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Assignment 1

Ans 1 $F = \{0, 1\}$ when addition $\Rightarrow$ XOR $(\oplus)$
multiplication $\Rightarrow$ AND $(.)$

We check properties of field for F

1) Additive identity : for 0 , $0 \oplus 0 = 0$
for 1 , $1 \oplus 0 = 1$

$\therefore$ here 0 is identity element for addition

2) Additive inverse : for 0 ; $0 \oplus 0 = 0$
for 1 , $1 \oplus 1 = 0$

There is additive inverse for all elements of F.

3) Associativity for addition : Here addition is represented by XOR operation which ~~is~~ follows associativity law . i.e.

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

$\therefore$ F is associative.

4) Commutativity for addition : Similar to above property XOR follows commutative property i.e. $a \oplus b = b \oplus a$

$\therefore$ F is commutative.

5) Multiplicative identity : for $0 \rightarrow 0 . 1 = 0$
for $1 \rightarrow 1 . 1 = 1$

$\therefore$ 1 is multiplicative identity for F.

6) Multiplicative inverse : for 1 , $1 . 1 = 1$

$\therefore$ for all non-zero (additive) elements inverse exists in F.

7) Associativity for multiplication : Here multiplication is represented with AND operation which follows associative law, i.e.

$$a . (b . c) = (a . b) . c$$

8) Commutativity for multiplication : Similar to above property AND operation also follows commutative law i.e. $(a . b) = (b . a)$

$\therefore$ F is commutative over multiplication.

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9) Distributive property: for distributive $a.(b \oplus c) = (a.b) \oplus (a.c)$ should be true. where $a, b, c \in F$.

i.e. $a.(b \oplus c) = a.(b\bar{c} + \bar{b}c)$

$= ab\bar{c} + a\bar{b}c$

$\Rightarrow (a\bar{a}b + ab\bar{c}) + (a\bar{a}c + \bar{b}ac)$ $[\because a.\bar{a} = 0]$

$\Rightarrow ab(\bar{a} + \bar{c}) + ac(\bar{a} + \bar{b})$

$\Rightarrow ab(\overline{ac}) + ac(\overline{ab})$

$\Rightarrow (a.b) \oplus (a.c)$

$\because F = \{0, 1\}$ follows all properties of field

$\therefore$ It is a field.

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Ans 2 (a) $V = R$ and $F = N$.

Check if F is field.

- Identity element: $\because 0 \notin N$ Therefore additive identity element is not in F. Hence $F = N$ is not a field and $V = R$ is NOT a ~~set~~ vector space.

b) $V = \mathbb{Q}$ $F = R$

Check if F is a field

- Identity element: Set of real numbers has 0 as additive identity & 1 as multiplicative identity.
  $\therefore \forall a \in (F = R)$ $(a + 0) = (0 + a) = a$
  and $(a . 1) = (1 . a) = a$
- Inverse property: For real numbers all elements $a \in (F = R)$ has additive inverse $(-a)$ and multiplicative inverse $(1/a)$. $\therefore \forall a \in (F = R)$
  $a + (-a) = (-a) + a = 0$
  and $a . \frac{1}{a} = \frac{1}{a} . a = 1$
- Associative property: Real number follow associativity i.e $\forall a, b, c \in (F = R)$ $(a + b) + c = a + (b + c)$
  and $(a . b) . c = a . (b . c)$
- Distributive property: Distributive law is valid for real nos. i.e $\forall a, b, c \in (F = R)$
  $a . (b + c) = (ab) + (ac)$.
- Commutative property: Real nos follow ~~associati~~ commutativity. i.e. $\forall a, b, c \in (F = R)$
  $(a + b) = (b + a)$ and $(a . b) = (b . a)$

$\therefore$ $(F = R)$ is a field.

Check if V is vector space

- Scalar multiplication: For V to be vector space it should follow below property.

