Assignment 4

Ans 1 a) $B = \{x^2, x, 1\}$

$T(x^2) = x + m = 0.x^2 + 1.x + m.1 = \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$

$T(x) = (m-1)x = 0.x^2 + (m-1)x + 0.1 = \begin{bmatrix} 0 \\ m-1 \\ 0 \end{bmatrix}$

$T(1) = x^2 + m = 1.x^2 + 0.x + m.1 = \begin{bmatrix} 1 \\ 0 \\ m \end{bmatrix}$

Using above we can find transformation matrix as

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & (m-1) & 0 \\ m & 0 & m \end{bmatrix}$$

b) Say $\{a, b, c\} \in Ker(T)$. Now according to the definition of kernel.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & (m-1) & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} c = 0 \\ a + bm - b = 0 \\ am + cm = 0 \end{matrix}$$

$\Rightarrow c = 0$ — I

$a + bm - b = 0$ — II

$am = 0$ — III

- If $m = 0$ then II becomes $a = b$

$\therefore$ Kernel space for $m = 0$ is $\begin{bmatrix} a \\ a \\ 0 \end{bmatrix}$

$\Rightarrow$ All polynomial of type $ax^2 + ax$ belong to kernel space where KER when $m = 0$.

• If $m \neq 0$ then according to III $a = 0$

$\therefore$ II becomes $bm = b$

Now for $m = 1$, Kernel space is $\begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$

$\therefore$ All polynomial of type bx are a part of kernel space when $m = 1$.

If $m \neq (1, 0)$ then $b = 0$ ($\because bm = b$). In that case kernel space is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

c) Similar to part (b) we have cases for image of T

• $m = 0$, then $T = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

Now since $c_1 + c_2 = 0$ $\therefore$ c_1, c_2, c_3 are not independent. Now by rank nullity theorem

(rank + nullity) = 3 $\Rightarrow$ rank = 2

(only two independent vectors)

$\therefore$ Range of T is span of vectors $\{(0,1,0), (1,0,0)\}$

• $m = 1$, $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Here nullity is 1 $\therefore$ rank = 2

$\therefore$ Image of T is span of vectors $\{(0,1,1), (1,0,1)\}$

• $m \neq (0, 1)$, then all c_1, c_2, c_3 are independent

$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}$ $\therefore$ Image is the span of $\{(0,1,m), (0, m-1, 0), (1, 0, m)\}$

Ans 2 Given $T(x,y,z) = (x+2y-z,\ 2x+3y+z,\ 4x+7y-z)$

$$\Rightarrow \begin{bmatrix} x+2y-z \\ 2x+3y+z \\ 4x+7y-z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}}_{\text{transformation matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Now let a, b, c be a vector in kernel space

$$\therefore \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} a+2b-c &= 0 \quad \text{-(I)} \\ 2a+3b+c &= 0 \quad \text{-(II)} \\ 4a+7b-c &= 0 \quad \text{-(III)} \end{aligned}$$

$2 \times \text{(II)} - \text{(III)} \Rightarrow b - 3c = 0 \Rightarrow b = 3c$

$3 \times \text{(I)} - 2 \times \text{(II)} \Rightarrow a + 5c = 0 \Rightarrow a = -5c$

Corresponding vector is $\begin{bmatrix} -5c \\ 3c \\ c \end{bmatrix}$ which forms kernel with basis $\{(-5,3,1)\}$

- For range of T we see span of column of T

$C_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $C_2 = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$ $C_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

we can see $-5C_1 + 3C_2 + C_3 = 0$

$\therefore$ C_1, C_2, C_3 are not independent. But C_1, C_2 are independent as their linear combination doesn't give 0.

$\therefore$ basis of Range of $T = \{(1,2,4), (2,3,7)\}$

Ans 3 To prove: A linear transformation $T: V \to W$ is one-one iff its kernel is singleton set of zero vector.

- First we prove if T is one-one then kernel is set of zero vector.

$\therefore$ Let T be one one. ~~State~~ Therefore zero vector of V maps to zero vector of W.

$$\Rightarrow T(0_V) = T(0_V - 0_V)$$
$$= T(0_V + (-1)0_V)$$
$$= T(0_V) + (-1)\,T(0_V) \quad \boxed{\text{linearity of } T}$$
$$= T(0_V) - T(0_V) = 0_W$$

Hence 0_V is in nullity of T.

Also if v is in null space of T then
$$T(v) = 0 = T(0_v)$$
Since T is one-one then $v = 0_v$

$\therefore$ Null space or kernel of $T = \{0_v\}$ and nullity of T is zero.

