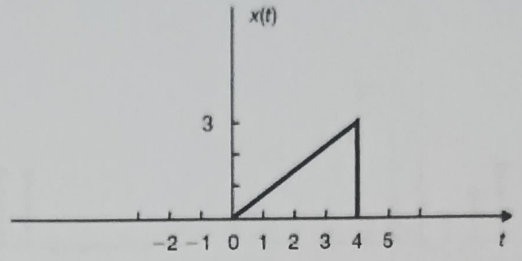
Signals and Systems

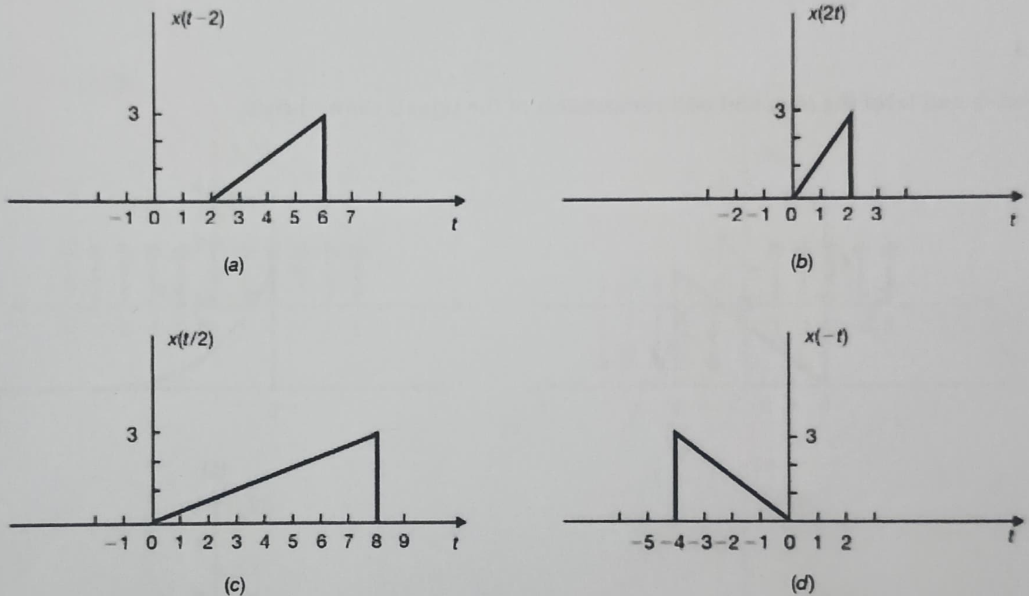
Tutorial – 1

1.1. **A continuous-time signal $x(t)$ is shown below. Sketch and label each of the following signals.**

(*a*) $x(t-2)$; (*b*) $x(2t)$; (*c*) $x(t/2)$; (*d*) $x(-t)$



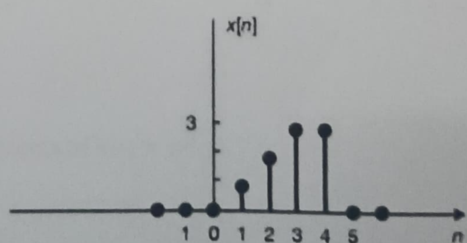
Sol^n.

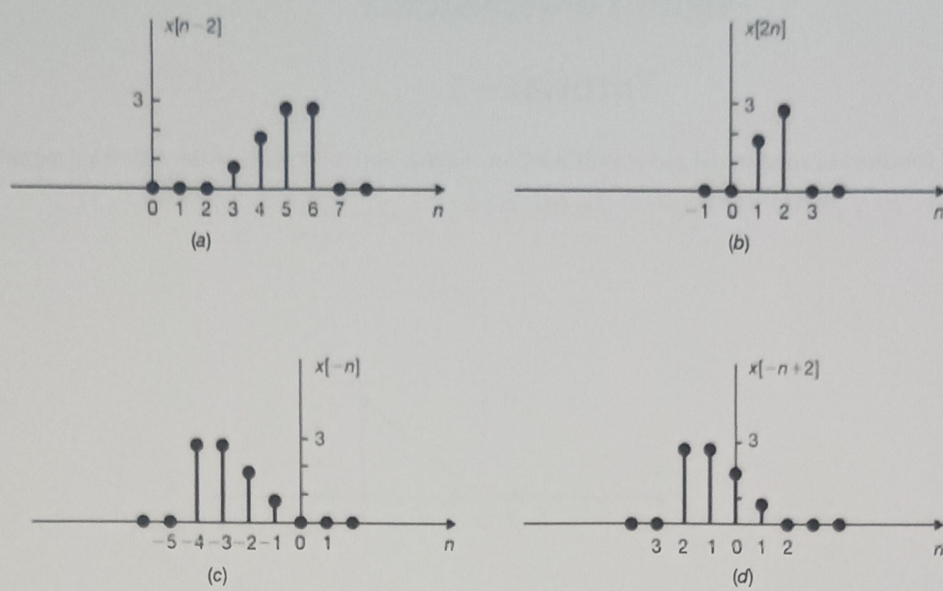


Sol^n.

1.2. **A discrete-time signal $x[n]$ is shown below. Sketch and label each of the following signals.**

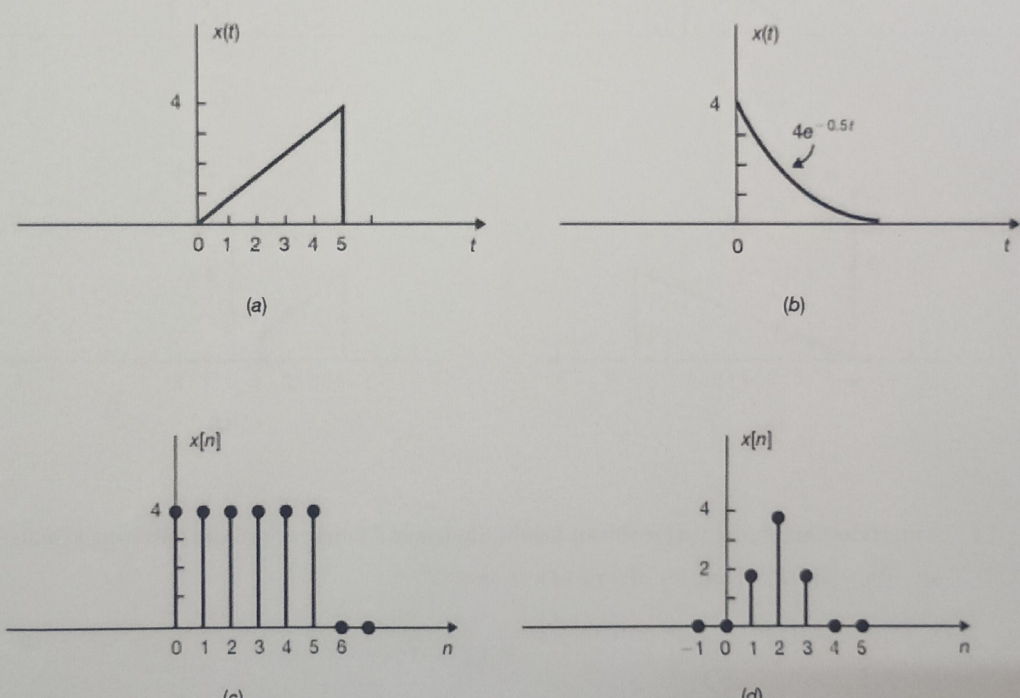
(*a*) $x[n-2]$; (*b*) $x[2n]$; (*c*) $x[-n]$; (*d*) $x[-n+2]$

