Assignment 4

LA Monsoon 2020, IIIT Hyderabad

November 7, 2020

Due date: November 12, 2020

General Instructions: All symbols have the usual meanings (example: F is an arbitrary field, $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of natural numbers, and so on.) Remember to prove all your intermediate claims, starting from basic definitions and theorems used in class to show whatever is being asked. You may use any other non-trivial theorems not used in class, as long as they are well known and a part of basic linear algebra texts. It is always best to try to prove everything from definitions. Arguments should be mathematically well formed and concise.

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1. [2 points] Define V as the vector space of all polynomials in x of degree ≤ 2 over $\mathbb{R}$. Define a basis $B = \{x^2, x, 1\}$. Define a linear transformation T as

$$T(x^2) = x + m$$
$$T(x) = (m-1)x$$
$$T(1) = x^2 + m$$

Answer the following:

(a) Find the matrix representation of T relative to the given basis.

(b) Find kernel(T) for all values of m.

(c) Find the image of T for all values of m.

2. [2 points] Define a linear transform $T : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

$$T(x, y, z) = (x + 2yz, 2x + 3y + z, 4x + 7yz)$$

What is the geometry of range and kernel of T? Find their equations.

3. [2 points] Prove that a linear transformation $T : V \to W$ between two vector spaces is one-to-one iff it's kernel is a singleton set of zero vector. Also show that such a transformation will preserve linear independence.

4. [1 point] Let V and W be 2 vector spaces. Define by $\Lambda(V, W)$ the set of all linear maps from V to W. Show that $\Lambda(V, W)$ forms a vector space under pointwise addition and scalar multiplication.

5. [4 points] Let V be a vector space and suppose that $V = M \bigoplus N$.

(A) Show that $\forall \mathbf{v} \in V \exists \mathbf{m} \in M, \mathbf{n} \in N s.t. \mathbf{v} = \mathbf{m} + \mathbf{n}$.

(B) Define $P : V \to V$, called (WLOG) a projection of V along M onto N as $P(\mathbf{v}) = \mathbf{n}$. Prove the following:

(a) P is linear.

(b) P is idempotent.

(c) $\text{range}(P) = N$

(d) $\text{kernel}(P) = M$

(C) Show that $I - P$ is the projection of V along N onto M.

6. [2 points] Suppose A is an $n \times n$ matrix, each of whose entries are blanks. Alice and Bob play a game where they pick a random real number and place it in one of the entries of the matrix. Alice wins if in the end, the determinant of the resulting matrix is non-zero, while Bob wins if it is zero. Find the conditions on n for Alice and Bob to have winning strategies. What are those strategies?

7. [2 points] Prove that:

(a) A linear transformation between vector spaces has a left inverse iff it is injective.

(b) A linear transformation between vector spaces has a right inverse iff it is surjective.