Ans 1 Let λ be an eigenvalue of Hermitian matrix A.
and X be an eigen vector corresponding to λ.
then $\Rightarrow AX = \lambda X$
multiplying $\bar{X}^T$ on both sides
$$\Rightarrow \bar{X}^T (AX) = \bar{X}^T (\lambda X)$$
$$= \lambda \bar{X}^T X$$
$$= \lambda \|X\|$$
Now we take conjugate transpose of both sides
$$\Rightarrow \bar{X}^T \bar{A}^T X = \bar{\lambda} \|X\| \quad — \text{I}$$
As A is Hermition
$\therefore \bar{A}^T = A$
$\therefore$ LHS becomes $\Rightarrow \bar{X}^T \bar{A}^T X = \bar{X}^T A X$
$$= \bar{X}^T \lambda X \quad [AX = \lambda X]$$
$$= \lambda \|X\| \quad — \text{II}$$
from I & II we get
$$\lambda \|X\| = \bar{\lambda} \|X\|$$
We know X is not zero vector & $\|X\| \neq 0$
$\therefore$ Dividing by length $\|X\|$ we get
$$\lambda = \bar{\lambda}$$
This means λ is a real number.
Since we assumed λ to be any eigen value
$\therefore$ All eigen value of Hermitian matrix A is a real number.

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Ans 2 (a) given $A, B \in \mathbb{C}^{n \times n}$ and $AB = BA$.
λ is eigen val of A and V_λ is subspace corresponding to λ
We have to show $\exists v \in V_\lambda$ such that v is eigen vector of B.
Let basis of V_λ be $(V_1, V_2 \cdots V_k)$
Now if we take V_1

$$AV_1 = \lambda V_1 \qquad [V_1 \in V_\lambda] \quad \text{--- (I)}$$

also

$$ABV_1 = BAV_1 \qquad [AB = BA]$$
$$= B\lambda V_1 \qquad [\text{from I}]$$
$$= \lambda BV_1 \qquad \text{--- (II)}$$

We can say that BV_1 is eigen vector of A. $\therefore BV_1 \in A$.
$\therefore BV_1$ can be written as

$$BV_1 = a_1 V_1 + a_2 V_2 \cdots a_k V_k$$

where $(V_1, V_2, \cdots V_k)$ are basis of V_λ as they are in A.
Since all V_i are linearly independent

$$BV_1 = a_1 V_1 \; ; \qquad a_2 = a_3 \cdots a_k = 0$$

$\therefore BV_1 = aV_1$ and V_1 is eigen vector of B

$\therefore$ Eigen values are not necessarily same.

b) Given spectrum is non-degenerate, i.e. there are n different eigen values.
Let V be an eigen vector of A corresponding to eigen value λ.

$$\therefore AV = \lambda V \qquad \text{--- (I)}$$

also $ABV = BAV \qquad [\because AB = BA]$

$\Rightarrow ABV = \lambda BV \qquad$ --- ~~(I)~~ [using (I)]

This means B is eigen vector of A for eigen value λ.

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Since there are n different eigen values of A, all of them have multiplicity 1 & hence they all are one dimensional.

Since Bv and v lie in ~~same~~ one dimensional eigen space, they should be dependent on each other.

Say $Bv = \mu v$ for some scalar μ

$\therefore$ B is eigen vector of B corresponding to eigenvalue μ.

Since A has n distinct eigen values which are also independent $\therefore$ it is diagonizable.

And as we have seen that eigen vectors of A are also eigen vectors of B

$\therefore$ A & B are simultaneously diagonal.

Ans 3 Let $S_K = \sum_{K=0}^{\infty} A^K = I + A + A^2 + A^3 \dots + A^K$

then $S_K A = A + A^2 + A^3 - - - + A^K + A^{K+1}$

subtracting we get

$$S_K - S_K A = S_K (I - A) = I - A^{K+1}$$

Now we know that

$$|A^{K+1}| \leq |A|^K |A| = |A|^{K+1}$$

$\therefore \lim_{K \to \infty} A^{K+1} = 0$ when $|A| < 1$

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$$\therefore S(I-A) = \lim_{K\to\infty} S_K(I-A)$$

$$= \lim_{K\to\infty} (I - A^{K+1})$$

$$= I - \underbrace{\lim_{K\to\infty} A^{K+1}}_{0} = I$$

Since $S(I-A) = I$, $\therefore$ S is left inverse of $(I-A)$. Similarly if we find AS_K and solve we will get $(I-A)S = I$ which means S is right inverse as well.

