Unit-3

Date
Page

"Algebraic Structure & Morphism"

★ **Boolean Algebra $(B, +, \cdot, ')$** -

$a \cup b \rightarrow$ LCM
$a \cap b \rightarrow$ HCF

B_1 Closer Law (If $a \in I, b \in I$ then $a * b \in I$)
B_2 Commutative law $(a * b = b * a)$
B_3 Distributive law $a \cdot (b+c) = (a+b) \cdot (a+c)$
B_4 Identity Law $a \cdot 1 = a$, $a + 0 = a$
B_5 Inverse Law $a \cdot a' = 0$, $a + a' = 1$

★ **Binary Operation** -

A non-empty set G equipped with one binary operation $(*)$ is called groupoid or binary operation.

i.e, G is groupoid if G is ~~called~~ closed for $*$ $(G, *)$ is a groupoid.

Ex.- $I = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\Rightarrow$ for $(I, +)$ -

Closure law

If -3 and $1 \in I$ then

$-3 + 1 = -2 \in I$

Piyush

⇒ <u>for $(I, \cdot)$</u>

Closure law

If $-1, -2 \in I$ then

$(-1) \cdot (-2) = 2 \in I$

★ <u>Semigroup</u> –

An algebric structure $(G, *)$ is called a semigroup, if the binary operation $(*)$ satisfy the following property.

① <u>Closure Law</u> – If $a, b \in G$, then $a * b \in G$

② <u>Associative Law</u> – If $a, b, c \in G$
$a * (b * c) = (a * b) * c$

<u>Ex.</u> → $G = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

(i) <u>Closure Law</u> – $(G, +)$

If $1, 2 \in G$, then
$2 + 1 = 3 \in G$

[∴ G is closed w.r.t (+)]

(ii) <u>Closure Law $(G, \cdot)$</u> –

If $2, 1 \in G$, then
$2 \cdot 1 = 2 \in G$

[$\therefore$ G is closed w.r.t $(\cdot)$]

★ Association law - (i) $(G, +)$

If $1, 2, 3 \in G$, then

$$1 + (2+3) = (1+2) + 3$$
$$1+5 = 3+3$$
$$6 = 6$$
$$6 \in G$$

(ii) Association law $(G, \cdot)$ -

If $1, 2, 3 \in G$, then

$$1 \cdot (2 \cdot 3) = (1 \cdot 2) \cdot 3$$
$$1 \cdot 6 = 2 \cdot 3$$
$$6 = 6$$
$$6 \in G$$

"Hence $(G, +)$ & $(G, \cdot)$ are Semi-Group".

★ Monoid Group - An algebric structure $(G, *)$ is called a monoid group, if the binary operation $(*)$ satisfy the following properties:-

1. Closure Law
2. Associative Law

} Same as Previous

M.I* → In Identity law in { Addition take $e=0$; Multiplication take $e=1$ }

Date ____ Page ____

③ **Identity Law** :-

If $a \in G$ and $e \in G$ then

$$a * e = e * a = a$$

where "e" is an identity element.

Ex. → $N = \{1, 2, 3, \ldots\}$

$(N, \cdot)$ → Identity Law

$1, 2 \in N$

$1 \cdot 2 = 2 \cdot 1 = 2$, where $(e = 1)$

★ **Group OR Abelian Group** :-

An algebric structure of set G & a binary operation (*) defined in G.

i.e, $(G, *)$ is called a group OR Abelian group if it satisfy following condition.

① **G_1, Closure Law** → If $a, b \in G$, then $a * b \in G$

② **G_2, Associative Law** → If $a, b, c \in G$, then $a * (b * c) = (a * b) * c$

③ **G_3, Identity Law** → If $a \in G$ & $e \in G$, then $a * e = e * a = a$

④ **G_4, Inverse Law** → If $a \in G$ and $a' \in G, e \in G$

$a * a' = a' * a = e$

Date
Page

⑤ G_5, Commutative law - If $a, b \in G$, then $a * b \neq b * a$

Ex. → Show that the set of integers form an abelian group under addition.

$$G = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \quad (G, +)$$

- G_1, Closure law ⇒ $2, 1 \in G$
  $2 + 1 = 3 \in G$

- G_2, Associative law ⇒ $-2, -1, 0 \in G$
  $-2 + (-1 + 0) = (-2 + (-1)) + 0$
  $-3 = -3$
  $-3 \in G$

- G_3, Identity law → The $2 \in G$ & $0 \in G$, then
  $2 + 0 = 0 + 2$
  $2 = 2$
  $2 \in G$

- G_4, Inverse law ⇒ $2 \in G$ & $-2 \in G$, then
  $2 + (-2) = (-2) + 2$
  $0 = 0$
  $0 \in G$

- G_5, Commutative law ⇒
  $2 \in G$ & $1 \in G$, then
  $2 + 1 = 1 + 2$
  $3 = 3$
  $3 \in G$

Hence, Set of integers form an Abelian Group. [∴ $(G, +)$ is an abelian group]

M.F.* → $a*b = a+b+ab$

Date ____
Page ____

Que. G is a set of rationals except '+' binary operation (*) is defined by

$$a*b = a+b+ab$$

Show that it is a group.

