Assignment - 3

1) (i) $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} = A$, $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$y_1 = AV_0 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\exists \quad y_1 = \begin{bmatrix} 5 \\ 22 \\ 5 \end{bmatrix}, \quad m_1 = 2\cancel{8}2$$

$$V_1 = \begin{bmatrix} \cancel{0.2}\ 0.2272 \\ \cancel{0.88}\ 1 \\ \cancel{0.2}\ 0.2272 \end{bmatrix}$$

$$y_2 = AV_1 = \begin{bmatrix} 1.909 \\ 20.454 \\ 1.909 \end{bmatrix}, \quad m_2 = 20.454$$

$$y_3 = AV_2 = \begin{bmatrix} 1.3733 \\ 20.187 \\ 1.3733 \end{bmatrix}, \quad m_3 = 20.187$$

$$y_4 = \begin{bmatrix} 1.2721 \\ 20.136 \\ 1.2721 \end{bmatrix}, \quad m_4 = 20.136$$

$$y_5 = \begin{bmatrix} 1.2527 \\ 20.126 \\ 1.2527 \end{bmatrix}, \quad m_5 = 20.126$$

$$y_6 = \begin{bmatrix} 1.249 \\ 20.125 \\ 1.249 \end{bmatrix}, \quad V_6 = \begin{bmatrix} 0.06206 \\ 1 \\ 0.06206 \end{bmatrix}$$

$$y_7 = \begin{bmatrix} 1.2482 \\ 20.1242 \\ 1.2482 \end{bmatrix}$$

$$|\lambda_1| = \max \left\{ \frac{1.2482}{0.6206}, \frac{20.1242}{1}, \frac{1.2482}{0.6206} \right\}$$

$$= 20.1242$$

$$\lambda_1 = 20.1242 \quad \text{or} \quad -20.1242$$

$|A - \lambda_1 I|$

If $\lambda_1 = 20.1242$

$|A - \lambda_1 I| = -0.042 \approx 0$

If $\lambda_1 = -20.1242$

$|A - \lambda_1 I| = 23303.114 \neq 0$

Eigenvalue $= 20.1242$

Eigenvector $= \begin{bmatrix} 0.06206 \\ 1 \\ 0.06206 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} = A, \quad V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$y_1 = AV_0 = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 4 \end{bmatrix}, \quad m_1 = 4$$

$$y_2 = \begin{bmatrix} 3.5 \\ 2.75 \\ 2.75 \\ 3.5 \end{bmatrix}, \quad m_2 = 3.5$$

$$y_3 = \begin{bmatrix} 3.5714 \\ 2.7857 \\ 2.7857 \\ 3.5714 \end{bmatrix}, \quad m_3 = 3.5714$$

$$y_4 = \begin{bmatrix} 3.56 \\ 2.78 \\ 2.78 \\ 3.56 \end{bmatrix}, \quad m_4 = 3.56$$

$$y_5 = \begin{bmatrix} 3.5618 \\ 2.7809 \\ 2.7809 \\ 3.5618 \end{bmatrix}, \quad m_5 = 3.5618$$

$$y_6 = \begin{bmatrix} 3.5615 \\ 2.7807 \\ 2.7807 \\ 3.5615 \end{bmatrix}, \quad m_6 = 3.5615$$

$$y_7 = \begin{bmatrix} 3.5616 \\ 2.7808 \\ 2.7808 \\ 3.5616 \end{bmatrix}, \quad V_6 = \begin{bmatrix} 1 \\ 0.7807 \\ 0.70\text{[illegible]}7 \\ 1 \end{bmatrix}$$

$$|\lambda| = \max\left\{ \frac{3.5616}{1}, \frac{2.7808}{0.7807} \right\} = 3.5619$$