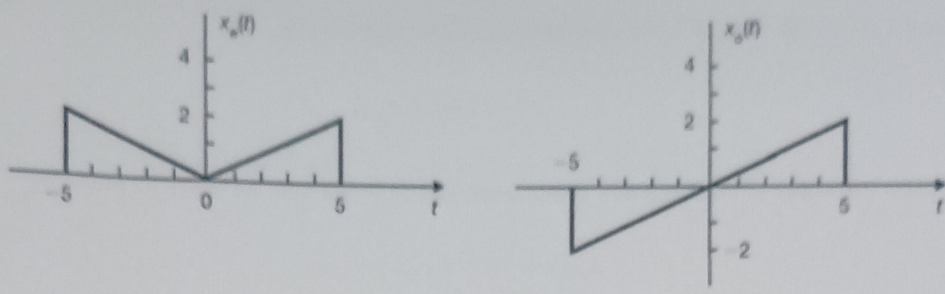




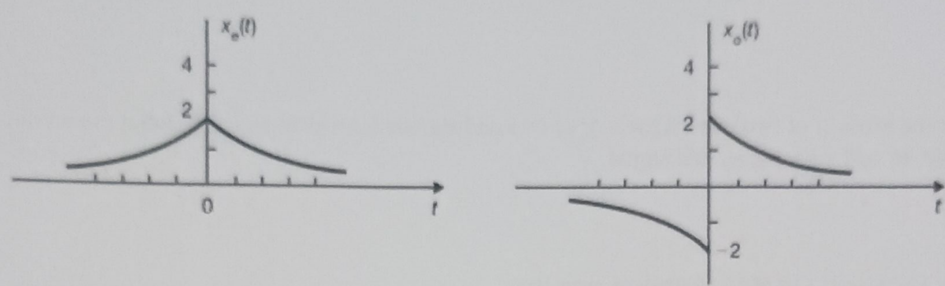
1.3

Sketch and label the even and odd components of the signals shown below.

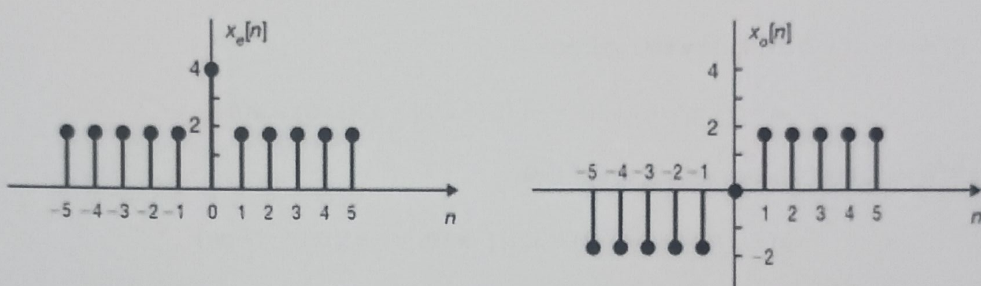




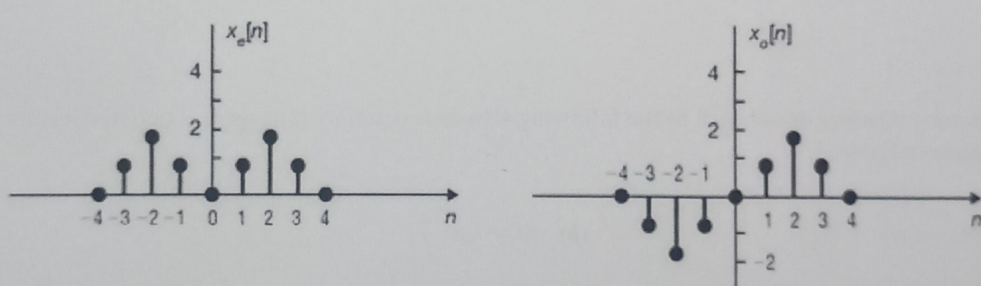
(a)



(b)



(c)



(d)

1.4

Find the even and odd components of $x(t) = e^{jt}$.

Sol^n.

Let $x_e(t)$ and $x_o(t)$ be the even and odd components of e^{jt}, respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

From the basic definition of even and odd components of a function and the Euler formula, we

$$x_e(t) = \frac{1}{2}(e^{jt} + e^{-jt}) = \cos t$$

$$x_o(t) = \frac{1}{2}(e^{jt} - e^{-jt}) = j\sin t$$

1.5

Show that the product of two even signals or of two odd signals is an even signal and that the product of an even and an odd signal is an odd signal.

Sol^n.

Let $x(t) = x_1(t)x_2(t)$. If $x_1(t)$ and $x_2(t)$ are both even, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ and $x_2(t)$ are both odd, then

$$x(-t) = x_1(-t)x_2(-t) = -x_1(t)\,[-x_2(t)] = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ is even and $x_2(t)$ is odd, then

$$x(-t) = x_1(-t)\,x_2(-t) = x_1(t)\,[-x_2(t)] = -x_1(t)x_2(t) = -x(t)$$

and $x(t)$ is odd. Note that in the above proof, variable t represents either a continuous or a discrete variable.

1.6

Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$ (b) $x(t) = \sin\frac{2\pi}{3}t$

(c) $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$ (d) $x(t) = \cos t + \sin\sqrt{2}t$

(e) $x(t) = \sin^2 t$ (f) $x(t) = e^{j[(\pi/2)t - 1]}$

(g) $x[n] = e^{j(\pi/4)n}$ (h) $x[n] = \cos\frac{1}{4}n$

(i) $x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n$ (j) $x[n] = \cos^2\frac{\pi}{8}n$

Sol^n.

(*a*) $x(t) = \cos\left(t + \dfrac{\pi}{4}\right) = \cos\left(\omega_0 t + \dfrac{\pi}{4}\right) \to \omega_0 = 1$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

(*b*) $x(t) = \sin\dfrac{2\pi}{3}t \to \omega_0 = \dfrac{2\pi}{3}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

(*c*) $x(t) = \cos\dfrac{\pi}{3}t + \sin\dfrac{\pi}{4}t = x_1(t) + x_2(t)$

where $x_1(t) = \cos(\pi/3)t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

(*d*) $x(t) = \cos t + \sin\sqrt{2}\,t = x_1(t) + x_2(t)$

where $x_1(t) = \cos t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin\sqrt{2}\,t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\,\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

(*e*) Using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we can write

$$x(t) = \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t = x_1(t) + x_2(t)$$

where $x_1(t) = \frac{1}{2}$ is a dc signal with an arbitrary period and $x_2(t) = -\frac{1}{2}\cos 2t = -\frac{1}{2}\cos\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \pi$. Thus, $x(t)$ is periodic with fundamental period $T_0 = \pi$.

(*f*) $x(t) = e^{j[(\pi/2)t - 1]} = e^{-j}e^{j(\pi/2)t} = e^{-j}e^{j\omega_0 t} \to \omega_0 = \frac{\pi}{2}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 4$.

(*g*) $x[n] = e^{j(\pi/4)n} = e^{j\Omega_0 n} \to \Omega_0 = \frac{\pi}{4}$

Since $\Omega_0/2\pi = \frac{1}{8}$ is a rational number, $x[n]$ is periodic, and by Eq. (1.55) the fundamental period is $N_0 = 8$.

(*h*) $x[n] = \cos\frac{1}{4}n = \cos\Omega_0 n \to \Omega_0 = \frac{1}{4}$

Since $\Omega_0/2\pi = 1/8\pi$ is not a rational number, $x[n]$ is nonperiodic.