Since S is right & left inverse of $(I-A)$

$\therefore$ S is the inverse of $(I-A)$

Ans 4 (a) To prove trace(A) is sum of its eigen values.
Let us see the minor cofactor expansion of $(A-\lambda I)$ which gives a sum of terms.
Each term is a product of n factor having one entry from all rows & colums.
Now members containing diagonal elements of minor cofactor are
$(a_{11}-\lambda)(a_{22}-\lambda) \cdots (a_{nn}-\lambda)$
$\therefore$ coefficient of λ^{n-1} term will be

$$\Rightarrow (-1)^n \left(\sum_{i=1}^{n} -\lambda_i\right)$$

$$\Rightarrow (-1)^{n+1} \left(\sum_{i=1}^{n} \lambda_i\right)$$

This is the only term that will contribute to λ^{n-1} order terms. (because we skipped one row & one col). $\therefore$ coefficient of λ^{n-1} is trace of matrix.
$\therefore$ trace $(A) = f(\lambda) = (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2) \cdots (\lambda-\lambda_n)$
The λ^{n-1} term is:

$$(-1)^n \sum_{i=1}^{n} -\lambda = (-1)^{2n} \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i$$

$\Rightarrow$ Trace (A) is sum of eigen values.

b) To prove ~~trace~~ Det (B) is product of eigen value.
i.e. $\det(A) = \prod_{j=1}^{n} \lambda_j$

Since $(-1)^n$ is highest order coefficient and $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are solution to f we write. $f(\lambda) = (-1)^n (\lambda-\lambda_1) \cdots (\lambda-\lambda_n)$

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for $\lambda = 0$; $f(0) = \det(A - 0I) = \det(A)$

$\Rightarrow \det(A) = f(0) = (-1)^n (0-\lambda_1) \cdots (0-\lambda_n)$

$$= (-1)^{2n} \prod_{i=1}^{n} \lambda_i$$

$$= \prod_{i=1}^{n} \lambda_i$$

$\therefore$ Product of EV of complex matrix A is determinant of A.

Ans 5 Given that A is diagonizable $n \times n$ matrix we have to prove that $\det(\exp(A)) = \exp(\text{trace}(A))$

For this matrix A, there is a matrix P such that $A = P^{-1} A' P$. where A' is diagonal matrix.

So now we take exponential of A using exponential series.

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} \cdots$$

$I \to$ identity matrix.

Substituting $P^{-1} A' P$ to A^2

$$A^2 = (P^{-1} A' P)(P^{-1} A' P)$$
$$= P^{-1} A' (P P^{-1}) A' P$$
$$= P^{-1} A' A' P$$
$$= P^{-1} A'^2 P$$

After reducing all powers of A like this, we get

$$\exp(A) = P^{-1} \left[I + A' + \frac{(A')^2}{2!} + \frac{(A')^3}{3!} \cdots \right] P$$

$\therefore$ it is true that

$$\exp(A) = P^{-1} \exp(A') P$$

Also ~~produc~~ determinant of product of square matrix is product of their determinants.

$\therefore$

$$\begin{aligned}
\det(\exp(A)) &= \det(P^{-1} \exp(A') P) \\
&= \det(P^{-1}) \det(\exp(A')) \det(P) \\
&= \det(P^{-1}) \det(P) \det(\exp(A')) \\
&= \det(P^{-1}P) \det(\exp(A')) \\
&= \det(I) \det(\exp(A')) \\
&= \det(I \exp(A')) \\
&= \exp(tr(A'))
\end{aligned}$$

