Finite Differences and Interpolation

Forward and Backward Differences

1) Consider $y = x^3 - 2x + 5$

Form the table for $x = 0, 1, 2, 3, 4$.

Also form finite differential table and write the values of $\Delta^2 y_2$, $\nabla^3 y_3$, $\Delta^3 y_0$, $\Delta^2 y$.

x	0	1	2	3	4
y	5	4	9	26	61

Finite differential table.

x	y	Δ	Δ^2	Δ^3	Δ^4
0	5	-1	6	6	0
1	4	5	12	6	
2	9	17	18		
3	26	35			
4	61				

Forward difference table.

Backward difference table

$\Rightarrow \Delta^2 y_2 = 18$

$\Delta^2 y_0 = 6$

$\Delta^2 y_1 = 12$

$\nabla^3 y_3 = 6$

$\Delta^3 y_0 = 6$

Finite differences :-

Consider the table of values

x	x_0	x_1	...	x_n
y	y_0	y_1	......	y_n

which is defined for an unknown function $y = f(x)$ where the values of x are at equal intervals or they are equally spaced.

i.e $x_i = x_0 + ih$ where $i = 0, 1, 2, \ldots n$

and h is called the step length

Forward differences:

If we denote $y_1 - y_0, y_2 - y_1, y_3 - y_2 \ldots y_n - y_{n-1}$ resp. with $\Delta y_0, \Delta y_1, \Delta y_2 \ldots \Delta y_{n-1}$, then $\Delta y_0, \Delta y_1, \Delta y_2 \ldots \Delta y_{n-1}$ are called first forward differences. Here Δ is called the forward difference operator. The differences of first forward differences are called second forward differences denoted by $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1 \ldots \Delta^2 y_{n-2} = \Delta^2 y_{n-1} - \Delta y_{n-2}$.

Similarly we define the third and other higher order differences.

With the above differences, we form a table called forward difference table given by

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
				$\Delta^3 y_0$	
x_2	y_2	Δy_1			$\Delta^4 y_0$
			$\Delta^2 y_1$		
x_3	y_3	Δy_2		$\Delta^3 y_1$	
			$\Delta^2 y_2$		
x_4	y_4	Δy_3			

Here values of x are called arguements and values of y are called entries.

If we denote the differences $y_1 - y_0$, $y_2 - y_1$, ⋯ $y_n - y_{n-1}$ respectively with ∇y_1, ∇y_2 ⋯ ∇y_n, then ∇y_1, ∇y_2 ⋯ ∇y_n are called first backward differences and ∇ is called the backward difference operator. We define second backward differences as differences of first denoted by

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 \quad \cdots\cdots \quad \nabla^2 y_n = \nabla y_n - y_{n-1}$$

Similarly we define third and higher order differences; with all the above values, we form the backward difference table given by

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
x_1	y_1	∇y_1			
			$\nabla^2 y_2$		
x_2	y_2	∇y_2		$\nabla^3 y_3$	
x_3	y_3	∇y_3	∇y_3		$\nabla^4 y_4$
				$\nabla^3 y_4$	
x_4	y_4	∇y_4	∇y_4		

2) Form finite differences table for $y = x^4 - 3x^2 + 5$ where $x = 0, 1, 2, 3, 4, 5$. Also find $\Delta^3 y_0$, $\Delta^3 y_3$, $\nabla^2 y_1$, $\nabla^3 y_1$, $\nabla^2 y_3$, $\nabla^3 y_4$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	5					
		-2				
1	3					
			8			
2	9	6		36		
			44		24	
3	59	50				0
				60		
			104		24	
4	212	154		84		
			188			
		342				
5	555					

$\Delta^3 y_0 = 36$

$\nabla^2 y_3 = 44$

$\nabla^3 y_4 = 60$

The fourth diff. of 4th degree polynomial are constant & the 5th diff is zero. ∴ nth diff. of nth degree polynomial are constant & (n+1)th diff. are same.

Q) Find the missing ~~tat~~ term from the following table.

x	0	1	2	3	4
y	1	3	9	—	81

Soln →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
		6		$a-19$	
2	9		$a-15$		$124-4a$
		$a-9$		$105-3a$	
3	a		$90-2a$		
		$81-a$			
4	81				

$124 - 4a = 0$

$4a = 124 \Rightarrow a = 124/4 = \underline{31}$

Q) Find the missing terms from the following data.

x	45	50	55	60	65
y	3	—	2	—	-2.4

Soln →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	3				
		$a-3$			
50	a		$5-2a$		
		$2-a$		$-9+3a+b$	
55	2		$a+b-4$		$12.6-4b-4a$
		$b-2$		$3.6-3b-a$	
60	b		$-0.4-2b$		
		$-2.4-b$			
65	-2.4				

Quadratic polynomial can be formed from given 3 functional values

$\Rightarrow$ 2nd differences are constant

$\geq$ 3rd differences are 0

$$3a + b - 9 = 0$$

$$-a - 3b + 36 = 0$$

$\Rightarrow a = 2.925 \quad b = 0.225$

Interpolation: is the process of finding the functional value within the given range $[x_0, x_n]$ and extrapolation is the process of finding the func^{n}al value just outside the given range $[x_0, x_n]$.

Newton-Gregory forward interpolation formula:-

Consider the table of values defined for an unknown func^n $y = f(x)$ and the values of x are equally spaced i.e.; $x_i = x_0 + ih$, $i = 0, 1, 2, 3 \ldots$

to estimate the value of y near the begining of the table we use the newton's forward interpola^n formula and is given by,

$$y(x) = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + u(u-1)(u-2)\frac{\Delta^3 y_0}{3!} + \cdots$$

$$\cdots + u(u-1)(u-2)\cdots(u-(n-1))\frac{\Delta^n y_0}{n!}$$

, $u = \dfrac{x - x_0}{h}$, $h \rightarrow$ successive diff. b/w any 2 values of x.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

So we use the diff. along the diagonal as shown in the ab. table in Newton's forward interpolaⁿ formula.

Q) Using NFI formula find $f(4)$ from the fol. table

x	3	5	7	9
$f(x)$	2.7	12.5	34.3	72.9

Solⁿ →

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
3	2.7			
		9.8		
5	12.5		12	
		21.8		4.8
7	34.3		16.8	
		38.6		
9	72.9			

$$u = \frac{x - x_0}{h} = \frac{4-3}{2} = 1/2 = 0.5$$

$$y(4) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 2.7 + 0.5(9.8) + \frac{0.5(-0.5)(12)}{2} + \frac{0.5(-0.5)(-1.5)(4.8)}{6}$$

$= 2.7 + 4.9 - 1.5 + 0.3$

$= \underline{6.4}$.

Newton - Gregory Backward interpolation formula

Consider the table of values defined for an unknown function $y = f(x)$, where the values of x are at equal interval i.e; $x_i = x_0 + ih$, $i = 0, 1, 2, 3 \ldots$

To estimate the value of y near the bottom of the table we use a newton's backward interpolation formula and this formula is given by, ~~y(x)~~

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

$$+ \frac{v(v+1)(v+2)\cdots(v+n-1)}{n!}\nabla^n y_n$$

where, $v = \frac{x - x_n}{h}$

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
x_1	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_3$	
x_2	y_2	∇y_2	$\nabla^2 y_3$		$\nabla^4 y_4$
x_3	y_3	∇y_3		$\nabla^3 y_4$	
x_4	y_4	∇y_4	$\nabla^2 y_4$		

We use the differences along the diagonal as shown above in newton's Backward interpolation formula

Q)
using NBIF find interpolating polynomial for the function $y = f(x)$ given by $f(0) = 1$, $f(1) = 2$, $f(2) = 1$, $f(3) = 10$

also find $f(2.5)$.

Soln

x	y	∇	∇^2	∇^3
0	1			
1	2	1		
2	1	-1	-2	
3	10	9	10	12
	y_n	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$

$$v = \frac{x - x_n}{h} = \frac{2.5 - 3}{1} = -0.5$$

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

$$= 10 + (-0.5)\,9 + \frac{(-0.5)(0.5)}{2}\,10 + \frac{(-0.5)(0.5)(1.5)(12)}{6}$$

$$= 10 - 4.5 - 1.25 - 0.7$$

$$= \underline{3.55}$$

$$v = \frac{x-3}{1} = \underline{x-3}$$

$$y(x) = 10 + (x-3)9 + \frac{(x-3)(x-3+1) \times \cancel{10}^{5}}{\cancel{2}} + \frac{(x-3)(x-3+1)(x-3+2) \times \cancel{12}^{2}}{\cancel{6}}$$

$$= 10 + 9x - 27 + (x-3)(x-2)5 + (x-3)(x-2)(x-1)\,2$$

$$= 9x - 17 + 5(x^2 - 5x + 6) + 2(x^3 - 6x^2 + 11x - 6)$$

$$= 2x^3 + (5-12)x^2 + (9 - 25 + 22)x + (-17 + 30 - 12)$$

$$= 2x^3 - 7x^2 + 6x$$

$$y(2.5) = \underline{3.5}$$

Note :- $(x-a)(x-b)(x-c)$

$= x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$.

Q) $y = f(x)$ given by $f(2) = 4$, $f(4) = 56$, $f(6) = 204$

$f(8) = 496$ hence find $f(7)$

Soln →

x	y	∇	∇^2	∇^3
2	4			
		52		
4	56		96	
		148		48
6	204		144	
		292		
8	496			

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$u = \frac{x - x_0}{h} = \frac{x-2}{2}$$

$$y(x) = 4 + \left(\frac{x-2}{\cancel{2}}\right)\overset{26}{\cancel{52}} + \left(\frac{x-2}{2}\right)\left(\frac{x-2}{2} - 1\right)\frac{96}{2} + \left(\frac{x-2}{2}\right)\left(\frac{x-2}{2} - 1\right)\left(\frac{x-2}{2} - 2\right) \times \frac{48}{6}$$

$$y(x) = 4 + (x-2)26 + (x-2)\frac{(x-2-2)}{\cancel{2}}\overset{12}{\cancel{24}} + (x-2)(x-2-2)(x-2-4) \times \frac{\cancel{48}}{\cancel{6\times2\times2\times2}}$$

$$= 4 + 26x - 52 + (x-2)(x-4)12 + (x-2)(x-4)(x-6)$$

$$= 4 + 26x - 52 + (x^2 - 6x + 8)12 + (x^2 - 6x + 8)(x-6)$$

$$= 4 + 26x - 52 + (x^2 - 6x + 8)12 + (x^3 - 6x^2 + 8x - 6x^2 + 36x - 48)$$

$$= 4 + 26x - 52 + \cancel{12x^2} - 72x + 96 + (x^3 - \cancel{6x^2} + 8x - \cancel{6x^2} + 36x - 48)$$

$$= \underline{x^3 - 2x}$$

$y(x)$ at $x = 7$

$$y(7) = x^3 - 2x = 7^3 - 2\times7$$

$$= 343 - 14$$

$$= \underline{329}$$

Q) The table gives the distances in nautical miles of the visible horizon for the given heights in feet ab. the earth's surface. find the values of y when $x = 218$ ft. and $x = 392$ ft.

x	200	250	300	350	400
y	15.04	16.81	18.42	19.90	21.27

Soln →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
200	15.04				
		1.77			
250	16.81		-0.16		
		1.61		0.03	
300	18.42		-0.13		-0.01
		1.48		0.02	
350	19.90		-0.11		
		1.37			
400	21.27				

Forward

$$u = \frac{x - x_0}{h} = \frac{218 - 200}{50} = \frac{18}{50} = 0.36$$

@218

$$y(x) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)(-0.16)}{2} + \frac{0.36(-0.64)(-1.64)(0.02}{6}$$

$$+ \frac{0.36(-0.64)(-1.64)(-2.64)(-0.01)}{24}$$

$$= 15.04 + 0.6372 + 0.018432 + 0.00188 + 0.000415$$

$$= \underline{15.69}$$

Backward.

$$v = \frac{x - x_n}{h} = \frac{392 - 21.27}{50} = 7.4146$$

@392

$$y(x) = 21.87 + 0.36(1.37) + \frac{(0.36)(1.36)(-0.11)}{2} + \frac{(0.36)(1.36)(2.36)(0.0}{6}$$

$$+ \frac{(0.36)(1.36)(2.36)(3.36)(-0.01)}{24}$$

$$= \underline{21.067}$$

) From the following data estimate the no. of persons having income b/w 2000 & 2500

Income (I)	Below 500	500-1000	1000-2000	2000-3000	3000-4000
no. of persons	6000	4250	3600	1500	650

Sol^n.

x = Income below, y = no. of persons

x	y	Δy	$\nabla^2 y$	$\nabla^3 y$
1000	10250			
		3600		
2000	13850		-2100	
		1500		1250
3000	15350		-850	
		650		
4000	16000			

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + u(u-1)(u-2)\frac{\Delta^3 y_0}{3!}, \quad u = \frac{x-x_0}{h}$$

$$= \frac{2500-1000}{1000} = \frac{1500}{1000} = \underline{1.5}$$

$$= 10250 + 1.5(3600) + \frac{(1.5)(0.5)}{2}(-2100)$$

$$+ \frac{(1.5)(0.5)(-0.5)(1250)}{6}$$

$$= 10250 + 5400 - 787.5 - 78.125$$

$$\approx \underline{14784}$$

$y(2500) - y(2000)$

$= 14784 - 13850$

$= \underline{934}$

Interpolaⁿ with unequal intervals:

Divided differences: consider the table of values

x	x_0	x_1	---	x_n
y	y_0	y_1	---	y_n

, where the values of x are at unequal intervals, we define the first divided difference as follows.

$$\delta(x_0, x_1) = \frac{y_1 - y_0}{x_1 - x_0}$$

$$\delta(x_1, x_2) = \frac{y_2 - y_1}{x_2 - x_1}$$

Illy,

$$\boxed{\delta(x_{n-1}, x_n) = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}}$$

we define the second divided differences as follows.

$$\delta(x_0, x_1, x_2) = \frac{\delta(x_1, x_2) - \delta(x_0, x_1)}{x_2 - x_0}$$

$$\delta(x_1, x_2, x_3) = \frac{\delta(x_2, x_3) - \delta(x_1, x_2)}{x_3 - x_1}$$

& so on.

The 3rd divided diff. are defined as follows.

$$\delta(x_0, x_1, x_2, x_3) = \frac{\delta(x_1, x_2, x_3) - \delta(x_0, x_1, x_2)}{x_3 - x_0}$$

$$\delta(x_0, x_1, x_2, x_3, x_4) = \frac{\delta(x_2, x_3, x_4) - \delta(x_1, x_2, x_3)}{x_4 - x_1}$$

& so on.

with the ab. values we form a table called the divided difference table and it is given by

DDT

x	y	I DD	II DD	III DD	III DD
x_0	y_0	$\delta(x_0, x_1)$			
			$\delta(x_0, x_1, x_2)$		
x_1	y_1			$\delta(x_0, x_1, x_2, x_3)$	
		$\delta(x_1, x_2)$			
x_2	y_2		$\delta(x_1, x_2, x_3)$		$\delta(x_0, x_1, x_2, x_3, x_4)$
		$\delta(x_2, x_3)$			
x_3	y_3			$\delta(x_1, x_2, x_3, x_4)$	
			$\delta(x_2, x_3, x_4)$		
x_4	y_4	$\delta(x_3, x_4)$			

Newton's divided difference formula :-

It is given by,

$$y(x) = y_0 + (x-x_0)\,\delta(x_0, x_1) + (x-x_0)(x-x_1)\,\delta(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\,\delta(x_0, x_1, x_2, x_3)$$
$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\,\delta(x_0, x_1, \ldots, x_n)$$

In the formula we use the differences along the diagonal as shown in the ab. table.

Q) Find an interpolating polynomial for the following data using Newton's divided difference formula and hence estimate $y(2.5)$

x	-1	0	2	3
y	-2	3	1	12

Soln →

x	y	I DD	II DD	III DD
-1	-8			
		$3+8 = 11$	$\frac{-1-11}{2-(-1)} = -4$	
0	3			$\frac{4-(-4)}{3-(-1)} = 2$
		$\frac{1-3}{2} = -1$		
2	1		$\frac{11+1}{3-0} = 4$	
		$\frac{12-1}{1} = 11$		
3	12			

$$y(x) = y_0 + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$$= -8 + (x+1)(11) + (x+1)(x)(-4) + (x+1)(x)(x-2)(2)$$

$$= -8 + 11x + 11 - 4x^2 - 4x + 2x(x^2 - x - 2)$$

$$= -8 + 11x + 11 - 4x^2 - 4x + 2x^3 - 2x^2 - 4x$$

$$= 2x^3 - 6x^2 + 3x + 3$$

$$y(2.5) = 2(2.5)^3 - 6(2.5)^2 + 3(2.5) + 3$$

$$= 31.25 - 37.5 + 7.5 + 3$$

$$= \underline{4.25}$$

Q) Apply the method dividing differences for interpolation find the value of y when $x = 5$ given

x	4.5	4.55	4.7	4.9	5.15
y	1345	1470	2010	3815	10965

Soln →

x	y	I DD	II DD	III DD	IV DD
4.5	1345				
		2500			
4.55	1470		5500		
		3600		25000	
4.7	2010		15500		33333.32
		9025		46666.66	
4.9	3815		43500		
		28600			
5.15	10965				

$$y(5) = 1345 + 0.5(2500) + 0.5(0.45)(5500) + 0.5(0.45)(0.3)(25000)$$
$$+ 0.5(0.45)(0.3)(0.1)(33333.32)$$

$$= 1345 + 1250 + 1237.5 + 1687.5 + 224.99991$$

$$= \underline{5744.99991} \approx \underline{5745}$$

Lagranges Interpolation Formula: (LIF)

Consider table of values,

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

, defined for an unknown funcn $y=f(x)$, where the values of x not neccesarily at equal intervals.

The Lagranges interpolation formula is given by,

$$y(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots\cdots(x_2-x_n)} \times y_2$$

$$+ \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})} \times y_n.$$

$(x-a)(x-b)(x-c)$
$= x^3-(a+b+c)x^2+(ab+bc+ca)x - abc$

Q.) Given $\log_{10} 654 = 2.8156$.

$\log_{10} 658 = 2.8182$.

$\log_{10} 661 = 2.8202$.

Find $\log_{10} 656 = ?$ using LIF.

Soln. →

$y = \log_{10} x$

	x_0	x_1	x_2
x	654	658	661
y	2.8156	2.8182	2.8202.

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2.$$

$$= \frac{(656-658)(656-661)}{(654-658)(654-661)} \times 2.8156$$

$$+ \frac{(656-654)(656-661)}{(658-654)(658-661)} \times 2.8182$$

$$+ \frac{(656-654)(656-658)}{(661-654)(661-658)} \times 2.8202$$

(check once again)

$= 1.0055 + 2.3485 - 0.5371$

$= 2.8166$

2) $y(1) = -2$ $y(3) = 9$ $y(4) = 30$ & $y(6) = 136$, find LI polynomial.

Soln

	x_0	x_1	x_2	x_3
x	1	3	4	6
y	-3	9	30	136

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-3)(x-4)(x-6)}{(-2)(-3)(-5)} \times -3 + \frac{(x-1)(x-4)(x-6)}{(2)(-1)(-3)} \times 9$$

$$+ \frac{(x-1)(x-3)(x-6)}{(3)(1)(-2)} \times 30 + \frac{(x-1)(x-3)(x-4)}{(5)(3)(2)} \times 136$$

$$= (x^3 - 13x^2 + 54x - 72) \times \frac{1}{10}$$

$$+ \frac{3}{2}(x^3 - 11x^2 + 34x - 24)$$

$$- 5(x^3 - 10x^2 + 27x - 18)$$

$$+ \frac{66}{15}(x^3 - 8x^2 + 19x - 12)$$

$$= \frac{1}{10}x^3 + \frac{3}{2}x^3 - 5x^3 + \frac{66}{15}x^3$$

$$+ x^2\left(\frac{-13}{10} - \frac{33}{2} + 50 - \frac{528}{15}\right)$$

$$+ x\left(\frac{54}{10} + \frac{102}{2} - 135 + \frac{1254}{15}\right)$$

$$+ \left(\frac{-78}{10} - \frac{72}{2} + 90 - \frac{792}{15}\right)$$

$= x^3 - 3x^2 + 5x - 6$

Inverse Lagrange's interpolan formula.

In the LIF, if we interchange values of x & y, we get inverse LIF.

$$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 +$$

$$= \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} x_1 +$$

$$= \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 +$$

$$= \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3 .$$

Q) Find the root of the eq^n $f(x)=0$, $f(30) = -30$, $f(34) = -13$, $f(42) = 18$.

Sol^n:

x	30	34	42
y	-30	-13	18

since sign changes root lies betw 34 & 42

$f(x) = 0 \Rightarrow y = 0$

$$x(y) = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$= \frac{(13)(-18)}{(-17)(-48)}(30) + \frac{(30)(-18)}{(17)(-31)}(34) + \frac{(30)(13)}{(48)(31)}(42)$$

$$= -8.6 + 34.83 + 11.008 = \underline{37.238}$$

Q) Find LIF polynomial x in terms of y.

x	2	10	~~14~~ 16
y	1	3	4.

Sol^n →

$$x(y) = \frac{(y-3)(y-4)}{(-2)(-3)} \times 2 + \frac{(y-1)(y-4)}{(2)(-1)}(10) + \frac{(y-1)(y-3)}{(3)(1)}(16)$$

$$= (y^2-7y+12)\left(\frac{+1}{3}\right) + \frac{(y^2-5y+4)(-10)}{2} + (y^2-4y+3) \times \left(\frac{16}{3}\right)$$

$$= \frac{1}{3}(y^2-7y+12) - 5(y^2-5y+4) + \frac{16}{3}(y^2-4y+3)$$

$$= y^2\left(\frac{1}{3} - 5 + \frac{16}{3}\right) + y\left(\frac{-7}{3} + 25 - \frac{64}{3}\right) + \left(\frac{12}{3} - 20 + \frac{48}{3}\right)$$

$$= y^2(0.66) + y(-6.33) + 0$$

Numerical Integration:

~~$I = \int$~~ $I = \int_a^b f(x)\,dx$, consider a interval a to b

$f(x)\,dx$, we divide the $[a,b]$ into n numbers of sub intervals in such a way by taking $h = \frac{b-a}{n}$, then $x_i = x_0 + ih$, $i = 0, 1, \dots n$. Formula for the table of values for each value of x_i, then we have the

$x_0 = a$	x_1	x_2	$\dots$	$x_n = b$
y_0	y_1	y_2	$\dots$	y_n.

Then to evaluate the integral I numerically we use the following formula.

Trapezoidal rule:-

$$I = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right],$$ which gives numerical value of I.

Simpson's 1/3 rule:

$$I = \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + y_6 + \cdots) + 4(y_1 + y_3 + y_5 + \cdots)\right]$$

Simpson's 3/8th rule

$$I = \frac{3}{8}h\left[y_0 + y_n + 2(y_3 + y_6 + y_9 + \cdots) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots)\right]$$

Q) Evaluate $\int_2^4 \frac{1}{\log_{10} x}\,dx$ by using trapezoidal rule, simpson's 1/3 rule & simpson's 3/8th rule by dividing the [2, 4] into 6 sub-intervals.

sol^n →

$\int_2^4 \frac{1}{\log_{10} x}\,dx$ [2, 4], $h = \frac{b-a}{n} = \frac{4-2}{6} = 1/3 = 0.333$

$y = \frac{1}{\log_{10} x}$

x	2	$2+1/3 = 7/3$	$2+2/3 = 8/3$	$2+3/3 = 3$	$2+4/3 = 10/3$	$2+5/3 = 11/3$	$2+\frac{6}{3} = 4$
y	3.321	2.71	2.34	2.09	1.918	1.772	1.66
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Trapezoidal Rule:

$$I = \frac{h}{2}[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{6}[3.321 + 2.71 + 2.34 + 2.09 + 1.913 + 1.772 + 1.66]$$

$$= \underline{\underline{4.44}}$$

Simpson's 1/3 Rule.

$$I = \frac{h}{3}[y_0 + y_6 + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{1}{6}[3.321 + 1.66 + 2(2.34 + 1.913) + 4(2.71 + 2.09 + 1.77)]$$

$$= \underline{\underline{4.4261}}$$

Simpson's 3/8 th Rule:

$$I = \frac{3}{8}h[y_0 + y_6 + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3}{8} \times \frac{1}{3}[3.321 + 1.66 + 2(2.09) + 3(2.71 + 2.34 + 1.913 + 1.7$$

$$= \underline{\underline{4.422}}$$

Q) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by taking 6 sub-intervals using Simpson's 1/3 rule also approximate the value of y.

soln →

$\int_0^1 \frac{dx}{1+x^2}$ $\quad h = \frac{1-0}{6} = 1/6$. $\quad , y = \frac{1}{1+x^2}$.

x.	0	1/6	1/3	1/2	2/3	5/6	1
y.	1	36/37	9/10	4/5	9/13	36/61	1/2

$$I = \frac{h}{3}\left[y_0 + y_6 + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{1}{18}\left[1 + 0.5 + 2\left(\frac{9}{10} + \frac{9}{13}\right) + 4\left(\frac{36}{37} + \frac{4}{5} + \frac{36}{61}\right)\right]$$

$$= \frac{1}{18}\left[1.5 + \frac{207}{65} + 9.45\right]$$

$$= \underline{\underline{0.7853}} \longrightarrow ①$$

Integration:

$$I = \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \pi/4 - 0 = \pi/4$$

equating ① & 2

$\pi/4 = 0.7853 \Rightarrow \underline{\underline{\pi = 3.1412}}$ (approximately equal to π value)

Q) Evaluate $\int_0^1 \frac{dx}{1+x}$ by taking 7 ordinates using simpson's $3/8$ rule.

sol^n

$y = \frac{1}{1+x}$, $h = \frac{1-0}{6} = 1/6$

[7 ordinates means 6 sub intervals]

[$\therefore$ n ordinates $\Rightarrow$ $(n-1)$ sub-intervals]

x	0	$1/6$	$1/3$	$1/2$	$2/3$	$5/6$	1
y	1	$6/7$	$3/4$	$2/3$	$3/5$	$6/11$	$1/2$

$$I = \frac{3h}{8}\left[y_0 + y_6 + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8 \times 6}\left[1 + 1/2 + 2(2/3) + 3\left(\frac{6}{7} + \frac{3}{4} + \frac{3}{5} + \frac{6}{11}\right)\right]$$

$$= \underline{\underline{0.693}} \quad — ①$$

Exact integration

$$I = \int_0^1 \frac{dx}{1+x} = \log(1+x)\Big|_0^1 = \log(2) - \log(1)$$

$$= \log_e 2$$

equating ① & ②

$$\log_e 2 = 0.693$$

Q) Evaluate $I = \int_0^{\pi/3} \frac{d\theta}{\sqrt{\cos\theta}}$, $\cancel{4}$ 4 SI, using S 1/3 Rule.

soln →

$y = \frac{1}{\sqrt{\cos\theta}}$ $\quad h = \frac{\frac{\pi}{3} - 0}{4} = \frac{\pi}{12}$

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$
y	1	1.017	1.074	1.189	1.414
	y_0	y_1	y_2	y_3	y_4

S 1/3 Rule

$$I = \frac{1}{3}h\,[y_0 + y_4 + 2(y_2) + 3(y_1 + y_3)]$$

$$= \frac{\pi}{36}\,[1 + 1.414 + 2(1.074) + 4(1.017 + 1.189)]$$

$$= \frac{\pi}{36}\,[1 + 1.414 + 2.148 + 8.824] = \underline{1.168}$$

Numerical Differenciation:

Finding $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ using newton's forward interpolation formula.

