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Assignment 3

Ans 1 (a) $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \Leftrightarrow \begin{array}{l} x+3y+5z=14 \\ 2x-y-3z=3 \\ 4x+5y-z=7 \end{array}$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \xrightarrow{R_2=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 4 & 5 & -1 & 7 \end{array}\right] \xrightarrow{R_3=R_3-4R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right] \xrightarrow{R_3=R_3-R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$\Rightarrow x_1+3x_2+5x_3=14$

$-7x_2-13x_3=-25$ $\Rightarrow -7x_2 - 13\times3 = -25 \Rightarrow \boxed{x_2=-2}$

$-8x_3=-24$ $\boxed{x_3=3}$

$x_1 = 14 - 3\times(-2) - (5\times3)$

$\boxed{x_1=5}$

(b) $\begin{array}{l} y+z=4 \\ 3x+6y-3z=3 \\ -2x+(-3y)+7z=10 \end{array} \Leftrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_1} \left[\begin{array}{ccc|c} 3 & 6 & -3 & 3 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{R_1=R_1/3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right] \xrightarrow{R_3=R_3+(2)R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right] \longrightarrow$$

$$\xrightarrow[R_3 = R_3 - R_2]{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right] \xrightarrow{R_3 = R_3/4} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 + 3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \therefore \boxed{\begin{array}{l} x_1 = -1 \\ x_2 = 2 \\ x_3 = 2 \end{array}}$$

Ans 2 1) Absolutely summable sequence $\sum_{i=0}^{\infty} |x_i| < \infty$

- Zero vector is part of absolutely summable as $(0, 0, 0 \ldots 0) \Rightarrow \sum_{i=1}^{\infty} 0 = 0$

- Additive closure :

Let $A \in S$ and $\sum_{i=1}^{\infty} |a_i| < K_A$ where $a_i \in A$

Let $B \in S$ and $\sum_{i=1}^{\infty} |b_i| < K_B$ where $b_i \in B$

Now $A + B = (a_1, a_2, a_3 \ldots) + (b_1, b_2, b_3 \ldots)$

$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots \ldots)$

~~Now in absolutely summable~~

Now for absolute sum $|a+b| \leq |a| + |b|$
$(a, b) \in \mathbb{R}$

$\therefore$

$|a_1 + b_1| + |a_2 + b_2| \ldots \leq |a_1| + |b_1| + |a_2| + |b_2| \ldots$

$\leq |a_1| + |a_2| + \ldots + |b_1| + |b_2| \ldots$

$\leq K_A + K_B$

$\therefore$ It is closed under addition.

- Multiplicative closure :

Let $A \in S$ and $\sum_{i=1}^{\infty} |a_i| < K_A$ for $a_i \in A$

Now $KA = (Ka_1, Ka_2, Ka_3 \ldots)$ $K \in \mathbb{R}$ (scalar)

Since $|a.b| \leq |a|.|b|$ $(a, b) \in \mathbb{R}$

$\therefore \sum_{i=1}^{\infty} |Ka_i| = |Ka_1| + |Ka_2| + |Ka_3| \ldots$

$\leq |k||a_1| + |k||a_2| + |k||a_3| - - - -$
$\leq |k| (|a_1| + |a_2| + |a_3| - - -)$
$\leq |k| \, K_A$

$\therefore$ K_A is closed under scalar multiplication.
$\therefore$ Absolutely summable sequence form a subspace.

2) Bounded sequences : $\exists M > 0$ s.t. $|x_i| \leq M \, \forall i$

- Zero vector is part of bounded sequence
  $A = (0, 0 - - -)$ is bounded seq as all $a_i \leq M$, where $M > 0$

- Additive closure

Let $A \in S$ and $|a_i| < M_A$ $\forall a_i \in A$
and Let $B \in S$ where $|b_i| \leq M_B$ $\forall b_i \in B$

Now

$A + B = (a_1 + b_1, a_2 + b_2, - - - -)$

We know ~~let a_i + b_i~~

$|a_i + b_i| \leq |a_i| + |b_i|$
$|a_i + b_i| \leq |M_A| + |M_B|$

$\Rightarrow$ Bounded sequence is closed under addition.

- Multiplicative closure (scalar)

Let $A \in S$ and $|a_i| \leq M$ $\forall a_i \in A$

Now

$k.A = \{ka_1, ka_2 - - - \}$

where k is a scalar.

Now $|ka_i| = \{ka_1, ka_2, ka_3 - - -\}$
$|ka_i| \leq |k||a_i|$ $\forall ka_i \in kA$

$\therefore$ closed under scalar multiplication.
Hence bounded sequence form a subspace.

3) Arithimatic sequence : $x_i = a + di$

- Zero vector is part of arithimetic subsequence.
put $a=0$, $d=0$ $\{0, 0, ---\}$

- Additive closure

Let $A \in S$ where $a_i = a + di$ (a,d fixed val)
Let $B \in S$ where $b_i = b + ci$ (b,c fixed val)

Now

$$A+B = \{a+d+b+c,\ a+2d+b+2c,\ a+3d+b+3c\ ---\}$$

$$= \{(a+b)+(d+c),\ (a+b)+2(d+c),\ (a+b)+3(c+d)\ ---\ ---\}$$

$\therefore$ we see $A+B$ is arithimetic sequence with initial val $= (a+b)$ and common diff $= (c+d)$.
$\therefore$ closed under vector addition.

