.489) = 0.516$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$

$= 3 + \frac{1}{6}(2.928) = 3.488$

$\boxed{y(0.2) = 3.488}$

Finding $y_2$:

$k_1 = 0.516$

$k_2 = 0.543$

$k_3 = 0.544$

$k_4 = 0.571$

$y_2 = 4.031$

$\boxed{y(0.4) = 4.031}$

Finding $y_3$ :

$k_1 = 0.591$
$k_2 = 0.598$
$k_3 = 0.599$
$k_4 = 0.625$

$y_3 = 4.0314 + \frac{1}{6}(3.59)$

$\boxed{y_3 = 4.629}$

$\boxed{y(0.8) = 4.629}$

* Truncation Error in Taylor method $= \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(c)$

$= TE_T$

Order of a method for solving Differential eqn :

Let TE be the truncation error in the numerical method ⊛. If $\frac{TE}{h} = \alpha . h^n$, then 'n' is called order of the numerical method ⊛.

→ ∴ The order of Taylor method $= \frac{TE_T}{h}$

$$= \frac{(h)^{n+1}}{(n+1)!} \cdot \frac{1}{h} \cdot y^{(n+1)}(c)$$

$$= h^n . \alpha$$

where $\alpha = \frac{y^{(n+1)}(c)}{(n+1)!}$

∴ The order of Taylor method $= n$

→ The truncation error in the Euler method

$$= \frac{h^2 y^{(2)}(c)}{2!} = TE_E$$

$$\frac{TE_E}{h} = \frac{h . y^{(2)}(c)}{2!}$$

∴ The order of Euler method $= 1$

Similarly the order of the modified Euler method is 2.

The order of Runge - kutta method is 4

## Adam - Bashforth methods [explicit methods]:

Consider $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$, $t \in [a, b]$

Let 'h' be the step size


We have to find $y(t_1), y(t_2), \ldots\ldots y(t_{n-1}), y(t_n)$

$y(t_n) = y_n$ . $y(t_{n-1}) = y_{n-1}$ $y_0(t_1) = y_1$

The second order Adam - Bashforth (AB) method is

$$y_{i+1} = y_i + \frac{h}{2}\Big[f(t_i, y_i) - 59 f(t_{i-1}, y_{i-1}) + 37 f(t_{i-2}, y_{i-2}) - 9 f(t_{i-3}, y_{i-3})\Big]$$

$i = 3, 4, 5, 6, \ldots\ldots n$

$i = 3 \Rightarrow$ $$y_4 = y_3 + \frac{h}{2}\Big[f(t_3, y_3) - 59 f(t_2, y_2) + 37 f(t_1, y_1) - 9 f(t_0, y_0)\Big]$$

$i = 4 \Rightarrow$ $$y_5 = y_4 + \frac{h}{2}\Big[f(t_4, y_4) - 59 f(t_3, y_3) + 37 f(t_2, y_2) - 9 f(t_1, y_1)\Big]$$

The second order Adam - Bashforth methods needs the starting values $y_0, y_1, y_2$ and $y_3$. These starting values will be computed by the single step methods such as Euler method, Taylor method, or Runge - kutta Method.

The third order Adam - Bashforth (AB) method is

$$y_{i+1} = y_i + \frac{h}{12}\left[23 f(t_i, y_i) - 16 f(t_{i-1}, y_{i-1}) + 5 f(t_{i-2}, y_{i-2})\right]$$

$i = 2, 3, 4, 5, 6, \dots n$

$i=2 \Rightarrow y_3 = y_2 + \frac{h}{12}\left[23 f(t_2, y_2) - 16 f(t_1, y_1) + 5 f(t_0, y_0)\right]$

$i=3 \Rightarrow y_4 = y_3 + \frac{h}{12}\left[23 f(t_3, y_3) - 16 f(t_2, y_2) + 5 f(t_1, y_1)\right]$

$\therefore$ The starting values required for third order AB method are $y_0, y_1$ and $y_2$.

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Single Step Methods :

The methods which require only one starting value are called single step methods.

Ex: Euler method, Taylor method, Runge - kutta method

Multi step Methods:

The methods which require more than one starting value are called multi step methods.