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + u(u-1)(u-2)\frac{\Delta^3 y_0}{3!}$$

$$+ u(u-1)(u-2)(u-3)\frac{\Delta^4 y_0}{4!} \longrightarrow (1)$$

$$y(x) = y_0 + u\Delta y_0 + \left(\frac{u^2-u}{2}\right)\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{6}\Delta^3 y_0 +$$

$$\frac{(u^4-6u^3+11u^2-6u)}{24}\Delta^4 y_0$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \quad x = x_0 + uh$$

$$1 = \frac{du}{dx}(h)$$

$$\frac{du}{dx} = \frac{1}{h}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dy}$$

$$= \frac{d}{du}\left(\frac{1}{h}\frac{dy}{du}\right)\frac{1}{h}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\frac{d^2y}{du^2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{24}\Delta^4 y_0\right] \longrightarrow (2)$$

Q) Find $y'(0.5)$ $y''(0.5)$ from the table given.

x	0	1	2	3
y	1	3	7	13.

Sol^n →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

$u = \frac{x - x_0}{h} = \frac{0.5 - 0}{1} = \underline{\underline{0.5}}$

$$y'(0.5) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0\right]$$

$$= \frac{1}{1}\left[2 + \frac{(2\times 0.5 - 1)}{2}\times 2\right]$$

$$= \underline{\underline{2}}$$

$$y''(0.5) = \frac{1}{h^2}\left[\Delta^2 y_0\right] = \frac{1}{1}(2) = \underline{\underline{2}}$$

$$\therefore y'(0.5) = y''(0.5) = \underline{\underline{2}}$$

Q) A func^n $y = f(x)$ is given in the following table. Find $f'(1.9)$, $f''(1.9)$.

x	1	1.2	1.4	1.6	1.8	2.0.
y	0	0.128	0.544	1.296	2.432	~~3~~ 4

Sol^n →

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	0				
		0.128			
1.2	0.128		0.288		
		0.416		0.048	
1.4	0.544		0.336		0
		0.752		0.048	
1.6	1.296		0.384		0
		1.136		0.048	
1.8	2.432		0.432		
		1.568			
2	~~3~~ 4				

$v = \frac{x - x_n}{h}$

$= \frac{1.9 - 2}{0.2} = \underline{\underline{-0.5}}$

$$y'(1.9) = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n\right]$$

$$= \frac{1}{0.2}\left[1.568 + \left(\frac{2(-0.5)+1}{2}\right)(0.432) + \left(\frac{3(-0.5)^2+6(-0.5)+2}{6}\right)(0.048)\right]$$

$$= \frac{1}{0.2}\left[1.568 - 0.002\right] = \underline{\underline{7.83}}$$

$$y''(1.9) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n\right]$$

$$= \frac{1}{(0.2)^2}\left[0.432 + (-0.5+1)(0.048)\right]$$

$$= \underline{\underline{11.4}}$$

Q)

Find $y'(0)$, $y''(0)$

x	0	1	2	3	4	5
y	4	8	15	7	6	2

soln →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		4				
1	8		3			
		7		-18		
2	15		-15		40	
		-8		22		-72
3	7		7		-32	
		-1		-10		
4	6		-3			
		-4				
5	2					

Since $x=0$ is one of the tabular points we consider 0 as x_0

$$y'(0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0\right]$$

$$= \frac{1}{1}\left[4 - \frac{1}{2}(3) + \frac{1}{3}(-18) - \frac{1}{4}(40) + \frac{1}{5}(-72)\right]$$

$$= \underline{-27.9}$$

$$y''(0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0\right]$$

$$= \frac{1}{12}\left[3 + 18 + \frac{11}{12}(40)\right]$$

$$= \underline{57.6}$$

$y'(4)$ & $y''(4) = ?$

$$y'(4) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n\right]$$

$$= \frac{1}{1}\left[-1 + \frac{1}{2}(7) + \frac{1}{3}(22) + \frac{1}{4}(40)\right]$$

$$= \underline{19.83}$$

$$y''(4) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n\right]$$

$$= \frac{1}{12}\left[7 + 22 + \frac{11}{12}(40)\right] = \underline{65.66}$$

Q) Use an appropriate interpolaⁿ formula to find radius of curvature at $x = \cancel{0.3}$ 3.0 from the following data

x	3	5	7	9	11
y	28.27	78.54	153.93	254.47	380.13

Solⁿ →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	28.27				
		50.27			
5	78.54		25.12		
				0.03	
		75.39			−0.06
7	153.93		25.15		
		100.54		−0.03	
9	254.47		25.12		
		125.66			
11	380.13				

$$y'(3) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$$

$$= \frac{1}{2}\left[50.27 - \frac{25.12}{2} + \frac{0.03}{3} + \frac{0.06}{4}\right]$$

$$= 18.8675$$

$$y''(3) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0\right]$$

$$= \frac{1}{4}\left[25.12 - 0.03 + \frac{11}{12}(-0.06)\right]$$

$$= 6.25875$$

$$\boxed{\rho = \frac{[1 + y_1^2]^{3/2}}{y_2}}$$

$$= \frac{[1 + (18.8675)^2]^{3/2}}{6.25875} = 1077.66$$

Q) If θ is observed temperature of vessel cooling water. t is the time in minutes from the begining of the observa… Find the approximate rate of cooling when t = 3 and t = 3.5 from the following data

x	1	3	5	7	9	11
y	85.3	74.5	67	60.5	54.3	41.8

$t = 3$, $t = 3.5$

Sol^n →

The rate of cooling $= \frac{d\theta}{dt}$

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
1	85.3					
		-10.8				
3	74.5		3.3			
		-7.5		-2.3		
5	67		1		1.6	
		-6.5		-0.7		-7.5
7	60.5		0.3		-5.9	
		-6.2		-6.6		
9	54.3		-6.3			
		-12.5				
11	41.8					

$$\left.\frac{d\theta}{dt}\right|_{t=3} = \frac{1}{h}\left[\Delta\theta_0 - \frac{1}{2}\Delta^2\theta_0 + \frac{1}{3}\Delta^3\theta_0 - \frac{1}{4}\Delta^4\theta_0\right]$$

$$= \frac{1}{2}\left[-7.5 - \frac{1}{2}(1) + \frac{1}{3}(-0.7) - \frac{1}{4}(-5.9)\right]$$

$$= \underline{\underline{-3.379}}$$

$$\left.\frac{d\theta}{dt}\right|_{t=3.5} = \frac{1}{h}\left[\Delta\theta_0 + \frac{(2u-1)}{2}\Delta^2\theta_0 + \frac{(3u^2-6u+2)}{6}\Delta^3\theta_0 + \frac{(4u^3-18u^2+22u-6)}{24}\Delta^4\theta_0\right]$$

$$u = \frac{x - x_0}{h} = \frac{3.5-3}{2} = \underline{\underline{0.25}}$$

$$\left.\frac{d\theta}{dt}\right|_{t=3.5} = \frac{1}{2}\left[-7.5 + (-0.25)(1) + (0.1145)(-0.7) + (-0.065)(-5.9)\right]$$

$$= \underline{\underline{-3.72}}$$

Partial differential eqn: Any eqn which contain partial derivatives of a funcn $u(x,y)$ or $u=(x,y,z)$ is called a partial differential eqn.

Ex :- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ (It is first order, first degree D.E.)

The order of the D.E is the order of the highest order derivative.

The degree is the power of the highest order derivative that is present in the eqn.

Solun :-

Any funcn $u(x,y)$ which satisfies the equation is called a solution of partial D.E. (PDE) usually the following notans will be used in partial D.E.

$p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$, $r = \frac{\partial^2 u}{\partial x^2}$, $s = \frac{\partial^2 u}{\partial x \partial y}$ $t = \frac{\partial^2 u}{\partial y^2}$

Solun of PDE by the method of separation of variables (SOV)

Q.) Solve $\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$ by (SOV).

Soln →

$\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0 \rightarrow ①$

we assume soln of ① as,

$U = u(x,y) = X(x)\, Y(y) \rightarrow ②$

i.e, $u = XY$

Since u is a soln of ① we have $\frac{\partial}{\partial x}(XY) - y\frac{\partial}{\partial y}(XY) = 0$

$\Rightarrow Y \frac{\partial}{\partial x}(X) - y X \frac{\partial}{\partial y}(Y) = 0$

dividing by XY

(from here onwards it is $\frac{d}{d}$ not $\frac{\partial}{\partial}$)

$\Rightarrow \frac{1}{X}\frac{d}{dx}(X) - \frac{y}{Y}\frac{d}{dy}(Y) = 0$

$\Rightarrow \frac{1}{X}\frac{d}{dx}(X) = \frac{y}{Y}\frac{d}{dy}(Y) = k$

equating each term to k we get

$\frac{1}{X}\frac{d}{dx}(X) = k \qquad \frac{y}{Y}\frac{d}{dy}(Y) = k.$

Separating variables:

$\frac{dX}{X} = k\,dx \qquad \frac{dY}{Y} = \frac{k\,dy}{y}$

integrating

$\log X = kx + \log c_1 \qquad \log Y = k\log y + \log c_2$

$\log X - \log c_1 = kx \qquad \log Y - \log c_2 = \log y^k$

$\log \frac{X}{c_1} = kx \qquad \log \frac{Y}{c_2} = \log y^k$

$\frac{X}{c_1} = e^{kx} \qquad \frac{Y}{c_2} = y^k$

$X = c_1 e^{kx}, \quad Y = c_2 y^k$

$u(x,y) = c_1 e^{kx} c_2 y^k$

$= c_1 c_2 e^{kx} y^k, \quad = \underline{c e^{kx} y^k} \quad (\because c = c_1 c_2)$

Q) Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ by (SOV)

Soln →

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u \rightarrow (1)$$

we assume the soln of (1) in the form

$$U = u(x,y) = X(x)Y(y) \rightarrow (2)$$

i.e, $u = XY$

$$\frac{\partial}{\partial x}(XY) + \frac{\partial}{\partial y}(XY) = 2(x+y)XY$$

$$Y\frac{\partial}{\partial x}(X) + X\frac{\partial}{\partial y}(Y) = 2(x+y)XY$$

dividing by XY

$$\frac{1}{X}\frac{\partial}{\partial x}(X) + \frac{1}{Y}\frac{\partial}{\partial y}(Y) = 2(x) + 2(y)$$

$$\frac{1}{X}\frac{d}{dx}(X) - 2x = 2y - \frac{1}{Y}\frac{d}{dy}(Y) = k$$

equating each term to k,

$$\frac{1}{X}\frac{d}{dx}(X) - 2x = k \qquad 2y - \frac{1}{Y}\frac{d}{dy}(Y) = k$$

Separating variables:

$$\frac{1}{X}\frac{d}{dx}(X) = k + 2x \quad , \quad -\frac{1}{Y}\frac{d}{dy}(Y) = k - 2y$$

$$\frac{dX}{X} = (k+2x)dx \quad , \quad \frac{dY}{Y} = (2y-k)dy$$

integrating,

$\log X = kx + x^2 + \log c_1$

$\log X - \log c_1 = kx + x^2$

$\log \frac{X}{c_1} = kx + x^2$

$\frac{X}{c_1} = e^{kx + x^2}$

$\log Y = y^2 - ky + \log c_2$

$\log Y - \log c_2 = y^2 - ky$

$\log \frac{Y}{c_2} = y^2 - ky$

$\frac{Y}{c_2} = e^{y^2 - ky}$

$\therefore X = c_1 e^{kx + x^2}$, $Y = c_2 e^{y^2 - ky}$

$u = XY$

$= c_1 e^{kx + x^2} c_2 e^{y^2 - ky} = c_1 c_2 e^{kx + x^2} e^{y^2 - ky}$

$= c\, e^{kx - ky + x^2 + y^2}$ $[\because c = c_1 c_2]$

Q) Solve $py^3 - qx^2 = 0$

sol^n $\rightarrow$

$p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$

$\therefore \frac{\partial u}{\partial x}(y^3) - \frac{\partial u}{\partial y} x^2 = 0 \Rightarrow \frac{\partial u}{\partial x} y^3 = \frac{\partial u}{\partial y}(x^2)$ ①

We assume the sol^n of ① in the form

$U = u(x, y) = X(x)\, Y(y) \rightarrow$ ② i.e $u = XY$

$\frac{\partial}{\partial x}(XY) y^3 - \frac{\partial}{\partial y}(XY) x^2 = 0$

$Y \frac{\partial}{\partial x}(X) y^3 - X \frac{\partial}{\partial y}(Y) x^2 = 0$

dividing by XY

$\frac{1}{X} \frac{\partial}{\partial x}(X) y^3 - \frac{1}{Y} \frac{\partial}{\partial y}(Y) x^2 = 0$

$\frac{1}{X} \frac{\partial}{\partial x}(X) y^3 = \frac{1}{Y} \frac{\partial}{\partial y}(Y) x^2$

$$\frac{1}{x^2}\times\frac{1}{X}\frac{d}{dx}(X) = \frac{1}{y^3}\frac{1}{Y}\frac{d}{dy}(Y) = k$$

equating each term to k

$$\frac{1}{x^2}\frac{1}{X}\frac{d}{dx}(X) = k \qquad \frac{1}{y^3}\frac{1}{Y}\frac{d}{dy}(Y) = k.$$

<u>Separating variables</u>

$$\frac{d(X)}{X} = kx^2\,dx \qquad \frac{d(Y)}{Y} = ky^3\,dy.$$

$$\log X = \frac{kx^3}{3} + \log c_1 \qquad \log Y = \frac{ky^4}{4} + \log c_2$$

$$\log X - \log c_1 = \frac{kx^3}{3} \qquad \log Y - \log c_2 = \frac{ky^4}{4}$$

$$\log\frac{X}{c_1} = \frac{kx^3}{3} \qquad \log\frac{Y}{c_2} = \frac{ky^4}{4}$$

$$\frac{X}{c_1} = e^{kx^3/3} \qquad \frac{Y}{c_2} = e^{ky^4/4}$$

$$X = c_1 e^{kx^3/3} \qquad Y = c_2 e^{ky^4/4}$$

$$u = XY$$

$$= c_1 e^{kx^3/3}\, c_2 e^{ky^4/4}$$

$$= c_1 c_2\, e^{kx^3/3 + ky^4/4}$$

$$= c\, e^{kx^3/3 + ky^4/4}$$

5) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u \longrightarrow ①$, $u(x,0) = 6e^{-3x}$

sol^{n} →

$\boxed{\text{given that } u(x,0) = 6e^{-3x}}$

$u(x,y) = X(x)\,Y(y) \rightarrow ②$

$u = XY$

$\frac{\partial}{\partial x}(XY) = 2\frac{\partial}{\partial y}(XY) + XY$

$Y\frac{dX}{dx} = 2X\frac{dY}{dy} + XY$

dividing by XY

$\frac{1}{X}\frac{dX}{dx} = \frac{2}{Y}\frac{dY}{dy} + 1 = k$

Equating each term to k

$\frac{1}{X}\frac{dX}{dx} = k$, $\frac{2}{Y}\frac{dY}{dy} + 1 = k$.

$\frac{dX}{X} = k\,dx$, $\frac{dY}{Y} = \left(\frac{k-1}{2}\right)dy$

integrating

$\log X = kx + \log C$

$\log X - \log C_1 = kx$

$\log\frac{X}{C_1} = kx$

$\frac{X}{C_1} = e^{kx}$

$X = C_1 e^{kx}$

$\log Y = \left(\frac{k-1}{2}\right)y + \log C_2$

$\log Y - \log C_2 = \left(\frac{k-1}{2}\right)y$

$\log\frac{Y}{C_2} = \left(\frac{k-1}{2}\right)y$

$\frac{Y}{C_2} = e^{\left(\frac{k-1}{2}\right)y}$

$Y = C_2 e^{\left(\frac{k-1}{2}\right)y}$

$u(x,y) = XY$

$= C_1C_2\, e^{kx + \left(\frac{k-1}{2}\right)y} = C e^{kx + \left(\frac{k-1}{2}\right)y}$ $(\because C = C_1C_2)$

using the coordinates $(x, 0)$

put $y = 0$

$$u(x, 0) = 6e^{-3x} = ce^{kx}$$

equating corresponding constants

$$c = 6 \quad k = -3$$

$$u(x, y) = 6e^{-3x + \left(\frac{-3-1}{2}\right)y}$$

$$u(x, y) = 6e^{-3x - 2y}$$

Q) Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \longrightarrow (1)$

Soln.

$$z(x, y) = X(x)Y(y) \longrightarrow (2)$$

$$z = XY$$

$$\frac{\partial^2}{\partial x^2}(XY) - 2\frac{\partial(XY)}{\partial x} + \frac{\partial(XY)}{\partial y} = 0$$

$$Y\frac{\partial^2}{\partial x^2}(X) - 2Y\frac{\partial(X)}{\partial x} + X\frac{\partial(Y)}{\partial y} = 0$$

dividing by XY

$$\frac{1}{X}\frac{d^2(X)}{dx^2} - \frac{2}{X}\frac{d(X)}{dx} + \frac{1}{Y}\frac{d(Y)}{dy} = 0$$

$$\frac{1}{X}\frac{d^2(X)}{dx^2} - \frac{2}{X}\frac{d(X)}{dx} = -\frac{1}{Y}\frac{d(Y)}{dy} = k$$

equating each term to k.

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{d(X)}{dx} = k \qquad \text{and} \qquad \frac{-1}{Y}\frac{d(Y)}{dy} = k$$

$$\frac{d^2(X)}{dx^2} - \frac{2d(X)}{dx} - kX = 0$$

$$\frac{d(Y)}{Y} = -k\,dy$$

$$(D^2 - 2D - k)X = 0$$

$$\log Y = -ky + \log c_3$$

$$m^2 - 2m - k = 0$$

$$\log Y - \log c_3 = -ky$$

$$1 \pm \sqrt{1+k}$$

$$\log\frac{Y}{c_3} = -ky$$

$$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$$

$$\frac{Y}{c_3} = e^{-ky}$$

$$Y = c_3 e^{-ky}$$

Q) Solve $x^2\dfrac{\partial^2 z}{\partial x\partial y} + 3y^2 z = 0 \longrightarrow ①$

Sol^n $\longrightarrow$

$$z(x,y) = X(x)\,Y(y) \rightarrow ②$$

$$z = XY$$

$$x^2\frac{\partial^2(XY)}{\partial x\partial y} + 3y^2XY = 0$$

$$x^2\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(XY)\right] + 3y^2XY = 0$$

$$x^2\frac{\partial}{\partial x}\left[X\frac{\partial}{\partial y}(Y)\right] + 3y^2XY = 0$$

$$x^2\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} = -3y^2XY$$

$$\frac{x^2}{X}\frac{dX}{dx} = \frac{-3y^2 Y}{\left(\frac{dY}{dy}\right)} = k$$

equating each term to k

$$\frac{x^2}{X}\frac{dX}{dx} = k \quad , \quad -3y^2 Y = k\frac{dY}{dy}$$

$$\frac{dX}{X} = \frac{k\,dx}{x^2} \qquad \frac{dY}{Y} = \frac{-3y^2}{k}dy$$

integrating,

$$\log X = \frac{-k}{x} + \log c_1 \qquad \log Y = \frac{-y^3}{k} + \log c_2$$

$$\log\frac{X}{c_1} = \frac{-k}{x} \qquad \log\frac{Y}{c_2} = -\frac{y^3}{k}$$

$$\frac{X}{c_1} = e^{-k/x} \qquad \frac{Y}{c_2} = e^{-y^3/k}$$

$$u = XY = c_1 e^{-kx} c_2 e^{-y^3/k}$$

$$= c_1 c_2 e^{-kx - y^3/k}$$

$$= c\, e^{-(kx + y^3/k)}$$

Solution of partial D.E's by direct integration

Q) Solve $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

Solⁿ →

$$\frac{\partial z}{\partial y} = \frac{x^3 y}{3} + f_1(y)$$

integrating w.r.t y keeping x constant

$$z = \frac{x^3 y^2}{6} + \int f_1(y)\,dy + f_2(x)$$

Q) $\frac{\partial^2 z}{\partial x^2} = x + y$.

given that $z = y^2$ when $x = 0$ and $\frac{\partial z}{\partial x} = 0$ when $x = 2$.

Solⁿ →

integrating w.r.t x (twice) keeping y constant

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} + xy + f_1(y) \longrightarrow (2)$$

$$z = \frac{x^3}{6} + \frac{x^2 y}{2} + f_1(y)x + f_2(y) \longrightarrow (3)$$

when $x = 0$, $z = y^2$.

$$y^2 = f_2(y)$$

when $x = 2$, $\frac{\partial z}{\partial x} = 0$

$$0 = 2 + 2y + f_1(y)$$

$$f_1(y) = -2 - 2y.$$

Substituting in (3)

$$z = \frac{x^3}{6} + \frac{x^2y}{2} - x(2+2y) + y^2.$$

Q) Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$.

given that $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ $\rightarrow$ (2)

$z=0$ when y is an odd multiple of $\pi/2$. $\rightarrow$ (3)

Soln $\rightarrow$

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \longrightarrow (1)$$

integrating w.r.t. x keeping y constant

$$\frac{\partial z}{\partial y} = -\cos x \sin y + F_1(y) \longrightarrow (4)$$

$$z = +\cos x \cos y + \int F_1(y) + F_2(x) \longrightarrow (5)$$

using (2) & (4)

$$* \ \frac{\partial z}{\partial y} = -2\sin y \ , x=0 \quad (\text{given } (2))$$

$$-2\sin y = -\sin y + F_1(y)$$

$$-\sin y = F_1(y) \quad (\text{Substitute in } 4)$$

$$z = \cos x \cos y - \int \sin y \, dy + F_2(x)$$

$$= \cos x \cos y + \cos y + F_2(x)$$

using condiⁿ ③

$$0 = \cos x \cos (2n-1)\frac{\pi}{2} + \cos (2n-1)\frac{\pi}{2} + F_2(x) \quad \left[\because y \text{ is odd multiple of } \pi/2\right]$$

$$F_2(x) = 0$$

$$\therefore \quad z = \cos x \cos y + \cos y$$

Q) Solve $\dfrac{\partial^2 z}{\partial x \partial y} = 2x\left(\dfrac{\partial z}{\partial y} + 1\right)$

Solⁿ →

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2x\left(\frac{\partial z}{\partial y} + 1\right) \quad , \text{let } \frac{\partial z}{\partial y} = q$$

$$\frac{\partial q}{\partial x} = 2x(q+1)$$

seperating variables

$$\frac{\partial q}{q+1} = 2x\, dx$$

integrating

$$\log(q+1) = x^2 + \log c_1$$

$$\frac{\log(q+1)}{\log c_1} = x^2 \Rightarrow \frac{q+1}{c_1} = e^{x^2} \Rightarrow q+1 = c_1 e^{x^2}$$

$$\Rightarrow \frac{\partial z}{\partial y} = c_1 e^{x^2} - 1$$

integrating w.r.t y keeping x constant

$$\boxed{z = c_1 y e^{x^2} - y + c_2}$$

Q) $xy\frac{\partial^2 u}{\partial x \partial y} - \frac{x \partial u}{\partial x} = y^2$.

Sol^n →

let $\frac{\partial u}{\partial x} = p$

we have $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

$\Rightarrow xy\frac{\partial^2 u}{\partial y \partial x} - x\frac{\partial u}{\partial x} = y^2$

$xy\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) - x\frac{\partial u}{\partial x} = y^2$.

$xy\frac{\partial p}{\partial x} - xp = y^2$.

dividing by xy

$\frac{\partial p}{\partial x} - \frac{p}{y} = \frac{y}{x}$, $IF = e^{-\int 1/y \, dy}$.

$= e^{-\log y}$

$= e^{\log 1/y}$

$= 1/y$

$p\left(\frac{1}{y}\right) = \int \frac{\cancel{y}}{x} \times \frac{1}{\cancel{y}} dy + c_1$

$\frac{p}{y} = \frac{y}{x} + c_1$

$p = \frac{y^2}{x} + c_1 y$.

Note $\left[p(IF) = \int Q(IF) + c_1 \right]$.

$\frac{\partial u}{\partial x} = \frac{y^2}{x} + c_1 y$.

integrating w.r.t x keeping y constant

$u = y^2 \log x + c_1 xy + c_2$

Eq^{n}s containing one independent variable.