Also, since trace of matrix is equal to sum of its eigen values

$\therefore$ $$tr(A') = tr(A)$$

$\Rightarrow$ $$\det(\exp(A)) = \exp(tr(A'))$$

$$\det(\exp(A)) = \exp(tr(A))$$

Ans 6 Given $\left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ $\begin{array}{l} \rightarrow x \\ \rightarrow y \end{array}$

$$\mu_x = \frac{4-2-1}{3} = \frac{1}{3} \qquad \mu_y = \frac{4-3-1}{3} = 0$$

Covar $$\sigma_{xy} = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{n}$$

$$= \frac{(4-1/3)(4-0) + (-2-1/3)(-3-0) + (-1-1/3)(-1-0)}{3}$$

$$= \frac{23}{3}$$

$$var(x) = \sigma_x^2 = \sum_{i=1}^{n} \frac{(x_i - \mu_x)^2}{n}$$

$$= \frac{(4 - 1/3)^2 + (-2 - 1/3)^2 + (-1 - 1/3)^2}{3}$$

$$\sigma_x^2 = \frac{62}{9}$$

$$var(y) = \sigma_y^2 = \sum_{i=1}^{n} \left(\frac{y - \mu_y}{n}\right)^2$$

$$= \frac{(4-0)^2 + (-3-0)^2 + (-1-0)^2}{3} = \frac{26}{3}$$

$$\text{covariance matrix} = \begin{bmatrix} 62/9 & 23/3 \\ 23/3 & 26/3 \end{bmatrix} = \begin{bmatrix} 6.88 & 7.66 \\ 7.66 & 8.66 \end{bmatrix}$$

We need to find eigen vector of this matrix

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6.88 - \lambda & 7.66 \\ 7.66 & 8.66 - \lambda \end{vmatrix} = \lambda^2 - 15.5\lambda + 0.905 = 0$$

On solving we get $\lambda_1 = 0.0585$, $\lambda_2 = 15.5$

For λ_1 finding eigen vector.

$$|A - \lambda_1 I| \vec{V_1} = 0$$

$$\begin{bmatrix} 6.82 & 7.66 \\ 7.66 & 8.60 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 6.82 V_1 + 7.66 V_2 = 0$$
$$7.66 V_1 + 8.60 V_2 = 0$$

on solving

$V_1 = -1.12 V_2$

$V_2 = V_2$

$$\Rightarrow \vec{V_1} = \begin{bmatrix} -1.12 V_2 \\ V_2 \end{bmatrix}$$

equation of line traced by $\bar{v}_1$ is

$$\boxed{x = -1.12y}$$

For $\lambda_2 = 15.5$ eigen vectors ~~are~~ is:

$$(A - \lambda_2 I)\,\bar{V}_2 = 0$$

$$\begin{bmatrix} -8.6 & 7.66 \\ 7.66 & -6.82 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-8.6V_1 + 7.66V_2 = 0$$

$$7.66V_1 - 6.82V_2 = 0$$

on solving

$V_1 = 0.891\,V_2$ $\quad\therefore\quad$ $\bar{v}_2 = \begin{bmatrix} 0.891V_2 \\ V_2 \end{bmatrix}$

$V_2 = V_2$

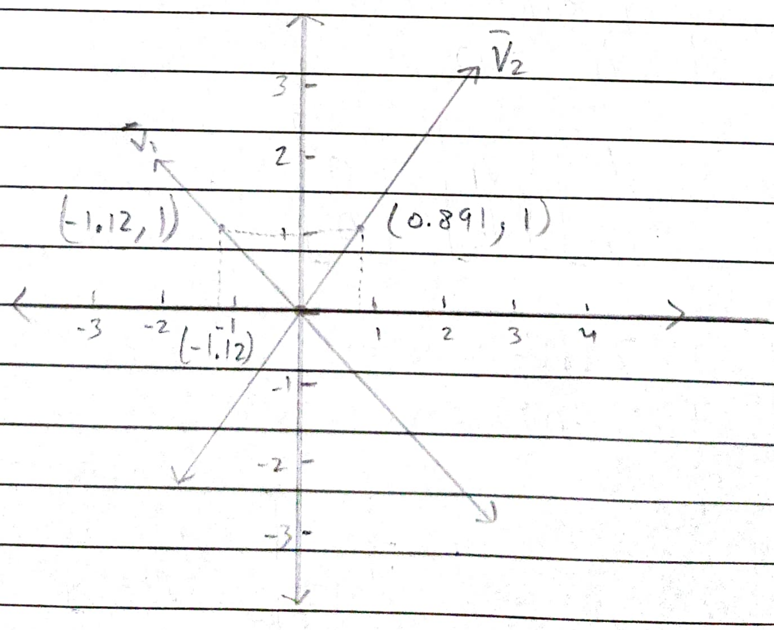
equation of line traced by $\bar{v}_2$ is

$$\boxed{x = 0.891\,y}$$

Now lines for V_1, V_2 both pass through $(0,0)$

in $\bar{V}_1$, for $y = 1$, $x = -1.12$

in $\bar{V}_2$, for $y = 1$, $x = 0.891$



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Ans 7 (a) Given matrix $A = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

It is clearly a symmetric matrix.