* Adams - Bashforth (AB) Numerical methods (Explicit and multi-step methods)

Solving $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$; $t \in [a, b]$

Let $h$ be the step size

$a = t_0 \quad t_1 \quad t_2 \quad \dots \quad t_{i-1} \quad t_i \quad \dots \quad t_{n-1} \quad t_n = b$

Second order Adams - Bashforth method:

$$y_{i+1} = y_i + \frac{h}{2}\left[3 f(t_i, y_i) - f(t_{i-1}, y_{i-1})\right]$$

$i = 1, 2, 3, \dots n-1$

Third order Adams-Bashforth method:

$$y_{i+1} = y_i + \frac{h}{12}\left[23 f(t_i, y_i) - 16 f(t_{i-1}, y_{i-1}) + 5 f(t_{i-2}, y_{i-2})\right]$$

$i = 2, 3, \dots n-1$

fourth order Adams-Bashforth method:

$$y_{i+1} = y_i + \frac{h}{24}\left[55 f(t_i, y_i) - 59 f(t_{i-1}, y_{i-1}) + 37 f(t_{i-2}, y_{i-2}) - 9 f(t_{i-3}, y_{i-3})\right]$$

$i = 3, 4, \dots n-1$

*Milne's Method of order-4 : (Explicit)

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \quad t \in [a, b]$$

$$y_{i+1} = y_{i-3} + \frac{4h}{3}\left[2 f(t_i, y_i) - f(t_{i-1}, y_{i-1}) + 2 f(t_{i-2}, y_{i-2})\right]$$

$i = 3, 4, \dots n-1$

*Adam's - Moultan (AM) Numerical methods (Implicit and multi-step method)

($y_{i+1}$ depends on $y_{i+1}$)

Solving $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$, $t \in [a, b]$

Let $h$ be the step size.

$a = t_0 \quad t_1 \quad t_2 \quad \dots \quad t_{i-1} \quad t_i \quad \dots \quad t_{n-1} \quad t_n = b$

Second order Adam's-Moultan method:

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_{i+1}, y_{i+1}) + f(t_i, y_i)\right], \quad i = 0, 1, 2, 3, \dots n-1$$

Third order Adams-Moultan method:

$$y_{i+1} = y_i + \frac{h}{2}\left[5 f(t_{i+1}, y_{i+1}) + 8 f(t_i, y_i) - f(t_{i-1}, y_{i-1})\right]$$

$i = 1, 2, \dots n-1$

Fourth order Adams-Moultan method:

$$y_{i+1} = y_i + \frac{h}{24}\left[9 f(t_{i+1}, y_{i+1}) + 19 f(t_i, y_i) - 5 f(t_{i-1}, y_{i-1}) + f(t_{i-2}, y_{i-2})\right]; \quad i = 2, 3, \dots n-1$$

Fourth order : Milne's Implicit method :

$$y_{i+1} = y_{i-1} + \frac{h}{3}\left[f(t_{i+1}, y_{i+1}) + 4f(t_i, y_i) + f(t_{i-1}, y_{i-1})\right]$$

$i = 1, 2, 3, \ldots, n-1$

Pb: Consider $\frac{dy}{dt} = t + y^2$, $y(0) = 1$

Let the step size be $0.1$. Find $y(0.4)$ and $y(0.5)$ by Adams - Bashforth method of order 4. compute the initial values which are required by Euler method.

Sol: $a = 0 = t_0$ $\quad t_1 = 0.1, \; t_2 = 0.2 \quad t_3 = 0.3 \quad t_4 = 0.4$

$t_5 = 0.5 = b.$

It is given that

$y(0) = 1 \Rightarrow y(t_0) = 1$

$\Rightarrow y_0 = 1$

$t_0 = 0 \quad t_1 = 0.1 \quad t_2 = 0.2 \quad t_3 = 0.3 \quad t_4 = 0.4 \quad t_5 = 0.5$

We need to calculate $y_4, y_5$ by AB - method of order 4.

$$y_{i+1} = y_i + \frac{h}{24}\left[55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1}) + 37f(t_{i-2}, y_{i-2}) - 9f(t_{i-3}, y_{i-3})\right] \quad \text{---} \; (*)$$

