Ans 1 Let $S = (s_1, s_2 \cdots s_m)$ and $\beta \in V$ but $\beta \notin \text{span}(S)$

We have to prove that $S' = (s_1, s_2 \cdots s_m, \beta)$ is linearly independent. For that, let

$$c_1 s_1 + c_2 s_2 \cdots c_m s_m + c\beta = 0$$

$$\Rightarrow \quad \beta = -\frac{1}{c}(c_1 s_1 + c_2 s_2 \cdots c_m s_m)$$

Here, if $c \neq 0$ then there is a solution for β such that $\beta \in \text{span}(S)$

But we know that $\beta \notin \text{span}(S)$. $\therefore c = 0$

$\therefore c_1 s_1 + c_2 s_2 + \cdots c_m s_m + c\beta = 0$ only when all c_i and c are zero (as S is linearly independent). Therefore $S' = (s_1, s_2 \cdots s_m, \beta)$ is linearly independent.

Ans 2 We know span is closed under linear combination.

Let $u, v \in \text{span}(S)$ and α, β be scalars.

By definition of span

$$u = c_1 S_1 + c_2 S_2 \cdots c_m S_m$$
$$v = d_1 S_1 + d_2 S_2 \cdots d_m S_m$$

For some scalars c, d

$$\Rightarrow \alpha u + \beta v = \alpha(c_1 S_1 + c_2 S_2 - c_m S_m) + \beta(d_1 S_1 \cdots d_m S_m)$$
$$\alpha u + \beta v = (\alpha c_1 + \beta d_1) S_1 + (\alpha c_2 + \beta d_2) S_2 \cdots \cdots (\alpha c_m + \beta d_m) S_m$$

As we can see in above equation, RHS is a linear combination of elements of S, and is therefore a span of S.

We know span is closed under linear combination and is a subspace of V.

Ans 3 (a)

$$\begin{bmatrix} 12 & 4 & 4 \\ 5 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{5R_1}{12}} \begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_3 = R_3 - \frac{R_1}{6}}$$

$$\begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 0 & 7/3 & 10/3 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 12 & 4 & 4 \\ 0 & 7/3 & 10/3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 = \frac{R_1}{4}]{R_2 = \frac{3R_2}{7}} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 10/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 3 & 0 & \frac{-3}{7} \\ 0 & 1 & 10/7 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1/3} \begin{bmatrix} 1 & 0 & -1/7 \\ 0 & 1 & 10/7 \\ 0 & 0 & 0 \end{bmatrix}$$

Now to solve

$$\begin{bmatrix} 1 & 0 & -1/7 \\ 0 & 1 & 10/7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + \left(-\frac{x_3}{7}\right) &= 0 \\ x_2 + \left(\frac{10x_3}{7}\right) &= 0 \\ 0 &= 0 \end{aligned}$$

$\Rightarrow x_3 = x$

$\Rightarrow x_2 = \left(\frac{-10}{7}\right)x$

$\Rightarrow x_1 = \left(\frac{x}{7}\right)$

$\therefore$ null space is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/7 \\ -10/7 \\ 1 \end{bmatrix} x$$

b) $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{(R_2 = R_2 - 6R_1)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{(R_3 = R_3 - 3R_1)}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & -2 & -5 \end{bmatrix} \xrightarrow[\left(R_2 = \frac{-R_2}{7}\right)]{\left(R_3 = R_3 - \frac{2R_3}{7}\right)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{(R_2 = R_2 + R_3)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\xrightarrow{R_3 = -R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\left(R_1 = R_1 - 3R_3\right)]{\left(R_1 = R_1 - 2R_2\right)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now solving,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}$ null space $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} 12 & 3 & 9 \\ 20 & 5 & 15 \\ 16 & 4 & 12 \end{bmatrix} \xrightarrow{\left(R_2 = R_2 - \frac{5R_1}{3}\right)} \begin{bmatrix} 12 & 3 & 9 \\ 0 & 0 & 0 \\ 16 & 4 & 12 \end{bmatrix} \xrightarrow{\left(R_3 = R_3 - \frac{4R_1}{3}\right)}$

$\begin{bmatrix} 12 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\left(R_1 = \frac{R_1}{12}\right)} \begin{bmatrix} 1 & 1/4 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now solving,

$\begin{bmatrix} 1 & 1/4 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + \frac{x_2}{4} + \frac{3x_3}{4} = 0, \quad 0 = 0, \quad 0 = 0$

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$$\Rightarrow \left.\begin{array}{l} x_2 = x_2 \\ x_3 = x_3 \end{array}\right\} \text{arbitrary values} \qquad x_1 = -\frac{1}{4}x_2 - \frac{1}{3}x_3$$

$$\therefore \text{null space} = \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix} x_3$$