$\therefore \quad |\lambda| = 3.5619$

$\lambda = 3.5619$ or -3.5619

$|A - \lambda I|$,

If $\lambda = 3.5619$,

$|A - \lambda I| = 5.72 \times 10^{-3} \approx 0$

If $\lambda = -3.5619$,

$|A - \lambda I| = 542 \neq 0$

So, Eigen value $= 3.5619$

Eigen vector $= \begin{bmatrix} 1 \\ 0.7807 \\ 0.7807 \\ 1 \end{bmatrix}$

2) $A = LL^t$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 6 \\ -7 \\ 3 \end{bmatrix}$$

$$y_1 = A^{-1} V_0 = \begin{bmatrix} 19 \\ -23 \\ 10 \end{bmatrix}, \quad m_1 = 23$$

$$y_2 = \begin{bmatrix} 2.6521 \\ -3.26 \\ 1.4347 \end{bmatrix}, \quad m_2 = 3.26$$

$$y_3 = \begin{bmatrix} 2.6273 \\ -3.254 \\ 1.4403 \end{bmatrix}, \quad m_3 = 3.254$$

$$y_4 = \begin{bmatrix} \cancel{0.887}\ 2.6149 \\ -3.25 \\ 1.4427 \end{bmatrix}, \quad m_4 = 3.2502$$

$$y_5 = \begin{bmatrix} 2.609 \\ -3.248 \\ 1.4438 \end{bmatrix}, \quad V_4 = \begin{bmatrix} 0.80453 \\ -1 \\ 0.4438 \end{bmatrix}$$

$$|\lambda| = \max\left\{ \frac{2.609}{0.80453}, \frac{-3.248}{-1}, \frac{1.4438}{0.4438} \right\}$$

$$= 3.2532$$

$\lambda = 3.2532$ or -3.2532

If $\lambda = 3.2532$
$|A^{-1} - \lambda I| = -0.03227 \approx 0$

If $\lambda = -3.252$
$|A^{-1} - \lambda I| = 107 \neq 0$

$\therefore$ Eigenvalue $= 3.2532$

Eigenvector $= \begin{bmatrix} 0.80453 \\ -1 \\ 0.4438 \end{bmatrix}$

Eigenvalue of A is $\frac{1}{\lambda} = 0.307389$

3) $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$y_1 = AV_0 = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$, $m_1 = 6$

$y_2 = AV_1 = \begin{bmatrix} 4.333 \\ 5.666 \\ 3.333 \end{bmatrix}$, $m_2 = 5.666$

$\cancel{y_3}$ $\cancel{\begin{bmatrix} 0.7647 \\ 1 \\ 0.7647 \end{bmatrix}}$ $\cancel{\begin{bmatrix} 4.333 \\ 5.666 \\ 4.333 \end{bmatrix}}$, $\cancel{m_3 = 5}$

$y_3 = \begin{bmatrix} 4.0588 \\ 5.5294 \\ 4.0588 \end{bmatrix}$, $m_3 = 5.5294$

$y_4 = \begin{bmatrix} 3.9362 \\ 5.4681 \\ 3.9362 \end{bmatrix}$, $m_4 = 5.4681$

$y_5 = \cancel{\begin{bmatrix} 0.7198 \\ 1 \\ 0.7198 \end{bmatrix}} \begin{bmatrix} 3.8794 \\ 5.4397 \\ 3.8794 \end{bmatrix}$, $m_5 = 5.4397$

$y_6 = \begin{bmatrix} 3.8526 \\ 5.4263 \\ 3.8526 \end{bmatrix}$, $m_6 = 5.4263$

$y_7 = \begin{bmatrix} 3.84 \\ 5.42 \\ 3.84 \end{bmatrix}$, $V_6 = \begin{bmatrix} 0.71 \\ 1 \\ 0.71 \end{bmatrix}$