(*i*) $x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n = x_1[n] + x_2[n]$

where

$$x_1[n] = \cos\frac{\pi}{3}n = \cos\Omega_1 n \to \Omega_1 = \frac{\pi}{3}$$

$$x_2[n] = \sin\frac{\pi}{4}n = \cos\Omega_2 n \to \Omega_2 = \frac{\pi}{4}$$

Since $\Omega_1/2\pi = \frac{1}{6}$ (= rational number), $x_1[n]$ is periodic with fundamental period $N_1 = 6$, and since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, from the result of Prob. 1.15, $x[n]$ is periodic and its fundamental period is given by the least common multiple of 6 and 8, that is, $N_0 = 24$.

(*j*) Using the trigonometric identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we can write

$$x[n] = \cos^2\frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{4}n = x_1[n] + x_2[n]$$

where $x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$ is periodic with fundamental period $N_1 = 1$ and $x_2[n] = \frac{1}{2}\cos(\pi/4)n = \frac{1}{2}\cos\Omega_2 n \to \Omega_2 = \pi/4$. Since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, $x[n]$ is periodic with fundamental period $N_0 = 8$ (the least common multiple of N_1 and N_2).

1.7

Determine whether the following signals are energy signals, power signals, or neither.

(*a*) $x(t) = e^{-at}u(t), \quad a > 0$ (*b*) $x(t) = A\cos(\omega_0 t + \theta)$

(*c*) $x(t) = tu(t)$ (*d*) $x[n] = (-0.5)^n u[n]$

(*e*) $x[n] = u[n]$ (*f*) $x[n] = 2e^{j3n}$

(*a*) $E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_0^{\infty} e^{-2at}\, dt = \frac{1}{2a} < \infty$

Thus, $x(t)$ is an energy signal.

(*b*) The sinusoidal signal $x(t)$ is periodic with $T_0 = 2\pi/\omega_0$. Then by the result from Prob. 1.18, the average power of $x(t)$ is

$$P = \frac{1}{T_0}\int_0^{T_0} [x(t)]^2\, dt = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} A^2\cos^2(\omega_0 t + \theta)\, dt$$

$$= \frac{A^2\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \frac{1}{2}[1 + \cos(2\omega_0 t + 2\theta)]\, dt = \frac{A^2}{2} < \infty$$

Thus, $x(t)$ is a power signal. Note that periodic signals are, in general, power signals.

(*c*) $E = \lim_{T\to\infty}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \lim_{T\to\infty}\int_0^{T/2} t^2\, dt = \lim_{T\to\infty}\frac{(T/2)^3}{3} = \infty$

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} t^2\, dt = \lim_{T\to\infty}\frac{1}{T}\frac{(T/2)^3}{3} = \infty$$

Thus, $x(t)$ is neither an energy signal nor a power signal.

(*d*) By definition (1.16) and using Eq. (1.91), we obtain

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} 0.25^n = \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

Thus, $x[n]$ is an energy signal.

(*e*) By definition (1.17)

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N} 1^2 = \lim_{N\to\infty}\frac{1}{2N+1}(N+1) = \frac{1}{2} < \infty$$

Thus, $x[n]$ is a power signal.

(*f*) Since $|x[n]| = |2e^{j3n}| = 2|e^{j3n}| = 2$,

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} 2^2$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}4(2N+1) = 4 < \infty$$

Thus, $x[n]$ is a power signal.

Signals and Systems

Tutorial – 2

2.1

Show that

(*a*) If $x(t)$ and $x[n]$ are even, then

$$\int_{-a}^{a} x(t)\,dt = 2\int_{0}^{a} x(t)\,dt$$

$$\sum_{n=-k}^{k} x[n] = x[0] + 2\sum_{n=1}^{k} x[n]$$

(*b*) If $x(t)$ and $x[n]$ are odd, then

$$x(0) = 0 \qquad \text{and} \qquad x[0] = 0$$

$$\int_{-a}^{a} x(t)\,dt = 0 \qquad \text{and} \qquad \sum_{n=-k}^{k} x[n] = 0$$

Sol^n.

(a) We can write

$$\int_{-a}^{a} x(t)\,dt = \int_{-a}^{0} x(t)\,dt + \int_{0}^{a} x(t)\,dt$$

Letting $t = -\lambda$ in the first integral on the right-hand side, we get

$$\int_{-a}^{0} x(t)\,dt = \int_{a}^{0} x(-\lambda)(-d\lambda) = \int_{0}^{a} x(-\lambda)\,d\lambda$$

Since $x(t)$ is even, that is, $x(-\lambda) = x(\lambda)$, we have

$$\int_{0}^{a} x(-\lambda)\,d\lambda = \int_{0}^{a} x(\lambda)\,d\lambda = \int_{0}^{a} x(t)\,dt$$

Hence,

$$\int_{-a}^{a} x(t)\,dt = \int_{0}^{a} x(t)\,dt + \int_{0}^{a} x(t)\,dt = 2\int_{0}^{a} x(t)\,dt$$

Similarly,

$$\sum_{n=-k}^{k} x[n] = \sum_{n=-k}^{-1} x[n] + x[0] + \sum_{n=1}^{k} x[n]$$

Letting $n = -m$ in the first term on the right-hand side, we get

$$\sum_{n=-k}^{-1} x[n] = \sum_{m=1}^{k} x[-m]$$

Since $x[n]$ is even, that is, $x[-m] = x[m]$, we have

$$\sum_{m=1}^{k} x[-m] = \sum_{m=1}^{k} x[m] = \sum_{n=1}^{k} x[n]$$

Hence,

$$\sum_{n=-k}^{k} x[n] = \sum_{n=1}^{k} x[n] + x[0] + \sum_{n=1}^{k} x[n] = x[0] + 2\sum_{n=1}^{k} x[n]$$

Since $x(t)$ and $x[n]$ are odd, that is, $x(-t) = -x(t)$ and $x[-n] = -x[n]$, we have

$$x(-0) = -x(0) \qquad \text{and} \qquad x[-0] = -x[0]$$

Hence,

$$x(-0) = x(0) = -x(0) \Rightarrow x(0) = 0$$
$$x[-0] = x[0] = -x[0] \Rightarrow x[0] = 0$$

Similarly,

$$\int_{-a}^{a} x(t)\,dt = \int_{-a}^{0} x(t)\,dt + \int_{0}^{a} x(t)\,dt = \int_{0}^{a} x(-\lambda)\,d\lambda + \int_{0}^{a} x(t)\,dt$$
$$= -\int_{0}^{a} x(\lambda)\,d\lambda + \int_{0}^{a} x(t)\,dt = -\int_{0}^{a} x(t)\,dt + \int_{0}^{a} x(t)\,dt = 0$$

and

$$\sum_{n=-k}^{k} x[n] = \sum_{n=-k}^{-1} x[n] + x[0] + \sum_{n=1}^{k} x[n] = \sum_{m=1}^{k} x[-m] + x[0] + \sum_{n=1}^{k} x[n]$$
$$= -\sum_{m=1}^{k} x[m] + x[0] + \sum_{n=1}^{k} x[n] = -\sum_{n=1}^{k} x[n] + x[0] + \sum_{n=1}^{k} x[n]$$
$$= x[0] = 0$$

2.2

Show that the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

is periodic and that its fundamental period is $2\pi/\omega_0$.

Sol^n.

Any eq^n $x(t)$ **will be periodic if** $\quad x(t+T) = x(t)$

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t}$$

Since

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}$$

we must have

$$e^{j\omega_0 T} = 1$$

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T. If $\omega_0 \neq 0$,

$$\omega_0 T = m2\pi \qquad \text{or} \qquad T = m\frac{2\pi}{\omega_0} \qquad m = \text{positive integer}$$

Thus, the fundamental period T_0, the smallest positive T, of $x(t)$ is given by $2\pi/\omega_0$.