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Ans 4 Given: W is subspace of V and Basis $(W) = \{\alpha_i : i \in [m]\}$

- Say F is the field of V

Now

$$\text{span}(W) = c_1\alpha_1 + c_2\alpha_2 ---- c_m\alpha_m$$

where $(c_1, c_2 ---- c_m) \in F$ and $(\alpha_1, \alpha_2 --- \alpha_m) \in \text{Basis}(W)$

So now the span of $[\alpha_i + \beta : i \in [m]]$ becomes

$$S = c_1(\alpha_1 + \beta) + c_2(\alpha_2 + \beta) ----- c_m(\alpha_m + \beta)$$

$$= (c_1\alpha_1 + c_2\alpha_2 ---- c_m\alpha_m) + (c_1 + c_2 --- c_m)\beta$$

Let $(c_1 + c_2 --- c_m) = x$

Since Field is closed under addition, $x \in F$.

$\therefore$ $S = (c_1\alpha_1 + c_2\alpha_2 ---- c_m\alpha_m) + x\beta$

Now let $x\beta = a$ which is a vector (closed under scalar multiplication)

$$S = \underbrace{(c_1\alpha_1 + c_2\alpha_2 ---- c_m\alpha_m)}_{\text{span}(W)} + a$$

Above equation is a vector addition which doesn't change the number of dimensions in S.

Since span(W) is a m-dimensional subspace

$\therefore$ $S = (c_1\alpha_1 + c_2\alpha_2 ---- c_m\alpha_m) + a$

is also m-dimensional subspace.

$\Rightarrow$ $[\alpha_i + \beta : i \in [m]]$ where $\beta \in V/W$ is a m-dimensional subspace.

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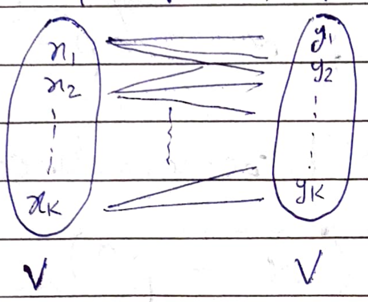
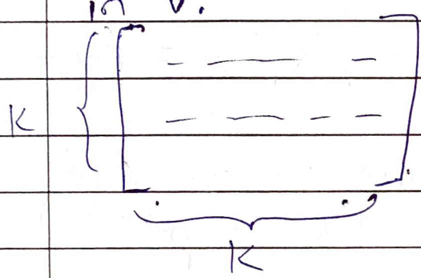
Ans 5 (a) Field F has p^n elements and V is a k-dimensional vector space over F.

$T: V \longrightarrow V$



This linear transformation can be represented as a K x K matrix which will have the elements of the field F.

It is k x k because there are k dimensions in V.



Here each element is a unique mapping in $T: V \to V$.
# of ways to set one mapping = p^n.

$\therefore$ for all k x k mappings on Linear transformation

$= (p^n)^{k^2} = \boxed{(p^{nk^2})}$

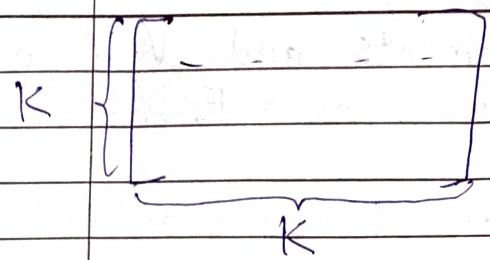
Note: By mappings I refer to linear transformation.

b) To find number of invertible linear transformation $T: V \to V$ we will use info in the previous part.

- For matrix to be invertible Determinant has to be non zero.

Therefore we will remove all combinations which can be linear ~~transformati~~ dependent.

In other words we will form the K x K matrix in such a way such that no row is linear ~~trans~~ combination of the other rows.



Number of ways to form first row $= (p^n)^k - 1$

Say, $\alpha = (p^{nk})$

Now for row 2 we will remove linear combination of row 1. i.e $= (\alpha - p^n)$.

Similarly for row 3, we remove combination of row 1, row 2 $= (\alpha - p^n \times p^n) = (\alpha - p^{2n})$

$\vdots$

Similary for k^{th} row we remove combination of first $(k-1)$ rows $= (\alpha - p^{(k-1)n})$

Multiplying above all to get total # of ways

$$= \alpha(\alpha - p^n)(\alpha - p^{2n}) \cdots (\alpha - p^{(k-1)n})$$

$$= \prod_{i=0}^{(k-1)} (\alpha - p^{in}) = \boxed{\prod_{i=0}^{k-1} (p^{nk} - p^{in})}$$