classmate
Date
Page

$$|\lambda| = \max\left\{\frac{3.84}{0.71}, 5.42\right\} = 5.42$$

$$\lambda = \pm 5.42$$

$$|A - \lambda I| = -0.0232 \approx 0 \quad \{\because \lambda = 5.42\}$$

$$|A - \lambda I| = 817 \neq 0 \quad \{\because \lambda = -5.42$$

~~Highest~~ Largest eigen value of A is 5.42

$$A^{-1} = \begin{bmatrix} 0.2678 & -0.071 & 0.0178 \\ -0.071 & 0.2857 & -0.071 \\ 0.0178 & -0.071 & 0.2678 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} 0.2142 \\ 0.1428 \\ 0.2142 \end{bmatrix}, \quad m_1 = 0.2142$$

$$y_2 = \begin{bmatrix} 0.238 \\ 0.0476 \\ 0.238 \end{bmatrix}, \quad m_2 = 0.238$$

$$y_3 = \begin{bmatrix} 0.2714 \\ -0.085 \\ 0.2714 \end{bmatrix}, \quad m_3 = 0.2714286$$

$$y_4 = \begin{bmatrix} 0.3082 \\ -0.233 \\ 0.3082 \end{bmatrix}, \quad m_4 = 0.3082707$$

$$y_5 = \begin{bmatrix} 0.3397 \\ -0.358 \\ 0.3397 \end{bmatrix}, \quad m_5 = 0.358885$$

$$y_6 = \begin{bmatrix} 0.3418 \\ -0.4209 \\ 0.3418 \end{bmatrix}, \quad \cancel{m_6 = 0.420943}$$

$$V_5 = \begin{bmatrix} 0.9488 \\ -1 \\ 0.9488 \end{bmatrix}$$

$$y_7 = \begin{bmatrix} 0.30034 \\ -0.401 \\ 0.3034 \end{bmatrix}, \quad m_7 = 0.303483$$

$$y_8 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$|\lambda| = \max \left\{ \frac{0.3418}{0.9488}, \ 0.4209 \right\}$$

$$= 0.4209$$

Largest eigen value of A^{-1} is 0.4209

Smallest eigen value of A is 2.37586

$|A - 2.37586 I| = 1.018 \approx 0$

$\lambda_1 = 5.42$, $\lambda_2 = 2.37586$, $\lambda_3 = 4+4+4 - \lambda_1 - \lambda_2$

$= 4.20414$

Assignment - 3 :

4) $y' = \frac{1}{x^2} - \frac{y}{x} - y^2$, $y(1) = -1$

1   1.25   1.5   1.75   2

$$y_1 = y_0 + h\, y_0' + \frac{h^2}{2!} y_0'' + \dots + \frac{h^n}{n!} y_0^{(n)} \quad - (1)$$