- Secondly we prove if ~~T~~ kernel(T) is set of zero vector then T is one-one.

Let v be in kernel of T then $T(v) = 0$ and $T(0) = 0$.

$\because$ T is one-one $\therefore$ $v = 0$

Now suppose $T(v_1) = T(v_2)$ for some $v_1 v_2 \in V$

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Then $0 = T(v_1) - T(v_2)$
$= T(v_1) + (-1)\,T(v_2)$ [By linearity of T]
$= T(v_1 - v_2)$

Hence $(v_1 - v_2)$ is in kernel of T, such that
$v_1 - v_2 = 0$ or $v_1 = v_2$

$\therefore$ Linear transformation is one-one.
Hence proved.

- Now let us see how independence is preserved.
Let $v = \{a_1, a_2 --- a_n\}$ be a set of independent vectors. So its linear combination
$(c_1 a_1 + c_2 a_2 --- c_n a_n) = 0$ iff all $c_i = 0$

Now for linear transformation of v linear combo. is

$\Rightarrow c_1 T(a_1) + c_2 T(a_2) ---- c_n T(a_n)$

$\Rightarrow T(c_1 a_1) + T(c_2 a_2) --- T(c_n a_n)$

$\Rightarrow T(c_1 a_1 + c_2 a_2 --- c_n a_n)$

$\because c_1 a_1 + c_2 a_2 --- c_n a_n = 0$ only if all c_i are 0 therefore $T(c_1 a_1 + c_2 a_2 --- c_n a_n)$ will only be zero if all c_i are zero.

This means
$c_1 T(a_1) + c_2 T(a_2) --- c_n T(a_n) = 0$ when all $c_i = 0$. which is the condition for independence.

$\therefore T: V \to W$ also preserves independence if it is one-one.

Ans 4 To prove: Set S of linear maps $\Lambda(V, W)$ from V to W, show that $\Lambda(V, W)$ forms vector space under pairwise addition & scalar multiplication.

So we prove all properties one by one.

- **Additive closure:** Let $T_1, T_2 \in S$ for any $a \in V$ where S is linear maps from V to W,
  Let $T_1(a) = b$ where $b \in W$
  $T_2(a) = b$

$(T_1 + T_2)a = T_1(a) + T_2(a) \in W$ ($\because T_1(a), T_2(a) \in W$)

$\therefore (T_1 + T_2) \in S$

- **Commutativity under addition:** Let $T_1, T_2 \in S$

$(T_1 + T_2)a = T_1(a) + T_2(a)$
$= T_2(a) + T_1(a) = (T_2 + T_1)a$

$\therefore$ closed under commutativity under addition

- **Associativity wrt addition:**

$((T_1 + T_2) + T_3)a = (T_1 + T_2)a + T_3(a)$
$= T_1(a) + T_2(a) + T_3(a)$

also

$(T_1 + (T_2 + T_3))a = T_1(a) + (T_2 + T_3)a$
$= T_1(a) + T_2(a) + T_3(a)$

equal

$\therefore$ Addition is commutative.

- **Additive identity:** Let $0: V \to W$ be a LT such that $0(V) = 0$ where 0 is additive identity of W.

Now let $u, v \in V$ and $\lambda \in F$, then
$o(u+v) = 0 = 0 + 0 = o(u) + o(v)$
$o(\lambda v) = 0 = \lambda 0 = \lambda o(v)$
So o is linear and belongs to transformation.
Now to see if it is identity
$(T+o)v = T(v) + o(v) = T(v) + 0 = T(v)$
$\therefore$ o is additive identity.

- Additive inverse:

~~$(T_1 + T_2)(u+v) = (T_1 + T_2)(u) + (T_1 + T_2)v$~~
~~$= (T_1 + T$~~

$(T_1 + T_2)(-v) = T_1(-v) + T_2(-v) = -T(v_1) - T_2(v)$
$= -(T_1 + T_2)v \in L$

- Closure under scalar multiplication:

Let $T \in S$ and $C \in$ Field, $a \in V$
$c\,T(v) = (cT)(v) \in L$
$\therefore$ closed under multiplication.

