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Q1) N letters are to be put in N separate envelops. (N=50)

P(at least one letter is in the correct envelop) = 1 − P(No letter is in the correct envelop)

$$\boxed{P(\text{at least one letter is in the correct envelop}) = 1 - \frac{D_n}{n!}}$$

$$D_n = \left[\frac{n!}{e} + \frac{1}{2}\right]$$

[ ] → greatest integer

$$\boxed{1 - \frac{D_{50}}{50!} \Rightarrow 1 - \frac{1}{50!}\left[\frac{50!}{e} + \frac{1}{2}\right] \simeq 1 - \frac{1}{e} \approx 0.6321}$$

This ... diagonal $x=y$ ...
... probability is ... $d/2 \,/\, d^2$

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(Q9) Convulation of 2 distribution function is also a distribution function

Let F & G be distribution function of 2 random variables X & Y

$Z = X+Y$

Convulation of F & G can be written as :-

$$H(x) = (F^*G)(x) = \int_{-\infty}^{\infty} F(x-y)\, d\, G(y)$$

(i) Let $x_1 < x_2$ for fixed y

$$F(x_1-y) \le F(x_2-y)$$

as F is ~~..~~ non-decreasing, integrating both sides w.r.t. G,

$$H(x_1) = \int F(x_1-y)\, d\, G(y) \le \int F(x_2-y)\, d\, G(y) = H(x_2)$$

$\Rightarrow$ $H(x)$ is non-decreasing

(ii) as $x \to -\infty$, for any y, $x-y \to -\infty \Rightarrow F(x-y) \to 0$

$$\lim_{x \to -\infty} H(x) = \int 0\, d\, G(y) = 0$$

as $x \to \infty$, $F(x-y) \to 1 \;\forall y$, $\lim_{x \to \infty} H(x) = \int 1\, dG(y) = 1$

(iii) To show: $H(x) \to H(x_0)$ as $x \to x_0^+$

$\because$ F is right continuous, the integrand $F(x-y)$ is also right-continuous in x for each y.

→ Thus, H(x) is right continuious.

→ As it obeys all properties of a p.d.f, hence a ~~d~~ distribution.

Q5. $1, 2, 3, \ldots\ldots N$; $n \leq N$ tickets are drawn with replacement. Only maximum prize ticket is to be kept.

M = prize money obtained $\quad E(M) = ?$

$$P(M \leq k) = P(\text{all } n \text{ draws} \leq k) = \left(\frac{k}{N}\right)^n$$

$$P(M = k) = P(M \leq k) - P(M \leq k-1) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

$$\Rightarrow E[M] = \sum_{k=1}^{N} k \cdot P(M = k) = \sum_{k=1}^{N} k\left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n\right]$$

Q6. Let X & Y be chose uniform on $[0,d]$

$P(|X-Y| < d/3)$ as X & Y are chose independently & uniformly, their joint distribution is uniform over squared $[0,d] \times [0,d]$

$$0 \le X, Y \le d$$
$$|X-Y| \le d/3$$

It is a square with area $d^2 \Rightarrow X - Y = \pm d/3$

The region is a band of width $2d/3$ along the diagonal $X=Y$ centered within the square, so desired probability is area of region where
$$|X-Y| < d/3$$

$$A_{Total} = d^2$$

Area outside band $|X-Y| \ge d/3 \rightarrow$ this consists of 2 right-angled triangles, one below $Y = X - d/3$ & one above $Y = X + d/3$

Each of Δ has length $= d - \frac{d}{3} = \frac{2d}{3}$

Area of each $\Delta = \frac{1}{2} \times \frac{2d}{3} \cdot \frac{2d}{3} = \frac{2d^2}{9}$

Area outside band $= \frac{4d^2}{9}$

Area inside band $= \frac{5d^2}{9}$

$\Rightarrow P(|X-Y| < d/3) = \left(\frac{5d^2}{9}\right) / d^2 = \frac{5}{9}$

Q2. Let G_i be gift in present i, $i = 1,2,3$
Let H_2 be host opens present 2 & shows it is empty.

I chose present 1, initially.

I need $E[\text{win}/H_2, \text{ I switch to present } 3]$

$$= 1000 \cdot P(G_3/H_2) + 0 \cdot P(G_1/H_2) = 1000 \cdot P(G_3/H_2)$$

→ All gifts are equally likely to be chosen
→ $P(G_1) = P(G_2) = P(G_3) = 1/3$

Case I → If gift in Present 1 ⇒ $P(H_2/G_1) = 1/2$

Case II → If gift in Present 2 ⇒ $P(H_2/G_2) = 0$

Case III → If gift in Present 3 ⇒ $P(H_2/G_3) = 1$

Now $P(H_2) = \sum_{i=1}^{3} P(H_2/G_i) \cdot P(G_i) = \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + \frac{1}{3} = 1/2$

$$P(G_3/H_2) = \frac{P(H_2/G_3) \cdot P(G_3)}{P(H_2)} = \frac{1 \cdot 1/3}{1/2} = \frac{2}{3}$$

Hence

$$\boxed{E[\text{win}/H_2, \text{switch}] = 1000 \cdot 2/3 = 666.67}$$

Q11) $u \in \mathbb{R}$. $\phi(x) = e^{ux}$ $\forall\, x \in \mathbb{R}$

X is a normal random variable, $\mu = E[X]$, $\sigma = [E(X-\mu)^2]^{1/2}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(i) $E[e^{uX}] = e^{u\mu + \frac{1}{2}\sigma^2 u^2}$

LHS $E[e^{ux}] = E[e^{ux - u\mu + u\mu}] = e^{u\mu} E[e^{u(x-\mu)}]$

$$E[e^{ux}] = e^{u\mu} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{u(x-\mu)} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma^2}\right)^2}$$

$$= e^{u\mu} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 u (x-\mu)\right]} dx$$

Now

$(x-\mu)^2 - 2\sigma^2 u (x-\mu) = (x-\mu)^2 - 2\sigma^2 u (x-\mu) + \sigma^4 u^2 - \sigma^4 u^2 =$

$\Rightarrow (x-\mu-\sigma^2 u)^2 - \sigma^4 u^2$

$$\therefore E(e^{ux}) = e^{u\mu} e^{\frac{\sigma^2 u^2}{2}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[(x-\mu-\sigma^2 u)^2]}{2\sigma^2}} dx$$

$$E(e^{ux}) = e^{u\mu + \frac{1}{2}\sigma^2 u^2} \quad \left(\because \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 u))^2}{2\sigma^2}} dx = 1\right)$$

$$\Rightarrow \boxed{E[e^{ux}] = e^{u\mu + \frac{1}{2}\sigma^2 u^2}} \quad \text{H.V.}$$

sists of two.

(iv) $E(\phi(X)) \geq \phi(E(X))$

Verification

$$E[\phi(X)] \Rightarrow e^{u\mu + \frac{1}{2}\sigma^2 u^2} = e^{u\mu} \cdot e^{\frac{1}{2}\sigma^2 u^2} = \phi(E[X]) \cdot e^{\frac{1}{2}\sigma^2 u^2}$$

Now

$\because$ $e^{x} \geq 1$ if $x \geq 0$

& clearly $\frac{1}{2}\sigma^2 u^2 \geq 0 \Rightarrow$ $\boxed{e^{\frac{1}{2}\sigma^2 u^2} \geq \cancel{\phi}\, 1}$

$\therefore$ $E[\phi(X)] \geq \phi(E[X])$ h.v.

$\Rightarrow$ Jensen's inequality holds