$y'(1) = \cancel{\frac{-2}{x^3}}\ 1 - \frac{-1}{1} - (-1)^2 = 1$

$$y'' = \frac{-2}{x^3} - \left(\frac{xy' - y}{x^2}\right) - 2yy'$$

$$\Rightarrow y''(1) = -2 - \left(\frac{1-(-1)}{1}\right) - 2\cdot(-1)\cdot 1$$

$$= \cancel{-4}\ -2$$

$$y_1 = -1 + \frac{1}{4}\cdot 1 + \frac{(1/4)^2}{\cancel{2}}\cdot(-\cancel{2})$$

$$\Rightarrow y_1 = -1 + \frac{1}{4} - \frac{1}{16} = \frac{-16+4-1}{16} =$$

$$\therefore y_1 = \frac{-13}{16} = -0.8125$$

$y_2 =$

5) $x y' = x - y$, $y(2) = 2$ at $x = 2.1$

$y_i^p = y_{i-1} + h y'_{i-1}$. $h = 0.1$

$y_1^p = y_0 + 0.1\, y'_0$

$\Rightarrow y_1^p = 2 + 0.1 \cdot \left(\frac{2-2}{2}\right)$

$\Rightarrow y_1^p = 2$

$y_1^c = y_0 + \frac{h}{2}\left[0 + \frac{2.1-2}{2.1}\right]$

$= 2 + \frac{0.1}{2} \cdot \frac{0.1}{2.1}$

$= 2.00238$

6) $y' = x y + 1$, $y(0) = 1$

$y_1^p = 1 + 0.1 \cdot 1 = 1.1$

$y_1^c = 1 + \frac{h}{2}\left\{1 + 0.1 \cdot 1.1 + 1\right\}$

$= 1.023696$

7) $h = 0.2$

(i) $y'(x) = (\cos y(x))^2$, $0 \leq x \leq 2$

$y(0) = 0$

5) $x y' = x - y$, $y(2) = 2$ at $x = 2.1$

$h = 0.1$ $y'' = 0 - \frac{x y' - y}{x^2}$

$y_0 = 2$

$\cancel{x}\, x_0 = 2$

$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0''$

$= 2 + 0.1 \cdot 0 + \frac{0.1^2}{2} \cdot \frac{1}{2}$

$= 2.0025$ at $x = 2.1$

6) $y' = x y + 1$, $y(0) = 1$

$h = 0.1$ $y'' = x y' + y$

$y_0 = 1$

$x_0 = 0$

$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0''$

$= 1 + 0.1 \cdot 1 + \frac{0.1^2}{2} \cdot 1$

$= 1.105$ at $x = 0.1$

7) $h = 0.2$

(i) $y(x) = (\cos y(x))^2$, $0 \le x \le 2$, $y(0) = 0$

$\cancel{x}$ $y_0 = 0$, $x_0 = 0$

$y_0' = 1$

$y_0'' = 2 (\cos y_0) \cdot y_0' = 2$

$y_1 = y_0 + h y_0' = 0 + 0.2 \cdot 1 = 0.2$

$y_2 = y_1 + h y_1' = 0.2 + 0.2 \cdot (\cos 0.2)^2 = 0.39\cancel{999}$ 2106

$y_3 = y_2 + h y_2' = 0.39\cancel{999}^{2106} + 0.2 (\cos 0.39\cancel{999}^{2106})^2$

$= \cancel{0.589384}$ 0.5629

~~$y_4 = y_3 + h y_3'$~~ $y_4 = 0.705941$

$y_5 = 0.821766$
$y_6 = 0.914449$
$y_7 = 0.988964$
$y_8 = 1.049366$
$y_9 = 1.098991$
$y_{10} = 1.140304$

(ii) $y' = \frac{1}{4} y \left(1 - \frac{1}{20} y\right), \; 0 \le x \le 2, \; y(0) = 1$

$y_1 = y_0 + h y_0'$

$\Rightarrow y_1 = y_0 + h \cdot \frac{1}{4} y_0 \left(1 - \frac{1}{20} y_0\right) = 1.0475$
$y_2 = 1.097131$
$y_3 = 1.148979$
$y_4 = 1.203127$
$y_5 = 1.259665$
$y_6 = 1.318681$
$y_7 = 1.38026$
$y_8 = 1.44451$
$y_9 = 1.511528$
$y_{10} = 1.58139$

~~8) (i) $y' = -10y + 11\cos x + 9\sin x, \quad y(0) = 1$~~
~~$h = \frac{1}{5}$~~

~~$y_1\left(\frac{1}{5}\right) = y_0 + \frac{h}{2}\left(y_0' + y_1'\right)$~~

~~$\Rightarrow y_1 = 1 + \frac{1}{10}($~~

8) (i) $y' = -10y + 11\cos x + 9\sin x$, $y(0) = 1$

$y_i^p = y_{i-1} + h\left(f(x_{i-1}, y_{i-1})\right)$

$y_i^c = y_{i-1} + \frac{h}{2}\left(f(x_{i-1}, y_{i-1}) + f(x_i, y_i^p)\right)$

$h = \frac{1}{5}$, $x_0 = 0$, $y_0 = 1$

$y(0.2)\begin{cases} y_1^p = y_0 + h\left(f(x_0, y_0)\right) \\ \quad = 1.2 \\ y_1^c = y_0 + \frac{h}{2}\left(f(x_0 + y_0) + f(x_1, y_1^p)\right) \\ \quad = 1.156875 \end{cases}$