2.3

Show that the complex exponential sequence

$$x[n] = e^{j\Omega_0 n}$$

is periodic only if $\Omega_0/2\pi$ is a rational number.

Sol^n. $x[n]$ will be periodic if

$$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} e^{j\Omega_0 N} = e^{j\Omega_0 n}$$

or

$$e^{j\Omega_0 N} = 1$$

Above eq^n holds only if

$$\Omega_0 N = m2\pi \qquad m = \text{positive integer}$$

or

$$\frac{\Omega_0}{2\pi} = \frac{m}{N} = \text{rational numbers}$$

Thus, $x[n]$ is periodic only if $\Omega_0/2\pi$ is a rational number.

2.4

Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2, respectively. Under what conditions is the sum $x(t) = x_1(t) + x_2(t)$ periodic, and what is the fundamental period of $x(t)$ if it is periodic?

Sol^n.

Since $x_1(t)$ and $x_2(t)$ are periodic with fundamental periods T_1 and T_2, respectively, we have

$$x_1(t) = x_1(t + T_1) = x_1(t + mT_1) \qquad m = \text{positive integer}$$
$$x_2(t) = x_2(t + T_2) = x_2(t + kT_2) \qquad k = \text{positive integer}$$

Thus,

$$x(t) = x_1(t + mT_1) + x_2(t + kT_2)$$

In order for $x(t)$ to be periodic with period T, one needs

$$x(t + T) = x_1(t + T) + x_2(t + T) = x_1(t + mT_1) + x_2(t + kT_2)$$

Thus, we must have

$$mT_1 = kT_2 = T$$

or

$$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number}$$

In other words, the sum of two periodic signals is periodic only if the ratio of their respective periods can be expressed as a rational number. Then the fundamental period is the least common multiple of T_1 and T_2, and it is given by **above eq^n** the integers m and k are relative prime. If the ratio T_1/T_2 is an irrational number, then the signals $x_1(t)$ and $x_2(t)$ do not have a common period and $x(t)$ cannot be periodic.

2.5

Let $x_1[n]$ and $x_2[n]$ be periodic sequences with fundamental periods N_1 and N_2, respectively. Under what conditions is the sum $x[n] = x_1[n] + x_2[n]$ periodic, and what is the fundamental period of $x[n]$ if it is periodic?

Since $x_1[n]$ and $x_2[n]$ are periodic with fundamental periods N_1 and N_2, respectively, we have

$$x_1[n] = x_1[n + N_1] = x_1[n + mN_1] \qquad m = \text{positive integer}$$
$$x_2[n] = x_2[n + N_2] = x_2[n + kN_2] \qquad k = \text{positive integer}$$

Thus,

$$x[n] = x_1[n + mN_1] + x_2[n + kN_2]$$

In order for $x[n]$ to be periodic with period N, one needs

$$x[n + N] = x_1[n + N] + x_2[n + N] = x_1[n + mN_1] + x_2[n + kN_2]$$

Thus, we must have

$$mN_1 = kN_2 = N$$

Since we can always find integers m and k to satisfy **above eq^n**, it follows that the sum of two periodic sequences is also periodic and its fundamental period is the least common multiple of N_1 and N_2.

2.6

Show that if $x(t+T) = x(t)$, then

$$\int_{\alpha}^{\beta} x(t)\,dt = \int_{\alpha+T}^{\beta+T} x(t)\,dt$$

$$\int_{0}^{T} x(t)\,dt = \int_{a}^{a+T} x(t)\,dt$$

for any real α, β, and a.

Sol^n.

If $x(t+T) = x(t)$, then letting $t = \tau - T$, we have

$$x(\tau - T + T) = x(\tau) = x(\tau - T)$$

and

$$\int_{\alpha}^{\beta} x(t)\,dt = \int_{\alpha+T}^{\beta+T} x(\tau - T)\,d\tau = \int_{\alpha+T}^{\beta+T} x(\tau)\,d\tau = \int_{\alpha+T}^{\beta+T} x(t)\,dt$$

Next, the RHS of the second eq^n can be written as

$$\int_{a}^{a+T} x(t)\,dt = \int_{a}^{0} x(t)\,dt + \int_{0}^{a+T} x(t)\,dt$$

From the first eq^n of the question, we have

$$\int_{a}^{0} x(t)\,dt = \int_{a+T}^{T} x(t)\,dt$$

Thus,

$$\begin{aligned}\int_{a}^{a+T} x(t)\,dt &= \int_{a+T}^{T} x(t)\,dt + \int_{0}^{a+T} x(t)\,dt \\ &= \int_{0}^{a+T} x(t)\,dt + \int_{a+T}^{T} x(t)\,dt = \int_{0}^{T} x(t)\,dt\end{aligned}$$

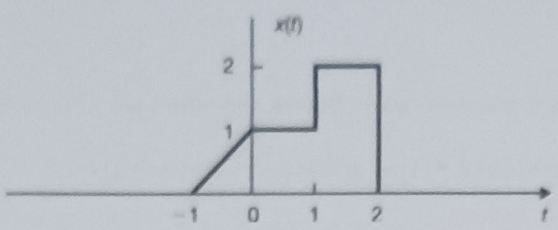
Signals and Systems

Tutorial – 3

3.1

A continuous-time signal $x(t)$ is shown below. Sketch and label each of the following signals.

(a) $x(t)u(1-t)$; (b) $x(t)[u(t)-u(t-1)]$; (c) $x(t)\delta(t-\frac{3}{2})$

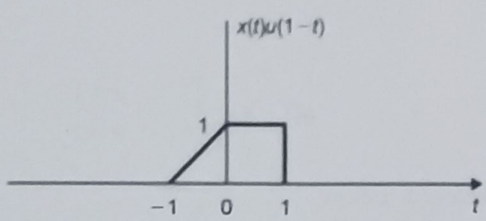


Sol^n.

By definition

$$u(1-t)=\begin{cases}1 & t<1\\ 0 & t>1\end{cases}$$

and $x(t)u(1-t)$ is sketched in the figure below.

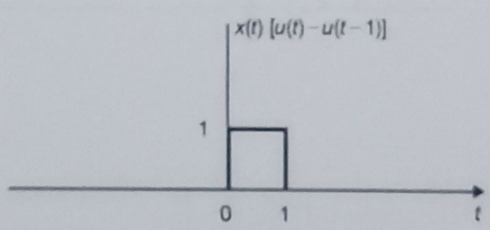


(a)

(b)

$$u(t)-u(t-1)=\begin{cases}1 & 0<t\le 1\\ 0 & \text{otherwise}\end{cases}$$

and $x(t)[u(t)-u(t-1)]$ is sketched below.

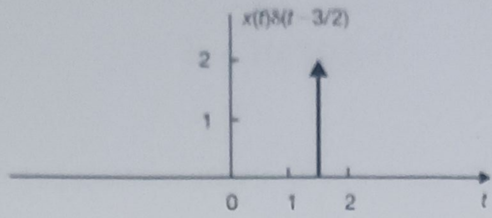


(b)

(c)

$$x(t)\delta\left(t-\frac{3}{2}\right)=x\left(\frac{3}{2}\right)\delta\left(t-\frac{3}{2}\right)=2\delta\left(t-\frac{3}{2}\right)$$

which is sketched below.