$i = 3, 4$

By Euler method, we get

$$y_i = y_{i-1} + h\, y_{i-1}^{(1)}$$

$\Rightarrow y_1 = y_0 + h\, y_0' = 1 + 0.1(1) = 1.1$

$\Rightarrow y_2 = y_1 + h\, y_1' = (1.1) + (0.1)(1.31) = 1.231$

$\Rightarrow y_3 = y_2 + h\, y_2' = 1.231 + (0.1)(1.71) = 1.4025361$

Taking $i = 3$ in (*) we get.

$$y_4 = y_3 + \frac{0.1}{24}\left[55f(t_3, y_3) - 59f(t_2, y_2) + 37f(t_1, y_1) - 9f(t_0, y_0)\right]$$

$$= 1.4025361 + \frac{0.1}{24}(63.27)$$

$$\boxed{y_4 = 1.666}$$

Similarly, taking $i=4$ in ⊛, we get

$$y_5 = y_4 + \frac{h}{24}\left[55f(t_4, y_4) - 59f(t_3, y_3) + 37f(t_2, y_2) - 9f(t_1, y_1)\right]$$

$$\boxed{y_5 = 2.049}$$

Aadams Bashforth and Aadams - Moultan Predictor - Collector methods :

| Predictor | Collector |
|---|---|
| $2^{nd}$ order AB-method ↔ | $2^{nd}$ order AM - method |
| $3^{rd}$ order AB | $3^{rd}$ order AM |
| $4^{th}$ order AB | $4^{th}$ order AM |

Pb: Consider $\frac{dy}{dt} = t + y^2$, $y(0) = 1$.
Find an approximation for $y(0.4)$ and $y(0.5)$ by Aadams - Moultan predictor- Adams - Bashforth and Aadams - Moultan predictor collector method of order 4 Taking step size as 0.1
Compute the initial values required by Euler method.

Sol: $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.3$, $t_4 = 0.4$, $t_5 = 0.5$

We have to compute $y(t_4)$ and $y(t_5)$ by predictor - collector method.

$$y_{i+1}^{p} = y_i + \frac{h}{24}\left[55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1}) + 37f(t_{i-2}, y_{i-2}) - 9f(t_{i-3}, y_{i-3})\right] \quad i = 3, 4 \quad —(1)$$

$$y_{i+1}^{c} = y_i + \frac{h}{24}\left[9f(t_{i+1}, y_{i+1}^{p}) + 19 f(t_i, y_i) - 5 f(t_{i-1}, y_{i-1}) + f(t_{i-2}, y_{i-2})\right] \quad - ②$$

$i = 3, 4$

$y_1 = 1.1$, $y_2 = 1.231$, $y_3 = 1.402536$ [By Euler method]

Taking $i=3$ in ①, we get

$$y_4^{p} = y_3 + \frac{h}{24}\left[55 f(t_3, y_3) - 59 f(t_2, y_2) + 37 f(t_1, y_1) - 9 f(t_0, y_0)\right]$$

$= 1.666$

Taking $i=3$ in ②, we get

$$y_4^{c} = y_3 + \frac{h}{24}\left[9 f(t_4, y_4^{p}) + 19 f(t_3, y_3) - 5 f(t_2, y_2) + f(t_1, y_1)\right]$$

$= 1.67067$

$\therefore \boxed{y_4 = 1.67067}$

Taking $i=4$ in ① we get

$y_5^{p} = 2.0519$

Taking $i=4$ in ② we get

$y_5^{c} = 2.0599$

Pb: Consider $\frac{dy}{dt} = e^{-\sin(y(t))}$, $y(1) = 2$, $y(1.1) = 2.4$, $y(1.2) = 2.7$, $y(1.3) = 0$.

Find an approximation for $y(1.4)$, $y(1.5)$ and $y(1.6)$ by Milne's predictor-collector method of order 4 with step-size $0.1$.