To prove $(x,y) \to x^T Ay$ gives a scalar product we need to prove three properties.

- We can write

$R^3 \to R : (x,y) = x^T Ay = x.(Ay)$ [dot product]

Since dot product is commutative

$\therefore (x,y) = x.(Ay) = (Ay).x$
$= (Ay)^T x = y^T A^T x$
$= y^T Ax$ [A is symm]
$= (y,x)$

$\therefore$ function is symmetric

- Next for vectors x, y, z and real no. r

$(rx, y) = (rx)^T Ay$

$= rx^T Ay = r(x,y)$

also

$(x+y, z) = (x+y)^T Az = (x^T + y^T) Az$

$= x^T Az + y^T Az = (x,z) + (y,z)$

Thus linearity is satisfied.

- If x is non-zero vector in R^3 then we have $(x,x) = x^T Ax > 0$

Also $(0,0) = 0^T A 0$
$= 0$

$\therefore (x,x) \geq 0$ for any vector $x \in R^n$

Suppose $(x,x) = 0$, then
$(x,x) = x^T Ax = 0$

~~Since A is a positive definite, this happens~~
~~$(x, x) = 0$ if and only if $x = 0$.~~

Since A is positive definite and $x^T A x = 0$
$\therefore$ x has to be 0.
This proves the positive definiteness of function.

Since $R^3 \to R : (x, y) \to x^T A y$ satisfies symmetry, linearity in first argument and positive definiteness ~~of the~~ properties.
$\therefore$ It gives a scalar product.

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b) Given $v_1 = (1,1,1)$, $v_2 = (2,2,0)$, $v_3 = (1,0,0)$

also $\alpha : (x_1\ x_2\ x_3) \to x_1 + x_2$
for $R^3 \to R$
α can be written as ~~$\alpha : [1\ 1\ 0]$~~
$\alpha : [1 \quad 1 \quad 0]$
Hence for $(x_1\ x_2\ x_3) \in R^3$

$$[1 \quad 1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + x_2 \quad [\alpha]$$

Now kernel of α will be 2D subspace
$\therefore$ Kernel $(\alpha) = K_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad K_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ — I

We have to find $e_1\ e_2\ e_3$ such that they are all orthonormal basis of R^3 and
$e_1 \in \text{span}(v_1)$
$\therefore$ $e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

now $e_2 = \ker(\alpha)$

from I we get $e_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

for e_3

$e_1^T e_3 = 0$, $e_2^T e_3 = 0$ (using part (a))

suppose

$$e_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then $e_1^T e_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow x + y + z = 0$ — II

$e_2^T e_3 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow x - y = 0$ — III

I + II	I − II	
$2x + z = 0$	$2y + z = 0$	
$\Rightarrow x = \frac{-z}{2}$	$y = \frac{-z}{2}$	$z = z$

hence $e_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $e_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

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Ans 8 given $A = \begin{bmatrix} 15 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 27 \end{bmatrix}$

a) To convert to $A = UDU'$ we find Eigen values.

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 15-\lambda & 0 & 6 \\ 0 & 15-\lambda & 3 \\ 6 & 3 & 27-\lambda \end{vmatrix} = \lambda^3 + 57\lambda^2 - 990\lambda + 5400 = 0$

on solving weget $\lambda_1 = 12,\ \lambda_2 = 15,\ \lambda_3 = 30$

Now finding eigen vectors.