Q) Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ —① , given that $z = e^y$ —② , $\frac{\partial z}{\partial x} = 1$, when $x = 0$. —③

Sol^n →

Let $D = \frac{\partial}{\partial x}$, $D^2 \frac{\partial^2}{\partial x^2}$.

① taken the form.

$D^2 z + z = 0$.

$(D^2+1) z = 0$.

Auxiliary eq^n is $(m^2+1) = 0 \Rightarrow m = \pm i$

$z = f_1(y) \cos x + f_2(y) \sin x$. —— ④

Applying condition ② ($z = e^y$, when $x = 0$)

$e^y = f_1(y)$ —— ⑤

Applying condi^n ③.

$1 = f_2(y)$ —— ⑥

substituting ⑤ & ⑥ in ④

$z = e^y \cos x + \sin x$

Q) $\frac{\partial^2 z}{\partial y^2} = z$ —① given that $z = 0$ when $y = 0$ —② , $\frac{\partial z}{\partial y} = \sin x$ when $y = 0$ —③

Sol^n →

Let $D = \partial/\partial y$, $D^2 = \partial^2/\partial y^2$.

then ① takes the form.

$D^2 z - z = 0$.

$(D^2 - 1)z = 0$

$\therefore$ Auxiliary eqn $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$z = F_1(x) e^{y} + F_2(x) e^{-y}$ — (4)

Applying condin (2)

$0 = F_1(x) + F_2(x)$ — (5)

$\frac{\partial z}{\partial y} = F_1(x) e^{y} - F_2(x) e^{-y}$

Applying condin (3)

$\sin x = F_1(x) - F_2(x)$ — (6)

Solving (5) & (6)

$2F_1(x) = \sin x$

$F_1(x) = \frac{1}{2} \sin x$

$F_2(x) = -\frac{1}{2} \sin x$

Substituting in (4)

$z = \frac{1}{2} \sin x \left[e^{y} - e^{-y} \right]$

$= \frac{1}{\cancel{2}} \sin x \, (\cancel{2} \sinh y)$

$= \sin x \sinh y$

Q) Solve $\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial z}{\partial x} - 4z = 0$ — (1)

given that $z = 1$, when $x = 0$ — (2)

$\frac{\partial z}{\partial x} = y$, when $x = 0$ — (3)

Solⁿ → Let $D = \frac{\partial}{\partial x}$, $D^2 = \frac{\partial^2}{\partial x^2}$

then ① takes the form

$$D^2 z + 3Dz - 4z = 0$$

$$(D^2 + 3D - 4) z = 0$$

A.E, $m^2 + 3m - 4 = 0$

$$m^2 + 4m - m - 4 = 0$$

$$m(m+4) - (m+4) = 0$$

$$(m+4)(m-1) = 0$$

$$m = -4, m = 1$$

$$z = f_1(y) e^{-4x} + f_2(y) e^{x} \quad \text{—④}$$

Applying condiⁿ ②

$$1 = f_1(y) + f_2(y) \quad \text{—⑤}$$

$$\frac{\partial z}{\partial x} = -4 f_1(y) e^{-4x} + f_2(y) e^{x}$$

Applying condiⁿ ③

$$y = -4 f_1(y) + f_2(y) \quad \text{—⑥}$$

Solving $4 \times ⑤ + ⑥ \Rightarrow$

$$4 + y = 5 f_2(y)$$

$$f_2(y) = (4+y)/5$$

$$f_1(y) = 1 - f_2(y) = 1 - \frac{4+y}{5} = \frac{5-4-y}{5} = \frac{(1-y)}{5}$$

Substitute $f_1(y)$ & $f_2(y)$ in ④

$$z = \frac{(1-y)}{5} e^{-4x} + \frac{(4+y)}{5} e^{x}$$

Q) Find the max & min values of x from the following table.

x	0	1	2	3	4
y	6	-5	-16	-12	-10

Sol^n →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	6				
		-11			
1	-5		0		
		-11		15	
2	-16				-32
			15		
3	-12	4		-17	
			-2		
4	-10	2			

$u = \frac{x - x_0}{h}$, $x_0 = 0$, $h = 1$

$u = x$, $\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \frac{1}{6}(3u^2 - 6u + 2)\Delta^3 y_0 + \frac{1}{24}(4u^3 - 18u^2 + 22u - 6)\Delta^4 y_0\right]$

$\frac{dy}{dx} = \left[-11 + \left(\frac{2x-1}{2}\right)0 + \frac{1}{6}(3x^2 - 6x + 2)15 + \frac{1}{24}(4x^3 - 18x^2 + 22x - 6)(-32\right.$

To find extreme values of the function, we first we need to find critical values of station points by making $\frac{dy}{dx} = 0$

$0 = -11 + \frac{15}{8}x^2 - \frac{30x}{2} + \frac{10}{2} - \frac{16x^3}{3} + 24x^2 - \frac{88x}{3} + \frac{24}{3}$

$0 = \frac{-16}{3}x^3 + \frac{63}{2}x^2 - \frac{133x}{3} + 2$

$x_1 = 3.6678$ $x_2 = 0.0466$ $x_3 = 2.1919$

which are stationary points or critical points.

The func^n is having maximum at $x = 0.0466$.

$$\frac{d^2y}{dx^2} = -16x^2 + 63x - \frac{133}{3}.$$

for $x = 0.0466$ $\frac{d^2y}{dx^2} < 0$

at $x = 3.6667$.

$$\frac{d^2y}{dx^2} = -16x^2 + 63x - \frac{133}{3}.$$

$$= -16(3.6667)^2 + 63(3.6667) - \frac{133}{3}$$

$\frac{d^2y}{dx^2} < 0$

at $x = 2.1918$

$$\frac{d^2y}{dx^2} = -16(2.1918)^2 + 63(2.1918) - \frac{133}{3}$$

$$\frac{d^2y}{dx^2} > 0$$

2)

Unit - 2

Random Variables & Probability Distribution

Experiment: Any action which results in some outcome, it is called an experiment.

Deterministic experiment: If the outcome of an experiment is one and only one is called deterministic exp.

non-Deterministic or Stochastic Experiment or Random Experiment: If an exp. contains more than one outcome we call it as random exp. (probability deals with uncertaincy)

. The numerical value of uncertainity is called probability

Random variable: Random variable is a real valued funcⁿ which assigns a real no. to each outcome of random exp.

Ex :-

Consider tossing a coin thrice. the sample space of this random exp. is $S = \{HT, TH, TT, HH\}$

Suppose X = Outcome of getting H and X is random variable, then $X = \{0, 1, 2\}$.

We assign real values for each of X

$P(X=0) = 1/4$ $P(X=1) = 2/4$ $P(X=2) = 1/4$.

Then

X	0	1	2
$P(X)$	$1/4$	$2/4$	$1/4$

The set of values $\{(x_i, P_{x_i}), i = 0, 1, \ldots n\}$ is called the probability distribuⁿ of random variable 'X'.

Discrete Random Variable (DRV) :

A Random Variable which takes some finite or countable finite number of values is called a discrete random variable.

note : We talk of (DRV) whenever there is a counting process involved.

Probability Mass funcⁿ (PMF) :

Let X be a discrete random variable which takes the values $x_1, x_2 \ldots x_n$ then for each x_i we assign a real no. P_{x_i} such that $P_{x_i} \geq 0$ for all i.

(ii) $\sum_{i=1}^{n} P(x_i) = 1$.   (i) $P(x_i) \geq 0$, $\forall i$.

Probability distribution function or cummulative distribuⁿ funcⁿ. (CDF) :

If X is a (DRV) then CDF of X is denoted by $F(t)$ and is defined as $\sum_{x_i \leq t} P(x_i)$ i.e., $F(t) = \sum_{x_i \leq t} P(x_i)$

Mean and variance : If X is a random variable then $E(X)$ is called the expected value or the mean of the random variable and it is defined as

$$E(X) = \mu = \sum_{i=1}^{n} x_i P(x_i)$$

Variance $= E(X^2) - [E(X)]^2$

$$E[X^2] = \sum_{i=1}^{n} x_i^2 \, (P(x_i))$$

Standard deviaⁿ = $\sqrt{\text{Variance}}$

Q.) Find the value of k such that the following represents the finite probability distribuⁿ and find mean & variance. Also find (i) $P(X \le 1)$ (ii) $P(X > 1)$ (iii) $P(1 < X \le 2)$.

X	-3	-2	-1	0	1	2	3
$P(X)$	k	$2k$	$3k$	$4k$	$3k$	$2k$	k.

Solⁿ → Since the ab. distribuⁿ represents a discrete probability distribuⁿ then,

(i) $p(x_i) \ge 0 \ \forall x_i$

(ii) $\sum_{i=1}^{n} P(x_i) = 1$

$$\sum_{i=1}^{n} P(x_i) = 1$$

$k + 2k + 3k + 4k + 3k + 2k + k = 16k = 1 \Rightarrow k = 1/16$

$\sum P(x_i) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=-3)$
$+ P(X=-2) + P(X=-1)$

X	-3	-2	-1	0	1	2	3
$P(X)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$.

Mean $= E(X) = \sum_{i=1}^{n} x_i P(x_i)$

$$= -\frac{3}{16} - \frac{4}{16} - \frac{3}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16} = 0$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$E(x^2) = \Sigma x_i^2 (p(x_i))$$

$$= \frac{(9)}{16} + \frac{8}{16} + \frac{3}{16} + \frac{3}{16} + \frac{8}{16} + \frac{9}{16}.$$

$$= \frac{1}{16}(9 + 8 + 3 + 3 + 8 + 9)$$

$$= \frac{40}{16} = 2.5$$

$$\text{Variance} = \underline{\underline{2.5}}$$

(i) $P(X \leq 1) = P(-3) + P(-2) + P(-1) + P(0) + P(1)$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16} = \underline{\underline{\frac{13}{16}}}$$

(ii) $P(X > 1) = P(X=2) + P(X=3)$

$$= \frac{2}{16} + \frac{1}{16} = \underline{\underline{3/16}}$$

(iii) $p(-1 < x \leq 2) = P(x=0) + P(x=1) + P(x=2)$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16} = 9/16.$$

Q) find the Mean & SD. of the following distribution also find (i) $P(x > 3)$ (ii) $P(X < 4)$. (iii) $P(2 < x \leq 4)$

x	∅	1	2	3	4	5
$P(x)$	0.2		0.35	0.25	0.15	0.05

Sol^n →

Mean $= 1(0.2) + 2(0.35) + 3(0.25) + 4(0.15) + 5(0.05)$

$E(X) = 2.5$

$S.D^2 = \Sigma(x^2) - [E(x)]^2$

$\Sigma(x^2) = 1(0.2) + 4(0.35) + 9(0.25) + 16(0.15) + 25(0.05)$

$= 7.5$

$SD = \sqrt{7.5 - (2.5)^2}$

$= \sqrt{1.25} = \underline{\underline{1.118}}$

(i) $P(X > 3) = P(X = 4) + P(X = 5)$

$= 4(0.15) + 5(0.05)$

$= \underline{\underline{0.2}}$

(ii) $P(X < 4) = 1 - P(X = 4) - P(X = 5)$

$= 1 - 0.2 = \underline{\underline{0.8}}$

(iii) $P(2 < x \leq 4) = \underline{0.4}$

Q) A Random Variable x has the following probability distribu^n.

X	1	2	3	4	5	6	7
P(X)	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) k (ii) $P(X \geq 6)$ (iii) $P(X < 6)$ (iv) $P(1 \leq X < 5)$

(v) $E(X)$ (vi) Mean (vii) Variance

Sol^n →

(i) $k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$9k + 10k^2 = 1 \Rightarrow 10k^2 + 9k - 1 = 0$

solving $k = \frac{1}{10}$, $k = -1$

x	1	2	3	4	5	6	7
$P(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(ii) $P(X \geq 6) = P(6) + P(7)$

$= \underline{\underline{\frac{19}{100}}}$

(iii) $P(X < 6) = 1 - P(X \geq 6)$

$= 1 - \frac{19}{100} = \frac{100-19}{100} = \underline{\underline{\frac{81}{100}}}$

(iv) $P(1 \leq X < 5) = P(1) + P(2) + P(3) + P(4)$

$= 1 \times \frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{12}{10}$

$= \underline{\underline{\frac{23}{10}}}$

(v) $E(X) = \frac{10}{100} + \frac{40}{100} + \frac{60}{100} + \frac{120}{100} + \frac{5}{100} + \frac{12}{100} + \frac{119}{100}$

$= \frac{366}{100} = \underline{\underline{3.66}}$

$E(X^2) = \Sigma x^2 p(x) =$

$= \frac{10}{100} + \frac{80}{100} + \frac{180}{100} + \frac{480}{100} + \frac{25}{100} + \frac{72}{100} + \frac{833}{100}$

$= \frac{168\cancel{0}}{10\cancel{0}} = \underline{\underline{16.8}}$

(vii) Variance $= E(X^2) - [E(X)]^2$

$= 16.8 - (3.66)^2$

$= \underline{3.4044}$

Continuous Random Variable: (CRV)

A random variable which takes all possible values in an interval is called a continuous Random variable.

note: whenever a measuring process is involved we talk of CRV.

pdf (probability density func^n): Let X be a (CRV). Let $f(x)$ be a continuous and differenciable func^n int the interval $(-\infty, \infty)$. $f(x)$ is said to be pdf if it satisfies the foll. condi^ns.

(i) $f(x) \geq 0 \quad \forall x$.

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Cummulative distributive function: (CDF)

The CDF of $f(x)$ is defined as,

$$F(t) = \int_{-\infty}^{t} f(x)dx.$$

Mean & Variance:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

$$\text{Variance} = E(X^2) - [E(X)]^2.$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx.$$

Note: Probability that random variable X lies in the interval $[a, b]$ is given by

$$P(a \leq X \leq b) = P(a < X < b)$$

$$\Rightarrow P(a \leq X < b) = P(a < X \leq b).$$

$$= \int_a^b f(x)\,dx.$$

Q.) If a CRV X is given by. $P(x) = \begin{cases} 1/4 & -2 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$

find (i) $P(X<1)$ (ii) $P(|X|>1)$ (iii) $P[2X+3 > 5]$.

Soln:

(i) $P(X<1) = \int_{-2}^{1} \frac{1}{4}\,dx = \frac{1}{4}(x)_{-2}^{1} = \frac{1}{4}(1-(-2)) = \frac{1}{4}(3) = 3/4$

(ii) $P[|X|>1] = 1 - P[|X|<1]$.

$$= 1 - \int_{-1}^{1} \frac{1}{4}\,dx.$$

$$= 1 - \frac{1}{4}(x)_{-1}^{1} = 1 - \frac{1}{4}[1-(-1)]$$

$$= 1 - \frac{2}{4} = 1/2$$

(iii) $P(2X+3>5) = P(X>1)$

$$= 1 - P(X \leq 1)$$

$$= 1 - \frac{3}{4} = 1/4$$

$$2k\left(\frac{\pi}{2} - 0\right) = 1$$

$$k\pi = 1 \Rightarrow k = 1/\pi$$

$$P(X \geq 0) = \frac{1}{\pi}\int_0^\infty \frac{dx}{1+x^2}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_0^\infty$$

$$= \frac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{1}{\pi}\left[\pi/2 - 0\right) = 1/2$$

4.) $f(x) = y_0\, e^{-|x|}$ $\quad -\infty < 0 < \infty$. P.T. $y_0 = 1/2$. find mean and variance for the distributions.

Soln →

Since given $f(x)$ is a pdf,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} y_0\, e^{-|x|}\, dx = 1$$

$$2\int_0^{\infty} y_0\, e^{-|x|}\, dx = 1$$

$$2\int_0^{\infty} y_0\, e^{-x}\, dx = 1$$

$$2 y_0\left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$

$$-2y_0\left[e^{-\infty} - e^0\right] = 1$$

$$2y_0 = 1 \Rightarrow y_0 = 1/2$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x e^{-|x|}\, dx$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \int_{0}^{\infty} x e^{-x}\, dx$$

$$= \int_{0}^{\infty} \underset{u}{x}\, \underset{v}{e^{-x}}\, dx$$

$$= \left[-xe^{-x} + e^{-x}\right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= \underline{\underline{1}}$$

X

$x e^{-|x|}$ is odd function

$\therefore E(X) = 0$

$$\text{Variance} = E(x2) - [E(x)]^2$$

$$E(x2) = \frac{1}{2}\int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\, dx$$

$$= 2 \times \frac{1}{2}\int_{0}^{\infty} x^2 e^{-x}\, dx$$

$$= \left[-x^2 e^{-x} - (2x)e^{-x} + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$$

$$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty}$$

$$= \left[-(-2)\right] = \underline{\underline{2}}$$

$$\text{Variance} = \underline{\underline{2 - 0 = 2}}$$

Q) The pdf of a random variable x is given by.

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2. \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \ge 1.5)$ & cdf (CDF)

Soln.

$$P(X \ge 1.5) = \int_{1.5}^{2} (2-x)\, dx.$$

$$= 2x - \frac{x^2}{2} \Big|_{1.5}^{2}.$$

$$= (4-2) - \left(3 - \frac{9}{8}\right)$$

$$= 2 - \frac{15}{8} = \underline{0.125} = 1/8$$

$$CDF = F(t) = \int_{\infty}^{t} f(x)\, dx.$$

for $-\infty < x < 0$.

$$f(t) = \int_{-\infty}^{t} 0\, dx = 0$$

for $0 \le x \le 1$

$$F(t) = \int_{-\infty}^{t} x\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{t} x\, dx$$

$$= 0 + \frac{x^2}{2} = \underline{\underline{\frac{t^2}{2}}}$$

for $-\infty < x < 2$

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{t} f(x)\,dx$$

$$= \int_{0}^{1} x\,dx + \int_{1}^{t} (2-x)\,dx$$

$$= \frac{1}{2} + 2t - \frac{t^2}{2} - \frac{3}{2}.$$

$$= 2t - \frac{t^2}{2} - 1$$

for $-\infty < x < \infty$

$$F(t) = \int_{-\infty}^{0} 0 + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx + \int_{2}^{t} 0\,dx$$

$$= \left.\frac{x^2}{2}\right|_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + \left[(4-2) - (2 - 1/2)\right]$$

$$= \frac{1}{2} + \left[2 - \left(\frac{3}{2}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2} = \underline{\underline{1}}$$

$$F(t) = \begin{cases} 0 \\ t^2/2 \\ 2t - t^2/2 - 1 \\ 1 \end{cases}$$

Binomial Distribution: (B.D)

Binomial distribution is a discrete probability distribution. We use this distribution in the following case

(i) There should be finite no. of trial.

(ii) All the trials are independent.

(iii) There should be only 2 outcomes p & q ~~where p is~~

~~(iv) the pdf of the bin~~

The pdf of B.D is defined as,

$$\boxed{P(X = r) = {}^nC_r \, p^r q^{n-r}}$$

where

$p \rightarrow$ probability of success.

$q = 1-p \rightarrow$ probability of failure.

$n \rightarrow$ no. of trials.

The pdf $P(X=r)$ represents probability of r^{th} success out of n trials.

${}^nC_r$ is the binomial co-efficient which is given

$$\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$$

If we know n & p we can form the pdf of the B.D so, therefore n & p are called the parameters of B.D.

Mean and Variance :-

$\mu = E(X) = \Sigma x \cdot p(x)$

$p(X=x) = p(x) = {}^nC_x p^x q^{n-x}$ $\quad \left[(p+q)^n = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} \right.$

$$E(x) = \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x\, {}^nC_x p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} \frac{x\, n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} = \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x}.$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^{x-1+1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^{x-1} q^{n-1-(x-1)}$$

$$= np(p+q)^{n-1} = np \qquad (p+q)=1.$$

Variance $= E(x^2) - [E(x)]^2$.

$$E(x^2) = \sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} x(x-1+1)\, p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, p(x) + \sum_{x=0}^{n} x\, p(x)$$

$$E(x^2) = \sum_{r=0}^{n} r(r-1)\,p(r) + \sum_{r=0}^{n} r\,p(r)$$

$$= \sum_{r=0}^{n} r(r-1)\,p(r) + np$$

$$\sum_{r=0}^{n} r(r-1)\,p(r) = \sum_{r=0}^{n} r(r-1)\,{}^{n}C_{r}\,p^{r} q^{n-r}$$

$$= \sum_{r=0}^{n} r(r-1) \frac{n!}{(n-r)!\,r!} p^{r} q^{n-r}$$

$$= \sum_{r=0}^{n} \frac{n!\,p^{r} q^{n-r}}{(r-2)!\,(n-r)!}$$

$$= \sum_{r=2}^{n} \frac{n(n-1)(n-2)!\,p^{r-2+2}\,q^{n-2-(r-2)}}{(r-2)!\,[(n-2)-(r-2)]!}$$

$$= n(n-1)\,p^2 (p+q)^{n-2}$$

$$= n(n-1)\,p^2 \quad [\because (p+q)=1]$$

Variance $= E(x^2) - [E(x)]^2$

$$E(x^2) = n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

Variance $= npq$

mean $= np$

3) The probability that a pin manufactured by a company will be defective is 0.1. If 12 such pins are selected at random, find the probability that (i) Exactly 2 will be defective (ii) Atmost 2 will be defective (iii) Atleast 2 will be defective (iv) None is defective.

Soln →

$p = 0.1$, $n = 12$, $q = 0.9$

$P(r) = {}^{n}C_{r}\, p^{r} q^{n-r}$

PDF, $p(r) = {}^{12}C_{r} (0.1)^{r} (0.9)^{12-r}$

i) Exactly two will be defective

$P(r=2) = {}^{12}C_{2} (0.1)^{2} (0.9)^{10}$

$= 0.230$

2.) Atmost 2 will be defective

$p(r = 0, r = 1, r = 2)$

$= {}^{12}C_{0} (0.1)^{0} (0.9)^{12} + {}^{12}C_{1} (0.1)^{1} (0.9)^{11} + {}^{12}C_{2} (0.1)^{2} (0.9)^{10}$

$= \underline{0.8890}$

3.) $P(r \geq 2) = 1 - P(r < 2)$

$= 1 - [P(0) + P(1)]$

$= 0.34$

4.) $P(\text{none}) = P(r=0)$

$= \underline{0.0282}$

2) In hundered sets of 10 tosses of an unbiased coin in how many tosses should we expect

(i) 7 heads 3 tails

(ii) Atleast 7 heads.

Soln →

$p = 0.5$ (H) $\quad q = 0.5$ (T)

$$p(x) = {}^{10}C_x (0.5)^x (0.5)^{10-x}$$

$$= {}^{10}C_x (0.5)^{10}$$

(i) $$P(x=7) = {}^{10}C_7 (0.5)^{10}$$

$$= {}^{10}C_7 (1/1024)$$

$$= 0.1171$$

No. of sets with 7H & 3T $= 100 \times 0.117 \simeq 12$

(ii) $$P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$$

$$= 0.171$$

No. of sets $= 100 \times 0.1719$

$= 17.1$

$= 17$

Poisson's distribution :-

It is a discrete probability distribution. We use the distribution in the foll. situation

(i) The no. of trials n is very large.

(ii) All the trials are independent.

(iii) Probability of success p is very small.

pdf of poisson distribuⁿ is given by,

$$\boxed{P(x) = \frac{e^{-m} m^x}{x!}}$$

m is mean

Q) P.T the poisson distribution is limiting case of binomial distribution.

Solⁿ → The pdf of binomial distribution is given by.

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$\text{Mean} = m = np$$

$$p = \frac{m}{n}, \quad q = 1 - \frac{m}{n}.$$

$$\therefore P(x) = {}^nC_x \left(\frac{m}{n}\right)^x \left(1 - \frac{m}{n}\right)^{n-x}.$$

$$P(x) = \frac{n!}{x!(n-x)!} \frac{m^x}{n^x} \frac{\left(1 - \frac{m}{n}\right)^n}{\left(1 - \frac{m}{n}\right)^x}$$

$$= \frac{m^x}{x!} \left[\frac{n(n-1)(n-2) \cdots (n-(x-1))\cancel{(n-x)!}}{\cancel{(n-x)!}\, n^x \left(1 - \frac{m}{n}\right)^x} \left(1 - \frac{m}{n}\right)^n \right]$$

$$= \frac{m^x}{x!} \left[\frac{n^x\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{(x-1)}{n}\right)\left(1 - \frac{m}{n}\right)^n}{n^x \left(1 - \frac{m}{n}\right)^x} \right]$$

Taking limit as $n \to \infty$ and using

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}$$

$$P(r) = \frac{m^r}{r!}\left[\lim_{n\to\infty}\frac{\left(1-\cancel{\frac{1}{n}}^{0}\right)\left(1-\cancel{\frac{2}{n}}^{0}\right)\cdots\left(1-\cancel{\frac{(r-1)}{n}}^{0}\right)\left(1-\frac{m}{n}\right)^n}{\left(1-\cancel{\frac{m}{n}}^{0}\right)^r}\right]$$

$$\therefore \boxed{P(r) = \frac{m^r e^{-m}}{r!}}$$

Q) Mean and Variance :-

$$E(X) = \text{Mean} = \sum r\,p(r)$$

$$P(r) = \frac{e^{-m} m^r}{r!}$$

$$E(X) = \sum_{r=0}^{\infty}\frac{r\,e^{-m} m^r}{r!}$$

$$= \sum_{r=0}^{\infty}\frac{r\,e^{-m} m^r}{r(r-1)!}$$

$$= e^{-m}\sum_{r=1}^{\infty}\frac{m^r}{(r-1)!}$$

$$= e^{-m}\left[m + \frac{m^2}{1!} + \frac{m^3}{2!} + \frac{m^4}{3!} + \cdots\right]$$

$$= m e^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right]$$

mean of poisons distribution is same as mean of binomial distribution

$$= m e^{-m} e^{m}$$

$$E(X) = m$$

Variance = $E(x^2) - [E(x)]^2$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1+1)\, p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) p(x) + \sum_{x=0}^{\infty} x\, p(x).$$

$$= \sum_{x=0}^{\infty} x(x-1) p(x) + m \rightarrow (1)$$

$$\sum_{x=0}^{\infty} x(x-1) p(x) = \sum_{x=0}^{\infty} \frac{x(x-1)\, e^{-m} m^x}{x!} = \sum_{x=0}^{\infty} \frac{\cancel{x(x-1)}\, e^{-m} m^x}{\cancel{x(x-1)}\,(x-2)!}$$

$$= e^{-m} \sum_{x=2}^{\infty} \frac{m^x}{(x-2)!}$$

$$= e^{-m}\left[m^2 + \frac{m^3}{1!} + \frac{m^4}{2!} + \frac{m^5}{3!} + \cdots\right]$$

$$= e^{-m} m^2 \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right]$$

$$= m^2 e^{-m} e^{m} = \underline{\underline{m^2}} \text{ --- } (2)$$

substitute (2) in (1).

$$E(x^2) = m^2 + m$$

$$\text{Variance} = \underbrace{m^2 + m}_{E(x^2)} - \underbrace{m^2}_{[E(x)]^2}$$

$$= \underline{\underline{m}}.$$

Q) A communication channel recieves independent pulses at the rate of 12 pulses per micro second. The probability of transmission error is 0.001 for each microsecond. Compute the probability of (i) no error during a micro second. then (ii) one error per micro second. (iii) atleast one error per micro second. (iv) two errors (v) Atmost 2 errors.