Sol: $t_0 = \cancel{0}1, \quad t_1 = \cancel{0}.1, \quad t_2 = 1.2, \quad t_3 = 1.3, \quad t_4 = 1.4, \quad t_5 = 1.5, \quad t_6 = 1.6$

$\cancel{t_7 = 1.}$

$y_0 = 2, \quad y_1 = 2.4, \quad y_2 = 2.7, \quad y_3 = 0$

∴ We have to calculate $y_4, y_5, y_6$.

$$y_{i+1}^{P} = y_{i-3} + \frac{4h}{3}\left[2f(t_i, y_i) - f(t_{i-1}, y_{i-1}) + 2f(t_{i-2}, y_{i-2})\right]$$

$$i = 3, 4, 5$$

$$y_{i+1}^{c} = y_{i-1} + \frac{h}{3}\left[f(t_{i+1}, y_{i+1}^{P}) + 4f(t_i, y_i) + f(t_{i-1}, y_{i-1})\right]$$

$$i = 3, 4, 5$$

Taking $i = 3$ we get

$$y_4^{P} = 2 + \frac{4(0.1)}{3}\left[2\underbrace{f(1.3, 0)} - f(1.2, 2.7) + 2f(1.1, 2.4)\right]$$

$\cancel{2 + \frac{0.4}{3}}$ $\quad f(t, y) = e^{-\sin(y)}$

$$y_4^{P} = \cancel{2.7736}\ 2.315$$

$$y_4^{c} = y_2 + \frac{0.1}{3}\left[f(1.4, 2.315) + 4f(1.3, 0) + f(1.2, 2.7)\right]$$

$$= 2.871$$

$$\therefore \boxed{y_4 = 2.871}$$

Taking $i = 4$ we get

$$y_5^{P} = y_1 + \frac{4(0.1)}{3}\left[2f(1.4, 2.871) - f(1.3, 0) + 2f(1.2, 2.7)\right]$$

$$y_5^{P} = 2.6447$$

$$y_5^{c} = y_3 + \frac{0.1}{3}\left[f(1.5, 2.6447) + 4f(1.4, 2.871) + f(1.3, 0)\right]$$

$$= 0.156$$

taking $i=5$ we get

$$y_6^p = y_2 + \frac{0.4}{3}\left[2f(1.5, 0.158) - f(1.4, 2.871) + 2f(1.3, 0)\right]$$

$$= 3.093$$

$$y_6^c = y_4 + \frac{0.1}{3}\left[f(1.6, 3.093) + 4f(1.5, 0.158) + f(1.4, 2.871)\right]$$

$$y_6^c = 3.042$$

$$\boxed{y_6 = 3.042}$$

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* Let $y$ be a $(n+1)$ times differentiable function on $[t, t+h]$, $h>0$. Then by Taylor's theorem, we get

$$y(t+h) = y(t) + \frac{h}{1!}y'(t) + \frac{h^2}{2!}y''(t) + \frac{h^3}{3!}y'''(t) + \frac{h^4}{4!}y^{(4)}(t) + \cdots + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(c_1)$$

$$c_1 \in (t, t+h) \quad \text{--- (1)}$$

Similarly,

$$y(t-h) = y(t) - \frac{h}{1!}y'(t) + \frac{h^2}{2!}y''(t) - \frac{h^3}{3!}y'''(t) + \frac{h^4}{4!}y^{(4)}(t) + \cdots + (-1)^{n+1}\frac{h^{n+1}}{(n+1)!}y^{(n+1)}(c_2) \quad \text{--- (2)}$$

$$c_2 \in (t-h, t)$$

Adding (1) and (2) we get

$$y(t+h) + y(t-h) = 2y(t) + h^2 y''(t) + 2\frac{h^4}{4!}y^{(4)}(t) + \cdots$$

By neglecting the terms containing $h^4, h^6, \ldots$ we get

$y(t+h) + y(t-h) \approx 2y(t) + h^2 y''(t)$

$\rightarrow \boxed{\dfrac{y(t+h) - 2y(t) + y(t-h)}{h^2} \approx y''(t)}$ : finite difference

① - ② $\rightarrow$ $y(t+h) - y(t-h) = 2h \cdot y'(t) + 2\frac{h^3}{3!} y'''(t)$
$+ 2\frac{h^5}{5!} y^{(5)}(t) + \ldots$

By neglecting terms containing $h^3, h^5, \ldots$ we get

$y(t+h) - y(t-h) \approx 2h\, y'(t)$

$\rightarrow \boxed{y'(t) \approx \dfrac{y(t+h) - y(t-h)}{2h}}$ : central difference approximation of $y'$ at point $t$

by neglecting $h^2, h^3, \ldots$ terms in ①, ② we get

$y'(t) \approx \dfrac{y(t+h) - y(t)}{h}$ : forward difference

$y'(t) \approx \dfrac{y(t) - y(t-h)}{h}$ : backward difference.