for $\lambda_1 = 12$

$[A - \lambda I]\ \bar{V}_1 = 0 \Rightarrow \begin{bmatrix} 3 & 0 & 6 \\ 0 & 3 & 3 \\ 6 & 3 & 15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$

on solving equations

$$\left.\begin{aligned} V_1 &= -2V_3 \\ V_2 &= -V_3 \\ V_3 &= V_3 \end{aligned}\right. \Leftarrow \left\{\begin{aligned} 3V_1 + 6V_3 &= 0 \\ 3V_2 + 3V_3 &= 0 \\ 6V_1 + 3V_2 + 15V_3 &= 0 \end{aligned}\right.$$

$\bar{V}_1 = \begin{bmatrix} -2V_3 \\ -V_3 \\ V_3 \end{bmatrix}$ for $\lambda = 12$, $EV = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda_2 = 15$

$[A - \lambda I]\ \bar{V}_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 3 \\ 6 & 3 & 12 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$

on multiplying we get

$$\left.\begin{aligned} 6V_3 &= 0 \\ \Rightarrow 3V_3 &= 0 \\ 6V_1 + 3V_2 + 12V_3 &= 0 \end{aligned}\right\} \xrightarrow{\text{solving}} \begin{aligned} V_1 &= -\tfrac{1}{2}V_2 \\ V_2 &= V_2 \\ V_3 &= 0 \end{aligned}$$

$\therefore$ for $\lambda = 15$, $EV = \begin{bmatrix} -\tfrac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

for $\lambda_3 = 30$

$$[A - \lambda_3 I]\bar{V} = 0 \Rightarrow \begin{bmatrix} -15 & 0 & 6 \\ 0 & -15 & 3 \\ 6 & 3 & -3 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

on multiplying

$$\Rightarrow \left.\begin{aligned} -15V_1 + 6V_3 &= 0 \\ -15V_2 + 3V_3 &= 0 \\ 6V_1 + 3V_2 + (-3)V_3 &= 0 \end{aligned}\right\} \xrightarrow{\text{solving}} \begin{aligned} V_1 &= 2/5\, V_3 \\ V_2 &= 1/5\, V_3 \\ V_3 &= V_3 \end{aligned}$$

$$\bar{V}_1 = \begin{bmatrix} 2/5\, V_3 \\ 1/5\, V_3 \\ V_3 \end{bmatrix} \quad \therefore \text{ for } \lambda = 30, \ EV = \begin{bmatrix} 2/5 \\ 1/5 \\ 1 \end{bmatrix}$$

Now D is diagonal matrix with Eigen value.

$$D = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 30 \end{bmatrix} \qquad U = \begin{bmatrix} -2 & -1/2 & 2/5 \\ -1 & 1 & 1/5 \\ 1 & 0 & 1 \end{bmatrix}$$

and U is a ~~vec~~ matrix with Eigen vector as colums. We get U^{-1} by calculating as:

$$U^{-1} = \begin{bmatrix} -1/3 & -1/6 & 1/6 \\ -2/5 & 4/5 & 0 \\ 1/3 & 1/6 & 5/6 \end{bmatrix}$$

$$A = U D U^{-1}$$

$$= \cancel{\begin{bmatrix} -2 & -1/2 & 2/5 \\ -1 & 1 & 1/5 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1/3 & -1/6 & 1/6 \\ -2/5 & 4/5 & 0 \\ 1/3 & 1/6 & 5/6 \end{bmatrix}}\Big[\quad\Big]$$

$$A = \begin{bmatrix} -2 & -1/2 & 2/5 \\ -1 & 1 & 1/5 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 30 \end{bmatrix}\begin{bmatrix} -1/3 & -1/6 & 1/6 \\ -2/5 & 4/5 & 0 \\ 1/3 & 1/6 & 5/6 \end{bmatrix}$$

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b) Writing in form of $A = \sum_{i=0}^{rank} \lambda_i v_i v_i^T$

Our eigen values, their eigen vectors & normalized eigen vectors are as follows.

$$\text{for } \lambda_1 = 12, \quad \bar{V}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}\right\} \text{normalized}$$

$$\text{for } \lambda_2 = 15, \quad \bar{V}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}\right\} \text{normalized.}$$

$$\text{for } \lambda_3 = 30, \quad \bar{V}_3 = \begin{bmatrix} 2/5 \\ 1/5 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{bmatrix} 2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}\right\} \text{normalized}$$

$$\Rightarrow A = 12 \begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} + 15 \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix}$$

$$+ 30 \begin{bmatrix} 2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{bmatrix}$$