Soln. →

Since the probability is very small, we can apply Poisson distribution.

$$m = np$$

$$= 12 \times 0.001$$

$$= \underline{0.012}$$

$$P(x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.012} (0.012)^x}{x!}$$

(i) $P(x) = P(0) = \dfrac{e^{-0.012} (0.012)^0}{0!} = \underline{0.988}$

(ii) $P(\text{one error}) = P(1) = \dfrac{e^{-0.012} (0.012)^1}{1!} = \underline{0.01185}$

(iii) $P(x \geq 1) = 1 - P(0) = 1 - 0.988 = \underline{0.012}$

(iv) $P(2) = \dfrac{e^{-0.012} (0.012)^2}{2!} = \underline{0.0000714}$

(v) $P(x \leq 2) = P(0) + P(1) + P(2)$

$= 0.988 + 0.01185 + 0.0000714$

$= \underline{0.99992}$

Q°) 2% of the fuses manufactured by a firm are found to be defective. Find the probability that a box containing 200 fuses contains.

(i) No defective fuses

(ii) 3 or more defective fuses.

~~(iii)~~

Solⁿ →

$p = 2\%$ $\qquad n = 200$ $\qquad \therefore m = np$

$= 0.02$

$= 200 \times 0.02$

$= 4$

$$P(x) = \frac{e^{-m}\, m^{x}}{x!} = \frac{e^{-4}(4)^{x}}{x!}$$

(i) $P(\text{no defective}) = P(0) = \dfrac{e^{-4}\, 4^{0}}{0!} = \underline{0.0183}$

(ii) $P(x \geq 3) = 1 - P(x < 3)$

$= 1 - P(0) - P(1) - P(2)$

$= 1 - 0.0183 - \dfrac{e^{-4}\, 4^{1}}{1!} - \dfrac{e^{-4}\, 4^{2}}{2!}$

$= 1 - 0.0183 - 0.0732 - 0.1465$

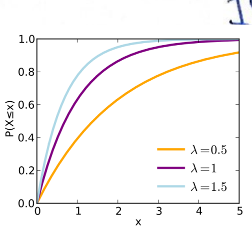
$= \underline{0.762}$

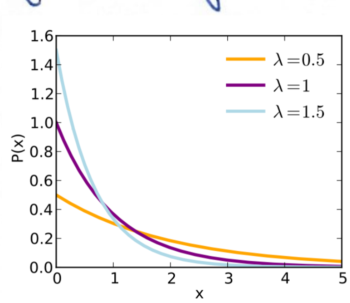
Exponential Distribution:

higher the value of α, faster the curve drops, and mean is closer to 0

The pdf of exponential distribuⁿ is defined by.

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$





Q) where α is a parameter.

Find Mean and variance of exponential distribution.

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x \alpha e^{-\alpha x}\, dx$$

$$= \alpha \left[x\left(\frac{e^{-\alpha x}}{-\alpha}\right) - (1)\left(\frac{e^{-\alpha x}}{\alpha^2}\right)\right]_0^{\infty} = \alpha\left[\infty \cdot \frac{1}{e^{\infty}} - \frac{1}{e^{\infty}}\right] - \left[0 - \frac{1}{\alpha^2}\cdot 1\right]$$

$$= +\frac{1}{\alpha^2}\cdot\alpha$$

$$= \frac{-1}{\alpha}\left[e^{-\infty} - e^{0}\right]$$

$$\boxed{E(X) = \frac{1}{\alpha}}$$

$$E(X) = \frac{1}{\alpha}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{0}^{\infty} x^2 \alpha e^{-\alpha x}\, dx$$

$$= \alpha\left[x^2\left(\frac{e^{-\alpha x}}{-\alpha}\right) - (2x)\left(\frac{e^{-\alpha x}}{\alpha^2}\right) + 2\left(\frac{e^{-\alpha x}}{-\alpha^3}\right)\right]_0^{\infty}$$

$$= \frac{-2\alpha}{\alpha^3}\left[e^{-\infty} - e^{0}\right] = \frac{2}{\alpha^2}$$

$$\text{Variance} = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

note: The exponential distribun may be applied to situations whenever poisson probability distribun is applicable. The Poisson's distribution gives the occurance of events per unit time,
whereas the exponential distribun gives the time btw 2 successive events.

Q) The length of telephone conversation in a booth has been an exponential distribution an found on an avg to be 5 minutes. Find the probability that a random call made from this booth (i) ends less than 5 minutes (ii) between 5 and 10 mins.

Soln →

Mean = 5 $\quad , \frac{1}{\alpha} = 5 \quad \Rightarrow \alpha = \frac{1}{5}$

$$F(x) = \alpha e^{-\alpha x}$$

$$f(x) = \frac{1}{5} e^{-\frac{x}{5}}$$

(i) P(ends less than 5 minutes) = P (X < 5)

$$P(X<5) = \int_0^5 \frac{1}{5} e^{-\frac{x}{5}} dx$$

$$= \frac{1}{5} \left. \frac{e^{-x/5}}{-1/5} \right|_0^5 = -\left[e^{-1} - e^0\right]$$

$$= \underline{0.63}$$

(ii) P(between 5 & 10 mins) = P(5 < x < 10)

$$= \int_5^{10} \frac{1}{5} e^{-x/5} dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_5^{10}$$

$$= -\left[e^{-2} - e^{-1}\right]$$

$$= \underline{0.2325}$$

Q. 8) The mileage which a car owner get with a certain kind radial tyre is a random variable having an exponential distribun with mean 40,000 kms. Find the probabilities that one of these tyres will last

(i) Atleast 20,000 kms. ~~to~~ (ii) utmost 30,000 kms.

soln →

$$M = 40000 \Rightarrow \alpha = 1/M = 1/40000$$

$$= 0.000025$$

$$f(x) = \alpha e^{-\alpha x} = 0.000025\, e^{-0.000025 x}$$

$$= \frac{1}{40000} e^{-\frac{x}{40000}}$$

(i) $$P(x \geq 20000) = \int_{20000}^{\infty} f(x)\,dx$$

$$= \int_{20000}^{\infty} \frac{1}{40000} e^{-\frac{x}{40000}}\, dx$$

$$= \frac{1}{40000} \left[\frac{e^{-x/40000}}{-1/40000} \right]_{20000}^{\infty}$$

$$= -\left[0 - e^{-1/2}\right] = e^{-1/2} = 0.6065$$

(ii) $$P(\text{Atmost } 30000 \text{ kms}) = P(x < 30000)$$

$$= \int_{0}^{30000} f(x)\,dx$$

$$= \int_{0}^{30000} \frac{1}{40000} e^{-x/40000}\, dx$$

$$= \frac{1}{\cancel{40000}} \left. \frac{e^{-x/40000}}{-1/\cancel{40000}} \right|_{0}^{30000}$$

$$= -\left[e^{-3/4} - e^{0} \right]$$

$$= \underline{\underline{0.5276}}$$

Q.) The sales per day in a shop are exponentially distributed with average sale amounting to Rs. 100 and net profit 8%. Find the probability that net profit exceeds Rs 30 on 2 consecutive days.

soln →

Let random variable x indicate sales per day which follows an exponential distribution

Avg sale $= 100$ $\quad$ $1/\alpha = 100$ $\quad$ $\alpha = 1/100$.

net profit $= 8\%$.

Let A be amount of sales per day

8% of $A = 30$

$$\frac{8}{100} A = 30$$

$$A = \underline{\underline{375}}$$

P(net profit exceeds Rs 30)

$=$ P(sales per day exceeds 375)

$$= \int_{375}^{\infty} \frac{1}{100} e^{-x/100} \, dx \quad = \frac{1}{\cancel{100}} \left[\frac{e^{-x/100}}{-1/\cancel{100}} \right]_{375}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-375/100} \right]$$

$$= \underline{\underline{0.0235}}$$

For two consecutive days

$= 0.0235 \times 0.0235$

$= 5.5225 \times 10^{-4}$

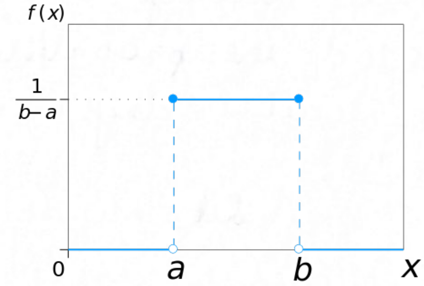
$= 0.00055225$

Uniform Distribution

this is also a continuous probability distribution.

pdf is given by,

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{elsewhere} \end{cases}$$



Mean & Variance:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\text{Mean} = \int_a^b \frac{1}{b-a} x\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2 - a^2}{2}\right]$$

$$= \frac{(a+b)\cancel{(a-b)}^{(b-a)}}{2(b-a)} = \frac{(a+b)}{2}$$

ariance $= E(x^2) - [E(x)]^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_a^b \frac{1}{b-a} x^2\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3 - a^3}{3}\right]$$

$$= \frac{1}{3(b-a)}[b^3 - a^3]$$

$$= (\cancel{b-a})(a^2 + ab + b^2) \times \frac{1}{3(\cancel{b-a})}$$

$$= \frac{a^2 + ab + b^2}{3}$$

$$\text{Variance} = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(a^2 + ab + b^2) - 3(a^2 + 2ab + b^2)}{12}$$

$$= \frac{a^2 - 2ab + b^2}{12} = \frac{(a-b)^2}{12}$$

Q) A RV x is uniformly distributed over the interval $(-2, 2)$. Find.

(i) $P(X < 1)$ (ii) $P(|X-1| \geq 1/2)$

Sol^n →

$$f(x) = \frac{1}{b-a} \quad a < x < b$$

i.e., $$f(x) = \frac{1}{4} \quad -2 < x < 2$$

(i) $$P(X < 1) = \int_{-2}^{1} f(x)\,dx = \int_{-2}^{1} \frac{1}{4}\,dx = \frac{3}{4}$$

(ii) $P(|X-1| \geq 1/2)$

$$P[|X-1| \geq 1/2] = 1 - P[|X-1| \leq 1/2]$$

$$= 1 - P\left[\frac{-1}{2} < (X-1) < 1/2\right]$$

$$= 1 - P\left[-1/2 + 1 < X < 1/2 + 1\right]$$

$$= 1 - P\left[1/2 < X < 3/2\right]$$

$$= 1 - P\left[\frac{1}{2} < X < \frac{3}{2}\right]$$

$$= 1 - \int_{1/2}^{3/2} \frac{1}{4}\,dx$$

$$= 1 - \frac{1}{4}\left[x\right]_{1/2}^{3/2}$$

$$= 1 - \frac{1}{4}\left[\frac{3}{2} - \frac{1}{2}\right] = 1 - \frac{1}{4}\left[\frac{2}{2}\right]$$

$$= 1 - \left(\frac{1}{4} \times 1\right)$$

$$= \frac{3}{4} //$$

8) On a certain city transport buses ply every 30 mins btw 6 am & 10 pm. If a person reaches a bus stop on this route at a random time during this period, what is the probability that he will have to wait for atleast 20 mins.

Soln →

$$f(x) = \begin{cases} 1/30 & 0 < x < 30 \\ 0 & \text{else where} \end{cases}$$

P(person has to wait for atleast 20 mins)

$$= P(X \geq 20)$$

$$= \int_{20}^{30} \frac{1}{30}\, dx = \frac{1}{3}$$

Revision :- (For 1st internals)

Unit - 2.

Joint probability distribution

Discrete probability distribution

Let X & Y be 2 discrete Random variable defined on the same sample space of a random experiment

Let $X = \{x_1, x_2 \dots x_m\}$, $i = 1, 2 \dots m$

$Y = \{y_1, y_2 \dots y_n\}$, j

The probability mass function pmf is denoted by

$P(X = x, Y = y) = P(x, y)$

The probability $P(x,y)$ represents simultaenous occurance of events x & y.

Let P_{ij} represents probabilities corresponding to $i = 1, 2, \dots m$ $j = 1, 2 \dots n$ and pmf is defined as

$P(x, y) = P(x_i, y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} P_{ij} = 1$, where $P_{ij} \geq 0$ for all i & j

All the above P_{ij}'s we represent in a joint probability distributen and it is ~~known~~ shown as here.

X \ Y	y_2	y_2 -	-	-	y_n	
x_1	P_{11}	P_{12}	-	-	P_{1n}	$f(x_1)$
x_2	P_{21}	P_{22}	-	-	P_{2n}	$f(x_2)$
⋮	⋮					⋮
x_m	P_{m1}	P_{m2}	-	-	P_{mn}	$f(x_m)$
	$g(y_1)$	$g(y_2)$	-	-	$g(y_n)$	1

In the above table last column entries are row-wise summations of n. The last row entries are columnwise summation of y and these are called marginal probability distributions of X & Y respectively

∴ Marginal probability distribution (MPD) of X is given by

X	x_1	x_2	$\cdots$	x_m
$f(x)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_m)$

MPD of Y is given by

Y	y_1	y_2	$\cdots$	y_n
$g(y)$	$g(y_1)$	$g(y_2)$	$\cdots$	$g(y_n)$

Mean, Variance & Co-variance :-

$E(X) = \mu_x = \Sigma x f(x)$

$E(Y) = \mu_y = \Sigma y g(y)$

$E(X^2) = \Sigma x^2 f(x)$

$E(Y^2) = \Sigma y^2 g(y)$

$Var(X) = E(X^2) - [E(X)]^2$

$Var(Y) = E(Y^2) - [E(Y)]^2$

$cov(X,Y) = E(XY) - E(X)E(Y)$

correlation co-efficient $= \rho(X,Y) = \dfrac{cov(X,Y)}{\sigma_x \sigma_y}$

$\sigma_x = \sqrt{Var(X)}$ $\quad \sigma_y = \sqrt{Var(Y)}$

$E(XY) = \sum_i \sum_j x_i y_j p_{ij}$

mean of Joint probability:

Note: If 2 variables x & y are said to be independent

then (i) $p_{ij} = f(x_i)\, g(y_j)$

Eg: (i) $p_{12} = f(x_1) g(y_2)$

(ii) $cov(X,Y) = 0$ ($\Rightarrow \rho(X,Y) = 0$)

Problems:

1) The joint probability distribution of 2 RV X & Y is given as follows:

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Find (i) MPD of X & Y. (ii) COV (X, Y) (iii) $\rho(X, Y)$

Check if X & Y are independent

Soln

X \ Y	1	3	9	f(x)
2	1/8	1/24	1/12	6/24
4	1/4	1/4	0	1/2
6	1/8	1/24	1/12	6/24
g(y)	1/2	1/3	1/6	1

(i) MPD of X

X	2	4	6
f(x)	1/4	1/2	1/4

MPD of Y

Y	1	3	9
g(y)	1/2	1/3	1/6

(ii) $\text{COV}(X, Y) = E(XY) - E(X)E(Y)$

$E(XY) = \sum_i \sum_j x_i y_j p_{ij}$

$= 2(1)(1/8) + (2)(3)(1/24) + 2(9)(1/2) + 1 + 3 + 0 + 6/8 + 18/24 + 54/12$

$= 19.5 = 12$

$E(X) = 2/4 + 4/2 + 6/4 = 4$

$E(Y) = 1/2 + 1 = 3$

$\text{COV}(X, Y) = 0$ $\therefore$ X & Y are independent

(iii) $\rho(X, Y) = 0$

Condiⁿal probability distribuⁿ

The condiⁿal probability of y given x is given

$$P(y|x) = \frac{P(x,y)}{f(x)} = \frac{\text{Joint probability of } x \,\&\, y}{f(x)}$$

CP of x given y is defined as

$$P(x,y) = \frac{P(x,y)}{g(y)}$$

Problems:

1) The joint probability distribuⁿ of discrete RV x & y is as follows. Evaluate CPD, P(X|1) P(y|2) S.T. X & Y are not independent

$$\frac{P(x,1)}{P(1)}$$

Solⁿ →

X \ Y	1	2	3	f(x)
1	1/12	1/6	0	1/4
2	0	1/9	1/5	14/45
3	1/8	1/4	2/15	79/180
g(y)	5/36	19/36	1/3	1

x	1	2	3
P(x|1)			

MPD of X

X	1	2	3
f(x)	1/4	14/45	79/180

MPD of Y

Y	1	2	3
g(y)	5/36	19/36	1/3

CPD of P(X/1)

$$P(x/y) = \frac{P(x,y)}{g(y)} \Rightarrow P(X/1) = \frac{P(x,1)}{g(1)}$$

$$= \frac{P(x,1)}{5/36} = \frac{36\, P(x,1)}{5}$$

x	1	2	3
P(x|1)	3/5	0	2/5

= 1

$$P(y|2) = \frac{P(2,y)}{f(2)} = 45 \times \frac{P(2,y)}{14}$$

y	1	2	3	
$P(y\|2)$	$\frac{45\times 0}{14}$	$\frac{45\times 1}{14\times 9}$	$\frac{45\times 1}{5\times 14}$	
	$=0$	$=5/14$	$=9/14$	$=1.$

$$P(2,2) \neq f(2)\,g(2)$$

Hence variables are dependent

2.) given the following bi-variant PD obtain (i) MPD of x & y (ii) CPD of y given $x=2$ (iii) $P(x|-1)$. Also find $\text{cov}(x,y)$, $\rho(x,y)$

Soln →

X \ Y	-1	0	1	$f(x)$
0	1/15	2/15	1/15	4/15
1	3/15	2/15	1/15	6/15
2	2/15	1/15	2/15	5/15
$g(y)$	6/15	5/15	4/15	1

(i) MPD of x

x	0	1	2
$f(x)$	4/15	6/15	5/15

MPD of y

y	-1	0	1
$g(y)$	6/15	5/15	4/15

(ii) $$P(y|2) = \frac{P(2,y)}{f(2)} = \frac{3\,P(2,y)}{1}$$

y	-1	0	1	
$P(y/2)$	$3 \times \frac{2}{15}$	$3\,P(2,0)$	$3\,P(2,1)$	
	$= \frac{6}{15}$	$3/15$	$\frac{6}{15}$	$= 1$

(iii) $P(x/-1) = \dfrac{P(x,-1)}{g(-1)} = \dfrac{15\,P(x,-1)}{6}$

x	0	1	2
$P(x\mid -1)$	$\frac{15 \times P(0,-1)}{6}$	$\frac{15 \times P(1,-1)}{6}$	$\frac{15 \times P(2,-1)}{6}$
	$= \frac{15}{6} \times \frac{1}{15}$	$= \frac{15}{6} \times \frac{3}{15}$	$= \frac{15}{6} \times \frac{2}{15}$
	$= \frac{1}{6}$	$= \frac{3}{6}$	$= 2/6$

$\text{COV}(x,y) = E(XY) - E(X)\,E(Y)$

$E(XY) = \sum_i \sum_j x_i y_j P_{ij}$

$= 0 + 0 + 0 + (1)(-1)(3/15) + 0 + (1)(1)(1/5)$

$+ 2(-1)(2/15) + 2(0) + 2(1)(2/15)$

$= \dfrac{-3}{15} + \dfrac{1}{15} - \dfrac{4}{15} + \dfrac{4}{15}$

$= \dfrac{-2}{15}$

$$E(X) = \frac{6}{15} + \frac{10}{15} = \frac{16}{15}$$

$$E(Y) = \frac{-6}{15} + \frac{4}{15} = \frac{-2}{15}$$

$$E(X)\,E(Y) = \frac{-32}{(15)(15)}$$

$$\text{cov}(x,y) = \frac{-2}{15} + \frac{32}{(15)(15)} = \frac{2}{225}$$

$$\rho(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{E(x^2) - [E(x)]^2}$$

$$E(x^2) = \frac{6}{15} + \frac{20}{15} = \frac{26}{15}$$

$$\sigma_x = \sqrt{\frac{26}{15} - \left(\frac{16}{15}\right)^2} = \frac{\sqrt{134}}{15}$$

$$\sigma_y = \sqrt{E(Y^2) - [E(Y)]^2}$$

$$E(Y^2) = \frac{6}{15} + \frac{4}{15} = \frac{10}{15}$$

$$\sigma_y = \sqrt{\frac{10}{15} - \frac{4}{225}} = \frac{\sqrt{146}}{15}$$

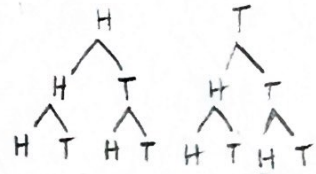
$$\rho(x,y) = \frac{2 \times 15 \times 15}{225 \times \sqrt{134} \times \sqrt{146}} = \underline{\underline{0.014}}$$

8) A fair coin is tossed 3 times. Let X denote 0 or 1 according as head or tail occurs on the 1st toss. Let Y denote the no of heads which occur. Then find (i) Marginal distribution of X & Y
(ii) Joint distribution of X & Y
(iii) Co-variance of X & Y.

Soln

Coin is tossed 3 times & no. of out comes $= 2^3 = 8$

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$



$X = \{0, 1\}$

Marginal distribution of X

X	0	1
P(X)	1/2	1/2

Y = no. of heads
$= \{0, 1, 2, 3\}$

Y	0	1	2	3
P(Y)	1/8	3/8	3/8	1/8

X \ Y	0	1	2	3	f(x)
0	0	1/8	2/8	1/8	4/8
1	1/8	2/8	1/8	0	4/8
g(y)	1/8	3/8	3/8	1/8	1

$COV(XY) = E(XY) - E(X)E(Y)$

$E(XY) = 0 + \cancel{(1)(0)(1/8)} + (1)(1)(2/8) + (1)(2)(1/8) + 0$

$= \frac{2}{8} + \frac{2}{8} = \frac{4}{8} = \underline{\underline{1/2}}$

$E(X) = 1/2$

$E(Y) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = 3/4$

$COV = \underline{\underline{-1/4}}$

*Q) 2 cards are selected at random from a box containing 5 ~~parts~~ cards ~~in which 2~~ numbered 1, 1, 2, 2 & 3. Find the joint distribution of X & Y where X = denote the sum and Y = Max of 2 no.s drawn. Also determine COV (X, Y), P(X, Y) & P(X/2), P(Y/1)

Soln →

$S = \{(1,1)(1,2)(1,2),(2,2),(2,1)$
$(2,1),(1,3),(2,3),(3,1),$
$(3,2)\}$

total no. of pairs $= {}^5C_2 = \frac{5 \times \cancel{4}^{2}}{\cancel{2}} = 10$

X = Denote the sum   Y = max of 2 no.s drawn

$X = \{2, 3, 4, 5\}$ $Y = \{1, 2, 3\}$

X \ Y	1	2	3	f(x)
2	1/10	0	0	1/10
3	0	4/10	0	4/10
4	0	1/10	2/10	3/10
5	0	0	2/10	2/10
g(y)	1/10	5/10	4/10	1

To find cov:

$cov(XY) = E(XY) - E(X)E(Y)$

$E(XY) = \Sigma\Sigma x_i y_j P_{ij}$

$E(XY) =$

Continued in next page →

Q) A bag contains 3 white, 2 red and 2 green bulbs are contained and randomly 2 bulbs are selected without replacement. If X & Y are discrete random variable, denoting the number of white and red bulbs respectively. determine (i) Joint distribuⁿ of X & Y
(ii) COV (X, Y)

Solⁿ →

3 white, 2 Red, 2 Green

Total no. of outcomes → $^7C_2 = \frac{7 \times 6}{2} = \underline{21}$

X = no. of white bulbs.