* We use central difference approximation for $y'(t)$

**Boundary value problem:**

Consider $y'' + a_1(t) \cdot y' + a_2(t) \cdot y = r(t)$, $t \in [a, b]$ — ①

The general solution of ① consists two arbitary constants. Therefore, to determine these arbitary constants, we need two conditions on $y$. If both the conditions are given at $t = a$, then ① with these conditions is called is called Initial value problem.

Ex: $y'' + (\sin t) y' + \cos t = t^2$, $t \in [1, 2]$
$y(1) = 3$, $y'(1) = 5$ } IVP

If one condition is given at $t=a$ and second condition is given at $t=b$, then ① with these conditions is called Boundary value problem.

Ex: $$\left.\begin{aligned} & y'' + (\sin t)\, y' + (\cos \cancel{y} t)\, y = t^2, \quad t \in [1,2] \\ & y(1) = 0, \quad y(2) = 100 \end{aligned}\right\} \text{BVP}$$

$$\left.\begin{aligned} & y'' + (\sin t)\, y' + (\cos t)\, y = t^2 \quad t \in [1,2] \\ & y'(1) = 0, \quad y(2) = 12 \end{aligned}\right\} \text{BVP}$$

Pb: Solve $y'' + e^t y' + (\sin t) y = \cos t$, $y(0) = 0$, $y(1) = 2$ by finite difference method with step size $h = \frac{1}{4}$.

Sol:

$t_0 = 0$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{4}$, $t_4 = 1$

It is given that $y(0) = 0 \Rightarrow y(t_0) = 0 \Rightarrow y_0 = 0$

$y(1) = 2 \Rightarrow y(t_4) = 2 \Rightarrow y_4 = 2$

$\therefore$ We have to calculate $y_1, y_2, y_3$

But $$\left.\begin{aligned} y''(t) &\approx \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} \\ y'(t) &\approx \frac{y(t+h) - y(t-h)}{2h} \end{aligned}\right\} \text{—①}$$

Using ① in the given ODE we get

$$\frac{y(t+h) - 2y(t) + y(t-h)}{h^2} + e^t \left[\frac{y(t+h) - y(t-h)}{2h}\right] + (\sin t)\, y(t) = \cos t$$

Taking $t = t_i$

$$\frac{y(t_i+h)-2y(t_i)+y(t_i-h)}{h^2}+e^{t_i}\left[\frac{y(t_i+h)-y(t_i-h)}{2h}\right]$$

$$+(\sin t_i)\,y(t_i)=\cos t_i$$

$$\Rightarrow \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+e^{t_i}\left[\frac{y_{i+1}-y_{i-1}}{2h}\right]+\sin(t_i)\,y_i=\cos t_i$$

$$\Rightarrow y_{i-1}\left[\frac{1}{h^2}-\frac{e^{t_i}}{2h}\right]+y_i\left[\sin(t_i)-\frac{2}{h^2}\right]+y_{i+1}\left[\frac{1}{h^2}+\frac{e^{t_i}}{2h}\right]=\cos t_i$$

$$i=1\Rightarrow y_0\left(\frac{1}{h^2}-\frac{e^{t_1}}{2h}\right)+y_1\left(\sin(t_1)-\frac{2}{h^2}\right)+y_2\left(\frac{1}{h^2}+\frac{e^{t_1}}{2h}\right)=\cos t_1$$

$$\Rightarrow y_1\left(\sin(t_1)-\frac{2}{h^2}\right)+y_2\left(\frac{1}{h^2}+\frac{e^{t_1}}{2h}\right)=\cos t_1 \quad [\because y_0=0] \quad -(*)$$

$$i=2\Rightarrow \left(\frac{1}{h^2}-\frac{e^{t_2}}{2h}\right)y_1+\left(\sin(t_2)-\frac{2}{h^2}\right)y_2+y_3\left(\frac{1}{h^2}+\frac{e^{t_2}}{2h}\right)=\cos t_2 \quad -(**)$$

