Assignment 1

LA Monsoon 2020, IIIT Hyderabad

October 12, 2020

Due date: October 18, 2020

General Instructions: All symbols have the usual meanings (example: F is an arbitrary field, $\mathbb{R}$ is the set of reals, $\mathbb{N}$ the set of natural numbers, and so on.) Remember to prove all your intermediate claims, starting from basic definitions and theorems used in class to show whatever is being asked. You may use any other non-trivial theorems not used in class, as long as they are well known and a part of basic linear algebra texts. It is always best to try to prove everything from definitions. Arguments should be mathematically well formed and concise.

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1. [1 point] Prove that the set $F = \{0, 1\}$ is a field if we define addition as the boolean **XOR** and multiplication as the boolean **AND** gates.

2. [2 points] Which of the following pairs of sets V and F form a valid vector space? Prove all your claims. (Addition and multiplication operations are defined as usual arithmetic on real numbers. Note that you have to check if V is a vector space over F.)

   (a) $V = \mathbb{R}$ and $F = \mathbb{N}$
   (b) $V = \mathbb{Q}$ and $F = \mathbb{R}$
   (c) $V = \mathbb{R}$ and $F = \mathbb{Q}$
   (d) $V = \mathbb{R}$ and $F = \mathbb{C}$

3. [3 points] Prove that a field F is a vector space over itself. Also show that the direct sums of a field F will form a vector space V over F. *Unmarked: Do you see a pattern here relating to the previous question?*

4. [1.5 points] Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$\begin{aligned}(x, y) + (x_1, y_1) &= (x + x_1, 0) \\ c(x, y) &= (cx, 0)\end{aligned}$$

   Is V, with these operations a vector space? Prove or Disprove.

5. [2.5 points] Which of the following sets of vectors $\alpha = (a_1, \dots, a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geq 3)$?

   (a) All α such that $a_1 \geq 0$
   (b) All α such that $a_1 + 3a_2 = a_3$
   (c) All α such that $a_2 = a_2^2$
   (d) All α such that $a_1 a_2 = 0$
   (e) All α such that a_2 is rational.