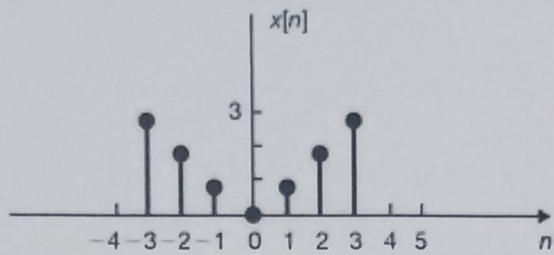


(c)

3.2

A discrete-time signal $x[n]$ is shown below. Sketch and label each of the following signals.

(*a*) $x[n]u[1-n]$; (*b*) $x[n]\{u[n+2]-u[n]\}$; (*c*) $x[n]\delta[n-1]$

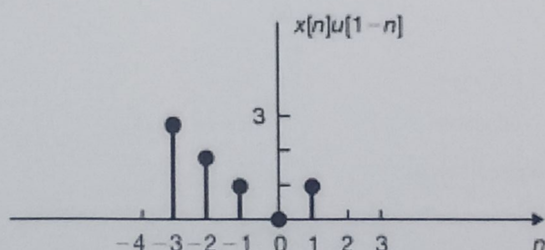


Solⁿ.

(*a*) By definition

$$u[1-n]=\begin{cases}1 & n\le 1\\ 0 & n>1\end{cases}$$

and $x[n]u[1-n]$ is sketched below.

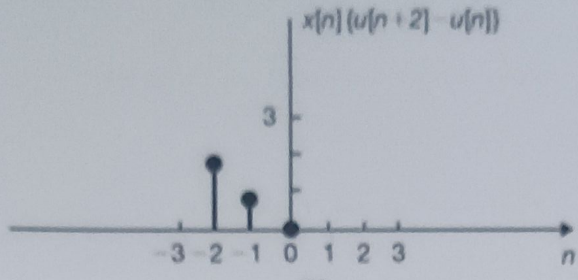


(a)

(*b*) By definitions

$$u[n+2]-u[n]=\begin{cases}1 & -2\le n<0\\ 0 & \text{otherwise}\end{cases}$$

and $x[n]\{u[n+2]-u[n]\}$ is sketched below.

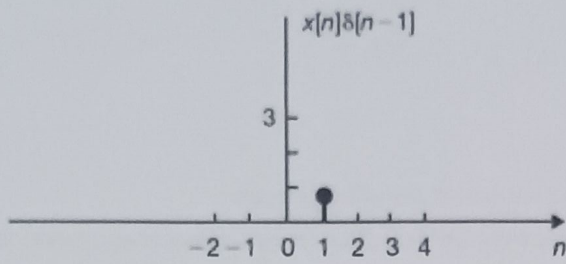


(b)

(c) By definition

$$x[n]\delta[n-1] = x[1]\delta[n-1] = \delta[n-1] = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

which is sketched below.



(c)

3.3

The unit step function $u(t)$ can be defined as a generalized function by the following relation:

$$\int_{-\infty}^{\infty} \phi(t)u(t)\,dt = \int_{0}^{\infty} \phi(t)\,dt$$

where $\phi(t)$ is a testing function which is integrable over $0 < t < \infty$. Using this definition, show that

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Sol^n.

Rewriting above eq^n.

$$\int_{-\infty}^{\infty} \phi(t)u(t)\,dt = \int_{-\infty}^{0} \phi(t)u(t)\,dt + \int_{0}^{\infty} \phi(t)u(t)\,dt = \int_{0}^{\infty} \phi(t)\,dt$$

we obtain

$$\int_{-\infty}^{0} \phi(t)u(t)\,dt = \int_{0}^{\infty} \phi(t)[1 - u(t)]\,dt$$

This can be true only if

$$\int_{-\infty}^{0} \phi(t)u(t)\,dt = 0 \quad \text{and} \quad \int_{0}^{\infty} \phi(t)[1 - u(t)]\,dt = 0$$

These conditions imply that

$$\phi(t)u(t) = 0, t < 0 \quad \text{and} \quad \phi(t)[1 - u(t)] = 0, t > 0$$

Since $\phi(t)$ is arbitrary, we have

$$u(t) = 0, t < 0 \quad \text{and} \quad 1 - u(t) = 0, t > 0$$

that is,

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

3.4

Verify the following.

(a) $\delta(at) = \dfrac{1}{|a|}\delta(t)$; (b) $\delta(-t) = \delta(t)$

Sol^n.

The proof will be based on the following *equivalence* property:

Let $g_1(t)$ and $g_2(t)$ be generalized functions. Then the equivalence property states that $g_1(t) = g_2(t)$ if and only if

$$\int_{-\infty}^{\infty} \phi(t)\, g_1(t)\, dt = \int_{-\infty}^{\infty} \phi(t)\, g_2(t)\, dt$$

for all suitably defined testing functions $\phi(t)$.

(a) With a change of variable, $at = \tau$, and hence $t = \tau/a$, $dt = (1/a)\, d\tau$, we obtain the following equations:

If $a > 0$,

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\, d\tau = \frac{1}{a}\phi\left(\frac{\tau}{a}\right)\Bigg|_{\tau=0} = \frac{1}{|a|}\phi(0)$$

If $a < 0$,

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{a}\int_{\infty}^{-\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\, d\tau = -\frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\, d\tau$$

$$= -\frac{1}{a}\phi\left(\frac{\tau}{a}\right)\Bigg|_{\tau=0} = \frac{1}{|a|}\phi(0)$$

Thus, for any a

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{|a|}\phi(0)$$

Now, for $\phi(0)$, we obtain

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{|a|}\phi(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt$$

$$= \int_{-\infty}^{\infty} \phi(t)\frac{1}{|a|}\delta(t)\, dt$$

for any $\phi(t)$. Then, by the equivalence property **, we obtain**

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

(*b*) Setting $a = -1$ in the above equation, we obtain

$$\delta(-t) = \frac{1}{|-1|}\delta(t) = \delta(t)$$

which shows that $\delta(t)$ is an even function.

3.5

(*a*) Verify

$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

if $x(t)$ is continuous at $t = t_0$.

(*b*) Verify

$$x(t)\,\delta(t) = x(0)\,\delta(t)$$

if $x(t)$ is continuous at $t = 0$.

Sol^n.

(*a*) If $x(t)$ is continuous at $t = t_0$, then by definition , **we have**

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t)[x(t)\delta(t - t_0)]\,dt &= \int_{-\infty}^{\infty} [\phi(t)x(t)]\delta(t - t_0)\,dt = \phi(t_0)x(t_0) \\ &= x(t_0)\int_{-\infty}^{\infty} \phi(t)\delta(t - t_0)\,dt \\ &= \int_{-\infty}^{\infty} \phi(t)[x(t_0)\delta(t - t_0)]\,dt\end{aligned}$$

for all $\phi(t)$ which are continuous at $t = t_0$. Hence, by the equivalence property **we conclude that**

$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

(*b*) Setting $t_0 = 0$ in the above expression, we obtain

$$x(t)\delta(t) = x(0)\delta(t)$$

Signals and Systems

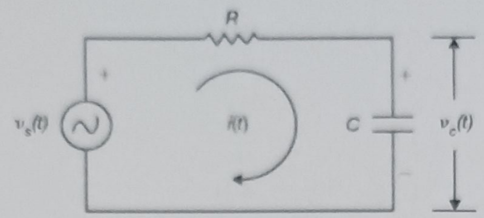
Tutorial – 4

4.1

Consider the RC circuit shown below. Find the relationship between the input $x(t)$ and the output $y(t)$

(*a*) If $x(t) = v_s(t)$ and $y(t) = v_c(t)$.

(*b*) If $x(t) = v_s(t)$ and $y(t) = i(t)$.