$a.\bar{\alpha} \in V$ where $a \in F$ and $\bar{\alpha} \in V$

But if $a = \sqrt{2}$ (irrational) ~~and~~ then

$\sqrt{2}.\bar{\alpha} \notin Q$ as it is irrational.

$\therefore$ $a.\bar{\alpha} \notin V$ Hence it is not a vector space.

c) ~~Take~~ $V = R$ and $F = Q$

Check for field Q.

- Identity element: Set of rational numbers have additive identity 0 and multiplicative identity 1.

$\therefore \forall\ a \in (F = Q)$ $a + 0 = 0 + a = a$

and $a.1 = 1.a = a$

- Inverse property: For all $a \in (F = Q)$ there exist an additive inverse $(-a)$ and multiplicative inverse $(1/a)$ such that

$a + (-a) = (-a) + a = 0$ and $a.\frac{1}{a} = \frac{1}{a}.a = 1$

- Commutative property: Rational nos follow commutativity i.e. $\forall\ a, b \in Q$ $(a + b) = (b + a)$ and $(a.b) = (b.a)$

- Associativity: Rational nos follow associativity i.e. $\forall\ a, b, c \in Q$ $a + (b + c) = (a + b) + c$

and $a.(b.c) = (a.b).c$

- Distributive property: Rational nos follow distributive law. i.e $\forall\ a, b, c \in (F = Q)$

$a.(b + c) = (a.b) + (a.c)$

$\therefore$ $(F = R)$ is a field

Check for vector space V

i) For any 2 vectors $\bar{\alpha}, \bar{\beta} \in V$, $\bar{\alpha} + \bar{\beta} \in V$

As ~~we~~ we know real nos are closed under addition

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2) For any 2 vectors $\bar{\alpha}, \bar{\beta} \in V$ then $\bar{\alpha}+\bar{\beta} = \bar{\beta}+\bar{\alpha}$
As we know real numbers are commutative w.r.t addition.

3) for any 3 vectors $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in V$
$(\bar{\alpha}+\bar{\beta})+\bar{\gamma} = \bar{\alpha}+(\bar{\beta}+\bar{\gamma})$ as we know that real numbers follow associativity.

4) There should be a unique vector $\phi \in V$ such that
$\bar{\alpha}+\phi = \phi+\bar{\alpha}$, Here $\phi = 0 \in R$

5) For every $\bar{\alpha} \in V$ there is a unique vector $(-\bar{\alpha})$ such that $\bar{\alpha}+(-\bar{\alpha}) = (-\bar{\alpha})+\bar{\alpha} = \phi$
Here $\bar{\alpha} \in R$, $\therefore$ $(-\bar{\alpha})$ exists for all elements.

prop 1 • We know that $Q \subset R$. We will use this fact for the following properties

6) For any $a \in F$ and any $\bar{\alpha} \in V$, $a\bar{\alpha} \in V$
Here $a\bar{\alpha} \in R$, where $a \in Q$ and $\bar{\alpha} \in R$
(using prop 1)

7) For any $a \in Q$ and $\bar{\alpha}, \bar{\beta} \in R$
$a.(\bar{\alpha}+\bar{\beta}) = a\bar{\alpha}+a\bar{\beta}$
($\because$ multiplication is distributive over addition)

8) For any $a, b \in Q$ and $\bar{\alpha} \in R$,
$(a+b)\bar{\alpha} = a\bar{\alpha}+b\bar{\alpha}$
(using prop 1 ~~and~~ we can see above property is valid)

9) For any two scalars $a, b \in F$ and $\bar{\alpha} \in V$
$(ab)\bar{\alpha} = a(b\bar{\alpha})$ ($Q \subset R$ and multiplication over R is associative)

10) For unit scalar $1 \in F$ and any vector $\bar{\alpha} \in V$
$1.\bar{\alpha} = \bar{\alpha}$. This is valid $\because$ $(R \times 1 = R)$

Since $V = R$ and $F = Q$ satisfy all properties
$\therefore$ It is a vector space.