G_1 Closure Law — If $a \in G$ & $b \in G$ then $a*b \in G$
G_2 Associative Law — $(a*b)*c = a*(b*c)$
G_3 Identity Law — $a*e = a = e*a$
G_4 Inverse Law — $a*a' = e = a'*a$
G_5 Commutative Law — $a*b = b*a$

Sol.ⁿ ⇒ The a & b is the element of G, then

① G_1 ⇒ $a*b = a+b+ab$

② G_2 ⇒ $a, b, c \in G$, then (Associative)

$$(a*b)*c = a*(b*c)$$

L.H.S ⇒

$$\begin{aligned}(a*b)*c &= (a+b+ab)*c \\ &= a+b+ab+c+(a+b+ab)c \\ &= a+b+c+ab+ac+bc+abc\end{aligned}$$

R.H.S ⇒

$$\begin{aligned}a*(b*c) &= a*(b+c+bc) \\ &= a+b+c+bc+a(b+c+bc) \\ &= a+b+c+bc+ab+ac+abc\end{aligned}$$

Hence,

$$L.H.S = R.H.S$$

Date
Page

(3) G_3 ⇒ Identity law ⇒ $a * e = a = e * a$

L.H.S ⇒ $a * e = a + e + ae$

$$\cancel{a} = \cancel{a} + e + ae$$

$$0 = e(1+a)$$

$$\boxed{e = 0}$$

(4) G_4 ⇒ Inverse Law ⇒ If $a \in G$ then $a * a' = e = a' * a$

L.H.S ⇒

$$a * a' = a + a' + a \cdot a'$$

$$e = a + a' + a \cdot a'$$

$\boxed{\because e = 0}$

$$0 = a + a' + a \cdot a'$$

$$-a = a' + a \cdot a'$$

$$-a = a'(1+a)$$

$$\boxed{\frac{-a}{(1+a)} = a'}$$

$\boxed{(1+a) \neq 0}$

(5) G_5 ⇒ Commutative Law — $a * b = b * a$

L.H.S ⇒ $a * b = a + b + ab$

R.H.S ⇒ $b * a = b + a + ab$

L.H.S = R.H.S

Hence, $(G, *)$ is a Group!

Que. In (I), we define $a * b = a + b + 1$ so that $(I, *)$ is an abelian group.

$\Rightarrow$ $G_1 \Rightarrow$ If a & b are the element of G, then
$$a * b = a + b + 1$$

$\Rightarrow$ $G_2 \Rightarrow$ $a, b, c \in I$, then
$$(a * b) * c = a * (b * c)$$

L.H.S
$$(a * b) * c = (a + b + 1) * c$$
$$= a + b + 1 + c + \cancel{(a+b+1) c}\ 1$$
$$= a + b + c + 2$$

R.H.S
$$a * (b * c) = a * (b + c + 1)$$
$$= a + b + c + 1 * 1$$
$$= a + b + c + 2$$

$$\text{L.H.S} = \text{R.H.S}$$

$\Rightarrow$ $G_3 \Rightarrow$ Identity law $\Rightarrow$ $a * e = a = e * a$

L.H.S $\Rightarrow$ $a * e = a + e + 1$
$$\cancel{a} = \cancel{a} + e + 1 \Rightarrow \boxed{e = -1}$$

~~R.H.S $\Rightarrow$~~

$\Rightarrow$ $G_4 \Rightarrow$ Inverse law -

If $a \in I$, then $(a * a' = a' * a = e)$
$$a * a' = a + a' + 1$$
$$e = a + a' + 1$$
$$-1 = a + a' + 1$$
$$a + a' = -2$$
$$\boxed{a' = -2 - a}$$

G_5 Commutative law :- $a * b = b * a$

L.H.S $\rightarrow$ $a * b = a+b+1$

R.H.S $\rightarrow$ $b * a = b+a+1$

L.H.S = R.H.S

Hence, $(I, *)$ is a "Abelian Group"