Sol^n.

(*a*) Applying Kirchhoff's voltage law to the *RC* Circuit, we obtain

$$v_s(t) = Ri(t) + v_c(t) \tag{1}$$

The current $i(t)$ and voltage $v_c(t)$ are related by

$$i(t) = C\frac{dv_c(t)}{dt} \tag{2}$$

Letting $v_s(t) = x(t)$ and $v_c(t) = y(t)$ and substituting eq. (2) into eq. (1), we obtain

$$RC\frac{dy(t)}{dt} + y(t) = x(t) \tag{3}$$

or

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

Thus, the input-output relationship of the *RC* circuit is described by a first-order linear differential equation with constant coefficients.

(*b*) Integrating Eq. (1.104), we have

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \tag{4}$$

Substituting eq. (4) into eq. (1) and letting $v_s(t)=x(t)$ and $i(t) = y(t)$, we obtain

$$Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t)$$

or

$$y(t) + \frac{1}{RC}\int_{-\infty}^{t} y(\tau)\,d\tau = \frac{1}{R}x(t)$$

Differentiating both sides of the above equation with respect to t, we obtain

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{R}\frac{dx(t)}{dt}$$

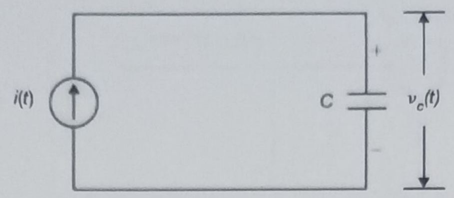
Thus, the input-output relationship is described by another first-order linear differential equation with constant coefficients.

4.2

Consider the capacitor shown below. Let input $x(t)$ and output $y(t) = v_c(t)$

(a) Find the input-output relationship.

(b) Determine whether the system is (i) memoryless, (ii) causal, (iii) linear, (iv) time-invariant, or (v) stable.



Sol^n.

(a) Assume the capacitance C is constant. The output voltage $y(t)$ across the capacitor and the input current $x(t)$ are related by

$$y(t) = \mathbf{T}\{x(t)\} = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

(b) (i) **From above eq.** it is seen that the output $y(t)$ depends on the past and the present values of the input. Thus, the system is not memoryless.

(ii) Since the output $y(t)$ does not depend on the future values of the input, the system is causal.

(iii) Let $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$. Then

$$\begin{aligned} y(t) = \mathbf{T}\{x(t)\} &= \frac{1}{C}\int_{-\infty}^{t} [\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)]\,d\tau \\ &= \alpha_1\left[\frac{1}{C}\int_{-\infty}^{t} x_1(\tau)\,d\tau\right] + \alpha_2\left[\frac{1}{C}\int_{-\infty}^{t} x_2(\tau)\,d\tau\right] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

Thus, the superposition property is satisfied and the system is linear.

(iv) Let $y_1(t)$ be the output produced by the shifted input current $x_1(t) = x(t - t_0)$.

Then

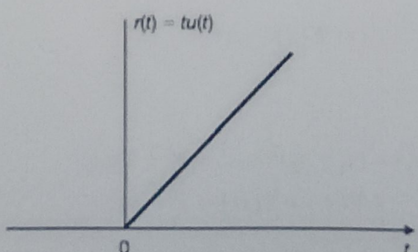
$$\begin{aligned} y_1(t) = \mathbf{T}\{x(t - t_0)\} &= \frac{1}{C}\int_{-\infty}^{t} x(\tau - t_0)\,d\tau \\ &= \frac{1}{C}\int_{-\infty}^{t-t_0} x(\lambda)\,d\lambda = y(t - t_0) \end{aligned}$$

Hence, the system is time-invariant.

(v) Let $x(t) = k_1 u(t)$, with $k_1 \neq 0$. Then

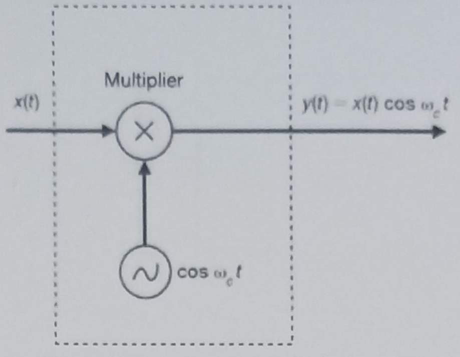
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} k_1 u(\tau)\,d\tau = \frac{k_1}{C}\int_{0}^{t} d\tau = \frac{k_1}{C} t u(t) = \frac{k_1}{C} r(t) \qquad (1$$

Where $r(t) = tu(t)$ is known as the *unit ramp* function (below fig.) Since $y(t)$ grows linearly in time without bound, the system is not BIBO stable.



4.3

Consider the system shown below. Determine whether it is (a) memoryless, (b) causal, (c) linear, (d) time-invariant, or (e) stable.



(a) From the above fig., we have

$$y(t) = \mathbf{T}\{x(t)\} = x(t) \cos \omega_c t$$

Since the value of the output $y(t)$ depends on only the present values of the input $x(t)$, the system is memoryless.

(b) Since the output $y(t)$ does not depend on the future values of the input $x(t)$, the system is causal.

(c) Let $x(t) = \alpha_1 x(t) + \alpha_2 x(t)$. Then

$$\begin{aligned} y(t) &= \mathbf{T}\{x(t)\} = [\alpha_1 x_1(t) + \alpha_2 x_2(t)] \cos \omega_c t \\ &= \alpha_1 x_1(t) \cos \omega_c t + \alpha_2 x_2(t) \cos \omega_c t \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

Thus, the superposition property is satisfied and the system is linear.

(d) Let $y_1(t)$ be the output produced by the shifted input $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = \mathbf{T}\{x(t - t_0)\} = x(t - t_0) \cos \omega_c t$$

But

$$y(t - t_0) = x(t - t_0) \cos \omega_c (t - t_0) \neq y_1(t)$$

Hence, the system is not time-invariant.

(e) Since $|\cos \omega_c t| \leq 1$, we have

$$|y(t)| = |x(t) \cos \omega_c t| \leq |x(t)|$$

Thus, if the input $x(t)$ is bounded, then the output $y(t)$ is also bounded and the system is BIBO stable.

4.4

A system has the input-output relation given by

$$y = \mathbf{T}\{x\} = x^2$$

Sol^n.

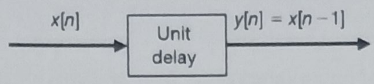
Show that this system is nonlinear.

$$\begin{aligned} \mathbf{T}\{x_1 + x_2\} &= (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1 x_2 \\ &\neq \mathbf{T}\{x_1\} + \mathbf{T}\{x_2\} = x_1^2 + x_2^2 \end{aligned}$$

Thus, the system is nonlinear.

4.5

The discrete-time system shown in fig. below is known as the *unit delay* element. Determine whether the system is *(a)* memoryless, *(b)* causal, *(c)* linear, *(d)* time-invariant, or (e) stable.



Solⁿ.

(a) The system input-output relation is given by

$$y[n] = \mathbf{T}\{x[n]\} = x[n-1]$$

Since the output value at n depends on the input values at $n-1$, the system is not memoryless.

(b) Since the output does not depend on the future input values, the system is causal.

(c) Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. Then

$$y[n] = \mathbf{T}\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]$$
$$= \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Thus, the superposition property is satisfied and the system is linear.

(d) Let $y_1[n]$ be the response to $x_1[n] = x[n-n_0]$. Then

$$y_1[n] = \mathbf{T}\{x_1[n]\} = x_1[n-1] = x[n-1-n_0]$$

and
$$y[n-n_0] = x[n-n_0-1] = x[n-1-n_0] = y_1[n]$$

Hence, the system is time-invariant.