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d) $V = \mathbb{R}$ and $F = \mathbb{C}$

check for field F

- Identity element: Set of complex nos has additive identity 0 and multiplicative identity 1.
  $\therefore \forall \; a \in (F = \mathbb{C}) \quad a + 0 = a$ and $a.1 = a$
- Inverse property: Every element $a \in (F = \mathbb{C})$ has additive inverse $(-a)$ and multiplicative inverse $(1/a)$ such that
  $a + (-a) = 0$ and $a.\left(\frac{1}{a}\right) = 1$
- Commutativity: Complex numbers are commutative over addition and multiplication. i.e $\forall \; a, b \in (F = \mathbb{C})$
  $(a + b) = (b + a)$ and $(a.b) = (b.a)$
- Associativity: Complex number are associative over addition & multiplication. i.e $\forall \; a, b, c \in (F = \mathbb{C})$
  $(a+b)+c = a+(b+c)$ and $a.(b.c) = (a.b).c$
- Distributive: Complex numbers follow distributive law. i.e $\forall \; a, b, c \in (F = \mathbb{C})$
  $a.(b+c) = (a.b) + (b.c)$.

Check for Vector space V

- Closure under scalar multiplication: Consider vector $\bar{a} \in V$ and complex no. $z \in F$.
  on multiplying $\Rightarrow z\bar{a}$ belongs to complex no.
  But $V = \mathbb{R}$, so $z\bar{a}$ is not in V and hence violates closure property.
  $\therefore$ It is not a vector space.

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Ans 3) Let F be a field. We have to show 'F' F_V is vector space over F.
i.e. (F, +, ·) is field. We check Vector space properties now.

1) Let $\bar{\alpha}, \bar{\beta} \in F_V$, then $(\bar{\alpha} + \bar{\beta}) \in F_V$ because a field is closed over addition.

2) Let $\bar{\alpha}, \bar{\beta} \in F_V$, then $(\bar{\alpha} + \bar{\beta}) = (\bar{\beta} + \bar{\alpha})$ because a field follows commutativity.

3) Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in F_V$, then $(\bar{\alpha} + \bar{\beta}) + \bar{\gamma} = \bar{\alpha} + (\bar{\beta} + \bar{\gamma})$ because a field follows associativity

4) Let $\bar{\alpha} \in F_V$ then there is $\phi \in V$ such that $\alpha + \phi = \phi + \alpha$. Here $\phi = 0$ exists as additive identity.

5) For any $\bar{\alpha} \in F_V$ a unique vector $(-\bar{\alpha})$ exists ~~for~~ such that $(-\alpha) + \alpha = (\alpha) + (-\alpha) = \phi$
This is true because field has additive inverse.

6) Let $a \in F$ and $\bar{\alpha} \in F_V$ then $a\bar{\alpha} \in F_V$ this is because field is closed under multiplication

7) For any $a \in F$ and $\bar{\beta}, \bar{\alpha} \in F_V$
$a(\bar{\alpha} + \bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$. This is because a field follows distributive property

8) Let $a, b \in F$ and $\bar{\alpha} \in F_V$, then
$(a+b)\bar{\alpha} = \bar{\alpha}a + \bar{\alpha}b$
This is because field is closed over distributive law.

9) Let $a, b \in F$ and $\bar{\alpha} \in F_V$ then
$a.(b.\bar{\alpha}) = (a.b)\bar{\alpha}$
This is because field follows associativity

10) For unit scalar $1 \in F$ and $\bar{\alpha} \in F_V$
$1.\bar{\alpha} = \bar{\alpha}$. This is because field has identity property.

$\therefore$ Every field makes vector space over itself.

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Direct Sum : Direct sums of a field F with itself n times gives us $\Rightarrow F + F \cdots F$ (n times) $= F^n$

F^n consists of n-tuples as orderd pairs $(a_1, a_2 \cdots a_n)$ where $a_1, a_2 \cdots a_n \in F$

To prove that direct sums gives vector space we apply & check properties of vector space on it.