Y = no. of red bulbs

$X = \{0, 1, 2, 3\}$ $Y = \{0, 1, 2\}$

X \ Y	0	1	2	P(x)
0	1/21	4/21	1/21	6/21
1	6/21	$\frac{^3C_1 \times {}^2C_1}{^7C_2}$	0	12/21
2	$\frac{^3C_2 \times {}^2C_0}{^7C_2}$	0	0	3/21
3	0	0	0	0
g(y)	10/21	10/21	1/21	1

$\frac{^3C_1 \times {}^2C_1}{^7C_2} = \frac{3 \times 2}{\frac{7 \times 6}{2}} = \frac{6}{21}$

$\frac{^3C_2 \times {}^2C_0}{21} = \frac{3}{21}$

To find COV :-

$COV(X, Y) = E(XY) - E(X)E(Y)$

$E(XY) = \Sigma\Sigma x_i y_j P_{ij}$

$E(XY) = 0 + (1)(0)\cancel{(6/21)}^0 + (1)(1)(6/21) + (1)\cancel{(2)(0)}^0 + (2)(0)\cancel{(3/21)}^0 + 2\cancel{(1)(0)}^0 + 2\cancel{(2)(0)}^0 + 0$

$= \underline{\underline{\frac{6}{21}}}$

MPD of x

X	0	1	2	3
$f(x)$	$6/21$	$12/21$	$3/21$	0

MPD of Y

Y	0	1	2
$g(y)$	$19/21$	$10/21$	$1/21$

$$E(X) = 0 + \frac{12}{21} + \frac{6}{21} + 0$$

$$= \frac{18}{21}$$

$$E(Y) = 0 + \frac{10}{21} + \frac{2}{21}$$

$$= \frac{12}{21}$$

$$COV(X,Y) = \frac{6}{21} - \left(\frac{18}{21}\right)\left(\frac{12}{21}\right) = \frac{6}{21} - \frac{216}{21\times 21}$$

$$= \frac{-10}{49}$$

* continuation

$$COV(XY) = E(XY) - E(X)E(Y)$$

$$E(XY) = 2(1)(1/10) + 3(2)(4/10) + 4(2)(1/10) + 4(3)(2/10) + (5)(3)(2/10)$$

$$= \frac{2}{10} + \frac{24}{10} + \frac{8}{10} + \frac{24}{10} + \frac{30}{10} = \frac{88}{10}$$

$$E(X) = 2(1/10) + 3(4/10) + 4(3/10) + 5(2/10)$$

$$= \frac{36}{10}$$

$$E(Y) = 1(1/10) + 2(5/10) + 3(4/10)$$

$$= \frac{23}{10}$$

$$\text{cov}(X,Y) = \frac{88}{10} - \left(\frac{36}{10}\right)\left(\frac{23}{10}\right)$$

$$= \frac{13}{25}$$

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{E(X^2) - [E(X)]^2}$$

$$\sigma_y = \sqrt{E(Y^2) - [E(Y)]^2}$$

$$E(X^2) = 4(1/10) + 9(4/10) + 16(3/10) + 25(2/10)$$

$$= \frac{138}{10}$$

$$E(Y^2) = 1(1/10) + 4(5/10) + 9(4/10)$$

$$= \frac{57}{10}$$

$$\sigma_x = \sqrt{138/10 - \frac{36 \times 36}{100}} = \frac{\sqrt{21}}{5}$$

$$\sigma_y = \sqrt{\frac{57}{10} - \frac{23 \times 23}{100}} = \frac{\sqrt{41}}{10}$$

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y} = \frac{13/25}{\frac{\sqrt{21}}{5} \times \frac{\sqrt{41}}{10}} = 0.89$$

Continuous Joint Probability distribn

If X & Y are 2 continuous random variables defined on sample space of a random experiment. P(X, Y) is said to be probability density funcn if it satisfies following condition

(i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y)\, dx\, dy = 1$

(ii) $P(x,y) \geq 0 \quad \forall\, x \,\&\, y$

$$P(a<x<b,\ c<y<d) = \int_{y=c}^{d}\int_{x=a}^{b} P(x,y)\,dx\,dy$$

Marginal probability distribution of X & Y.

$$f(x) = \int_{-\infty}^{\infty} P(x,y)\,dy$$

i.e. we integrate the ab. integral w.r.t y keeping x constant

Marginal distribution of Y.

$$g(y) = \int_{-\infty}^{\infty} P(x,y)\,dx$$

i.e. we integrate w.r.t. x, keeping y as constant

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, g(y)\,dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,P(x,y)\,dx\,dy$$

Conditional probability distribution

$$P(X|Y) = \frac{P(x,y)}{g(y)} \qquad \cancel{P(Y)}\ P(Y|X) = \frac{P(x,y)}{f(x)}$$

If X & Y are continuous random variables having the joint distribution function as follows.

$$P(x,y) = \begin{cases} c\,(x^2+y^2) & 0 \le x \le 4, \quad 0 \le y \le 4. \\ 0 & \text{otherwise.} \end{cases}$$

find (i) c (ii) $P(1 < X < 2, 2 < Y < 3)$

(iii) $P(X > 3)$ (iv) $P(Y < 2)$ (v) $P(X \ge 3, Y \le 2)$.

soln →

(i) Since $P(XY)$ is a pdf we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y)\,dx\,dy = 1.$$

$$\int_{y=0}^{4}\int_{x=0}^{4} c(x^2+y^2)\,dx\,dy = 1$$

$$c\int_{y=0}^{4}\left(\frac{x^3}{3} + xy^2\right)_0^4 dy = 1.$$

$$c\int_0^4\left(\frac{1}{3}(64) + 4y^2\right)dy = 1$$

$$c\left[\frac{64}{3}(y) + \frac{4y^3}{3}\right]_0^4 = 1$$

$$c\left[\frac{64\times4}{3} + \frac{4\times64}{3}\right] = 1.$$

$$\frac{2\times64\times4}{3}\times c = 1$$

$$c = \frac{3}{2\times64\times4} = \underline{\underline{\frac{3}{512}}}$$

(ii) $P(1 < x < 2,\ 2 < y < 3)$

$$= \frac{3}{512} \int_{y=2}^{3} \int_{x=1}^{2} (x^2+y^2)\, dx\, dy$$

$$= \frac{3}{512} \int_{y=2}^{3} \left. \frac{x^3}{3} + xy^2 \right|_1^2 dy$$

$$= \frac{3}{512} \int_{y=2}^{3} \left[\left(\frac{8}{3} + 2y^2\right) - \frac{1}{3} - y^2\right] dy$$

$$= \frac{3}{512} \int_{2}^{3} \frac{7}{3} + y^2 \, dy$$

$$= \frac{3}{512} \left[\frac{7}{3} y + \frac{y^3}{3}\right]_2^3$$

$$= \frac{3}{512} \left[(7 + 9) - (14/3 + 8/3)\right]$$

$$= \frac{3}{512} \left[\frac{16 \times 3}{3} - \frac{22}{3}\right] = \frac{\cancel{3}}{512} \times \frac{26}{\cancel{3}}$$

$$= \frac{26}{512} = \frac{13}{256}$$

(iii) $$P(X > 3) = \frac{3}{512} \int_{x=3}^{4} \left(4x^2 + \frac{64}{3}\right) dx$$

$$= \frac{\cancel{3}}{512} \left[\frac{4x^3}{\cancel{3}} + \frac{64x}{\cancel{3}}\right]_{x=3}^{4}$$

$$= \frac{1}{512} \left[4(64-27) + 64(4-3)\right]$$

$$= \frac{1}{512} \left[148 + 64\right]$$

$$= \frac{53}{128}$$

(iv) $P(Y<3) \Rightarrow g(y) = \frac{3}{512}\int_{x=0}^{4}(x^2+y^2)\,dx = \frac{3}{512}\left[\frac{x^3}{3} + xy^2\right]_{x=0}^{4}$

$$= \frac{3}{512}\left[\frac{64}{3} + 4y^2\right]$$

$$g(y<2) = \frac{3}{512}\int_{y=0}^{2}\left(\frac{64}{3} + 4y^2\right)dy$$

$$= \frac{3}{512}\left[\frac{64}{3}y + \frac{4y^3}{3}\right]_0^2$$

$$= \frac{\cancel{3}}{512}\left[\frac{64\times 2}{\cancel{3}} + \frac{4\times 8}{\cancel{3}}\right]$$

$$= \frac{1}{512}\left[128 + 32\right]$$

$$= \underline{\underline{5/16}}$$

(v)

Q) Suppose that the joint funcⁿ of 2 continuous random variable x & y is $f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & , 0<x<1, 0<y<2 \\ 0 & , \text{otherwise} \end{cases}$

find (i) $P(X > 1/2)$ (ii) $P(Y/X)$

Soln:

$$P(X) = \int_{y=0}^{2} \left(x^2 + \frac{xy}{3}\right) dy$$

$$= \left[x^2 y + \frac{xy^2}{6}\right]_{y=0}^{2}$$

$$P(X) = 2x^2 + \frac{4x}{6}$$

$$P(X > 1/2) = \int_{1/2}^{1} \left(2x^2 + \frac{4x}{6}\right) dx$$

$$= \left[2\frac{x^3}{3} + \frac{4x^2}{12}\right]_{x=1/2}^{1}$$

$$= \frac{2}{3}\left[1 - \frac{1}{8}\right] + \frac{1}{3}\left[1 - \frac{1}{4}\right]$$

$$= \frac{2}{3} \times \frac{7}{8} + \frac{\cancel{3}}{4} \times \frac{1}{\cancel{3}}$$

$$= \frac{2}{3} \times \frac{7}{8} + 1/4 =$$

$$P(Y|X) = \frac{f(x,y)}{P(x)} = \frac{x^2 + \frac{xy}{3}}{2x^2 + \frac{4x}{6}} = \frac{3x^2 + xy}{\cancel{3}} \times \frac{\cancel{6}2}{12x^2 + 4x}$$

$$= \frac{(3x^2 + xy)\cancel{2}}{\cancel{4}_2 (x + 3x^2)}$$

$$= \frac{1}{2} \frac{(3x^2 + xy)}{x + 3x^2}$$

$$= \frac{1}{2} \frac{(3x+y)}{(1+3x)}$$

Q) Let X & Y be continuous random variables with a joint distribution function $f(x,y) = \begin{cases} 4xy & 0<x<1, \quad 0<y<1 \\ 0 & , \text{otherwise} \end{cases}$

Find $\text{cov}(X,Y)$

Solⁿ → $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} 4x^2y^2\,dx\,dy$$

$$= \underline{\underline{\frac{4}{9}}}$$

$$E(X) = \int_{-\infty}^{\infty} xP(x)\,dx$$

$$P(x) = \int_{y=0}^{1} 4xy\,dy = \left(\frac{4xy^2}{2}\right)_{y=0}^{1} = 2x$$

$$g(y) = \int_{x=0}^{1} 4xy\,dx = 2y$$

$$E(X) = \int_{x=0}^{1} 2x^2\,dx = 2/3$$

$$E(Y) = \int_{y=0}^{1} 2y^2\,dy = 2/3$$

$$\text{cov}(X,Y) = 0$$

X & Y are independent

Stochastic Process:-

Many engineering problems are associated with time varying waveforms that have some element of chance or randomness associated with them.

Uncertainity ~~is~~ plays a keys role in engineering system designs.

For eg:- while conversing ~~from~~ over a telephone we may experience noise at some point of time instant dealing etc. In the ab. example the ~~signals~~ system parameters may change randomly and the signals may be corrupted. Signals are waveforms as in the case of ab. example is called a random signal. Random process are a stochastic process a probalistic model to characterise a random signal

Stochastic process definition: A stochastic process is a maping of the outcomes of random experiment to a set of waveform or func^n of time i.e., a random process is a collec^n of waveforms which is usually denoted by $x(s,t)$, where t is the time and s is the outcome of the random experiment. ~~the ran~~

State space: the set of values assumed by the random variable $x(t)$ is called the state space and the elements of state space are called states.

Index set : the parameter $t \in T \subset R$ is called the index set.

classification of stochastic process:

The state space may be continuous or discrete and the index set 'T' may be discrete or continuous.

1) If state space is discrete and index set T is discrete then, we say that a stochastic process is discrete state, discrete parameter process.

2) If state space is continuous and 'T' is discrete then we say that process is continuous ~~sp~~ state discrete parameter process.

3) If state space is discrete & 'T' is continuous, then we say the process is discrete state continuous parameter process

4) If state space is continuous & 'T' is continuous, then we say the process is continuous state continuous parameter process.

Statistical quantities:

$\mu_x(t) = E\{x(t)\}$, the mean of the stochastic process.

· the auto correlan funcn of ~~stat~~ stochastic process is,

$$R(t_1, t_2) = E\{x(t_1)\, x(t_2)\}$$

cov of stochastic process (auto cov)

$$c(t_1, t_2) = R(t_1, t_2) - \underset{E(x(t_1))}{\mu_x(t_1)}\,\underset{E(x(t_2))}{\mu_x(t_2)}$$

Correlan co-efficient:-

$$\rho(t_1, t_2) = \frac{c(t_1, t_2)}{\sqrt{c(t_1, t_1)\, c(t_2, t_2)}}$$

$$c(t_1\ t_1) = R(t_1, t_1) - E(x(t_1)) \cdot E(x(t_2))$$

Normal distribution :- (Gaussian distribution)

(*) $$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

$$-\infty < x < \infty$$

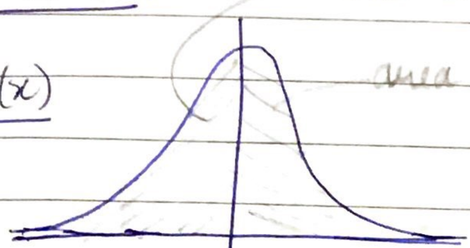
Parameters of Normal distribution $\begin{cases} \mu = \text{Mean} \\ \sigma = \text{Standard deviation} \end{cases}$

Normal curve :-

area = 0.5

$y = f(x)$

area = 0.5



Asymptotic curve.

$x = \mu$

Total area = 1

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx = 1$$

if we consider $z = \frac{x-\mu}{\sigma}$

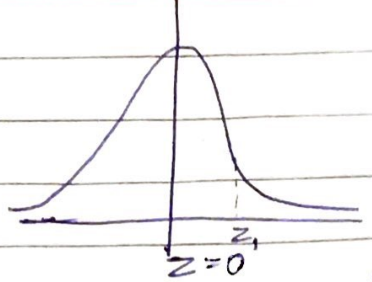
where $\mu = 0$, $\sigma = 1$; Then the distribution is called standard Normal distribution.

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}$$

$$\text{Area} = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \cdot \sigma(dz)$$

Consider $\sigma = 1$

$$\text{Area} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \cdot dz$$



$$P(0 < z < z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2}\, dz$$

Normal distribution is also known as (~~Partial~~ Gaussian distribution)
And this is a Continuous PDF.
The PDF of the Normal distribution is given by (*)

Properties of Graph :-

1> The graph of $y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is known as Normal curve shown in following figure. The Normal line is symmetrical above the line $x=\mu$ and its a bell shaped curve, ~~This curve is Symm~~

2> Mean ~~&~~, Median & mode coincide for the Normal curve.

3> The Tails of Normal curve is asymptotic along the horizontal axis.

4> The area under the normal curve $= 1$
Since the Normal curve is symmetrical about line $x=\mu$, on either side of line $x=\mu$, the area $=0.5$

5> Probability that x lies between x_1 and x_2 is given by

$$P(x_1 < x < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \text{--- (1)}$$

The probability lies on the values of μ and σ.
For different values of μ and σ, we have to evaluate the interval repeatedly. So that is why, the Normal distribution is standardized with the Transformation $z = \frac{x-\mu}{\sigma}$. Here

z = Standard Normal Variable.

For this $\mu = 0$ and $\sigma = 1$ and the standard normal distribution is given by

$$P(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}$$

The evaluation of the interval from $z=0$ to $z=z_1$ is

$\int_{z=0}^{z_1} e^{-z^2/2} dz$ is represented in a Table called Standard Normal Table which gives the area between $z=0$ to $z=z_1$ under the standard Normal curve

for $x = x_1$ $\quad z_1 = \frac{x_1-\mu}{\sigma}$

$x = x_2$ $\quad z_2 = \frac{x_2-\mu}{\sigma}$

Then (1) takes the form

$$P(x_1 < x < x_2) = P(z_1 < z < z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz$$

1> Find the following probabilities using the standard Normal Table. [Refer Table]

i) $P(0 < z < 3)$

ii) $P(z > 2)$

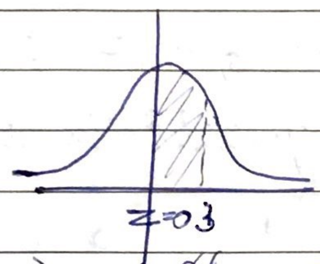
iii) $P(-2 < z < 2)$

iv) $P(-2 < z < 0)$

v) $P(z < -1)$

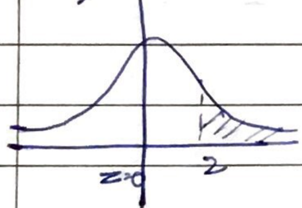
vi) $P(z < 1)$

A> i)



$$P(0 < Z < 3) = \frac{1}{\sqrt{2\pi}} \int_0^3 e^{-\frac{z^2}{2}}\, dz$$
$\hookrightarrow = 0.4987$

ii) $P(Z > 2) = \cancel{P}\ 0.5 - P(0 < Z < 2)$
$= 0.5 - 0.4772 = 0.0228$



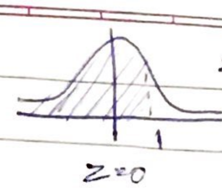
iii) $P(-2 < Z < 2) = P(-2 < Z < 0) + P(0 < Z < 2)$
$= P(0 < Z < 2) + P(0 < Z < 2)$
$= 2 \cdot P(0 < Z < 2)$
$= 2 \cdot (0.4772) = 0.9556$

iv) $P(-2 < Z < 0) = P(0 < Z < 2) = 0.4772)$

v) $P(Z < -1) = \cancel{P(Z > 1)}\ P(1 < Z) = P(Z > 1)$

$= 0.5 - P(0 < Z < 1)$
$0.5 - 0.3413 = \underline{0.1587}$

vi) $P(Z<1) = \cancel{P(Z}$



$= 1 - P(Z>1)$

$1 - [0.5 - P(0<Z<1)]$

$1 - 0.5 + 0.3413$

$= \underline{0.8413}$

2) For the Normal distribution with Mean $= 2 = (\mu)$, $\sigma = 4$, evaluate :-

i) $P(x \geq 5)$

ii) $P(|x|<4)$ iii) $P(|x|>3)$

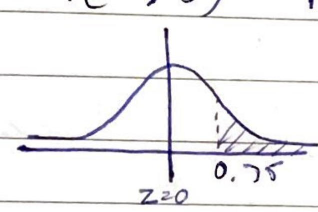
Now $Z = \dfrac{x-\mu}{\sigma}$

$Z = \dfrac{x-2}{4}$ [Normalized value]

i) for $x=5$; $Z = \dfrac{5-2}{4} = \dfrac{3}{4} = 0.75$

$P(x \geq 5) = P(Z>0.75) = 0.5 - P(0<Z<0.75)$

$= 0.5 - 0.27305$

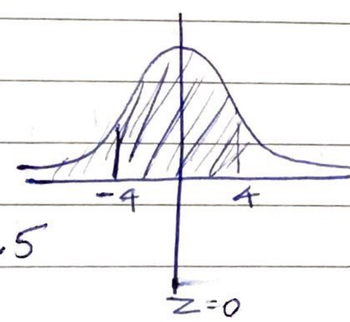


$P(x \geq 5) = \underline{0.22695}$

ii) $P(|x|<4)$

$\downarrow$

$-4 < x < 4$



$Z = \dfrac{4-2}{4} = 0.5$

$Z = \dfrac{-4-2}{4} = \dfrac{-3}{2} = -1.5$

$\Rightarrow P(-1.5<Z<0.5) = P(0<Z<1.5) + P(0<Z<0.5)$

$= 0.4332 + 0.1915 = \underline{0.6247}$

iii) $P(|x|>3) = 1 - P(|x|<3)$

$-3 < x < 3$

$1 - [P(-1.25 < Z < 0.25)]$

$Z = \dfrac{3-2}{4} = \underline{0.25}$

$1 - [P(0<Z<1.25) + P(0<Z<0.25)]$

$= 1 - [0.39405 + 0.0986]$

$Z = \dfrac{-3-2}{4} = -1.25$

$P(|x|>3) = \underline{0.50735}$

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3> For ~~\$~~ μ = Mean ; SD = σ . Evaluate

i) $P[|x-\mu| \leq \sigma)$

i) $Z = \dfrac{x-\mu}{\sigma}$

Now $P(|x-\mu| \leq \sigma) = P\left(\left|\dfrac{x-\mu}{\sigma}\right| \leq 1\right)$

Now $P(|z| \leq 1) = P(-1 \leq z \leq 1)$

$= P(-1 \leq z \leq 0) + P(0 \leq z \leq 1)$ *

* $= P(0 \leq z \leq 1) + P(0 \leq z \leq 1)$

$= 2 . P(0 \leq z \leq 1)$

ii) |||ly $P\left(\left|\dfrac{x-\mu}{\sigma}\right| \leq 2\right)$

$P(|z| \leq 2) = P(-2 \leq z \leq 2)$

$= P(-2 \leq z \leq 0) + P(0 \leq z \leq 2)$

$= 2 . P(0 \leq z \leq 2)$

iii) $P\left(\left|\dfrac{x-\mu}{\sigma}\right| \leq 3\right) = P(|z| \leq 3)$

$= P(-3 \leq z \leq 3) = P(-3 \leq z \leq 0) + P(0 \leq z \leq 3)$

$= 2 . P(0 \leq z \leq 3)$

4> The marks of 1000 students in an examination follows a normal distribution with mean ~~1~~70 and SD = 5. Find No of students whose marks are

i) < 65

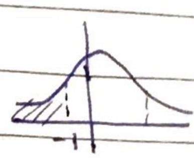
ii) > 75

iii) between 65 and 75

$$Z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$$

For $x = 65$, $z = -1$

$x = 75$, $z = 1$



~~between 65 and 75 ⇒~~

- $P(\text{less than } 65) = P(x<65) = P(z<-1)$

$= P(z>1) = 0.5 - P(0<z<1)$

$= 0.5 - 0.3413 = 0.1587$

No of students scoring less than 65 $= 10^3 \times 0.1587$

$\rightarrow = 1587$

- $P(\text{greater than } 75) = P(x>75) = P(z>1) = 0.1587$

No of students scoring greater than 75 $= 1587 \; (0.1587 \times 10^3)$

- $P(\text{marks between 65 an 75}) = P(65<x<75) = P(-1<z<1)$

$= 2.(0.1587) = 0.3174$

No of students scoring between 65 and 75 $= 3174$

$= (10^3 \times 0.3174)$

Unit IV

20

4) • $\mu = 2040$

$\sigma = 60$

$$z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$$

No of lamps = 3000 lamps.

i) More than 2150 hrs

ii) less than 1950 hrs

iii) between 1920 and 2160 hrs

given that $A(1.5) = 0.4332$

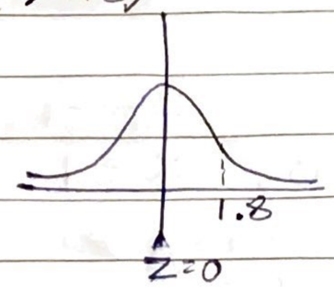
$A(1.8) = 0.4641$

$A(2) = 0.4772$

i) $P(x>2150)$ for $x = 2150$

$$z = \frac{\cancel{x}\, 2150 - 2040}{60} = 18$$

$P(x > 2150) = P(Z > 1.8)$

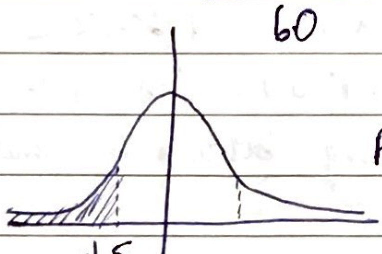


$P(x > 2150) = 0.5 - P(0 < Z < 1.8) = 0.5 - 0.4641$
$= \underline{0.0359}$

No of lamps expected to burn for more than 2150 hrs $= 0.0359 \times 3000 = \underline{107.7} \simeq \underline{108}$

ii) $P(x < 1950) \Rightarrow$

$$Z = \frac{1950 - 2040}{60} = -1.5$$



$P(Z < -1.5) =$
$P(Z > 1.5)$
$= 0.5 - P(0 < Z < 1.5)$
$= 0.5 - 0.4332$
$= \underline{0.0668}$

No of lamps expected to burn for more than 2150hrs
$= 0.0668 \times 3000 \simeq \underline{200}$

iii) $P(1920 < x < 2160)$

for 1920; $Z = \frac{1920 - 2040}{60} = -2$

for 2160; $Z = \frac{2160 - 2040}{60} = +2$

$P(-2 < x < 2) = P(-2 < x < 0) + P(0 < x < 2)$
$= 2 \cdot P(0 < x < 2)$
$= 2 \cdot 0.4772 =$

4) In a Normal distribution, 31% are under 45 & 8% are over 64. Find the mean & SD Given that

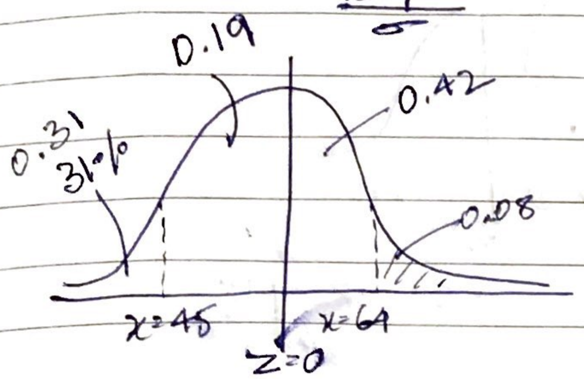
$A(0.5) = 0.19$

$A(1.4) = 0.42$

where $A(z)$ is the area under the Normal curve from 0 to z

$A(z) = ?$

$$Z = \frac{x-\mu}{\sigma}$$



Value of z corresponding to probability 0.19 = ~~0.19~~ 0.5

$$Z = \cancel{0.19} = \frac{0.5-\mu}{\sigma}$$

Since $x=45$ lies on left of $z=0$, we assign a negative sign. Therefore for $x=45$ corresponding value is

for $x=45$

$$-0.5 = \frac{45-\mu}{\sigma} \Rightarrow 45-\mu = (-0.5)\sigma$$

$$\mu-(0.5)\sigma = 45 \quad \text{—①}$$

The value of z corresponding to probability 0.42 is 1.4

$x=64$ lies on right of line $z=0$, we assign a positive value for z. Therefore for $x=64$,

$$Z = \frac{64-\mu}{\sigma} = 1.4$$

$$1.4 = \frac{64-\mu}{\sigma}$$

$$\mu+1.4\sigma = 64 \quad \text{—②}$$

from ① & ②, $\mu = 50$, $\sigma = 10$

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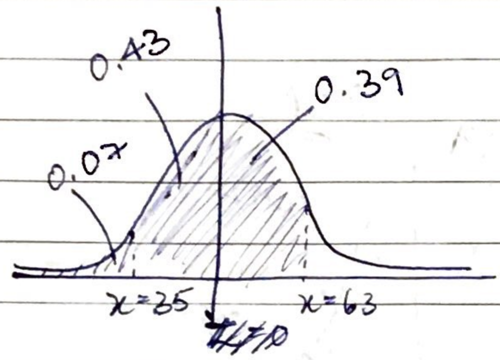
5) In a Normal distribution 7% (0.07) are under 35 and (0.89) 89% are under 63. Find the mean and SD given that $A(1.23) = 0.39$
$A(1.48) = 0.43$

In the original Notation



$$z = \frac{x-\mu}{\sigma}$$

Value of z corresponding to probability 0.39 is 1.23

for $x=35$ = ~~Z = 45 - μ~~

Since $x=35$ lies on left of line $z=0$, we get a ~~-ve sign~~

for $x=35$,

$$-1.48 = \frac{35-\mu}{\sigma}$$

$$\mu - 1.48\sigma = 35 \quad \text{--- (1)}$$

III^ly

$$z = 1.23$$

$$1.23 = \frac{63-\mu}{\sigma}$$

$$\mu + 1.23\sigma = 63 \quad \text{--- (2)}$$

Solving (1) & (2),

$$\mu = 50.49, \quad \sigma = 10.33$$

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6) Steel rods manufactured to be 3cms in diameter but they are acceptable if they are acceptable if they are inside the limits 2.299cm and 3.01cm. Its observed that 5% are rejected as oversized and 5% are rejected as undersized. Assuming that the diameters are normally distributed, find the SD of distribution.

