## LECTURE 1: Probability models and axioms

- Sample space
- Probability laws
  - Axioms
  - Properties that follow from the axioms
- Examples
  - Discrete
  - Continuous
- Discussion
  - Countable additivity
  - Mathematical subtleties
- Interpretations of probabilities

## Sample space

- Two steps:
  - Describe possible outcomes
  - Describe beliefs about likelihood of outcomes

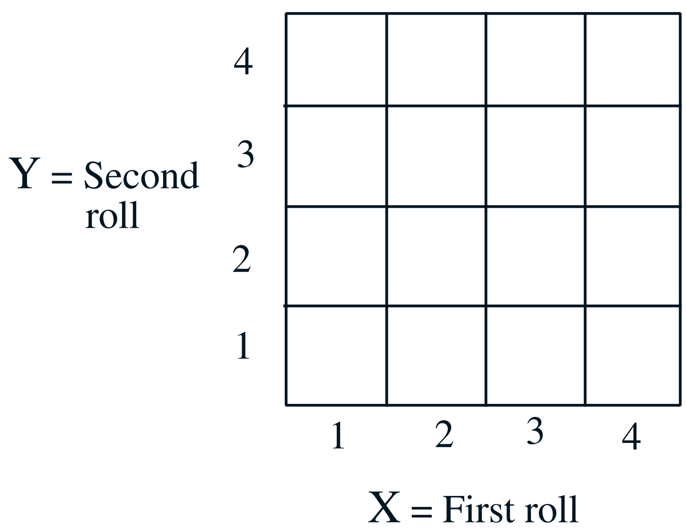
## Sample space

- List (set) of possible outcomes, $\Omega$

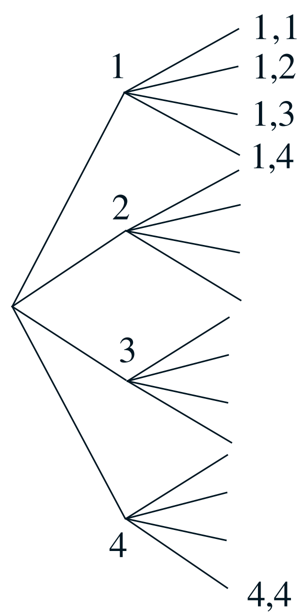
- List must be:
  - Mutually exclusive
  - Collectively exhaustive
  - At the “right” granularity

## Sample space: discrete/finite example

- Two rolls of a tetrahedral die

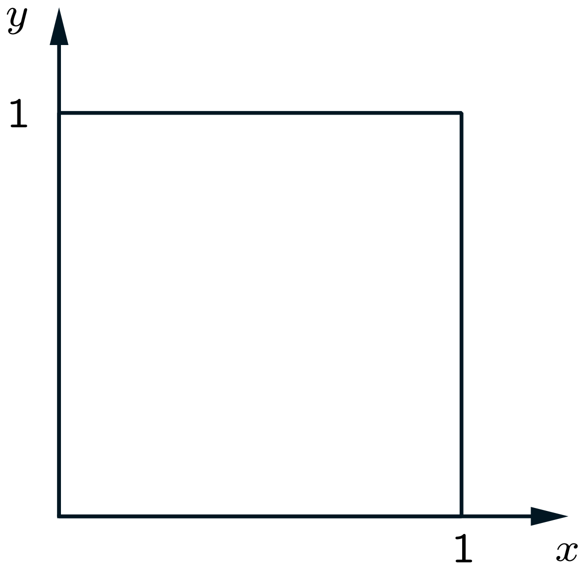



sequential description



## Sample space: continuous example

- $(x, y)$ such that $0 \leq x, y \leq 1$



## Probability axioms

- **Event:** a subset of the sample space
  - Probability is assigned to events

- **Axioms:**
  - Nonnegativity: $\mathbf{P}(A) \geq 0$
  - Normalization: $\mathbf{P}(\Omega) = 1$
  - (Finite) additivity: (to be strengthened later)
    If $A \cap B = \emptyset$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