$$i=3\Rightarrow y_2\left(\frac{1}{h^2}-\frac{e^{t_3}}{2h}\right)+y_3\left(\sin(t_3)-\frac{2}{h^2}\right)+y_4\left(\frac{1}{h^2}+\frac{e^{t_3}}{2h}\right)=\cos t_3$$

$$\Rightarrow y_2\left(\frac{1}{h^2}-\frac{e^{t_3}}{2h}\right)+y_3\left(\sin(t_3)-\frac{2}{h^2}\right)=\cos t_3-2\left(\frac{1}{h^2}+\frac{e^{t_3}}{2h}\right)$$

$$-(***) \quad [\because y_4=2]$$

So ~~these equations become~~

~~$y_1(-31.75)+y_2($~~

By solving (*), (**), (***), we get

$$\boxed{\begin{aligned} y_1&=0.775\\ y_2&=1.378\\ y_3&=1.787\end{aligned}}$$

19/9/24

Pb: Solve $y'' + e^{-t} y' + e^{t} y = t^2+1$, $y(0) = -1$

$y'(1) + y(1) = 0$ by finite difference method

with step size $\frac{1}{3}$.

Sol: Here $h = \frac{1}{3}$

$t_0 = 0$, $t_1 = \frac{1}{3}$, $t_2 = \frac{2}{3}$, $t_3 = 1$

$y(0) = -1 \Rightarrow y(t_0) = -1$

$\boxed{y_0 = -1}$

We have to calculate $y(t_1)$ $y(t_2)$ $y(t_3)$.

That is $y_1, y_2, y_3$

By using finite difference approximations for $y''$ and $y'$ in the given ODE, we get

$$\frac{y(t+h) - 2y(t) + y(t-h)}{h^2} + e^{-t}\left[\frac{y(t+h) - y(t-h)}{2h}\right] + e^{t} y(t) = t^2+1$$

Taking $t = t_i$

$$\frac{y(t_i+h) - 2y(t_i) + y(t_i-h)}{h^2} + e^{-t_i}\left[\frac{y(t_i+h) - y(t_i-h)}{2h}\right] + e^{t_i} y(t_i) = t_i^2+1$$

$i = 1, 2, 3$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + e^{-t_i}\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + e^{t_i} y_i = t_i^2+1 \quad i = 1, 2, 3$$

$$y_{i-1}\left[\frac{1}{h^2} - \frac{e^{-t_i}}{2h}\right] + y_i\left[\frac{-2}{h^2} + e^{t_i}\right] + y_{i+1}\left[\frac{1}{h^2} + \frac{e^{-t_i}}{2h}\right] = t_i^2+1$$

$i=1 \Rightarrow$ $y_0\left[\frac{1}{h^2} - \frac{e^{-t_1}}{2h}\right] + y_1\left[\frac{-2}{h^2} + e^{t_1}\right] + y_2\left[\frac{1}{h^2} + \frac{e^{-t_1}}{2h}\right] = t_1^2+1$

$\Rightarrow y_1\left[\frac{-2}{h^2} + e^{t_1}\right] + y_2\left[\frac{1}{h^2} + \frac{e^{-t_1}}{2h}\right] = t_1^2 + 1 + \frac{1}{h^2} - \frac{e^{-t_1}}{2h}$ —①

$i=2 \Rightarrow$ $y_1\left[\frac{1}{h^2} - \frac{e^{-t_2}}{2h}\right] + y_2\left[\frac{-2}{h^2} + e^{t_2}\right] + y_3\left[\frac{1}{h^2} + \frac{e^{-t_2}}{2h}\right] = t_2^2$ —②

$i=3 \Rightarrow y_2\left[\frac{1}{h^2}-\frac{e^{-t_3}}{2h}\right]+y_3\left[\frac{-2}{h^2}+e^{t_3}\right]+y_4\left[\frac{1}{h^2}+\frac{e^{-t_3}}{2h}\right]=t_3^2+1$