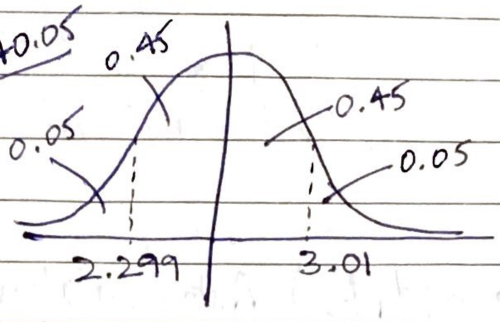
$\mu = 3$

from the table $1.6 + 0.05$

$A(1.65) = 0.45$

$x = 3.01$

$$1.65 = \frac{3.01 - 3}{\sigma}$$



The value of z corresponding to 0.45 is 1.65 & value of $x = 3.01$ is on right of the line $z = 0 \Rightarrow$ +ve sign for z

$$1.65 = \frac{3.01 - 3}{\sigma} \Rightarrow \sigma = \frac{0.01}{1.65} = 0.00606$$

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Stationary processes:-

A stochastic process is said to be stationary process if probability distribution fn and sum of its expected value are invarient under time.

A Stochastic process $x(t)$ is said to be WSS if it satisfies the following conditions:-

i) $\mu_x(t) = E\{X(t)\} =$ constant

ii) $R(t_1, t_2)$ is a function of $t_1 - t_2$

- $R(t_1, t_2)$ Autocorrelation fn is a function of $t_1 - t_2$
  $f(t_1 - t_2)$

Wide sense stationary process or weak sense stationary process (simply WSS):-

It is a stochastic process $x(t)$ is said to be WSS, if it satisfies the following conditions.

i) $\mu_x(t) = E\{x(t)\} =$ constant

ii) $R(t_1, t_2)$ is a function of $t_1 - t_2$.

* A SSS is always a WSS but not the converse.

1) A random process $X(t)$ is represented by the ensemble i.e $\{-K, -2K, -3K, K, 2K, 3K\}$, $K > 0$

↳ collection of random variables

; corresponding to the outcomes of an event which are equally probable. S.T the outcome is SSS.

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A) The probability distribution function is not containing the time variable 't'. Therefore the PDF is invariant under the time 't'.

x	$-K$	$-2K$	$-3K$	K	$2K$	$3K$
$p(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$\mu_x(t) = E(X(t)) = \sum x.p(x)$$
$$\mu_x(t) = 1/6[0] = \underline{\underline{0}} \Rightarrow [\text{constant}]$$

Now, $R(t_1, t_2) = E[X(t_1)\,X(t_2)]$
$$= \sum x(t_1)x(t_2)\,p(x)$$
$$= 1/6\left[K^2 + (2K)^2 + (3K)^2 + (-K)^2 + (-2K)^2 + (-3K)^2\right]$$
$$= 1/6\left[2K^2 + 8K^2 + 18K^2\right] = \frac{28K^2}{6} \Rightarrow [\text{constant}]$$

a function of $t_1 - t_2$

So $\mu_x(t) = E\{X(t)\} = 0 \rightarrow$ constant.
$R(t_1, t_2)$ is a function of $t_1 - t_2$
$\therefore$ Given fn is a $\underline{\underline{SSS}}$.

- Find Autocorrelation & correlation co-efficient of $X(t)$ with a random process and all the outcomes are equally probable. Also check whether the process is SSS or WSS.

outcome	1	2	3	4	5	6
$X(t)$	-4	-3	1	2	$-t$	t
$P(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$\mu_x(t) = E(X(t)) = \sum x.p(x)$$
$$\mu_x(t) = 1/6[-4] = \underline{\underline{-2/3}} \quad (\text{constant})$$

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$$R(t_1, t_2) = E[X(t_1)\,X(t_2)]$$
$$= \tfrac{1}{6}\left[(-4)(-4) + (-3)(-3) + (1)(1) + (2)(2) + (-t_1)(-t_2) + (t_1)(t_2)\right]$$

$$= \tfrac{1}{6}\left[16 + 9 + 1 + 4 + t_1 t_2 + t_1 t_2\right]$$

$$= \tfrac{1}{6}\left[30 + 2t_1 t_2\right]$$

* $\therefore R(t_1, t_2)$ is not a fn. of $t_1 - t_2$. This indicates its neither SSS or WSS.

$C(t_1, t_2)$ = Correlation coefficient $= R(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$

$$\mu_x(t_1) = -2/3 \qquad \mu_x(t_2) = -2/3$$

$$C(t_1, t_2) = \tfrac{1}{6}(30 + 2t_1 t_2) - \frac{4}{9}$$

$$= 5 - \frac{4}{9} + \frac{2}{3} t_1 t_2$$

$$= \frac{41}{9} + \frac{2}{3} t_1 t_2$$

$$r(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1)\,C(t_2, t_2)}}$$ = Correlation coefficient.

$$r(t_1, t_2) = \frac{\frac{41}{9} + \frac{2}{3} t_1 t_2}{\sqrt{\left(\frac{41}{9} + \frac{2}{3} t_1^2\right)\left(\frac{41}{9} + \frac{2}{3} t_2^2\right)}}$$

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3) Find the Autocorrelation for $R(t_1, t_2)$ of the stochastic process defined by $X(t) = A\cos(10t + \theta)$ where A is a random variable with mean 0 and variance 1 and θ is uniformly distributed in the interval ~~0 to 1~~ $[-\pi, \pi]$

Ans

$E(A) = 0$

$\text{Variance} = E(A^2) - [E(A)]^2 = 1 \quad \Rightarrow E(A^2) = 1$

$P(x) = \frac{1}{B-A} \; \forall \; A < x < B$ else 0

$P(\theta) = \frac{1}{\pi - (-\pi)} = \frac{1}{2\pi}$

$R(t_1, t_2) = E[X(t_1)X(t_2)]$

$= E[A\cos(10t_1 + \theta) + A\cos(10t_2 + \theta)]$

$= E[A^2 \cdot \frac{1}{2}\{\cos(10t_1 + \theta + 10t_2 + \theta) + \cos(10t_1 + \theta - 10t_2 - \theta)$

$\Rightarrow \frac{E(A^2)}{2}[\cos(10(t_1 + t_2) + 2\theta) + \cos(10(t_1 - t_2))]$

$E(A^2) = 1$

$\Rightarrow \frac{1}{2} E[\cos(10(t_1 + t_2) + 2\theta)] + \frac{1}{2} E[\cos(t_1 - t_2)]$

$= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos[(10(t_1 + t_2) + 2\theta] d\theta + \frac{1}{2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t_1 - t_2) d\theta$

$= \frac{-1}{4\pi}\left[\frac{\sin(10(t_1 + t_2) + 2\theta)}{2}\right]_{-\pi}^{\pi} + \frac{1}{4\pi}\cos(t_1 - t_2)[\theta]_{-\pi}^{\pi}$

$= \frac{-1}{8\pi}[\sin(10t_1 + 10t_2) + 2\pi - \sin(10t_1 + 10t_2) - 2\pi]$

$+ \frac{1}{2\pi}\cos(t_1 - t_2) = \frac{\cos(t_1 - t_2)}{2\pi}$

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- Find the autocorrelation for $R(t_1, t_2)$ of a stochastic process defined by a $A\cos(\omega t+\theta)$ where random variables A and θ are independent & θ is uniformly distributed in the interval $-\pi$ to π

$$X(t) = A\cos(\omega t+\theta) \quad [-\pi, \pi]$$

Since θ is uniformly distributed in the interval $-\pi$ to π,

$$P(\theta) = \frac{1}{2\pi}, \quad -\pi < \theta < \pi$$

$$R(t_1, t_2) = E[X(t_1)X(t_2)] = E[A\cos(\omega t_1+\theta)A\cos(\omega t_2+\theta)]$$

$$= E(A^2)E[\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)]$$

$$= E(A^2)E\left[\frac{1}{2}\left(\cos(\omega t_1+\theta+\omega t_2+\theta)+\cos(\omega t_1+\cancel{\theta}-\omega t_2-\cancel{\theta})\right)\right]$$

$$= \frac{E(A^2)}{2}E\left[\cos\{\omega(t_1+t_2)+2\theta\}+\cos\{\omega(t_1-t_2)\}\right]$$

$$= \frac{E(A^2)}{2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\{\omega(t_1+t_2)+2\theta\}d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\{\omega(t_1-t_2)\}d\theta\right]$$

$$= \frac{E(A^2)}{4\pi}\left[\left.\frac{\sin\{\omega(t_1+t_2)+2\theta\}}{2}\right|_{-\pi}^{\pi} + \cos\{\omega(t_1-t_2)\}(\theta)_{-\pi}^{\pi}\right]$$

$$\frac{E(A^2)}{8\pi}\left[\sin\{\omega(t_1+t_2)+2\pi\}-\sin\{\omega(t_1-t_2)-2\pi\}\right]+\frac{E(A^2)}{4\pi}\cos(\omega(t_1-t_2$$

$$= \frac{E(A^2)}{8\pi}\left[\sin\{\omega(t_1+t_2)\}-\sin\{-\{2\pi-\omega(t_1+t_2)\}\}\right]+\frac{E(A^2)}{4\pi}\cos\omega(t_1-t_2)$$

$\sin(-\theta) = -\sin\theta$
$\sin(2\pi-\theta) = -\sin\theta$

$$= \frac{E(A^2)}{8\pi}$$

$$= \frac{E(A^2)}{8\pi}\left[\sin(\omega t_1+\omega t_2)-\sin(\omega t_1+\omega t_2)\right]+\frac{E(A^2)}{2}\cos(\omega(t_1-t_2))$$

$$= \frac{E(A^2)}{2}\cos\{\omega(t_1-t_2)\}$$

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For SSS stochastic time

$\mu = \text{const}$ $\quad$ $ACF = f(t_1 - t_2)$

+ $X(t)$ not function of t

WSS

$\mu = \text{const}$ $\quad$ $R(t_1, t_2) = f(t_1 - t_2)$

- $X = A\cos\omega t + B\sin\omega t$

where A and B are uncorrelated random variables, each with mean zero & variance 1 & ~~ω~~ ω = +ve const

WSS

$E(A) = 0$, variance $= E(A^2) - [E(A)]^2$

$E(A^2) = 1$, $E(B) = 0$, $E(B^2) = 1$

$\mu_x(t) = E\{X(t)\} =$

$E\{A\cos\omega t + B\sin\omega t\}$

$= E(A)\,E(\cos\omega t) + E(B)\,E(\sin\omega t)$

$\mu_x(t) = \underline{\underline{0}}$

$R(t_1, t_2) = E[X(t_1)X(t_2)] = E[\{A\cos\omega t_1 + B\sin\omega t_1\}\{A\cos\omega t_2 + B\sin\omega t_2\}]$

$= E[A^2(\cos\omega t_1 \cos\omega t_2) + AB\cos\omega t_1 \sin\omega t_2 + AB\sin\omega t_1 \cos\omega t_2 + B^2\sin\omega t_1 \sin\omega t_2\}$

$= E(A^2)(\cos\omega t_1 \cos\omega t_2) + E(B^2)(\sin\omega t_1 \sin\omega t_2)\} + E(AB)\{\cos\omega t_1 \sin\omega t_2 + \sin\omega t_1 \cos\omega t_2\}$

$= \{\cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \sin\omega t_2\} + E(A)E(B)(\cos\omega t_1 \sin\omega t_2 + \sin\omega t_1 \cos\omega t_2)$

$= \cos(\omega(t_1 - t_2))$

$E(AB) = E(A) \cdot E(B)$

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- A random process $x(t)$ is described by ensemble $\{1, \sin t, -\sin t, \cos t, -\cos t, -1\}$

S.T that $x(t)$ is WSS but not SSS

Soln) Given ensemble is not invarient under time to $X(t)$

$X(t)$	1	$\sin t$	$-\sin t$	$\cos t$	$-\cos t$	-1
$P(X)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$\mu_x(t) = E\{X(t)\} = \frac{1}{6}\left[1+\sin t-\sin t+\cos t-\cos t-1\right]$$

$$= 0 \quad \text{constant.}$$

$$R(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \Sigma\, x(t_1)x(t_2)P(x)$$

$$= \frac{1}{6}\left[1+\sin t_1\sin t_2+\sin t_1\sin t_2+\cos t_1\cos t_2+\cos t_1\cos t_2+1\right.$$

$$= \frac{2}{6}\left[1+\sin t_1\sin t_2+\cos t_1\cos t_2\right]$$

$$R(t_1,t_2) = \frac{1}{3}\left[1+\cos(t_1-t_2)\right]$$

$\hookrightarrow \mu = 0 \rightarrow$ const & $R(t_1,t_2)$ is a fn of t_1-t_2.
Therefore the given process is a WSS.

- A Stochastic process with its ensemble function are assumed to have equal probabilities are given by

$X_1(t) = 3$
$X_2(t) = 3\sin t$
$X_3(t) = -3\sin t$
$X_4(t) = 3\cos t$
$X_5(t) = -3\cos t$
$X_6(t) = -3$

S.T process is WSS but not SSS.

$X(t)$	~~3sint~~ 3	3sint	-3sint	3cost	-3cost	-3
$P(X)$	1/6	1/6	1/6	1/6	1/6	1/6

$\mu_x(t) = E\{X(t)\} = 0$

$R(t_1, t_2) = E\{X\{t_1\} X\{t_2\}\} = \Sigma x(t_1) x(t_2) P(x)$

$X(t)$ is not invarient under time t. So process might not be a SSS. Since $\mu_x(t) = 0$ [const] & $R(t_1, t_2)$ is a function of $t_1 - t_2$, therefore its a WSS

- Consider a stochastic process defines on finite sample space with 3 sample points. It description is provided by the specification of 3 sample functions

$X(t, \lambda_1) = 3$, $X(t, \lambda_2) = 3\cos t$

$X(t, \lambda_3) = 3\sin t$ $\qquad P(\lambda_1) = P(\lambda_2) = P(\lambda_3) = \frac{1}{3}$

Decide whether process is SSS or WSS

$x(t)$	3	3cost	3sint
$P(x)$	1/3	1/3	1/3

$\mu_x(t) = E\{X(t)\} = [3 + 3(\cos t + \sin t)] 1/3$

$\Rightarrow \mu_x(t) \rightarrow$ neither SSS or WSS

constant mean.

We need not find even $R(t_1, t_2)$

Since $\mu_x(t)$ is a function of time, the process is neither SSS or WSS.

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Queueing Theory :-

Introduction: Waiting for a service is part of our daily life. Waiting is observed in the following situations:

1> There may be more demand for service than the facility available for service so that there is an excess of waiting time.
2> Shortage of servers
3> Limited space to the amounts of service that can be provided.

In the above situations, the problem of interest is either to schedule arrival / to provide facility or both, so as to achieve an optimal balance between the cost associated with waiting time and idle time in order to obtain max profit.

Queueing Theory attempts to solve through math Model. It concerns with statistical description of behaviours of arrivals i.e PD of the number of customers in the queue.
PD of waiting time of a customer, a queue system can be described as
i> Customers arriving for service
2> Waiting time for service.

The queuing system has the following basic characteristics:-

1> Arrival pattern of customers
2> Service ——"—— Servers
3> Queue discipline
4> System Capacity.

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1) Arrival pattern of customers through a Queing System is often measured in terms of avg no of arrivals for some unit of time / avg time between successive arrivals.

These two are not treated as const but a random variable.

*. usually arrival pattern follows poisson distribution with mean λ, and mean-inter-arrival time, follows exponential distribution with Mean $\frac{1}{\lambda}$.

2) Service pattern of servers :-

Service pattern can be described by a service rate or as a time required to service a customer.

In queing system, service pattern follows poisson distribution with service rate μ & inter service time has an exponential distribution with mean $1/\mu$.

3) Queue discipline :- This is the manner by which customers are selected for service when a Queue has formed.

Following are Queue disciplines :-

FIFO or FCFS

↳ This is common discipline that can be seen in everyday life.

LIFO or LCFS

↳ which is applicable to many inventory systems.

SIRO

↳ Selection for service in random order

PIR (Priority in Selection) where the customers are given priorities upon entering a system.

4) System Capacity :- Max No of customers in a system can be finite or infinite. In some situations, Only limited no of customers are allowed into system, no further customers are allowed to enter the system unless the NO becomes less than limiting value.

Symbolic representation of queuing model

Queuing model — KENDALL's model

↳ a/b/c : d/e

where

a = Type of distribution of No of arrivals / unit time (Inter arrival distribution.

b = Type of distribution of Service Time or departure distribution

c = No of Servers

d = Capacity of System

e = Queue discipline.

4. * following nota^{n}s will be used in the queing theory.

n = no of customers in the system.

$p(n)$ = probability of having n customers in the system.

λ = Mean arrival rate

μ = Mean service rate

$\rho = \frac{\lambda}{\mu}$ = probability that a service channel is busy or it is also known as traffic intensity

$$\rho < 1 \quad \text{i.e.,} \quad \frac{\lambda}{\mu} < 1 \Rightarrow \lambda < \mu.$$

if $\lambda > \mu$ the queue length becomes infinite.

L_s = expected no. of customers in the system

L_q = " " " " " queue

L_n = " " " " " non-empty queue

W_s = expected waiting time in the system

W_q = " " " " queue

~~W_n = " " " " non-empty queue~~

P_0 = probability that there are no customers in the queue.

$P(n>k)$ = " " no. of customers exceeds k

$P(n \geq k)$ = " " no. of customers are k or more

$P(W>t)$ = " " waiting time of customers in the queue exceeds t.

$L_s = \frac{\lambda}{\mu-\lambda}$	$W_s = \frac{1}{\mu-\lambda}$	$P(n>k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$
$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$	$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$	$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^{k}$
$L_n = \frac{\mu}{\mu-\lambda}$	$P_0 = 1-\rho = 1-\frac{\lambda}{\mu}$	$P(W>t) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$

Relation between L_s, L_q, W_s & W_q.

$$L_s = \lambda W_s \quad , \quad L_q = \lambda W_q .$$

The ab. formulae is known as LITTLE'S formulae

Q.) At ~~an~~ one-man barbar shop it takes on the avg. ½ hr for hair cut and customers arrive at an avg. rate of ⅓rd every 45 mins if the poisson exponential model is assumed at any point in time : On the avg.

(i) How many customers will be there in a shop

(ii) What time will a customer spend in a shop.

(iii) Of this time what part is ~~be~~ spent waiting to be ~~spent~~ served.

(iv) What portion of the day barbar will be idle.

(v) What is the prob. that the customer has to wait for 30 mins.

Soln →

Mean Service rate $= \dfrac{1}{\mu} = \dfrac{1}{2}$

$\mu = 2$ customers per hr.

Mean inter arrival time $= \dfrac{1}{\lambda} = 45$ mins

$$\frac{1}{\lambda} = \frac{45}{60}$$

$$\lambda = \frac{60}{45} = \frac{4}{3}$$

$$\rho = \frac{\lambda}{\mu} = \frac{4}{3} \times \frac{1}{2}$$

$$= 0.666$$

the no. of customers in the shop =
no. of customers waiting in the system = L_s

$$L_s = \frac{4/3}{2 - 4/3} = \underline{\underline{2}} \qquad \frac{\frac{4}{3}}{\frac{6-4}{3}} \quad \frac{4}{2} = 2$$

(b) $W_s = \frac{L_s}{\lambda} = \frac{2}{4/3} = \frac{6}{4} = \frac{3}{2} = \underline{1.5}$.

(c) $W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \cancel{\frac{4/3}{1/2(1/2 - 4/3)}} \quad \frac{4/3}{2(2-4/3)} = \frac{4/3}{2\left(\frac{6-4}{3}\right)} = \frac{4/3}{4/3} = \underline{\underline{1}}$.

$= \underline{\underline{1}}$. $\qquad = \cancel{\frac{4/3}{1/2\left(\frac{3-8}{6}\right)}} =$

(d) $P_0 = 1 - \rho = 0.334$.

(e) $P(W > t) = P(W > 0.5)$

$$= \frac{4}{3} \times \frac{1}{2} e^{-\left(2 - \frac{4}{3}\right)0.5}$$

$$= \frac{4}{6} e^{-1/3}$$

$$= \underline{\underline{0.477}}$$

Q) The customers arrive at a one-man barbar shop according to poisson process with μ ~~an~~ inter arrival time of 20 min. customers spent an avg of 15 mins in the barbar chair. If an ~~one~~ hour is used as a unit of time.

(i) what is the prob. that a customer need not wait for hair cut

(ii) what is the expected no. of customers in the barbar shop & in queue.

(iii) How much time can a customer expect to spend in a barbar shop

(iv) Find the avg. time the customer spends in the shop.

(v) The owner of the shop will provide another chair & hire another barbar. when customer's avg. time in shop exceeds 1.25 hrs. By how much should avg. rate of arrival increase in order to justify second barber.

(vi) Estimate the fracⁿ of the day that the barbar will be idle

(vii) what is the prob. that there will be more than 6 customers

(viii) " " " " 6 or more customers waiting for service

*(ix) Estimate the % of customers who have to wait prior to getting into barbar's chair

(x) what is the prob. that the waiting time in a system & in a queue is greater than 12 mins.

Soln →

Given $\frac{1}{\lambda} = \frac{20}{60} = \frac{1}{3}$ $\Rightarrow \lambda = 3$ customers / hr.

Service rate $\frac{1}{\mu} = \frac{15}{60} = \frac{1}{4}$.

$\mu = 4/\text{hr}$.

(ii) $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{4} = \frac{1}{4}$.

(iii) No. of customers in the shop $= L_s$

$= \frac{\lambda}{\mu - \lambda} = \frac{3}{4-3}$

$= 3$

(iv) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{9}{4} \simeq 2$.

W_s = Customers wait in the shop.

$= \frac{1}{\mu - \lambda} = 1$

*b: The type of distribution of the service time or departure distribution.

c: no. of servers

d: capacity of system

e: Queue discipline

*3.

$$\rightarrow N_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{3}{4(4-3)} = \frac{3}{4}$$

in order to justify the second barbar we need to find the expected no. of customers waiting for the service; so let

N_s = no. of customers waiting.

$$\frac{1}{\mu-\lambda_1} > 1.25$$

$$\frac{1}{1.25} > \mu - \lambda_1$$

$$\frac{1}{1.25} > 4 - \lambda_1$$

$$\lambda_1 > 4 - \frac{1}{1.25}$$

$$\lambda_1 > 3.2 \approx 3.$$

(vi) $P_0 = ?$, $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{4} = 1/4$

(vii) Probability that there will be more than 6 customers in the queue

$$P(x > 6) = \left(\frac{\lambda}{\mu}\right)^7$$

(viii) $P(x \geq 6) = \left(\frac{\lambda}{\mu}\right)^6$ (Probability of 6 or more than 6)

(v) $P(W > 12 \text{ min}) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$

$$P\left(W > \frac{12}{60}\right) = \frac{3}{4} e^{-(4-3)\left(\frac{12}{60}\right)}$$

Q) A machine repairing shop gets on an avg. 16 machines per day (of 8 hrs day) for repair and the arrival pattern poisson. at the moment there is no repair man available in the shop the shop owner has 2 applicants A & B for the job of repairman. Both A & B claim that the service times are exponentially distributed with means 20 & 15 mins resp. They demand salaries ₹500 & ₹600 ~~resp~~ /per resp. The loss ~~last~~ time cost ₹50/hr/machine. Assuming that the claims of applicants to be true, which one should be employed.

Soln →

$A = 20$ & $B = 15$

$\frac{1}{\mu_A} = \frac{20}{60}$ $\quad\quad$ $\frac{1}{\mu_B} = \frac{15}{60}$

$\mu_A = 3/\text{hr}$ $\quad\quad$ $\mu_B = 4/\text{hr}$

Average no. of machine that A will repair $= L_s = \frac{\lambda}{\mu_A - \lambda}$

$= \frac{2}{3-2} = 2.$

previous charges that company bearing per day = $2 \times 8 \times 50$
= 800.

charges for the repairer A = A = 800 + 500
= 1300

Avg no. of machine that B will repair/chr
$= \frac{2}{4-2} = 1$

previous charges that company
bearing per day = $1 \times 8 \times 50$
= 400

charge for the repairer B = 600 + 400
= 1000.