## Some simple consequences of the axioms

**Axioms**

$\mathbf{P}(A) \geq 0$

$\mathbf{P}(\Omega) = 1$

For disjoint events:

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

**Consequences**

$\mathbf{P}(A) \leq 1$

$\mathbf{P}(\varnothing) = 0$

$\mathbf{P}(A) + \mathbf{P}(A^c) = 1$

$\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$

and similarly for $k$ disjoint events

$$\mathbf{P}(\{s_1, s_2, \ldots, s_k\}) = \mathbf{P}(\{s_1\}) + \cdots + \mathbf{P}(\{s_k\})$$

$$= \mathbf{P}(s_1) + \cdots + \mathbf{P}(s_k)$$

## Some simple consequences of the axioms

### Axioms

$\mathbf{P}(A) \geq 0$

$\mathbf{P}(\Omega) = 1$

For disjoint events:

$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

### Some simple consequences of the axioms

- $A, B, C$ disjoint: $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$

- $\mathbf{P}(\{s_1, s_2, \ldots, s_k\}) =$

## More consequences of the axioms

- If $A \subset B$, then $\mathbf{P}(A) \le \mathbf{P}(B)$

- $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$

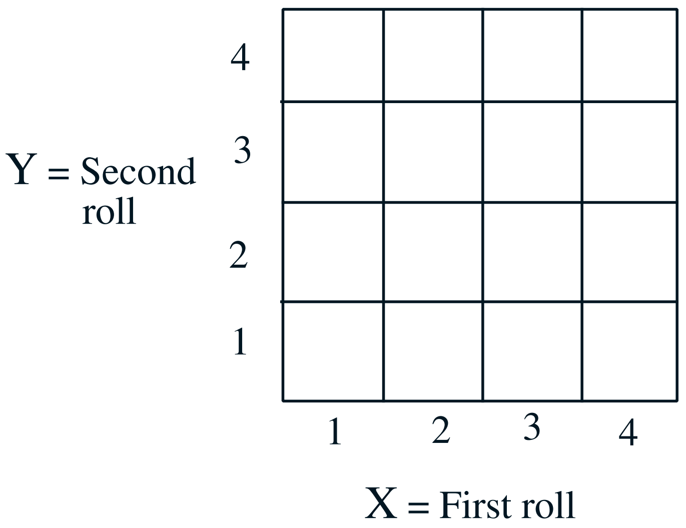
- $\mathbf{P}(A \cup B) \le \mathbf{P}(A) + \mathbf{P}(B)$