$y(0.4)\begin{cases} y_2^p = y_1 + h f(x_1, y_1) \\ \quad = 1.156875 + \frac{1}{5}\left(-10\times 1.156875 + 11\cos 0.2 + 9\sin 0.2\right) \\ \quad = 1.356875 \\ y_2^c = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^p)\right) \end{cases}$

$= \cancel{1.7421226}\ 1.2636442$

$\cancel{y_3^p = y_2 + h f(x_2, y_2)}$ $\cancel{-3.784789}$

$\cancel{= 1.7421226 + 0.2(\ldots)}$

$\cancel{= \ldots\ 0.985164}$

$\cancel{y_3^c = y_2 + \frac{h}{2}\left(f(x_2, y_2) + f(x_2, y_2^p)\right)}$

$=$

$y(0.6)$

$y(0.8)$

$y(1)$

$$y(0.6)\begin{cases} y_3^p = y_2 + h\, f(x_2, y_2) \\ \quad = 1.463642 \\ y_3^c = y_2 + \frac{h}{2}\left(f(x_2, y_2) + f(x_3, y_3^p)\right) \\ \quad = 1.316049 \end{cases}$$

$$y(0.8)\begin{cases} y_4^p = y_3 + h\, f(x_3, y_3) \\ \quad = 1.51604 \\ y_4^c = y_3 + \frac{h}{2}\left\{f(x_3, y_3) + f(x_4, y_4^p)\right\} \\ \quad = 1.312005 \end{cases}$$

$$y(1)\begin{cases} y_5^p = y_4 + h\, f(x_4, y_4) \\ \Rightarrow y_5^p = 1.51199 \\ y_5^c = y_4 + \frac{h}{2}\left(f(x_4, y_4) + f(x_5, y_5^p)\right) \\ \quad = 1.251664. \end{cases}$$

$$\therefore \; y(1) = 1.251664$$

(ii) $y' = -10y + \frac{1}{1+x^2} + 10\tan^{-1}x, \quad y(0) = 0$

$h = \frac{1}{5}, \quad x_0 = 0, \quad y_0 = 0$

$$y(0.2)\begin{cases} y_1^p = y_0 + h\, f(x_0, y_0) \\ \quad = \cancel{0.2}\ 0.2 \\ y_1^c = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_0, y_1^p)\right) \rightarrow 0.193549 \\ \quad = 0 + 0.1 \times (1 + 0.935494) = \cancel{0.387098} \end{cases}$$

$$y(0.4)\begin{cases} y_2^p = y_1 + h\, f(x_1, y_1) \\ \quad = 0.193549 + 0.2\left(-10 \times 0.193549 + \frac{1}{1+0.2^2} + 10\tan^{-1}0.2\right) \\ \quad = \cancel{0.387098}\ 0.393549 \\ y_2^c = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, x_2^p)\right) = 0.3667136 \end{cases}$$

$y_2 = 0.3667136$

$y(0.6)$

$y_3^p = y_2 + h\, f(x_2, y_2)$

$= 0.566712$

$y_3^c = y_2 + \frac{h}{2}\left(f(x_2, y_2) + f(x_3, y_2^p) \right)$

$= 0.513949$

$y(0.8)$

$y_4^p = y_3 + h\, f(x_3, y_3)$

$= 0.513949 + 0.2\left(-10\times 0.513949 + 10\tan^{-1} 0.6 + \frac{1}{1+0.6^2}\right)$

$= 0.713948$

$y_4^c = y_3 + \frac{h}{2}\left(0.999995 + -10\times 0.713948 + 10\tan^{-1} 0.8 + \frac{1}{1+0.8^2}\right)$

$= 0.635717$

$y_5^p = y_4 + h\left(f(x_4, y_4)\right)$

$= 0.835716$

$y_5^c = 0.635717 + \frac{h}{2}\left(0.999995 + -10\times 0.835716 + 10\tan^{-1} 1 + \frac{1}{2}\right)$

$= 0.7353986$

$\therefore\ y(1) = 0.7353986$