1) **Vector addition is closed :** Let $\bar{\alpha}, \bar{\beta} \in F^n$ where

$$\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \bar{\beta} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \bar{\alpha} + \bar{\beta} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \in F^n$$

This is because F is closed under addition.

2) **Vector addition is commutative :** Let $\bar{\alpha}, \bar{\beta} \in F^n$ then

$$\bar{\alpha} + \bar{\beta} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \\ \vdots \\ b_n + a_n \end{bmatrix} = \bar{\beta} + \bar{\alpha}$$

$(a_i + b_i = b_i + a_i)$ because Field is commutative.

3) **Addition is Associative :** Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in F^n$ and $\gamma = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$$\bar{\alpha} + (\bar{\beta} + \bar{\gamma}) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ \vdots \\ b_n + c_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 + c_1 \\ a_2 + b_2 + c_2 \\ \vdots \\ a_n + b_n + c_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = (\bar{\alpha} + \bar{\beta}) + \bar{\gamma}$$

4) **Additive identity :** Here $\phi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in F^n$

We can check that it is additive identity as

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad (\bar{\alpha} + \phi = \phi + \bar{\alpha} = \bar{\alpha})$$

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5) Additive Inverse: For every $\bar{\alpha} \in F^n$ there is $(-\bar{\alpha}) \in F$

$\alpha^n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then $-\alpha^n = \begin{bmatrix} -a_1 \\ \vdots \\ -a_n \end{bmatrix}$

$\bar{\alpha} + (-\bar{\alpha}) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} -a_1 \\ \vdots \\ -a_n \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 \\ \vdots \\ -a_n \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{((-\bar{\alpha}) + \bar{\alpha})} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \phi$

$\therefore \bar{\alpha} + (-\bar{\alpha}) = (-\bar{\alpha}) + \alpha = \phi, \quad \forall \bar{\alpha} \in F^n$

6) For any element $k \in F$, $\bar{\alpha} \in F^n$

$k\bar{\alpha} = \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix} \in F^n \quad \because \left(ka_i \in F \quad i \in \{1 \dots n\} \right.$ $\because$ field is closed over multiplication$\left. \right)$

$\therefore$ closed under scalar multiplication.

7) For $k \in F$ and $\bar{\alpha}, \bar{\beta} \in F^n$

$k(\bar{\alpha} + \bar{\beta}) = k \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} = \begin{bmatrix} ka_1 + kb_1 \\ \vdots \\ ka_n + kb_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix} + \begin{bmatrix} kb_1 \\ \vdots \\ kb_n \end{bmatrix}$

$= k\bar{\alpha} + k\bar{\beta}$

[Here $k(a_i + b_i) = ka_i + kb_i$
because F ~~is close~~ follows distributive property]

8) For $m, n \in F$ and $\bar{\alpha} \in F^n$

$(m+n)\bar{\alpha} = (m+n) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ma_1 + na_1 \\ \vdots \\ ma_n + na_n \end{bmatrix} = \begin{bmatrix} ma_1 \\ \vdots \\ ma_n \end{bmatrix} + \begin{bmatrix} na_1 \\ \vdots \\ na_n \end{bmatrix}$

[$(m+n)a_i = ma_i + na_i$
Because F follows distributive law] $= m\bar{\alpha} + n\bar{\alpha}$

9) For $a, b \in F$ and $\bar{\alpha} \in F^n$

$(a.b)\,\bar{\alpha} = a.b \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a.ba_1 \\ \vdots \\ a.ba_n \end{bmatrix} = a \begin{bmatrix} ba_1 \\ \vdots \\ ba_n \end{bmatrix} = a(b\bar{\alpha})$

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[$(ab)a_i = a(ba_i)$, $i \in \{1 \dots n\}$, $a_i \in F$
Because Field is closed over multiplicative distribution]

10) Here $1 \in F$ and let $\alpha \in F^n$.

$$1 . \bar{\alpha} = 1 . \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1.a_1 \\ \vdots \\ 1.a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

[$1 . a_i = a_i$ where $i \in \{1, 2 \dots n\}$ and $a_i \in F$
because 1 is multiplicative identity in F]

$\therefore 1 \in F$ is unit scalar over F.