- Associativity under multiplication: $u, v \in V$ & $\lambda \in F$

$(\lambda T)(u+v) = \lambda T(u+v)$
$= \lambda(T(u) + T(v)) = \lambda T(u) + \lambda T(v)$
$= (\lambda T)u + (\lambda T)v$

- Multipliclicative identity: Let $T \in S$

$(1.T)v = 1 \cdot T(v) = T(v)$
$\therefore$ multiplicative identity exists

- Distributive: Let $c \in$ Field F, $T \in S$, $a \in V$

then
$(cT)(a) = c\,T(a)$

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$$\Rightarrow (c(T_1+T_2))(a) = c((T_1+T_2)a)$$
$$= c(T_1(a)+T_2(a))$$
$$= cT_1(a) + cT_2(a)$$
$$\therefore c(T_1+T_2) = cT_1 + cT_2$$

$$\Rightarrow ((c+d)T)(a) = (c+d)T(a)$$
$$= cT(a) + d\{T(a)$$
$$= (cT + dT)(a)$$
$$\therefore (c+d)T = cT + dT$$

$\therefore$ closed under distributive property.

All properties of vector space are satisfied

$\therefore$ S is a vector space.

Ans 5 V is a vector space such that $V = M \oplus N$ (direct sum)

a) for all $m_1, m_2, m_3 \in M$ and $n, n_1, n_2 \in N$ and $c \in$ field F, direct sum of M & N follows.

$$(m_1, n_1) + (m_2, n_2) = (m, n)$$

and $c(m, n) = (cm, cn)$

Now for all $V = M \oplus N$ each element of V can be written as $v = m + n$ where $v \in V$

In direct sum this shows ordered pairs (m, n) in V

$\therefore \forall v \in V$, $\exists m \in M$ and $n \in N$ such that $v = m + n$.

b) $P: V \to V$ is a projection of V along M onto N as $P(v) = n$

$\Rightarrow v = m + n$, $m \in M$ and $n' \in N$

$\Rightarrow P(m + n') = n$

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a) Let $V_1, V_2 \in V$ $\therefore$ $V_1 = m_1 + n_1$

$V_2 = m_2 + n_2$

$m_2, m_1 \in M$ and $n_1, n_2 \in N$

$V_1 + V_2 = (m_1 + m_2) + (n_1 + n_2)$

Here $(m_1 + m_2) \in M$ and $(n_1 + n_2) \in N$ as it is closed under addition.

$\therefore P(V_1 + V_2) = n_1 + n_2 = P(V_1) + P(V_2)$

Hence closed under vector addition

Also for $c \in F$ (field)

$cV_1 = cn_1 + cm_1$

~~$\therefore cn_1, cm_1 \in$~~

$\therefore cn_1 \in N$ and $cm_1 \in M$

$\therefore P(cV_1) = cn_1 = cP(V_1)$

$\therefore$ Closed under scalar multiplication

$\therefore P$ is linear.

b) $P(v) = n$ and $v = m + n$

for $P^2 v = P(P(v)) = P(n) = n$

So for any $P^n v = n$ $\therefore$ it is idempotent.

c) V is defined on natural numbers N.

Let $y \in$ Range (P)

$\therefore \exists x \in V$ such that $y = P(x)$

and we already know $x = \underbrace{m + n}_{\text{also} \in N}$

Also,

$P(x) = n = y \in$ range (P)

$\therefore n \in$ range (P) $\therefore$ Range (P) $\Rightarrow N$

d) For kernel of P

$P(x) = 0$ where $x = n + m$ & $P(x) = n$

$\Rightarrow P(x) = n = 0 \Rightarrow n = 0$

$\therefore x = 0 + m = m \in M$

$\therefore P(x) = P(m) = 0$

$\therefore$ kernel of $P = m$

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c) To prove that $(I-P)$ is projection of V along N ~~and~~ onto M, first it has to be idempotent.

So for $(I-P)^2 = (I-P)(I-P)$

$$= I - 2P + P^2$$

$$= I - 2P + P \quad \left(P \text{ is idempotent already proved}\right)$$

$$= I - P$$

$\therefore (I-P)$ is idempotent. Now P is projection of v onto vector space N. i.e.

$P(v) = n$ and $v = m + n$

$$\Rightarrow m = v - n$$

$$m = v - P(v)$$

$$m = (I - P)v$$

As we can clearly see $(I-P)$ is projection of v onto vector space M where $m \in M$.

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Ans 6 Alice wins if determinant is non zero i.e none of the columns are linearly dependent.
Bob wins if determinant becomes zero i.e. some of the columns are linearly dependent.
Since Alice starts first, cases can be:
i) n is even ii) n is odd.

i) If n is even: For every move of Alice Bob responds by putting number in same row in such a way that it makes columns linearly dependent to other. eg say Alice puts 8 in $r_1 c_1$ then Bob can respond by putting 8 in $r_1 c_2$. In this way Bob will replicate two colums & wins the game always.
If both players are playing optimally, Alice will never win if n is even.
ii) If n is odd: In this case last entry will be done by ~~al~~ Alice. Alice tries to make each column independent after her first move. This independent value will be based on what Bob ~~wi~~ inputs. Now Alice will start from $r_1 c_1$, then Bob replicates in any other row, so now Alice will put in $r_n c_n$, then again Bob replicates but ~~not in same col~~ it cannot do it optimally in same column

$$\begin{bmatrix} A_1 & B_1 & B_4 \\ A_5 & A_3 & B_3 \\ B_2 & A_4 & A_2 \end{bmatrix}$$

$A_i \rightarrow$ Alice's ~~~~ entry in i^{th} turn
$B_i \rightarrow$ Bob's entry in i^{th} turn.