Repairer B is must be employed.

M/M/1 : ∞/FIFO.

The ab. M/M/1 : ∞/FIFO represents the queue system where M represents the avg. no. of customers in the system which follows the poisson distribution if inter arrival time is given b/w the customers it follows exponential distribution.

M - avg service time, which follows an exponential distribution

1 - single server.

∞ - queue length.

FIFO - queue discipline which is first in first out or (FCFS)

Time average : of random process is defined as

$$A[x(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

Ergodic process: if for a stationary process all the time averages are equal to statistical averages, then the process is called an ergodic process.

A fair ~~coin is tossed 3 times. Let X denote 0 or~~

MARKOV CHAINS:

Probability vector :-

consider vector, $V = \{v_1, v_2, \cdots v_m\}$

all $v_i \geq 0$, $i = 1, 2, \cdots m$

$\sum_{i=1}^{m} v_i = 1.$

V → A probability vector if

Example :- $\{1/3, 1/3, 1/3\}$. are all probability vectors

the vectors $\{1, 0, -1, 0, 1\}$ & $\{1/2, 0, -1/2, 1\}$. are not probability vectors even though sum of the components is 1, but a -ve component is present in both the vectors.

Stochastic matrix.

Let P = Square matrix of order m,

if each row of P is a probability vector, then P is said to be a stochastic matrix

Eg: consider

$$P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Regular stochastic matrix:

A stochastic matrix P is said to be a regular stochastic matrix if for some power of T; if all the elements are non-zero, then we say that P is regular stochastic matrix.

Fixed probability vector:

A vector $V = \{v_1, v_2 \ldots v_m\}$ is said to be a fixed probability vector of the stochastic matrix P if

$VP = V$

A stochastic matrix is said to be irreducable if it is a regular stochastic matrix.

Q) Show that the matrix $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is regular.

$$P^2 = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Elements in P^2 are non-zero. ∴ It is regular.

Q) Find fixed PV of $P = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Soln

$V = \{v_1, v_2\}$

$v_1 + v_2 = 1 \longrightarrow ①$

$VP = V$

$$[v_1 \quad v_2] \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [v_1 \quad v_2]$$

$$\begin{bmatrix} 3/4 v_1 + 1/4 v_2 & v_1/4 + v_2/2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

Equating elements on both sides.

$$\frac{3v_1}{4} + \frac{v_2}{4} = v_1$$

$$\frac{3v_1 + v_2}{4} = v_1 \Rightarrow 3v_1 + v_2 = 4v_1 \longrightarrow ②$$

$$\frac{v_1}{4} + \frac{v_2}{2} = v_2 \Rightarrow v_1 + 2v_2 = 4v_2 \longrightarrow ③$$

from ① $v_2 = 1 - v_1$

using in ②

$3v_1 + 2(1-v_1) = 4v_1$

$3v_1 + 2 - 2v_1 = 4v_1$

$3v_1 - 2v_1 - 4v_1 + 2 = 0$

$-3v_1 = -2 \Rightarrow v_1 = 2/3$

$v_2 = 1 - \frac{2}{3}$

$= 1/3$

$v = \left[2/3 \quad 1/3 \right]$

Q) Find the fixed probability matrix of $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$.

Soln

If v is a fixed PV then $vP = v$

$v = \{ v_1 \quad v_2 \quad v_3 \}$

$v_1 + v_2 + v_3 = 1$

$$[v_1 \quad v_2 \quad v_3] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [v_1 \quad v_2 \quad v_3]$$

$$\left[\frac{v_2}{6} \quad v_1 + \frac{v_2}{2} + \frac{2v_3}{3} \quad \frac{v_2}{3} + \frac{v_3}{3} \right] = [v_1 \quad v_2 \quad v_3]$$

$\frac{v_2}{6} = v_1$, $\quad v_1 + \frac{v_2}{2} + \frac{2v_3}{3} = v_2$, $\quad \frac{v_2}{3} + \frac{v_3}{3} = v_3$

$v_2 = 6v_1$, $\quad v_1 + 3v_1 + \frac{2v_3}{3} = 6v_1$, $\quad 2v_1 + \frac{v_3}{3} = v_3 \Rightarrow 6v_1 + v_3 - 3v_3 = 0$

$6v_1 - 2v_3 = 0$

$\Rightarrow v_3 = 3v_1$

$$v_1 + v_2 + v_3 = 1$$

$$v_1 + 6v_1 + 3v_1 = 1$$

$$10v_1 = 1 \Rightarrow v_1 = 1/10$$

$$\Rightarrow v_2 = \frac{6}{10} \Rightarrow v_3 = \frac{3}{10}.$$

$$v = \left[\frac{1}{10} \quad \frac{6}{10} \quad \frac{3}{10}\right]$$

Q.) Find FPM for PV, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$.

Soln

$$v = \{v_1 \quad v_2 \quad v_3\} \Rightarrow v_1 + v_2 + v_3 = 1 \longrightarrow (1)$$

$$vP = v$$

$$[v_1 \quad v_2 \quad v_3]\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} = [v_1 \quad v_2 \quad v_3].$$

$$\left[\frac{v_2}{2} + \frac{v_3}{2} \quad v_1 + \frac{v_3}{4} \quad \frac{v_2}{2} + \frac{v_3}{4}\right] = [v_1 \quad v_2 \quad v_3]$$

$$\frac{v_2}{2} + \frac{v_3}{2} = v_1 \qquad v_1 + \frac{v_3}{4} = v_2 \qquad \frac{v_2}{2} + \frac{v_3}{4} = v_3$$

$$\Rightarrow v_2 + v_3 = 2v_1 - (2), \quad 4v_1 + v_3 = 4v_2 - (3), \quad 2v_2 + v_3 = 4v_3 - (4)$$

$$3v_3 = 2v_2 \Rightarrow v_3 = 2/3\, v_2.$$

using (2)

$$2v_1 = v_2 + \frac{2}{3}v_2 = \frac{5v_2}{3} \Rightarrow v_1 = \frac{5v_2}{6}$$

Subs v_1 & v_3 in ①

$$\frac{5v_2}{6} + v_2 + \frac{2}{3}v_2 = 1$$

$$v_2 = \frac{6}{15}.$$

$$v_1 = \frac{5}{15}.$$

$$v_3 = \frac{4}{15}.$$

$$V = \begin{bmatrix} \frac{5}{15} & \frac{6}{15} & \frac{4}{15} \end{bmatrix}$$

Q) Find ~~FPV~~ FPM of the $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{bmatrix}$.

soln

$V = \{v_1 \quad v_2 \quad v_3\} \Rightarrow v_1 + v_2 = 1$

$VP = V.$

$$[v_1 \quad v_2] \begin{bmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{bmatrix} = [v_1 \quad v_2]$$

$$[0.7v_1 + 0.8v_2 \quad 0.3v_1 + 0.2v_2] = [v_1 \quad v_2]$$

$0.7v_1 + 0.8v_2 = v_1 \qquad 0.3v_1 + 0.2v_2 = v_2$

$0.8v_2 = 0.3v_1$

$$v_2 = \frac{0.3}{0.8}v_1$$

$v_1 + \frac{0.3}{0.8}v_1 = 1 \Rightarrow v_1 + \frac{3}{8}v_1 = 1$

$\frac{11v_1}{8} = 1 \Rightarrow v_1 = \frac{8}{11}$

$v_2 = \frac{3}{\cancel{8}} \times \frac{\cancel{8}}{11} = \frac{3}{11}$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \frac{8}{11} & \frac{3}{11} \end{bmatrix}$$

$\Rightarrow$

Markov chain:

A stochastic process of generation of probability distribution depends only on present process is markov process. If the state place is discrete we say that the cell is discrete state process or chain, then markov process is called markov chain.

Let x_1, x_2 & so on sequence of trials satisfies foll. properties.

(i) Each outcome of an trial depends on outcome of preceding trial. The probability P_{ij} is allocated with every pair of state (a_i, a_j) such that a_j occurs immediately after a_i occurs. Such a stochastic process is called markov chain.

The probability a_{ij} which are non-zero real no. are called transition probabilities and the formed matrix is called transition probability matrix and is given by,

$$P = [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & & & \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix}$$

The rows in matrix P satisfies,

(i) $0 \le P_{ij} \le 1$

(ii) $\sum_{j=1}^{m} P_{ij} = 1$

$i = 1, 2, 3, \ldots m$.

The ab. 2 properties are the requirement for a stochastic matrix. ∴ the transition probability matrix is a stochastic matrix.

Higher transition probabilities:

The entry P_{ij} in the transition matrix represents the probability that the system changes from the state A_i to the state A_j. The prob. that the system changes from a_i to a_j in n steps is $P_{ij}^{(n)}$. The matrix found by these probabilities is given by,

$[p^{(n)}] = [P_{ij}^{(n)}] = P^n$. i.e., n-step ~~conult~~ transition matrix is the n^{th} power of P.

Let $P = (P_i) = (P_1, P_2 \ldots P_m)$ represents the prob. distribution of a markov chain after some time, then pP, pP^2, pP^3 .... pP^n and so on resp. the prob. dis n of the system after one step, after 2 steps and so on after n steps ~~stps~~ resp. ~~If~~

If ~~$p^{(0)}$~~ $P^{(0)} = [p_1^{(0)}, p_2^{(0)} \ldots p_m^{(0)}]$ denote the initial prob. distribution at the start of the process then, $p^n = [p_1^{(n)}, p_2^{(n)} \ldots p_m^{(n)}]$ denote the step prob. distribu^n @ the end of n steps.

$p^{(1)} = p^{(0)}P$ represents the prob. distri^n of the markov chain after one step

$p^{(2)} = p^{(1)}P$ which represents the probability dis. ~~of the system~~ after 2 steps and ~~son~~ so on.

Stationary distribu^n of a regular markov chain. If T is a transi^n matrix of markov chain then the sequence of n step transi^n matrices. $P^2, P^3 \ldots P^n$ approaches a matrix V whose rows are each unique fixed prob. vector.

i.e., $p^{(n)} = [p_1^{(n)}, p_2^{(n)} \ldots p_m^{(n)}]$.

Therefore as $n \to \infty$,

$$p_i^{(n)} = v_i, \quad i = 1, 2, 3 \ldots m,$$

then this ~~st~~ distribu^n is called stationary distribu^n of the markov chain.

Q) A student's study habits are as follows if he studies one night, he is 70% sure not to study the ~~next the~~ next night, on the other hand if he doesn't study one night, he is 60% sure not to study next night. In the long run how often does he study

Soln → $S = \{s, ns\}$

$$P = \begin{array}{c c} & \begin{array}{cc} s & ns \end{array} \\ \begin{array}{c} s \\ ns \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

In order to find in the ~~run~~ long run how often does the student study it is enough to find the unique fixed prob vector such that, $VP = V$ & $v_1 + v_2 = 1$

$$[v_1 \ v_2]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [v_1 \ v_2] \Rightarrow [0.3v_1 + 0.4v_2 \quad 0.7v_1 + 0.6v_2] = [v_1 \ v_2]$$

equating corresponding elements

$0.3v_1 + 0.4v_2 = v_1 \rightarrow ②$ $\qquad$ $0.7v_1 + 0.6v_2 = v_2 \rightarrow ③$

from ① $v_1 = 1 - v_2$

using ②.

$0.3(1-v_2) + 0.4 v_2 - (1-v_2) = 0 \Rightarrow v_2 = \frac{7}{11}$ $\quad v_1 = \frac{4}{11}$.

$v = \left[\frac{4}{11} \quad \frac{7}{11}\right]$. , $\left(\frac{4}{11} \times 100\right) = 36.36\%$. he will study.

63.63%. he will not study

8) A salesman's territory consists of 3 cities A, B & C. He never sells in the same city on successive days. If he sells in city A then the next day he sells in city B. However if he sells in either ~~A &~~ B or C then the next day he is twice as ~~highly~~ likely to sell in city A as in other city. In the long run, how often does he sell in each of the cities.

sol^n

$S = \{A, B, C\}$. $P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 2/3 & 0 & 1/3 \\ C & 2/3 & 1/3 & 0 \end{array}$

$v = [v_1 \quad v_2 \quad v_3]$ s.t. $vP = v$, $v_1 + v_2 + v_3 = 1$

$$[v_1 \quad v_2 \quad v_3]\begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = [v_1 \quad v_2 \quad v_3].$$

$$\left[\frac{2v_2}{3} + \frac{2v_3}{3}, \quad v_1 + \frac{v_3}{3} \quad \frac{v_2}{3}\right] = [v_1 \quad v_2 \quad v_3]$$

equating corresponding elements

$$\frac{2v_2}{3} + \frac{2v_3}{3} = v_1$$

$$2v_2 + 2v_3 = 3v_1 \rightarrow ⑤$$

$$v_1 + \frac{v_3}{3} = v_2$$

$$3v_1 + v_3 = v_2 \rightarrow ③$$

$$\frac{v_2}{3} = v_3 \Rightarrow v_2 \Rightarrow 3v_3 \rightarrow ④$$

using ③ $\Rightarrow$ $v_1 = v_2 - \frac{v_3}{3} = 3v_3 - \frac{v_3}{3}$, $v_1 = \frac{1}{3}v_3 \rightarrow ⑤$

Sub. v_1, v_2 in ①

$$\frac{5}{3}v_3 + 3v_3 + v_3 = 1$$

$$\frac{(8+9+3)}{3}v_3 = 1$$

$$v_3 = \frac{3}{20} \Rightarrow v_1 = \frac{8}{20} \Rightarrow v_2 = \frac{9}{20}$$

$$[v_1 \quad v_2 \quad v_3] = [8/20 \quad 9/20 \quad 3/20]$$

$\left(\frac{8}{20} \times 100\right) = 40\%$ he sells in city A

$\left(\frac{9}{20} \times 100\right) = 45\%$ he sells in city B

$\left(\frac{3}{20} \times 100\right) = 15\%$ he sells in city C

Q) 3 boys A, B, C are throwing ball to each other. A always throws the ball to B & B always throws the ball to C. & C is just likely to throw the ball ~~to~~ to B as to A. If C was the 1st person to throw the ball find the prob. that

(i) A has the ball

(ii) B "

(iii) C " , for the 4th ~~to~~ throw.

Soln

$S = \{A, B, C\}$

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Since C was the 1st person to throw the ball, then the initial prob distribn

$p^{(0)} = [0 \quad 0 \quad \boxed{1}] \rightarrow$ C is holding the ball

$$p^{(1)} = p^{(0)}P = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [1/2 \quad 1/2 \quad 0]$$

$$p^{(2)} = p^{(1)}P = [1/2 \quad 1/2 \quad 0]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [0 \quad 1/2 \quad 1/2]$$

$$p^{(3)} = p^{(2)} P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}}}.$$

∴ After 3 throws the probability that A has the ball is $\frac{1}{4}$, $B \to 1/4$, $C \to 1/2$.

Q.) 2 boys b_1 & b_2 and 2 girls g_1 & g_2 are throwing the ball from one to the other. each boy throws the ball to the other with prob. $1/2$ and to each girl with prob. $1/4$. On the other hand each girl throws the ball to each boy with prob. $1/2$ and never to the girl. In the long run how often does each receive the ball.

<u>soln</u>

$S = \{b_1, b_2, g_1, g_2\}$

$$P = \begin{array}{c} \\ b_1 \\ b_2 \\ g_1 \\ g_2 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & g_1 & g_2 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$

$V = [v_1 \quad v_2 \quad v_3 \quad v_4]$.

$VP = V \quad , \quad v_1 + v_2 + v_3 + v_4 = 1$

$$[v_1 \ v_2 \ v_3 \ v_4] \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [v_1 \quad v_2 \quad v_3 \quad v_4]$$

$$\left[\frac{V_2}{2}+\frac{V_3}{3}+\frac{V_4}{2},\ \frac{V_1}{2}+\frac{V_3}{2}+\frac{V_4}{2},\ \frac{V_1}{4}+\frac{V_2}{4},\ \frac{V_1}{4}+\frac{V_2}{4}\right]=[V_1\ V_2\ V_3\ V_4]$$

Equating corresponding elements;

$$\frac{V_2}{2}+\frac{V_3}{2}+\frac{V_4}{2}=V_1$$

$$V_2+V_3+V_4=2V_1 \rightarrow (2)$$

$$V_1+V_3+V_4=2V_2 \rightarrow (3)$$

$$\frac{V_1}{4}+\frac{V_2}{4}=V_3 \Rightarrow V_1+V_2=4V_3 \rightarrow (4)$$

$$V_1+V_2=4V_4 \rightarrow (5)$$

from 4 & (5)

$$V_3=V_4.$$

$$V_1+V_4+V_4=2V_2$$

$$V_1+2V_4=2V_2 \rightarrow (6)$$

$$V_1=2V_2-2V_4.$$

$$V_2+2V_4=2V_1 \rightarrow (7)$$

$$V_1+V_2+2V_4=1 \rightarrow (8)$$

$$V_1-2V_2+2V_4=0$$

$$2V_1-V_2-2V_4=0$$

$$V_1+V_2+2V_4=1$$

solving,

$$[V_1\ V_2\ V_3\ V_4]=[1/3\ 1/3\ 1/6\ 1/6].$$

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1) There are 2 white marbles in bag A and 3 red marbles in bag B. At each step of process, a marble is selected at random from each bag & the 2 marbles selected are interchanged then find

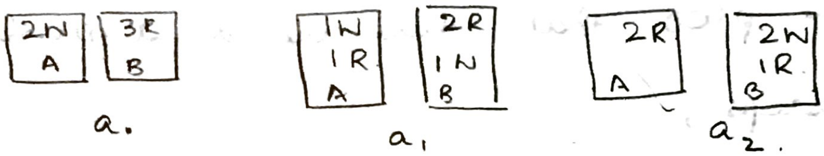
(i) transⁿ prob. matrix.

(ii) what is prob. that there are 2 Red marbles in A after 3 steps.

(iii) In long run, what is prob. that there are 2 Red in A ?.

Solⁿ →

A - 2W B - 3R



$S = \{a_0, a_1, a_2\}$

$$P = \begin{array}{c} \\ a_0 \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} \end{array}$$

Row-wise discussions.

Row 1: The system has to move from a_0 to a_1

Row 2: 1) Prob. of a system moving from a_1 to a_0 choosing 1 R marble from A & 1W - from B and interchanging, we get. $1/2 \times 1/3 = 1/6$.

2) Prob. that the system remains in a_1 posiⁿ choosing 1W from A & B or choosing 1R from A & B. The system remains in the a_1 posiⁿ $= (1/2 \times 1/3) \cancel{1/6} + (1/2 \times 2/3) = 1/2$.

3) System moves from a_1 to a_2 ⇒ choosing 1W from A & 1R from B. $1/2 \times 2/3 = 1/3$

System never moves from a_2 to a_1

Row 3 : Prob. system moves from a_2 to a_1 ~~= choosing~~

~~from a_2 to a_1~~ = choosing 1R from A & 1W from B $\frac{2}{2} \times \frac{2}{3} = \frac{2}{3}$

Prob. system remains in $a_2 = 1 - 2/3 = 1/3$.

The initial prob. distribution.

$$p^{(0)} = [1 \quad 0 \quad 0]$$

To find the prob. that there are 2 red marbles in A after 3 steps,

$$p^{(1)} = p^{(0)} P$$

$$= [0 \quad 1 \quad 0]$$

$$p^{(2)} = p^{(1)} P$$

$$= [1/6 \quad 1/2 \quad 1/3]$$

$$p^{(3)} = p^{(2)} P$$

$$= [1/6 \quad 1/2 \quad 1/3] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

$$= [1/12 \quad 1/6 + 1/4 + 2/9 \quad 1/6 + 1/9]$$

$$= [1/12 \quad 23/36 \quad 5/18]$$

The prob. that there will be 2 red marbles in A = 5/18. (corresponds to state a_2)

$$VP = V$$

$$[v_1 \quad v_2 \quad v_3]\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [v_1 \quad v_2 \quad v_3].$$

$$\frac{v_2}{6} = v_1, \quad v_1 + \frac{v_2}{2} + \frac{2v_3}{3} = v_2, \quad \frac{v_2}{3} + \frac{v_3}{3} = v_3.$$

$\boxed{v_2 = 6v_1}$

$$v_1 + 3v_1 + \frac{2v_3}{3} = 6v_1$$

$$\cancel{3v_1 + 3v_1 +}\ 12v_1 + 2v_3 = 18v_1$$

$$2v_3 = 6v_1$$

$$\boxed{v_3 = 3v_1}$$

$$2v_1 + \frac{v_3}{3} = v_3.$$

$$6v_1 + v_3 = 3v_3.$$

$$v_1 + v_2 + v_3 = 1 \Rightarrow v_1 + 6v_1 + 3v_1 = 1 \Rightarrow 10v_1 = 1 \Rightarrow v_1 = 1/10.$$

$$v_2 = \frac{6}{10}, \quad v_3 = \frac{3}{10}$$

$$\therefore [v_1 \quad v_2 \quad v_3] = [1/10 \quad 6/10 \quad 3/10].$$

) A player has ₹300. At each play of a game he loses ₹100 with prob. 3/4 but wins ₹200 with prob. 1/4. He stops playing if he has lost his money of ₹300 or he has won atleast ₹300. Determine the transition prob. matrix. Find the prob. that there are atleast 3 plays to the game.

Soln →

$$S = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$$

$$\begin{array}{c|c} & \begin{array}{ccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \hline \begin{array}{c} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$p(0) = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$

$p(1) = p(0)\, P$

$= [0 \quad 0 \quad 3/4 \quad 0 \quad 0 \quad 1/4 \quad 0]$

$p(2) = p(1)\, P$

$= \left[0 \quad 9/16 \quad 0 \quad 0 \quad \frac{6}{16} \quad 0 \quad 3/16\right]$

$p(3) = p(2)\, P$

$= \left[27/64 \quad 0 \quad 0 \quad 27/64 \quad 0 \quad 0 \quad 10/64\right]$

If the player is in a_0 and a_6 positions, he is not going to play. $\therefore$ prob. that there are at least 3 plays in the game $= 0 + 0 + \frac{27}{64} + 0 + 0$

$= \frac{27}{64}$

Sampling Distributions:

Sampling theory is a study of relaⁿship existing b/w popⁿ & sample drawn from popⁿ.

Objective of sampling:

The main objective of sampling is that

1) Aims at gathering the max info. abt the popⁿ with min effort ~~or~~, cost & time 2.) to obtain the best possible values of the parameters under specific condiⁿs

3) Determine reliability of these estimates.

Population: A finite or infinite collecⁿ of items under consideraⁿ.

The total no. of objects or observaⁿ in the popⁿ.

Parameter: A popⁿ parameter is a statistical measure or constant obtained from the ~~constant~~ popⁿ:

Eg: popⁿ mean $- \mu$.

popⁿ variance $- \sigma^2$.

It is a finite subset of popⁿ selected from

For estimating the popⁿ characteristics a part of the popⁿ will be studied; that part is called sample.

Sample size: is the n no. of observaⁿs in the sample.

Statistic: A statistic is a statistical measure computed from sample observaⁿ. Ex: Sample mean $- \bar{x}$

Sample variance $- s^2$.

Sampling: The process of drawing samples from the popⁿ is called sampling.

Large & Small small: If $n \geq 30$, we call sample is large.

If $n < 30$ —— " —— small.

Random sampling: Sampling in which each member of the pop n has equal chances or probablity of being included into the sample is called random sampling

Random Sample: A sample obtained through the random sampling is called random sample.

Sampling with replacement: If the items are selected one by one in such a way that an item drawn at a time is replaced back to the pop n before the next or ~~ate~~ subsequent drawn, such a sampling is called sampling with replacement.

Sampling without replacement: A sampling in which an item of the pop n cannot be chosen more than once is known as sampling without replacement

Standard error: is the standard devian of the sampling distribun.

Testing of hypothesis: An assumpn made abt the pop n parameter which may be true or false is called hypothesis. A process to decide whether to reject or accept hypothesis is called testing of hypothesis

Tests of hypothesis are also called test of significance.

null hypothesis : The hypothesis formulated for the sake of rejecⁿ under the assumpⁿ that it is true is called null hypo & it is usually denoted by $\underline{H_0}$.

alternate hypo : The hypo which is contradictory to the null hypo is called alternate hypo, denoted by $\underline{H_1}$

note : The null hypo will be rejected in favour of the alternate hypo only when sample evidence suggests that H_0 is false.

The 2 possible concluⁿs form a hypo. is to reject H_0 or accept H_0.

Type 1 & Type 2 errors : Type one error → rejecting the null hypothesis when it is true.
Type 2 error : not rejecting null hypo. when it is false.

Level of significance (LOS) : ~~not~~ The probability ~~that~~ of type 1 error before which we reject the null hypo is known as LOS.

In practice 0.05 & 0.01 or 5% & 1% are commonly accepted values of the level of significance.
If ~~te~~ LOS is 0.05 it means that on an avg. 5 chances out of 100 we are likely to reject a correct H_0

Confidence interval :-

Confidence interval is an interval within which the true value of parameter is excepted to lies & it constructed on sample statistic

Degree of freedom :-

The no of degrees of freedom is usually denoted by ν & it is no of values which may be assigned arbitrary. It can be interpreted as a no of independent values generated by a sample of small size for estimating the population parameter.

If there are k constants is fixed. the no of degrees of freedom = n-k. for example. In finding the Variance we use the mean so that the no of degrees of freedom = n-1.

Tests of hypothesis for small samples & the s.a

t-test:-

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \qquad \bar{x} = \frac{\sum x_i}{n}.$$

$$s^2 = \frac{\sum (x - \bar{x})^2}{n-1}.$$

$\nu = n-1$. Calculated $|t| < t_\alpha$ table for $\alpha = n-1$.

$H_0 : \mu = \bar{x}$. Accept H_0.

$H_1 : \mu \neq \bar{x}$.

~~t - test~~ This t-test we use in the following cases.

1. To check whether there is a significance difference b/w a sample mean & population mean.

The t-test we will apply when the S.D of population is unknown. The test statistics for t-test is given by. $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$.