$y'(1)+y(1)=0 \rightarrow \frac{y(1+t)-y(1-t)}{2h}+y(1)=0$

$$\frac{y_4-y_2}{2h}+y_3=0$$

$$\rightarrow y_4=y_2-2h\,y_3$$

$\rightarrow$ ~~$y_2\left[\frac{2}{h^2}\right]-2h\left(\frac{1}{h^2}+\frac{e^{-t_3}}{2h}\right)y_1+y_3\left(-\frac{2}{h^2}+e^{t_3}\right)=t_3^2+1$ — (3)~~

~~Solving (1), (2), (3) we get~~

$$y_2\left[\frac{2}{h^2}\right]+y_3\left[\frac{-2}{h^2}+e^{t_3}-\frac{2}{h}-e^{-t_3}\right]=t_3^2+1$$

$$\left(\frac{2}{h^2}\right]y_2+\left[-\frac{2}{h}-\frac{2}{h^2}+e^{t_3}-e^{-t_3}\right]y_3=t_3^2+1 \quad -(3)$$

By solving (1), (2), (3) we get

$$\boxed{\begin{aligned} y_1 &= \cancel{-1.28}\ -2 \\ y_2 &= \cancel{-1.84}\ -2.37 \\ y_3 &= \cancel{-1.46}\ -2.06 \end{aligned}}$$

Pb: 1) Solve $y''+(t^2+2)y'+ty=\sin t$, $y'(0)=3$, $y(1)=5$ by the finite difference method with $h=\frac{1}{3}$

2) Solve $y''+(\sin t)y'+y=\log_e(t^2+1)$, $y'(0)=5$, $y'(1)=-1$ by the finite difference method with $h=\frac{1}{2}$

Sol: 1). here $h=\frac{1}{3}$ $y_0, y_1, y_2$ — ?

$t_0=0$, $t_1=\frac{1}{3}$, $t_2=\frac{2}{3}$, $t_3=1$

$y(1)=y_3=5$

By using finite differential approximations for $y''$, $y'$ in the given ODE, we get

$$\frac{y(t+h) - 2y(t) + y(t-h)}{h^2} + (t^2+2)\left(\frac{y(t+h) - y(t-h)}{2h}\right) + ty = \sin t$$

Taking $t = t_i$

$$\frac{y(t_i+h) - 2y(t_i) + y(t_i-h)}{h^2} + (t_i^2+2)\left(\frac{y(t_i+h) - y(t_i-h)}{2h}\right) + t_i y_i = \sin t_i \quad -①$$

$i = 0, 1, 2$

Given $y'(0) = 3$ ($t_0$)

$$\Rightarrow \frac{y(t_0+h) - y(t_0-h)}{2h} = 3 \Rightarrow \frac{y_1 - y_{-1}}{2/3} = 3$$

$$\boxed{y_{-1} = y_1 - 2}$$

~~Put t=0~~

⇒ ① becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{(t_i^2+2)}{2h}(y_{i+1} - y_{i-1}) + t_i y_i = \sin(t_i)$$

$$y_{i-1}\left(\frac{1}{h^2} - \frac{(t_i^2+2)}{2h}\right) + y_i\left(\frac{-2}{h^2} + t_i\right) + y_{i+1}\left(\frac{1}{h^2} + \frac{(t_i^2+2)}{2h}\right) = \sin(t_i)$$

$$i=0 \Rightarrow y_{-1}\left(\frac{1}{h^2} - \frac{t_0^2+2}{2h}\right) + y_0\left(\frac{-2}{h^2} + t_0\right) + y_1\left(\frac{1}{h^2} + \frac{(t_0^2+2)}{2h}\right) = \sin(t_0)$$

$$y_{-1} = y_1 - 2$$

$$\Rightarrow y_0\left(\frac{-2}{h^2} + t_0\right) + y_1\left(\frac{2}{h^2}\right) = \sin t_0 + \frac{2}{h^2} - \frac{t_0^2+2}{h} \quad -①$$

$$i=1 \Rightarrow y_0\left(\frac{1}{h^2} - \frac{t_1^2+2}{2h}\right) + y_1\left(\frac{-2}{h^2} + t_1\right) + y_2\left(\frac{1}{h^2} + \frac{t_1^2+2}{2h}\right) = \sin(t_1) \quad -②$$