As F^n follows all properties of vector space, therefore we can say that F^n is a vector space.

Hence direct sums of a field F will form a vector space V over F.

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Ans 4 V → set of pairs (x,y)
F → field of real nos

$$(x,y) + (x_1, y_1) = (x+x_1, 0)$$
$$c(x,y) = (cx, 0)$$

V is not a vector space.

proof: Let us assume V is vector space. ∴ there will be an identity element 'e' for it, such that
$$v + e = v \quad , \text{where } v \in V$$
Now for an arbitrary pair $(x,y) \in V$ where $x \neq 0$ and $y \neq 0$ and the identity element $e \Rightarrow (e_1, e_2) \in V$ the relation
$$(x,y) + (e_1, e_2) = (x,y)$$
should hold true.
But based on the properties given
$$(x,y) + (e_1, e_2) = (x+e_1, 0)$$
which contradicts the identity element property
∴ V is not a vector space.

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Ans 5 a) $\alpha = (a_1, a_2 --- a_n)$ in R^n and $a_1 \geq 0$

Let us take a vector $V = (c_1, c_2 --- c_n)$ and an element k in field F such that $k < 0$.

Now $\because$ vector subspace is closed under scalar multiplication

$$K.V = (kc_1, kc_2 --- kc_n)$$ should belong to subspace.

But since $c_1 > 0 \therefore$ ~~and~~ $kc_1 < 0$ which violates the $a_1 \geq 0$ condition

$\therefore$ This is not a subspace.

b) All α such that $a_1 + 3a_2 = a_3$

Let us take two vectors

$$\alpha = (a_1, a_2 --- a_n)$$
$$\beta = (b_1, b_2 --- b_n)$$

then $(\alpha + \beta) \in S$ (closed under addition)

and $c\alpha \in S$ (closed under scalar mult.)

where c is a scalar

Here $(a_1 + b_1) + 3(a_2 + b_2)$

$= (a_1 + 3a_2) + (b_1 + 3b_2)$

$= a_3 + b_3$ $\Rightarrow$ (closed under addition)

and also

$\cancel{c}$ $ca_1 + 3(ca_2$

$= c(a_1 + 3a_2) = ca_3$

$\Rightarrow$ closed under scalar multiplication.

$\therefore$ This is a subspace.

c) Similar to part (a) we take a vector $V = (c_1, c_2 --- c_n)$ and a scalar K from field such that $k < 0$

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now $K.V = (KC_1, KC_2 --- KC_n)$
Since
$a_2 = a_1^2$ $\therefore$ $a_2 \geq 0$ always
but in $K.V$ it is not the case as
KC_1 is less than 0.
$\therefore$ It violates scalar multiplication closure
and is not a subspace.

d) All α such that $a_1 . a_2 = 0$.
$\therefore$ either $a_1 = 0$ or $a_2 = 0$ or $a_1 = a_2 = 0$
Let us take two vectors
$V_x = [x_1, 0, 0 ---- 0]$
$V_y = [0, y_2, 0 ---- 0]$

where $x_1, y_2 > 0$
Now $(V_x + V_y) = [x_1, y_2, 0, 0 ---- 0]$

Here $(V_x + V_y)$ ~~stor~~ is violating the property
$a_1 . a_2 = 0$ $\therefore$ $(V_x + V_y) \notin S$. which
means that it is not closed under
addition. $\therefore$ It is not a subspace.

e) Given that a_2 is rational let us take
a vector $V = (a_1, a_2 ------ a_2)$
and a scalar K such that K is irrational
So now for subspace KV all Ka_i
should be rational.
i.e. $KV = (Ka_1, Ka_2 ---- Ka_n)$
But $\because$ K is irrational Ka_2 is irrational
Hence not in vector space.
$\therefore$ V is not closed under scalar multiplication
and is not a subspace.