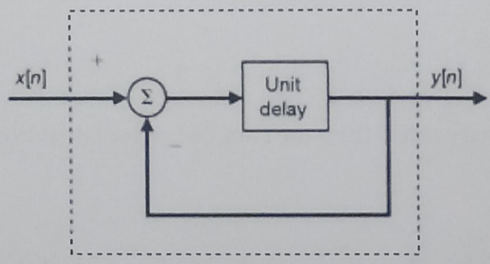
(e) Since

$$|y[n]| = |x[n-1]| \le k \qquad \text{if } |x[n]| \le k \text{ for all } n$$

the system is BIBO stable.

4.7

Find the input-output relation of the feedback system shown in Fig below.



Solⁿ.

From above fig. the input to the unit delay element is $x[n]-y[n]$. Thus, the output $y[n]$ of the unit delay element is

$$y[n] = x[n-1] - y[n-1]$$

Rearranging, we obtain

$$y[n] + y[n-1] = x[n-1] \tag{1.112}$$

Thus, the input-output relation of the system is described by a first-order difference equation with constant coefficients.

Signals and Systems

Tutorial – 5

5.1

We call a set of signals $\{\Psi_n(t)\}$ *orthogonal* on an interval (a, b) if any two signals $\Psi_m(t)$ and $\Psi_k(t)$ in the set satisfy the condition

$$\int_a^b \Psi_m(t)\,\Psi_k^*(t)\,dt = \begin{cases} 0 & m \neq k \\ \alpha & m = k \end{cases} \tag{5.95}$$

where $*$ denotes the complex conjugate and $\alpha \neq 0$. Show that the set of complex exponentials $\{e^{jk\omega_0 t}: k = 0, \pm 1, \pm 2, \ldots\}$ is orthogonal on any interval over a period T_0, where $T_0 = 2\pi/\omega_0$.

Sol^n.

For any t_0 we have

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\,dt = \frac{1}{jm\omega_0} e^{jm\omega_0 t}\Big|_{t_0}^{t_0+T_0} = \frac{1}{jm\omega_0}\left(e^{jm\omega_0(t_0+T_0)} - e^{jm\omega_0 t_0}\right)$$

$$= \frac{1}{jm\omega_0} e^{jm\omega_0 t_0}\left(e^{jm2\pi} - 1\right) = 0 \qquad m \neq 0$$

since $e^{jm2\pi} = 1$. When $m = 0$, we have $e^{jm\omega_0 t}\big|_{m=0} = 1$ and

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\,dt = \int_{t_0}^{t_0+T_0} dt = T_0$$

From the above, we conclude that

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\left(e^{jk\omega_0 t}\right)^*\,dt = \int_{t_0}^{t_0+T_0} e^{j(m-k)\omega_0 t}\,dt = \begin{cases} 0 & m \neq k \\ T_0 & m = k \end{cases}$$

which shows that the set $\{e^{jk\omega_0 t}: k = 0, \pm 1, \pm 2, \ldots\}$ is orthogonal on any interval over a period T_0.

5.2

Derive the trigonometric Fourier series from the complex exponential Fourier series.

Sol^n.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}\right)$$

and using Euler's formulas

$$e^{\pm jk\omega_0 t} = \cos k\omega_0 t \pm j \sin k\omega_0 t$$

we have

$$x(t) = c_0 + \sum_{k=1}^{\infty}\left[(c_k + c_{-k})\cos k\omega_0 t + j(c_k - c_{-k})\sin k\omega_0 t\right]$$

Setting

$$c_0 = \frac{a_0}{2} \qquad c_k + c_{-k} = a_k \qquad j(c_k - c_{-k}) = b_k$$

The above eq^n. becomes

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

5.3

Determine the complex exponential Fourier series representation for each of the following signals:

(a) $x(t) = \cos \omega_0 t$

(b) $x(t) = \sin \omega_0 t$

(c) $x(t) = \cos\left(2t + \dfrac{\pi}{4}\right)$

(d) $x(t) = \cos 4t + \sin 6t$

(e) $x(t) = \sin^2 t$

Sol^n.

(a Evaluate the complex Fourier coefficient c_k using Euler's formula, we get

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\cos \omega_0 t$ are

$$c_1 = \frac{1}{2} \qquad c_{-1} = \frac{1}{2} \qquad c_k = 0, |k| \neq 1$$

(b) In a similar fashion we have

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) = -\frac{1}{2j}e^{-j\omega_0 t} + \frac{1}{2j}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\sin \omega_0 t$ are

$$c_1 = \frac{1}{2j} \qquad c_{-1} = -\frac{1}{2j} \qquad c_k = 0, |k| \neq 1$$

(c) The fundamental angular frequency ω_0 of $x(t)$ is 2. Thus,

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Now

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2}(e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)})$$

$$= \frac{1}{2}e^{-j\pi/4}e^{-j2t} + \frac{1}{2}e^{j\pi/4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos(2t + \pi/4)$ are

$$c_1 = \frac{1}{2}e^{j\pi/4} = \frac{1}{2}\frac{1+j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1+j)$$

$$c_{-1} = \frac{1}{2}e^{-j\pi/4} = \frac{1}{2}\frac{1-j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1-j)$$

$$c_k = 0 \qquad |k| \neq 1$$

(*d*) The fundamental period T_0 of $x(t)$ is π and $\omega_0 = 2\pi/T_0 = 2$. Thus,

$$x(t) = \cos 4t + \sin 6t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we have

$$x(t) = \cos 4t + \sin 6t = \frac{1}{2}(e^{j4t} + e^{-j4t}) + \frac{1}{2j}(e^{j6t} - e^{-j6t})$$

$$= -\frac{1}{2j}e^{-j6t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j4t} + \frac{1}{2j}e^{j6t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos 4t + \sin 6t$ are

$$c_{-3} = -\frac{1}{2j} \qquad c_{-2} = \frac{1}{2} \qquad c_2 = \frac{1}{2} \qquad c_3 = \frac{1}{2j}$$

and all other $c_k = 0$.

(*e*) The fundamental period T_0 of $x(t)$ is π and $\omega_0 = 2\pi/T_0 = 2$. Thus,

$$x(t) = \sin^2 t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we get

$$x(t) = \sin^2 t = \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^2 = -\frac{1}{4}(e^{j2t} - 2 + e^{-j2t})$$

$$= -\frac{1}{4}e^{-j2t} + \frac{1}{2} - \frac{1}{4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\sin^2 t$ are

$$c_{-1} = -\frac{1}{4} \qquad c_0 = \frac{1}{2} \qquad c_1 = -\frac{1}{4}$$

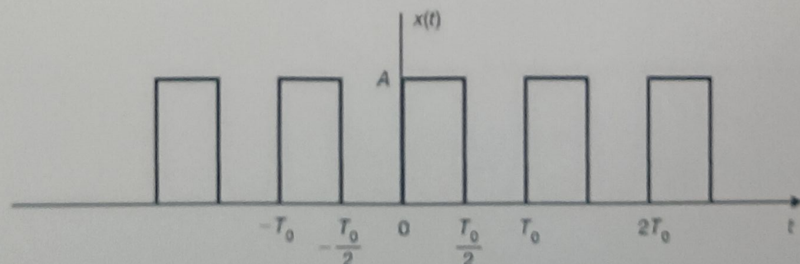
and all other $c_k = 0$.

5.4

Consider the periodic square wave $x(t)$ shown below.

(*a*) Determine the complex exponential Fourier series of $x(t)$.