- Scalar multiplication closure

Let $A \in S$ with $a_i = a + di$ (a,d fixed val)

now $cA = \{ca + cd,\ ca + 2cd,\ ca + 3cd\ ---\ ---\}$
(scalar)

we can see that cA is still an arithimatic sequence. Just the common difference and initial values (cdi and ca respectively) are changed. So it is closed under scalar multiplication.

Since above properties are satisfied, therefore Arithimetic sequence is a subspace.

4) Geometric sequence : $x_i = ar^i$

It violates vector addition property.

eg $A = \{1, 2, 4, 8\ ---\}$
$B = \{1, 3, 9, 27\ ---\}$

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Now $A+B = \{2, 5, 13, 35 ----\}$

On checking common ratio we see that

$\frac{5}{2} \neq \frac{13}{5}$. which means $(A+B)$ is not a geometric sequence.

$\therefore$ Not closed under vector addition and hence it is <u>not a subspace</u>.

Ans 3 Let S be set of all functions $f:[0,1]\to\mathbb{R}$ where f is continuous.

i) for some $f, g \in S$

$(f+g)(x) = f(x) + g(x)$

$\because f(x) \in \mathbb{R}$ and $g(x) \in \mathbb{R}$ $\therefore f(x)+g(x) \in \mathbb{R}$

$\therefore f+g \in S$ hence closed under addition.

ii) for $f, g \in S$

$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$

$\therefore$ Addition is commutative.

iii) for some $f, g, h \in S$

$(f + (g+h))(x) \Rightarrow f(x) + (g+h)(x)$

$\Rightarrow f(x) + g(x) + h(x)$

$\Rightarrow (f+g)(x) + h(x)$

$\Rightarrow ((f+g) + h)(x)$

$\therefore$ Associativity is followed under addition.

iv) Let $\phi \in S$ such that $\phi(x) = 0$

now $(\phi + f)(x) = \phi(x) + f(x) = 0 + f(x) = f(x)$

also $(f + \phi)(x) = f(x) + \phi(x) = f(x) + 0 = f(x)$

$\therefore$ ~~$\phi(x)$ is~~ $\phi(x) = 0$ is identity.

5) Additive inverse: $\forall f \in S$

$f(x) + (-f)(x) = f(x) - f(x) = \phi(x) = 0$

$\therefore$ Additive inverse exists $((-f)(x)$ or $f(-x))$

6) Scalar multiplication: $\forall$ $c \in R$ and $f \in S$

$c f(x) = (cf)(x) = f'(x)$

$\because f'(x)$ is continuous $\therefore$ closed under scalar multiplication.

7) For any $a \in R$ and $f, g \in S$

$a(f+g)(x) = a(f(x) + g(x)) = af(x) + ag(x)$

$\therefore$ Distributive property closed under vector addition and scalar multiplication.

8) For any $a, b \in R$ and $f \in S$

$(a+b) f(x) = af(x) + bf(x)$

$\therefore$ Closed under distributive property for scalar addition and vector to scalar multiplication.

9) For any $a, b \in R$ and $f \in S$

$(ab) f(x) = a(bf(x))$

$\therefore$ Closed under associative property for scalar multiplication.

10) For unit scalar $1 \in R$ and $f \in S$

$(1 . f)(x) = 1 . f(x) = f(x)$

$\therefore$ Multiplicative identity exists.

$\because$ All properties of vector space are satisfied $\therefore$ S is a vector space over R.

Ans 4 Given vectors $f(x) = x$, $g(x) = e^x$, $h(x) = e^{-x}$

we have to prove that they are linearly independent.

Let us assume that f, g, h are dependent.

then $a_1 x + a_2 e^x + a_3 e^{-x} = 0$

for all $x \in [0,1]$ and a_i $(i = \{1,2,3\})$ for

Scalars in a_i

Putting $x = 0$ $\quad a_2 = -a_3$ — I

Putting $x = 1/2$ $\quad \dfrac{a_1}{2} + \sqrt{e}\, a_2 + \dfrac{a_3}{\sqrt{e}} = 0$ — II

Putting $x = 1$ $\quad a_1 + a_2 e + \dfrac{a_3}{e} = 0$ — III

Put $a_2 = -a_3$ in II

$$\frac{a_1}{2} + a_2\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0 \quad \text{— IV}$$

Put $a_2 = -a_3$ in III

$$a_1 + a_2\left(e - \frac{1}{e}\right) = 0 \Rightarrow a_1 = -a_2\left(e - \frac{1}{e}\right)$$

Put val of a_1 in IV

$$-\frac{a_2}{2}\left(e - \frac{1}{e}\right) + a_2\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) = 0 \Rightarrow \boxed{a_2 = 0}$$

$\therefore$ $a_2 = -a_3 \Rightarrow \boxed{a_3 = 0}$

put a_2, a_3 in III we get $\boxed{a_1 = 0}$

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put a_2, a_2, in equn III we get

$\boxed{a_1 = 0}$

Since all a_i need to be 0 for linear combination to be 0 therefor our assumption that f, g, h are dependent is false.

$\therefore$ f, g, h are linearly independent.