Basically what we are doing above is that putting Alice's elements in priciple diagonal's ends alternatively. This puts Bob in a position where it has to decide whether to replicate Alice or Break its alternating turns on the diagonal.

In this way Alice will win because it controls the last entry ~~at~~ and the diagonal elements. Which means Bob is unable to create dependence for any two colums.

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Ans 7 a) • To prove that if left inverse of Transformation exist then it is injective

Let transformation be $S: V \to X$ where $S(T(x)) = x \quad \forall x \in X$. S is the left inverse of T.

Let $x, y \in X$ such that

$T(x) = T(y)$ then

$x = S(T(x)) = S(T(y)) = y$

$\therefore$ for $T(x) = T(y) \Rightarrow x = y$ $\therefore$ T is injective.

• Now to prove if Transformation is injective then left inverse exists.

Let $\{x_i : i \in X\}$ be a basis of x

Then $\alpha = \{T(x_i) : i \in X\}$ is linearly independent subset of U.

So there exists a basis β of U such that $\alpha \subset \beta$.

Let $S: U \to X$ be linear map defined on β by

$$S(u) = \begin{cases} v & \text{if } u \in \alpha \text{ with } u = T(v) \\ 0 & \text{if } u \notin \alpha \end{cases}$$

Then for $x = \sum \lambda_i x_i \in X$ we have

$$S(T(x)) = S(T(\sum \lambda_i x_i)) = \sum \lambda_i S(T(x_i))$$

$$= \sum \lambda_i x_i = x$$

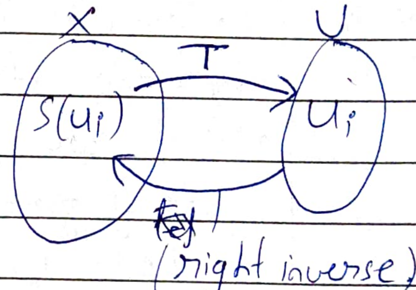
$\therefore$ S is left inverse of T.

b) Let $T : X \to U$. Then if there exists an operator $S : U \to X$ such that $T(S(u)) = u$ for all $u \in U$, then S is called right inverse of T.

- To prove: Right inverse exists then T is surjective. [right inverse]

As ~~T~~ S, exists for every $u_i \in U$ we have $S(u_i) \in X$ such that $T(S(u_i)) = u_i$

∴ T is surjective.



- To prove: If T is surjective then right inverse exists.
- Let $B = \{u_i : i \in I\}$ be basis of U
- As T is surjective, there exists $\{x_i : i \in I\}$ where $x_i \in X$ such that $T(x_i) = u_i$

$\Rightarrow \{x_i : i \in I\}$ defined above independent as shown below.

Let $\alpha_1 \; \alpha_2 \; \dots \; \alpha_n$ be scalars such that

$$\alpha_1 x_1 + \alpha_2 x_2 \; \dots \; \alpha_n x_n = 0$$
$$T(\alpha_1 x_1 + \alpha_2 x_2 \; \dots \; \alpha_n x_n) = 0$$
$$\alpha_1 T(x_1) + \alpha_2 T(x_2 \; \dots \; + \alpha_n T(x_n) = 0$$
$$\alpha_1 u_1 + \alpha_2 u_2 \; \dots \; \alpha_n u_n = 0$$

As u_i is basis for u $\alpha_1 = \alpha_2 \; \dots = \alpha_n = 0$
(∵ because of independence)

∴ $\{x_i : i \in I\}$ are independent.

Now Let $S : U \to X$ be a linear map defined as

$$S(u_i) = \begin{cases} v & , \text{if } u \in B \text{ with } u = T(v) \\ 0 & , \text{if } u \notin B \end{cases}$$

where B is basis of u.

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Then for $u = \sum \lambda_i u_i$ we have

$$T(S(u)) = T(S(\sum u_i \lambda_i))$$

$$= \sum \lambda_i T(S(u_i)) = \sum \lambda_i u_i = u$$

$\therefore T(S(u)) = u$ and hence it is right invertible.