(*b*) Determine the trigonometric Fourier series of $x(t)$.



Sol^n.

(*a*) Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

the complex coefficient of the Fourier series is given by

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_0^{T_0/2} A\, e^{-jk\omega_0 t}\, dt \\ &= \frac{A}{-jk\omega_0 T_0} e^{-jk\omega_0 t}\bigg|_0^{T_0/2} = \frac{A}{-jk\omega_0 T_0}\left(e^{-jk\omega_0 T_0/2} - 1\right) \\ &= \frac{A}{jk2\pi}\left(1 - e^{-jk\pi}\right) = \frac{A}{jk2\pi}\left[1 - (-1)^k\right] \end{aligned}$$

since $\omega_0 T_0 = 2\pi$ and $e^{-jk\pi} = (-1)^k$. Thus,

$$\begin{aligned} c_k &= 0 & k &= 2m \neq 0 \\ c_k &= \frac{A}{jk\pi} & k &= 2m+1 \\ c_0 &= \frac{1}{T_0}\int_0^{T_0} x(t)\, dt = \frac{1}{T_0}\int_0^{T_0/2} A\, dt = \frac{A}{2} \end{aligned}$$

Hence,

$$c_0 = \frac{A}{2} \qquad c_{2m} = 0 \qquad c_{2m+1} = \frac{A}{j(2m+1)\pi} \tag{1}$$

and we obtain

$$x(t) = \frac{A}{2} + \frac{A}{j\pi}\sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

(*b*) **From the eq. (1), we have**

$$\frac{a_0}{2} = c_0 = \frac{A}{2} \qquad a_{2m} = b_{2m} = 0,\ m \neq 0$$

$$a_{2m+1} = 2\,\mathrm{Re}[c_{2m+1}] = 0 \qquad b_{2m+1} = -2\,\mathrm{Im}[c_{2m+1}] = \frac{2A}{(2m+1)\pi}$$

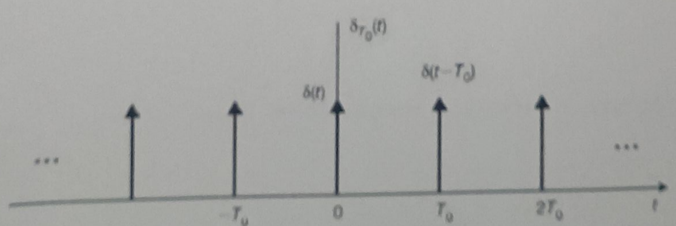
Substituting these values in Eq. (5.8), we get

$$\begin{aligned} x(t) &= \frac{A}{2} + \frac{2A}{\pi}\sum_{m=0}^{\infty} \frac{1}{2m+1}\sin(2m+1)\omega_0 t \\ &= \frac{A}{2} + \frac{2A}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right) \end{aligned}$$

5.5

Consider the periodic impulse train $\delta_{T0}(t)$ shown in fig. below and defined by

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$



(a) Determine the complex exponential Fourier series of $\delta_{T_0}(t)$.
(b) Determine the trigonometric Fourier series of $\delta_{T_0}(t)$.

Sol^n.

(a) Let

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Since δ(t) is involved, the Fourier coefficients are given by:

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}$$

Hence, we get

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

(b) Let

$$\delta_{T_0}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

Since $\delta_{T_0}(t)$ is even, $b_k = 0$,

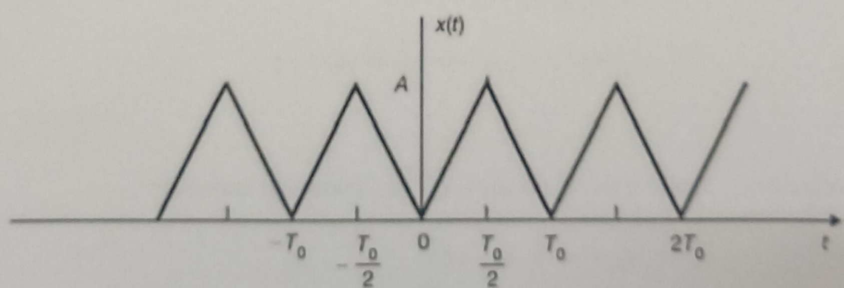
$$a_k = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) \cos k\omega_0 t\, dt = \frac{2}{T_0}$$

Thus, we get

$$\delta_{T_0}(t) = \frac{1}{T_0} + \frac{2}{T_0}\sum_{k=1}^{\infty} \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

5.6

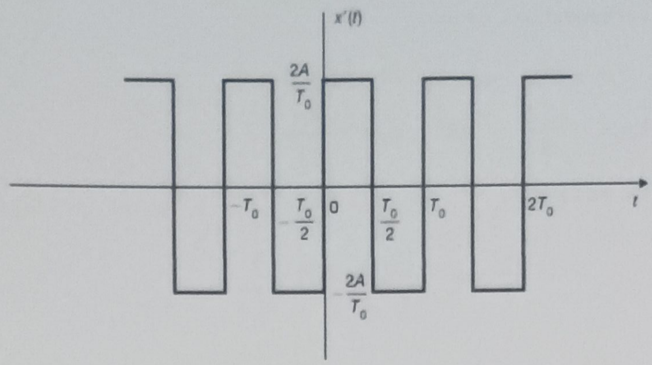
Consider the triangular wave $x(t)$ shown below. Using the differentiation technique, find (a) the complex exponential Fourier series of $x(t)$, and (b) the trigonometric Fourier series of $x(t)$.



(a)

Sol^n.

The derivative $x'(t)$ of the triangular wave $x(t)$ is a square wave as shown below.



(a) Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Differentiating the above eq., we obtain

$$x'(t) = \sum_{k=-\infty}^{\infty} jk\omega_0 c_k e^{jk\omega_0 t}$$

Above equation shows that the complex Fourier coefficients of $x'(t)$ equal $jk\omega_0 c_k$. Thus, we can find c_k $(k \neq 0)$ if the Fourier coefficients of $x'(t)$ are known.

The Fourier series for the periodic signal *x'(t)*, is given by the following relation

$$x'(t) = \frac{4A}{j\pi T_0} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

Equating both the eq. for *x'(t)*, we obtain

$$c_k = 0 \qquad k = 2m \neq 0$$

$$jk\omega_0 c_k = \frac{4A}{j\pi k T_0} \quad \text{or} \quad c_k = -\frac{2A}{\pi^2 k^2} \qquad k = 2m+1$$

Hence, we have

$$c_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt = \frac{A}{2}$$

Substituting these values into the equation for *x(t)*, *we obtain*

$$x(t) = \frac{A}{2} - \frac{2A}{\pi^2} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)^2} e^{j(2m+1)\omega_0 t}$$

In a similar fashion, differentiating Eq. (5.8), we obtain

$$x'(t) = \sum_{k=1}^{\infty} k\omega_0 (b_k \cos k\omega_0 t - a_k \sin k\omega_0 t)$$

Above equation shows that the Fourier cosine coefficients of $x'(t)$ equal to $k\omega_0 b_k$ and that the sine coefficients equal to $-k\omega_0 a_k$. Hence,

$$x'(t) = \frac{8A}{\pi T_0} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\omega_0 t$$

Equating both the above equations, we have

$$b_k = 0 \qquad a_k = 0 \qquad k = 2m \neq 0$$

$$-k\omega_0 a_k = \frac{8A}{\pi k T_0} \qquad \text{or} \qquad a_k = -\frac{4A}{\pi^2 k^2} \qquad k = 2m + 1$$

To find the value of C_0, we have

$$\frac{a_0}{2} = c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{A}{2}$$

Hence, we have

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\,\omega_0 t$$