3.

(a) $P(A \cap B | C) = P(A | B \cap C) P(B | C)$

$$P(A | B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

multiply $\to P(B|C)$

$$P(A|B \cap C) P(B|C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)}$$

Hence, True

(b) $P(A \cap B | C) = P(A|C) P(B|C)$ (A & B are independent)

$$P(A \cap B \cap C) = \frac{P(A \cap C) P(B \cap C)}{P(C)} \quad ; P(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(C)}$$

not always true, independent events A & B does not garantee independence of C,

Hence, False.

(c) A is more likely to occur when B is absent. i.e B^c

- for D : $A | B^c > A | B$
- for D^c : $A | B^c > A | B$

this probability is just weighted sum of regional probability

hence, $P(A|B^c) > P(A|B)$

is True

4)

let x be random variable such that we define $x \in N$

$$N = \{1, 2, 3 \ldots\} \quad (\text{finite})$$

$$P(x = n) = c/n^P \quad \boxed{n \geq 1}$$

Now,

$$E(x) = \sum_{n=1}^{\infty} n \cdot \frac{c}{n^P} = c \sum_{1}^{\infty} 1/n^{P-1}$$

$$E(x^2) = \sum_{1}^{\infty} n^2 \frac{c}{n^P} = c \sum_{1}^{\infty} \frac{1}{n^{P-2}}$$

let $P \geq 2$

therefore, $\sum \frac{1}{n^{P-1}}$ will converge

but $\sum \frac{1}{n^{P-2}}$ will diverge.

ex) say $P = 3$

$$c \sum_{1}^{\infty} \frac{1}{n} \longrightarrow \infty \quad \text{but} \quad c \sum_{1}^{\infty} \frac{1}{n^{-2}} \text{ will}$$

be finite.

b) $f(x) = \frac{c}{x^P} \quad x \geq 1 \ \& \ P > 1$

$$E(x) = \int_{1}^{\infty} \frac{x \cdot c}{x^P} dx = c \int_{1}^{\infty} x^{1-P} dx$$

$$E(x^2) = \int_{1}^{\infty} \frac{x^2 c}{x^P} dx = c \int_{1}^{\infty} x^{2-P} dx$$

now, let $P > 2$ say, $P = 2.5$

then, $E(x) < \infty$
but $E(x^2) \Rightarrow \infty$ | thus, exist

c) $E(e^{-x}) \geq e^{-E(x)} \geq e^{-1} \geq 0.3679$

thus, for random variable with $E(x) = 1$ we must have $E(e^{-x}) \geq 1/3$

thus, does not exist

7)

(a) people - P_i $\quad \because i = 0, 1, 2, \ldots n$

starting @ "P_0"

event - 'no one ever tells the rumor back to P_0 in first r steps'

At any step (i), there are $(n+1)$ people one of which is P_0, so, probability of "P_0" is not chosen.

$$P(A) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Also, every step is independent of past, future or any other step, thus according to law of independent events.

Ans: $$P_r(E) = [P(A)]^r$$
$$= \left[\frac{n}{n+1}\right]^r.$$

(b) Probability that rumor is told r times without being repeated to any person

$P(E)$ after 1^{st} step $= 1$ $\quad \because n/n.$

$P(E)$ " 2^{nd} " $= n-1/n.$

thus, $P(E) = \frac{n-(r-1)}{n} = \frac{n-r+1}{n.}$

$$P(E) = \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right) \ldots \left(\frac{n+1-r}{n}\right)$$
$$= \frac{n!}{(n-r)!\,n^r}$$

(a) when each step tells to group of 'N' random people,

$$P(E) = \frac{{}^{n}C_N}{{}^{n+1}C_N} = \frac{n(n-1)\ldots(n-N+1)}{(n+1)(n)\ldots(n-N+1)}$$

(b) No one is told twice.

$$P_r = \frac{\prod_{k=1}^{r} \left({}^{n+1-N(k-1)-1}C_N \right)}{{}^{n+1}C_N}.$$

8) independent events have independent components.

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c)$$

$$\because P(A_i^c) = 1 - P(A_i)$$

so,

$$\prod_{i=1}^{n} P(A_i^c) = \prod_{i=1}^{n} (1 - P(A_i))$$

for any real no. $\underline{\underline{1 - x \leq e^{-x}}}$

this gives,

$$\prod_{i=1}^{n} (1 - P(A_i)) \leq \prod_{i=1}^{n} e^{-P(A_i)}$$

$$\Rightarrow \prod_{i=1}^{n} e^{-P(A_i)} = e^{-\sum_{i=1}^{n} P(A_i)}$$

thus,

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq e^{-\sum_{i=1}^{n} P(A_i)}$$

10)

Show : $E[x] = \int_0^{\infty} [1 - F(x)]\, dx$

by showing:

$$\int_{\Omega} \int_0^{\infty} 1_{[0,\, X(\omega)} \, dx \, dP(\omega)].$$

$$\Rightarrow E[X] = \int_0^{\infty} [1 - F(x)]\, dx$$

$$X(\omega) = \int_0^{\infty} 1_{[0,\, X(\omega)]}(x) dx$$

taking expectation value of both sides

$$E[X(\omega)] = E\left[\int_0^{\infty} 1_{[0, X(\omega)]}(x) dx \right]$$

applying 'Fubini's theorem'

$$E(x) = \int_{\Omega} \int_0^{\infty} 1_{[0, X(\omega)}(x) dx \, dP(\omega)$$

$$= \int_0^{\infty} \left(\int_{\Omega} 1_{[0,\, X(\omega)}(x) dP(\omega) \right)$$

Now,

$$\int_{\Omega} 1_{[0,\, x(\omega)}(x) dP(x) = P(X > x)$$
$$= 1 - F(x)$$

$E(x) = \int_0^{\infty} (1 - F(x))\, dx$

$E(x) = \int_0^{\infty} (1 - F(x))\, dx$

Therefore, $E(x) = \int_{\Omega} \int_0^{\infty} 1_{[0,\, X(\omega)} x \, dx \, dP(x))$

$$= \int_0^{\infty} (1 - F(x)) dx$$