Here $\bar{x}$ = sample mean.

μ = population mean.

s = S.D of sample mean.

n = no of observations

The S.D ~~a~~ s is calculated as,

$$s^2 = \frac{\sum (x-\bar{x})^2}{n-1}.$$

For t-test then ~~a~~ no of degrees of freedom = $n-1$.

we take another hypothesis as $H_0 : \mu = \bar{x}$.

this indicates there is no significance difference bez the population mean & sample mean.

Than $H_1 : \mu \neq \bar{x}$. ~~dw~~

Decisions

Let α = level of significance. Find the α for $\lambda = n-1$. from t-distribution table.

If we calculated the ~~value~~ less than $|t| < t_\alpha$. we accept the null hypothesis H_0.

If $|t| > t_\alpha$. we reject the null hypothesis H_0.

Test of significance b/w sample mean :-

Consider two independent samples x_i for $i = 1, 2, \ldots n_1$ & y_i for $i = 1, 2, \ldots n_2$ drawn from the same population then the test statistic is calculated using the formula

$$t = \frac{\bar{x} - \bar{y}}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

where $s^2 = \frac{1}{n_1 + n_2 - 2} \left[\Sigma (x_i - \bar{x})^2 + \Sigma (y_i - \bar{y})^2 \right]$.

$\nu = n_1 + n_2 - 2$.

$H_0 : \bar{x} = \bar{y}$. this indicate there is no significant difference b/w the sample means. then calc

Confident limits :-

For 5% level of significance. we find confident limits as. $|t| \leq t_{0.05}$ for $n-1$.

i.e. $\left| \frac{\bar{x} - \mu}{s/\sqrt{n}} \right| \leq t_{0.05}$. for $\nu = n-1$.

① The 9 items of a sample has the following values.

x : 45 47 50 52 48 47 49 53 51.

Thus the mean of this is differ significantly from the ~~...~~ assumed mean of 47.5

Sol:

$$\bar{x} = \frac{\sum x_i}{n}$$

$\sum x_i = 442$

$\mu = 47.5$.

$$= \frac{442}{9}$$

$= 49.11 \simeq \underline{49}$

$(x-\bar{x})$: -4 -2 1 3 -1 -2 0 4 2.

$(x-\bar{x})^2$: 16 4 1 9 1 4 0 16 4.

$$s^2 = \frac{\sum (x-\bar{x})^2}{n-1}$$

$$= \frac{55}{8}$$

$s^2 = 6.87. \Rightarrow s = \sqrt{6.87}$

$= 2.62$.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$= \frac{49 - 47.5}{2.62/\sqrt{9}}$$

$$= \frac{1.5}{0.87}$$

$$= 1.7241$$

$\nu = 8$.

$t_{0.05} = 1.86$.

$|t| \leq t_{0.05}$.

$1.7 < 1.86$.

$\therefore$ H_0 is accepted. the null hypothesis H_0.

2. A machinist is making engine parts with axial diameter of 0.7 inch. A random sample of 10 parts shows mean diameter is 0.742 inch with S.D is 0.04 inch. On the basis of the sample would u say that work is inferior.

Sol:

$\mu = 0.7$. $n = 10$.

$\bar{x} = 0.742$.

$s = 0.04$.

Consider H_0: The work is not inferior.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$= \frac{0.742 - 0.7}{0.04/\sqrt{10}}$$

3.16.

$$= \frac{0.042}{0.0126}$$

$= 3.33$.

$\nu = 9$

$t_{0.05} = 1.833$ for 9 df.

$|t| > t_{0.05}$. $\therefore$ H_1 is accepted. the work is inferior.

$3.33 > 1.833$.

3. The mean lifetime of 25 bulbs is found as 1550 hrs. the S.D. 120 hrs. the Company manifacturing the bulbs claims that the average life of their bulbs is 1600 hrs is that claim acceptable at 5% los.

sol

$\bar{x} = 1550$.

$s = 120$.

$\mu = 1600$.

$n = 25$.

H_0: The claim is acceptable

H_0: $\mu = \bar{x}$.

$\nu = n-1$,

$= 25-1$

$\nu = 24$

$t_{0.05} = 1.711$.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$= \frac{1550 - 1600}{120/\sqrt{25}} \Rightarrow \frac{50}{120/5} \Rightarrow \frac{50}{24}.$$

$= 2.08$

$|t| > t_{0.05}$.

The claim is not acceptable.

4. The heights of 10 males of a given locality are found to be ~~175~~, ~~168~~.

x 175 168 155 170 152 170 175 160 160 165

based on this sample. find the 95% Confidence limits for the heights of males in the locality.

Sol

Confidence limits = $\left|\dfrac{\bar{x}-\mu}{s/\sqrt{n}}\right| \leq t_{0.05}$.

Since the population mean μ is not given. we Consider μ. is 0 a find $\bar{x}$.

$\bar{x} = \dfrac{1650}{10} = 165$. $\mu = 0$.

Calculate $x-\bar{x}$. $s^2 = \dfrac{\Sigma(x-\bar{x})^2}{n-1}$.

$(x-\bar{x})$ 10. -3. 10 5 -13. 5 -10 -5 -5. 0.

$(x-\bar{x})^2$ 100. 9 100 25. 169 25 100 25 25 0.

$s^2 = \dfrac{578}{9}$

$s^2 = 64.22$ or 64.

$s = \sqrt{64.2} \Rightarrow 8.01$.

$\nu = n-1$,
$= 9$.

$t_{0.05} = 1.833$.

$$\left|\frac{165-\mu}{8.01/\sqrt{10}}\right| < 1.833.$$

$$|165-\mu| < 1.833 \times 8.01/3.16.$$

$$|165-\mu| < 1.833 \times 2.5348.$$

$$|165-\mu| < 4.6463.$$

$|x| < 1$

$-1 < x < 1$

$$-4.6463 < 165-u < 4.6463$$

$$-169.6463 < -u < -160.3537$$

8. Two independent samples of 8 & 7 contained the following values.

Sample 1 19 17 15 21 16 18 16 14.

Sample 2. 15 14 15 19 15 18 16.

with the different size in the difference b/w the sample mean is significant.

Sample 1 = x_1 , Sample 2 = x_2.

H_0 : $\overline{x_1} = \overline{x_2}$ i.e. There is no significant difference b/w the two samples.

$$\overline{x_1} = \frac{136}{8} = 17.$$

$$\overline{x_2} = \frac{112}{7} = 16.$$

$x_1 - \overline{x_1}$: 2 0 -2. 4. -1. 1 -1. -3.

$x_2 - \overline{x_2}$: -1. -2 -1. 3. -1. 2 0.

$(x_1 - \overline{x_1})^2$: 4 0 4 16 1 1 1 9.

$(x_2 - \overline{x_2})^2$: 1 4 1 9 1 4 0.

$$\cancel{s = \frac{\sum (x-\bar{x})^2}{n-1}}.$$

$$s^2 = \frac{1}{n_1+n_2-2}\left[\sum (x_i-\bar{x})^2 + \sum (y_i-\bar{y})^2\right].$$

$n_1+n_2-2 \Rightarrow 8+7-2 = 13.$

$$s^2 = \frac{1}{13}[36+20].$$

$s^2 = 4.3076.$

$s = 2.075.$

$t_{0.05} = 1.771.$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}.$$

$$= \frac{17-16}{2.075\sqrt{0.125+0.142}}$$

$$= \frac{1}{1.0721}.$$

$t = 0.9327.$

$|t| < t_{0.05}$

so there is no significant difference b/w the samples so, H_0 is accepted.

6. 11^{th} school boys ~~were~~ were given test in drawing they were given a months for ~~the~~ the tuition of A 2^{nd} test ~~of~~ of equal difficulty was held at the end of it. Do the marks give evidence that the students have ~~the~~ benefited by extra coaching.

Boys	1	2	3	4	5	6	7	8	9	10	11.
Test 1 marks.	23	20	19	21	18	20	18	17	23	16	19.
Test 2 marks.	24	19	22	18	20	22	20	20	23	20	17.

Sol:

H_0: The students have benefited by extra coaching. $\bar{x}_1 = \bar{x}_2$.

H_1: The students have not benefited by extra coaching.

Test 1 marks = x_1

Test 2 marks = x_2.

$n = 11$.

$$\bar{x}_1 = \frac{\Sigma x_1}{n} = \frac{214}{11} = 19.45.$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n} = \frac{225}{11} = 20.45.$$

$(x_1-\bar{x}_1)$	$(x_2-\bar{x}_2)$	$(x_1-\bar{x}_1)^2$	$(x_2-\bar{x}_2)^2$
3.55.	3.55.	12.60	12.60.
0.55.	-1.45.	0.3025.	2.1025.
-0.45.	1.55.	0.2025.	2.4025.
1.55.	-2.45.	2.4025.	6.0025.
-1.45.	-0.45.	2.1025.	0.2025.
0.55.	1.55.	0.3025.	0.2025.
-1.45.	-0.45.	2.1025.	0.2025.
-2.45.	-0.45.	6.0025.	0.2025.
3.55.	2.55.	12.60.	6.5025.
-3.45	-0.45.	11.90.	0.2025.
-0.45.	-3.45.	0.2025.	11.9025.

$\Sigma(x_1-\bar{x}_1)$ = 3.55. 0.55. -0.45. 1.55. -1.45. 0.55. -1.45.
-2.45. 3.55. -3.45. -0.45.

$\Sigma(x_2-\bar{x}_2)$ = 3.55. -1.45. 1.55. -2.45. -0.45. 1.55. ~~-0.45~~.
~~1.55~~ ~~1.55~~ 2.55 -0.45. -3.45.
-0.45. -0.45.

$\Sigma(x_1-\bar{x}_1)^2$ = 12.60. 0.30. 0.20. 2.40. 2.10. 0.30. 2.10.
6.00. 12.60. 11.90. 2.10.

$\Sigma(x_2-\bar{x}_2)^2$ = 12.60 2.10. 2.40. 6. 0.20 2.40. 0.20.
0.20. 6.5. 0.20. 11.90.

$$\Sigma(x_1 - \bar{x}_1)^2 = 52.6.$$

$$\Sigma(x_2 - \bar{x}_2)^2 = 44.7.$$

$$s^2 = \frac{1}{n_1+n_2-2.}\left[\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2.\right.$$

$$n_1+n_2-2 = -24.$$

$$s^2 = \frac{1}{-2.}\left[52.6 + 44.7\right].$$

$$= -48.65.$$

F-test :-

In testing the significance difference of two sample we assumed that two samples we have same population & population with same variance. The object of the F-test is to whether two indipendent two population in variance differs significantly or whether two samples may be reworded as ~~drawn~~ drawn from the normal population having the same variance therefore before applying the t-test for the significance of two means we have to test for the equality of population variance for F-test by using F-test the ~~statistic~~ statistic $F_{1,2} = \frac{s_2^2}{s_1^2}$ such is given by.

That $s_1^2 > s_2^2$ when s_1 & s_2 are the S.D of the two samples & a numerator s_1^2 should always be greater than denominator s_2^2 & ~~no~~ of degrees of freedom for F-test is n_1-1, n_2-1 & we assume the null hypothesis $H_0: \sigma^2 = s_1^2 = s_2^2$ where σ^2 is the population variance. We accept the null hypothesis if ~~calculated~~ F is less than ~~calculated~~ tabulated F at n_1-1, n_2-1 degree of freedom. At 5% level of 1% level significance. If $s_1^2 = \frac{(x_1-\bar{x}_1)^2}{n_1-1}$ & $s_2^2 = \frac{(x_2-\bar{x}_2)^2}{n_2-1}$

1. Two independent samples of sizes 7 & 6 have the following values.

Sample A	28	30	32	33	29	34	33.
Sample B	29	30	30	24	27	29.	

Examine whether the samples have been drawn from normal population having the same variance.

Sol:

Sample A = x_1 , Sample B = x_2.

x_1	x_2	$(x_1-\bar{x}_1)^2$	$(x_2-\bar{x}_2)^2$
28	29	10.7584	0.7056
30	30	1.6384	3.3856
32	30	0.5184	3.3856
33	24	2.9584	17.3056
29	27	5.1984	1.3456
34	29	7.3984	0.7056
33		2.9584	

$$\bar{x}_1 = \frac{\Sigma x_1}{n} = \frac{219}{7} = 31.28$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n} = \frac{169}{6} = 28.16$$

$$s_1^2 = \frac{(x_1 - \bar{x}_1)^2}{n_1 - 1} \qquad s_2^2 = \frac{(x_2 - \bar{x}_2)^2}{n_2 - 1}.$$

$$\Sigma (x_1 - \bar{x}_1)^2 = 31.4288.$$

$$\Sigma (x_2 - \bar{x}_2)^2 = 26.83.$$

$$s_1^2 = \frac{31.4288}{6} = 5.2381.$$

$$s_2^2 = \frac{26.83}{5}.$$

$$s_2^2 = 5.3667.$$

$$\cancel{F = \frac{5.2381}{5.3667} = 0.9760}.$$

$$F = \frac{s_2^2}{s_1^2} = \frac{5.3667}{5.2381} = 1.0245.$$

$df = 5, 6.$

$F_{5,6} = 4.95.$

Calculated $F < F_{5,6}$ at 5% level of significance.

so, H_0 is accepted.

2. Two random samples drawn two normal population are given below. Test whether two populations have the same variance.

Sample 1. 20 16 26 27 23 22 18 24 25 19 -- $n=10$

Sample 2 17 23 32 25 22 24 28 6 31 33 20 27 $n=12$

$H_0 : \sigma_1^2 = \sigma_2^2$

Sol

Sample 1 = x_1 & Sample 2 = x_2.

$$\bar{x}_1 = \frac{\Sigma x_1}{n} = \frac{220}{10} = 22.$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n} = \frac{288}{12} = 24.$$

$\Sigma(x_1 - \bar{x}_1)^2 = 120$.

$\Sigma(x_2 - \bar{x}_2)^2 = 614$.

$(x_1 - \bar{x}_1)^2$	$(x_2 - \bar{x}_2)^2$
4.	49.
36.	1.
16.	64.
25.	1.
1	4.
0.	16.
16.	324.
4.	49.
9.	81
9.	16.
	9.

$$s_1^2 = \frac{\Sigma(x_1 - \bar{x}_1)^2}{n_1 - 1}. \qquad s_2^2 = \frac{\Sigma(x_2 - \bar{x}_2)^2}{n_2 - 1}.$$

$$s_1^2 = \frac{\cancel{119}\ 120}{9}. \qquad s_2^2 = \frac{614}{11}.$$

$$s_2^2 = 55.81.$$

$$s_1^2 = 13.33.$$

$$F = \frac{s_2^2}{s_1^2}$$

$$= \frac{55.81}{13.33}$$

$$F = \cancel{4.2246}\ 4.1867.$$

$F_{11,9} = 2.9$ at 5% level of significance.

Calculated $F > F_{11,9}$ so H_0 is rejected.

3. Test the equality of S.D. for the data given below at 5% level of significance for $n_1 = 10$, $n_2 = 14$, $s_1 = 1.5$, $s_2 = 1.2$.

sol.

$$s_1^2 = \frac{n_1 s_1^2}{n_1 - 1} \qquad s_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

$$= \frac{10 \times (1.5)^2}{9} \qquad = \frac{14 \times (1.2)^2}{13}$$

$$= \frac{22.5}{9} \qquad = \frac{20.16}{13}$$

$$s_1^2 = 2.5 \qquad s_2^2 = 1.550$$

$$F = \frac{s_1^2}{s_2^2}$$

$$= \frac{2.5}{1.5}$$

$$= 1.66$$

Calculated F is less than $F_{9,13}$. so H_0 is accepted.

$$F_{9,13} = \cancel{}\ 2.71$$

(4) The mean & S.D height of randomly of 8 randomly choosen are 166.9 & 8.29 cm. the according values of 6 randomly choosen 170.3 & 8.5cm respectively.

based on this data can we compute that the rotiolers of in general shorter than sailers.

3)

$\overline{x}_1 = 166.9$

$\sigma_1 = 8.29$

$\overline{x}_2 = 170.3$

$\sigma_2 = 8.5$

$n_1 = 8$

$n_2 = 6$

$$\sigma_1^2 = \frac{\Sigma(x_1 - \overline{x}_1)^2}{n_1}$$

$$n_1\sigma_1^2 = \Sigma(x_1 - \overline{x}_1)^2$$

$$n_2\sigma_2^2 = \Sigma(x_2 - \overline{x}_2)^2$$

$$s^2 = \frac{\Sigma(x_1 - \overline{x}_1)^2 + \Sigma(x_2 - \overline{x}_2)^2}{n_1 + n_2 - 2}$$

$$\Sigma(x_1 - \overline{x}_1)^2 = 8 \times (8.29)^2$$
$$= 549.79$$

$$\Sigma(x_2 - \overline{x}_2)^2 = 6 \times (8.5)^2$$
$$= 433.5$$

$$s^2 = \frac{549.79 + 433.5}{8 + 6 - 2}$$

$$= 81.9408 \quad \Rightarrow s = 9.05$$

$$t = \frac{\overline{x}_1 - \overline{x}_2}{s\sqrt{1/n_1 + 1/n_2}} \qquad \frac{166.9 - 170.3}{9.05\sqrt{1/8 + 1/6}}$$

$$= \frac{-3.4}{4.88} = -0.6967$$

The queue. the system can accommodate max no of customers limited to k. let average arrival rate $\lambda_n = \lambda$. for $n \geq k$.

Average service time $\mu_n = \mu$. $n = 1, 2, \ldots k$.

$$P_0 = p(\text{no customers}) = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{k+1}} \quad \lambda \neq \mu.$$

$$= \frac{1}{k+1} \quad \lambda = \mu.$$

$P_n = p(n \text{ customers in queue})$.

$= (\lambda/\mu)^n P_0 \quad \lambda \neq \mu$.

$= (1/k+1) \quad \lambda = \mu$.

Note 1.

For this model λ didn't be less than μ.

Since queue cannot ~~filled~~ build up without bound.

Ls | no of customer.

$$L_s = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)(\lambda/\mu)^{k+1}}{1 - (\lambda/\mu)^{k+1}} \quad \lambda \neq \mu.$$

$= k/2$

Effective arrival rate $\lambda' = \mu(1 - P_0)$.

Average no customer in the queue

$L_q = L_s - \frac{\lambda}{\mu}$.

Average waiting time in the queue.

$$w_q = \frac{L_q}{\lambda}.$$

$$w_q = \frac{L_q}{\lambda'}.$$

1. Patients arrive at a clinic according to a poisson distribution at a rate of 30 person per hour the waiting room doesn't accomodate 14 person. Examination time per patient is exponentially with mean rate 25 per hr find the effective arrival rate at the clinic. What is the probability that the arriving patient

ii. what is the excepted time rate until a patient is discharged by the clinic.

Sol
k = no of patients in the clinic $= 14+1 = 15$.

$\mu = 20$. $\lambda = 30$.

$$P_0 = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{k+1}}.$$

probabality that the patient didn't wait.

$$= \frac{1 - 30/20}{1 - (30/20)^{16}}$$

$$= \frac{-0.5}{1 - (3/2)^{15}}$$

$$= \cancel{\frac{-0.5}{-436.89}} = \frac{1 - 3/2}{1 - (3/2)^{16}}$$

$$= \frac{\frac{2-3}{2}}{(-655.84)}$$

$$= \frac{-1/2}{-655.84}$$

$$P_0 = 7.62 \times 10^{-4}$$

Average arrival $= \lambda' = 20(1 - 7.62 \times 10^{-4})$

$= 19.98$

Average no of patients in the clinic $= L_s$:

$$L_s = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)(\lambda/\mu)^{k+1}}{1 - (\lambda/\mu)^{k+1}}$$

$$= \frac{30}{20-30} - \frac{(16)\left(30/20\right)^{16}}{1-\left(30/20\right)^{16}}.$$

$$= \cancel{3}0/-\cancel{1}0 - \frac{16\left(3/2\right)^{16}}{1-\left(3/2\right)^{16}}.$$

$$= -3 - \frac{10509.45}{-6,55.84}.$$

$$= 13.024.$$

$$W_q = \frac{L_q}{\lambda'} = \frac{13.024}{19.98} = 0.6518.$$

$$\lambda' = \mu(1-P_0)$$

$$= 20(1-7.62\times10^{-4}).$$

$$= 19.98.$$

M/G/1 queueing system:-

The arrival process is a poisson process & the service time is generalized distribution

1. Average no of customers in the system.

$$L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho.$$

σ^2 = Variance of the service time.

$\rho = \lambda/\mu.$

2. Average no of customers in the queue.

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

3. Average waiting time the customer spent ~~time~~ in the queue.

$$W_s = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} + 1/\mu.$$

4. Average waiting time in the queue.

$$W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}.$$

1. Automatic car wash facility operates with only one bay car arriving ~~correspo~~ according to poisson with mean of 4 cars/hr. & may wait in the facilities parking lot, if the bay is busy. If the service time is constant for all cars & equal to 10 mins. then find L_s, L_q, W_s, W_q, in its usual notations.

$\lambda = 4$.

$\mu = 10$ mins. $\mu = 60/10 = 6$ /hr. $\rho = \frac{\lambda}{\mu} = \frac{4}{6} = \frac{2}{3} = 0.66$.

Si it is mentioned the service time for all Cars is Constant $\sigma^2 = 0$, $\rho = \frac{\lambda}{\mu} = 2/3 = 0.66$.

$$L_s = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)} + \rho.$$

$$= \frac{(4)^2(0) + (0.66)^2}{2(1-0.66)} + 0.66.$$

$$= \frac{0 + 0.4356}{0.68} + 0.66.$$

$$= 1.30.$$

$L_q = 1.30$ $\cancel{\lambda^2\sigma^2}$

$W_s = \frac{(4)^2(0) + (0.66)^2}{2(4)(1-0.66)} + \frac{1}{(0.66)}$

$W_s = 0.7400$.

$W_q = 0.0800$.

3. In a heavy machine shop, the overhead crane is 75% ~~utti~~ utilized. Time study observation gives the average service time is 10.5 min with a S.D 8.8. What is the average calling rate for the service of the ~~b~~ crane & what is the Avg. delay in the service. If the average service time is cut to ~~of~~ 8 min with S.D of 6 min. How ~~th~~ much reduction will occur on a average in the delay of getting serve.

Sol

Since the crane is utilized for 75%.

i.e. the traffic intensity $\rho = 0.75$.

the $\mu = 1/10.5$ per hour $60/10.5 = 5.71$ /hr

$\lambda = \rho\mu$.

$= 0.75 \times 5.71$

$= 4.28$.

$\sigma = 8.8$ min.

~~$= 60/8.8 = 6.8181$~~

$= 8.8/60 = 0.1466$ hr

$$W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}$$

$$= \frac{(4.28)^2(0.1466) + (0.75)^2}{2(4.28)(1-0.75)}$$

$$= \frac{3.2479}{2.14}$$

$$= 1.5177 \text{ the.}$$

$\mu = 60/8 = 7.5$.

$\sigma = 0.25$

The propability of single head $p = 1/2$ & $q = 1/2$.

$n = 5$, $N = 320$.

The ~~to~~ value 5 Coins are formed the excepted frequencies 0, 1, 2, 3, 4, 5. $p(x) = {}^nc_r (p)^r (q)^{n-r}$.

In 320 trails are $e_0 = {}^5C_0 (1/2)^0 (1/2)^{5-0} = 10$.

$e_1 = 320 \times {}^5C_1 (1/2)^1 (1/2)^{5-1} = 50$

$e_2 = 320 \times {}^5C_2 (1/2)^2 (1/2)^{5-2} = 100$.

$e_3 = 320 \times {}^5C_3 (1/2)^3 (1/2)^{5-3} = 100$.

$e_4 = 320 \times {}^5C_3 (1/2)^4 (1/2)^{5-4} = 50$

$e_5 = 320 \times {}^5C_3 (1/2)^5 (1/2)^{5-5} = 50$.

O_i	15	45	85	65	60	20
e_i	10	50	100	100	50	10

$\Sigma o_i = \Sigma e_i = 320$.

H_0: Coin is unbiased.

$$\chi^2 = \sum_{i=1}^{7} \frac{(O_i - e_i)^2}{e_i}.$$

$$= \frac{1}{10}(15-10)^2 + \frac{1}{50}(45-50)^2 + \frac{1}{100}(85-100) + \frac{1}{100}(65-100)^2 + \frac{1}{50}(60-50) + \frac{1}{10}(20-10)^2.$$

$\chi^2 = 12.6$.

$\chi^2_{0.05}$ at 4 deg of freedom.

$\chi^2 > \chi^2_{0.05}$: H_0 is rejected.

2. Fit the poisson distribution of data & test the goodness of it.

x	0	1	2	3	4	5	6	Total.
f	273	70	30	7	7	2	1	390.

$\boxed{\bar{x} = m}$

$$\bar{x} = \frac{\sum f x}{\sum f} = \frac{\cancel{1195}}{}$$

$$= \frac{1}{390}[0 + 70 + 60 + 21 + 28 + 10 + 6]$$

$m = 1/2.$

$$p(x) = \frac{e^{-m}\, m^{x}}{x!}$$

$$e_0 = \frac{390 \times e^{-1/2} (1/2)^0}{0!} = 236.54$$

$$e_1 = \frac{390 \times e^{(-1/2)} (1/2)^1}{1!} = 118.2$$

$$e_2 = \frac{390 \times e^{(-1/2)} (1/2)^2}{2!} = 29.56$$

$$e_3 = \frac{390 \times e^{(-1/2)} (1/2)^3}{3!} = 4.9$$

$$e_4 = \frac{390 \times e^{(-1/2)} (1/2)^4}{4!} = 0.6$$

$$e_5 = \frac{390 \times e^{(-1/2)} (1/2)^5}{5!} = 0.1$$

$$e_6 = \frac{390 \times e^{(-1/2)} (1/2)^6}{6!} = 0.005$$