## More consequences of the axioms

- $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$

## Probability calculation: discrete/finite example

- Two rolls of a tetrahedral die




- Let every possible outcome have probability $1/16$

- $\mathbf{P}(X = 1) =$

Let $Z = \min(X, Y)$

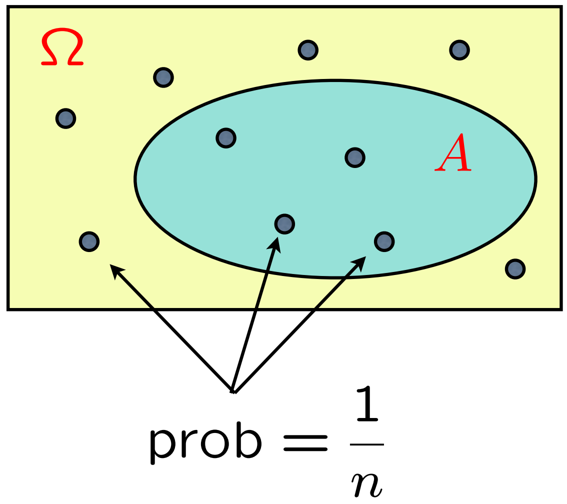
- $\mathbf{P}(Z = 4) =$

- $\mathbf{P}(Z = 2) =$

## Discrete uniform law

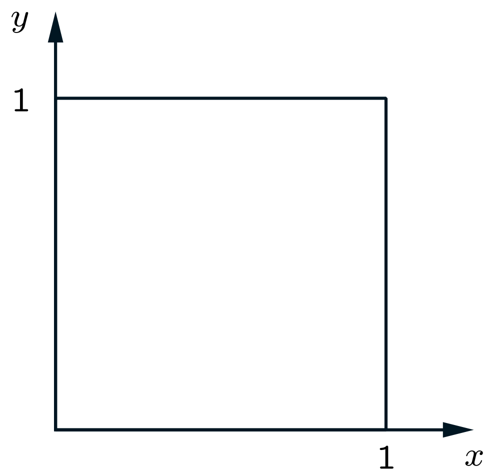
- Assume $\Omega$ consists of $n$ equally likely elements
- Assume $A$ consists of $k$ elements

$$\mathbf{P}(A) =$$



## Probability calculation: continuous example

- $(x, y)$ such that $0 \leq x, y \leq 1$




- **Uniform** probability law: Probability = Area

$$\mathbf{P}\Big(\{(x,y) \mid x + y \leq 1/2\}\Big) =$$

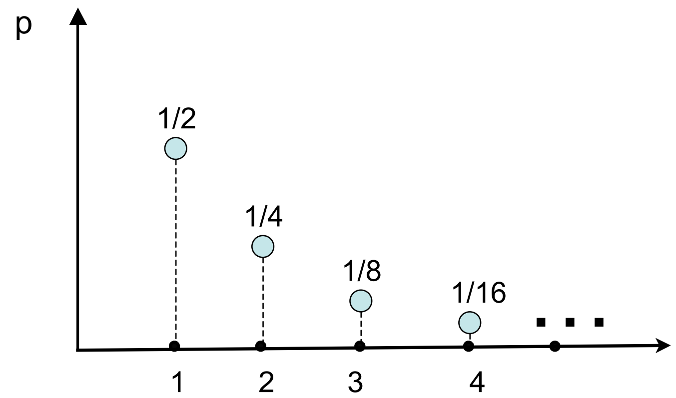
$$\mathbf{P}\Big(\{(0.5, 0.3)\}\Big) =$$

## Probability calculation steps

- Specify the sample space
- Specify a probability law
- Identify an event of interest
- Calculate...

**Probability calculation: discrete but infinite sample space**

- Sample space: $\{1, 2, \ldots\}$
  - We are given $\mathbf{P}(n) = \dfrac{1}{2^n}, \quad n = 1, 2, \ldots$




- $\mathbf{P}$(outcome is even) $=$

## Countable additivity axiom

- Strengthens the finite additivity axiom

**Countable Additivity Axiom:**
If $A_1$, $A_2$, $A_3$,... is an infinite **sequence** of disjoint events,
then $\mathbf{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) + \cdots$

## Mathematical subtleties

**Countable Additivity Axiom:**
If $A_1, A_2, A_3, \ldots$ is an infinite **sequence** of disjoint events,
then $\mathbf{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) + \cdots$

- Additivity holds only for "countable" sequences of events
- The unit square (similarly, the real line, etc.) is **not countable** (its elements cannot be arranged in a sequence)
- "Area" is a legitimate probability law on the unit square, as long as we do not try to assign probabilities/areas to "very strange" sets

## Interpretations of probability theory

- A narrow view: a branch of math
  - Axioms $\Rightarrow$ theorems

**"Thm:"** "Frequency" of event $A$ "is" $\mathbf{P}(A)$

- Are probabilities frequencies?
  - $\mathbf{P}(\text{coin toss yields heads}) = 1/2$
  - $\mathbf{P}(\text{the president of \ldots will be reelected}) = 0.7$

- Probabilities are often intepreted as:
  - Description of beliefs
  - Betting preferences

## The role of probability theory

- A framework for analyzing phenomena with uncertain outcomes
  - Rules for consistent reasoning
  - Used for predictions and decisions