#What is Linear Algebra in Data Science?

Linear Algebra is the mathematical backbone of machine learning models. Data is often represented as vectors and matrices, and algorithms like neural networks use matrix operations for computation.

- **Vectors:** An ordered list of numbers representing data points. Ex: A 3-dimensional data point → **x = [5, 3, 1]**

- **Matrices:** A rectangular array of numbers (rows × columns). Ex: Dataset with 3 samples and 2 features:
  X = [[1, 2],
  [3, 4],
  [5, 6]]

#Matrix Operations

- **Addition / Subtraction:** Only possible for matrices of the same dimensions. Ex:
  A = [[1, 2], [3, 4]] & B = [[5, 6], [7, 8]] → **A + B = [[6, 8], [10, 12]]**

- **Multiplication:** Matrix A (m×n) × Matrix B (n×p) → Result (m×p). Example:
  A = [[1, 2],
  [3, 4]] (2×2)
  B = [[5],
  [6]] (2×1)
  A × B = [[1*5 + 2*6], [3*5 + 4*6]] = [[17], [39]]

- **Transpose:** Flips rows ↔ columns. Ex:
  A = [[1, 2, 3],
  [4, 5, 6]]
  A^T = [[1, 4],
  [2, 5],
  [3, 6]]

- **Inverse (A^{-1}):** $A \times A^{-1}$ = Identity matrix I.
  Formula:

  1.
  $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
  The inverse of a matrix is found using the following formula:
  $$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$
  $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  2.
  - The inverse matrix is also found using the following equation:
  $$A^{-1} = adj(A)/det(A),$$
  where *adj(A)* refers to the adjoint of a matrix A, *det(A)* refers to the determinant of a matrix A.
  - The adjoint of a matrix A or adj(A) can be found using the following method.
  In order to find the adjoint of a matrix A first, find the cofactor matrix of a given matrix and then
  take the transpose of a cofactor matrix.
  - The cofactor of a matrix can be obtained as
  $$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

Ex: (2x2)

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$$

Solution: Let A = IA

Or $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $R_1 \rightarrow (1/2)R_1$, we have

$$\begin{bmatrix} 1 & 1/2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2 - 7R_1$

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -7/2 & 1 \end{bmatrix} A$$

Applying $R_2 \rightarrow 2R_2$, we have

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -7 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - (1/2)R_2$, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} A$$

Thus, the inverse of matrix A is given by:

$I = A^{-1} A$

Therefore,

$$A^{-1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

Ex: (3x3)

Find the inverse of a matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix}$

Solution:

Determinant of the given matrix is

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix} = 1 \cdot 33 - 2(-6) + 3(-27)$$
$$= -36$$

Let us find the minors of the given matrix as given below:

$$M_{1,1} = \det\begin{pmatrix} 5 & 6 \\ 2 & 9 \end{pmatrix} = 33$$

$$M_{1,2} = \det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -6$$

$$M_{1,3} = \det\begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} = -27$$

$$M_{2,1} = \det\begin{pmatrix} 2 & 3 \\ 2 & 9 \end{pmatrix} = 12$$

$$M_{2,2} = \det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = -12$$

$$M_{2,3} = \det\begin{pmatrix} 1 & 2 \\ 7 & 2 \end{pmatrix} = -12$$

$$M_{3,1} = \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = -3$$

$$M_{3,2} = \det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = -6$$

$$M_{3,3} = \det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}: \quad -3$$

cofactors: $\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}$

Now, find the adjoint of a matrix by taking the transpose of cofactors of the given matrix.

$$\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}^T = \begin{pmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{pmatrix}$$

Now,

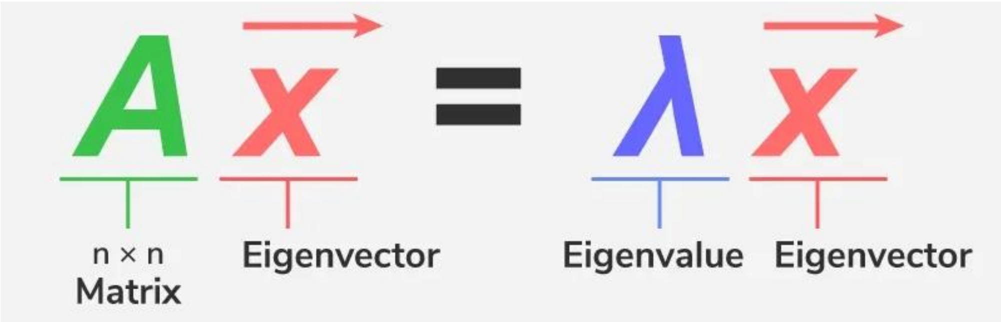
$A^{-1} = (1/|A|)$ Adj A

Hence, the inverse of the given matrix is:

$$= \begin{pmatrix} -\frac{11}{12} & \frac{1}{3} & \frac{1}{12} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{3}{4} & -\frac{1}{3} & \frac{1}{12} \end{pmatrix}$$

#Eigenvalues & Eigenvectors:

Eigenvalues and eigenvectors are fundamental concepts in linear algebra, used in various applications such as matrix diagonalization, stability analysis, and data analysis (e.g., PCA). They are associated with a square matrix and provide insights into its properties.



#Eigenvalues:

Eigenvalues are unique scalar values linked to a matrix or linear transformation. They indicate how much an eigenvector gets stretched or compressed during the transformation. The eigenvector's direction remains unchanged unless the eigenvalue is negative, in which case the direction is simply reversed.

The equation for eigenvalue is given by: $Av=\lambda v$

Where: A is the matrix, v is associated eigenvector and λ is scalar eigenvalue.

#Eigenvectors

Eigenvectors are non-zero vectors that, when multiplied by a matrix, only stretch or shrink without changing direction. The eigenvalue must be found first before the eigenvector. For any square matrix A of order n × n, the eigenvector is a column matrix of size n × 1. This is known as the right eigenvector, as matrix multiplication is not commutative.

Alternatively, the left eigenvector can be found using the eqn $vA=\lambda v$, where v is a row matrix of size 1 × n.

Example: Find the eigenvalues and the eigenvector for the matrix A = $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$

Solution:

If eigenvalues are represented using λ and the eigenvector is represented as v = $\begin{bmatrix} a \\ b \end{bmatrix}$

Then the eigenvector is calculated by using the equation,

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix} = 0$$

$(1-\lambda) \times (4-\lambda) - (2 \times 5) = 0$

$\Rightarrow 4 - \lambda - 4\lambda + \lambda^2 - 10 = 0$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow \lambda^2 - (6\lambda - \lambda) - 6 = 0$

$\Rightarrow \lambda^2 - 6\lambda + \lambda - 6 = 0$

$\Rightarrow (\lambda-6)(\lambda+1) = 0$

$\lambda = 6$ and $\lambda = -1$

- **For λ = 6**

$(A-\lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 1-6 & 2 \\ 5 & 4-6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -5 & 2 \\ 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$\Rightarrow 5a - 2b = 0$

Simplifying the above equation we get, 5a=2b

The required eigenvector is, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

- **For λ = -1**

$$\Rightarrow \begin{bmatrix} 1-(-1) & 2 \\ 5 & 4-(-1) \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$\Rightarrow 2a + 2b = 0, \Rightarrow 5a + 5b = 0$

simplifying the above equation we get, a = -b

The required eigenvector is, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then the eigenvectors of the given 2 × 2 matrix are $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

#Ex (3x3): https://metric.ma.ic.ac.uk/metric_public/matrices/eigenvalues_and_eigenvectors/eigenvalues2.html