$$i=2 \Rightarrow y_1\left(\frac{1}{h^2} - \frac{t_2^2+2}{2h}\right) + y_2\left(\frac{-2}{h^2} + t_2\right) = \sin t_2 - 5\left(\frac{1}{h^2} + \frac{t_2^2+2}{2h}\right) \quad -③$$

by solving ①, ②, ③ we get

$y_0 = 4.28$, $y_1 = 4.9$, $y_2 = 5.1$

2) $y'' + (\sin t)\, y' + y = \log_e^{(t^2+1)}$ $\quad y'(0)=5,\ y'(1)=-1$

Given $h=\frac{1}{2}$

We have calculate $y_0, y_1, y_2$ $\quad t_0=0 \quad t_1=\frac{1}{2} \quad t_2=1$

By using finite difference approximations for $y'', y'$ in the given ODE we get

$$\frac{y(t+h)-2y(t)+y(t-h)}{h^2} + (\sin t)\left(\frac{y(t+h)-y(t-h)}{2h}\right) + y = \log_e^{(t^2+1)}$$

put $t = t_i$

$$\Rightarrow \frac{y(t_i+h)-2y(t_i)+y(t_i-h)}{h^2} + \sin t_i\left(\frac{y(t_i+h)-y(t_i-h)}{2h}\right) + y_i = \log_e^{(t_i^2+1)}$$

$$\Rightarrow \frac{y_{i+1}-2y_i+y_{i-1}}{h^2} + \sin t_i\left(\frac{y_{i+1}-y_{i-1}}{2h}\right) + y_i = \log_e^{(t_i^2+1)}$$

$$\Rightarrow y_{i-1}\left(\frac{1}{h^2}-\frac{\sin t_i}{2h}\right) + y_i\left(-\frac{2}{h^2}+1\right) + y_{i+1}\left(\frac{1}{h^2}+\frac{\sin t_i}{2h}\right) = \log_e^{(t_i^2+1)}$$

$i=0,1,2$

Given $y'(0)=5 \Rightarrow \frac{y_1-y_{-1}}{2h} = 5 \Rightarrow \boxed{y_{-1} = y_1 - 5}$

$y'(1)=-1 \Rightarrow \frac{y_3-y_1}{2h} = \cancel{5}\,-1 \Rightarrow \boxed{y_3 = y_1 - 1}$

$i=0$;

$$y_{-1}\left(\frac{1}{h^2}-\frac{\sin t_0}{2h}\right) + y_0\left(1-\frac{2}{h^2}\right) + y_1\left(\frac{1}{h^2}+\frac{\sin t_0}{2h}\right) = \ln(t_0^2+1)$$

$$\Rightarrow y_0\left(1-\frac{2}{h^2}\right) + y_1\left(\frac{2}{h^2}\right) = \ln(t_0^2+1) + 5\left(\frac{1}{h^2}-\frac{\sin t_0}{2h}\right) \quad -(1)$$

$i=1$;

$$y_0\left(\frac{1}{h^2}-\frac{\sin t_1}{2h}\right) + y_1\left(-\frac{2}{h^2}+1\right) + y_2\left(\frac{1}{h^2}+\frac{\sin t_1}{2h}\right) = \ln(t_1^2+1) \quad -(2)$$

$i=2$;

$$y_1\left(\frac{1}{h^2}-\frac{\sin t_2}{2h}\right) + y_2\left(-\frac{2}{h^2}+1\right) + y_3\left(\frac{1}{h^2}+\frac{\sin t_2}{2h}\right) = \ln(t_2^2+1)$$

$$\Rightarrow y_1\left(\frac{2}{h^2}\cancel{-\frac{\sin t_2}{2h}}\right) + y_2\left(1-\frac{2}{h^2}\right) = \ln(t_2^2+1) + \left(\frac{1}{h^2}+\frac{\sin t_2}{2h}\right) \quad -(3)$$

the equations become :

$$-7y_0 + 8y_1 = 20$$

$$3.5y_0 - 7y_1 + 4.48y_2 = 0.223$$

$$8y_1 - 7y_2 = 5.346$$

by solving these we get

$$\boxed{\begin{aligned} y_0 &= 4.57 \\ y_1 &= 6.45 \\ y_2 &= 6.58 \end{aligned}}$$